ANNALES DE L'INSTITUT FOURIER

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*Annales de l'institut Fourier*, tome 23, n° 1 (1973), p. 135-156.

<http://www.numdam.org/item?id=AIF_1973__23_1_135_0>

Ann. Inst. Fourier, Grenoble
23,1 (1973), pp. 135-156

# DIRICHLET FORMS ON SYMMETRIC SPACES

by Christian BERG

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## 0. Introduction.

Beurling and Deny have reduced the problem of determining all Dirichlet forms on a locally compact space X endowed with a positive Radon measure $\xi$ to the question of determining contraction-semigroups of hermitian operators on $L^2(X, \xi)$, which moreover are submarkovian, cf. [3]. In order to obtain a complete solution of the last problem, it is certainly necessary to impose further structure, and for example in the case of a locally compact abelian group X with Haar measure $\xi$, the translation invariant semigroups of the above type are characterized by the so-called negative definite functions on the dual group ([3]).

In the present paper we shall extend this result to a more general setting including the symmetric spaces. The results are most satisfactory in the compact case, where we generalize results obtained for the sphere in [1]. Furthermore there is a surprising analogy between the compact case and the symmetric spaces of noncompact type of rank one : The potentials of finite energy in invariant regular Dirichlet spaces are all square integrable with respect to Riemannian measure.

## 1. Description of the scope.

Let G be a locally compact group with left Haar measure $dg$, $L^1(G)$ the Banach algebra of complex functions integrable with respect to $dg$, considered as a subalgebra of $M^1(G)$, which consists $f$ all bounded complex measures on G.

If K denotes a compact subgroup of G, $M^1(G)^\natural$ denotes the subalgebra of measures $\mu \in M^1(G)$, which are bi-invariant under K, i.e.

$$\varepsilon_k * \mu = \mu * \varepsilon_k = \mu \quad \text{for every } k \in K.$$

In general, for a subset $A \subseteq M^1(G)$ we write $A^\natural$ for the set of elements of A, which are bi-invariant under K. For functions $f$ the bi-invariance amounts to

$$f(gk) = f(kg) = f(g) \quad \text{for } g \in G,\ k \in K,$$

because the modulus $\Delta$ of G is constant 1 on K.

For any continuous function $f$ on G we define a continuous bi-invariant function $f^\natural$ on G as

$$f^\natural(g) = \int_K \int_K f(kgl)\, dk\, dl,$$

where $dk$, $dl$ denote normalized Haar measure on K.

Notice that $(f^\natural)^\vee = (\check{f})^\natural$, where $\check{f}(g) = f(g^{-1})$.

We now fix a compact subgroup K of G and will always assume the fundamental hypothesis :

$M^1(G)^\natural$ *is commutative.* (1)

(1) This implies that G *is unimodular.* In fact if $\mathcal{K}(G)$ denotes the set of continuous functions with compact support, it suffices to verify $\int f(g)\,dg = \int f(g^{-1})\,dg$ for any $f \in \mathcal{K}(G)^\natural$. We choose $\varphi \in \mathcal{K}(G)^\natural$ to be 1 on the compact set $\mathrm{supp}(f) \cup \mathrm{supp}(f)^{-1}$ and get

$$\int f(g)\,dg = f * \varphi(e) = \varphi * f(e) = \int f(g^{-1})\,dg.$$

This hypothesis is fullfilled in the following three situations :

a) (G , K) is a Riemannian symmetric pair.

b) G is abelian, K $=\{e\}$.

c) G = $\mathcal{U} \times \mathcal{U}$ where $\mathcal{U}$ is any compact group, and K is the diagonal in G.

Let now X = G/K be the homogeneous space of left cosets, $\pi : G \mapsto X$ the canonical surjection. The action of G on X is denoted $(g, x) \mapsto g \cdot x$, where $g \cdot x = \pi(gg_1)$ if $x = \pi(g_1)$. Functions on G/K are identified with right invariant functions on G. If F is a function on G and $s \in G$, we let $\lambda(s)$ F denote the function $g \mapsto F(s^{-1} g)$, which is right invariant if F is so.

On X there is a unique G-invariant measure $\xi$ fixed by the formula

$$\int_G F(g)\, dg = \int_X \left(\int_K F(gk)\, dk\right) d\xi(\pi(g)) \ . \qquad (1)$$

Our main purpose is to characterize Dirichlet forms (Q , V) on $L^2(X, \xi)$ which are *G-invariant*, that is

$$\forall s \in G \quad \forall F \in V \ (\lambda(s)\, F \in V \ , \ Q(\lambda(s)\, F) = Q(F)) \ .$$

Here V is a dense subspace of $L^2(X, \xi)$ and Q is a closed positive hermitian form on V, and we suppose that the normal contractions operate in (Q , V), cf. [3].

In § 2 we give a brief summary of the harmonic analysis of the algebra $L^1(G)^{\natural}$. Everything is more or less in the article of Godement [9]. In the next § we prove that the G-invariant semigroups involved can be viewed as convolution semigroups of bi-invariant measures. The Fourier transformation of such semigroups leads to the notion of negative definite functions defined on the set $\Omega^+$ of positive definite spherical functions.

In the abelian case b), this reduces to Schoenberg's notion of negative definite functions on an abelian group (viz. the dual group of G).

The G-invariant Dirichlet forms are in one to one correspondence with the real negative definite functions on $\Omega^+$. This main result is established in § 4.

Next we give an intrinsic characterization of the negative definite functions. The result (theorem 5.2) is not true in full generality because the function 1 can be an isolated point in $\Omega^+$.

Finally we consider the problem : When does a G-invariant Dirichlet form give rise to a Dirichlet space ? A complete answer is obtained in the compact case and for symmetric spaces of noncompact type of rank one.

## 2. Harmonic analysis on symmetric spaces.

We shall now summarize, how the Gelfand theory applies to the commutative Banach algebra $L^1(G)^\natural$, which has an involution $*$, $f^*(g) = \overline{f(g^{-1})}$, cf. [9], [11].

A (zonal) spherical function is any non-zero continuous solution $\omega : G \to C$ to the equation

$$\int_K \omega(g_1 k g_2)\, dk = \omega(g_1)\, \omega(g_2) \;,\; g_1, g_2 \in G \;. \tag{2}$$

A spherical function $\omega$ is bi-invariant and $\omega(e) = 1$. The maximal ideal space of $L^1(G)^\natural$ can be identified with the set $\Omega$ of bounded spherical functions, and the Gelfand transform of $f \in L^1(G)^\natural$ takes the form

$$\mathfrak{F}f(\omega) = \int_G f(g)\, \overline{\omega(g)}\, dg \;,\; \omega \in \Omega \;.$$

We also introduce the co-transform $\overline{\mathfrak{F}} f$ defined as

$$\overline{\mathfrak{F}} f(\omega) = \int_G f(g)\, \omega(g)\, dg \;.$$

Now let $\Omega^+$ be the set of positive definite spherical functions. Then $\Omega^+$ is a closed subset of $\Omega$, and the topology on $\Omega^+$ can be described as the topology of compact convergence over G, [4, th. 13.5.2]. This implies that $(g, \omega) \to \omega(g)$ is a continuous mapping from $G \times \Omega^+$ into C.

The main result in the whole theory is the *Godement-Plancherel theorem* :

For every positive definite measure $\mu$ on G there is a uniquely determined positive measure $\hat{\mu}$ on $\Omega^+$ such that $\mathcal{F}\varphi \in \mathcal{L}^2(\Omega^+ , \hat{\mu})$ for all $\varphi \in \mathcal{K}(G)^\natural$, and such that for all $\varphi , \psi \in \mathcal{K}(G)^\natural$.

$$\mu(\varphi * \psi^*) = \int_{\Omega^+} \overline{\mathcal{F}}\varphi(\omega) \overline{\overline{\mathcal{F}}\psi(\omega)} \, d\hat{\mu}(\omega) \ . \qquad (3)$$

(Note that $\overline{\mathcal{F}}\varphi(\omega) = \mathcal{F}\check{\varphi}(\omega)$ for $\omega \in \Omega^+$).

In particular if $\varphi$ is a continuous bi-invariant positive definite function, we obtain the *Bochner theorem* :

$$\varphi(g) = \int_{\Omega^+} \omega(g) \, d\sigma(\omega) \qquad (4)$$

where $\sigma = (\varphi dg)$ is a uniquely determined positive bounded measure on $\Omega^+$.

Furthermore, for $\mu = \varepsilon_e$ the measure $\hat{\mu}$ is called *the Plancherel measure* and is denoted $d\omega$. Much trouble is caused by the fact, that the support $\Omega_0^+$ of $d\omega$ can be a proper subset of $\Omega^+$. The formula (3) is specialized to

$$\int_G \varphi(g) \overline{\psi(g)} \, dg = \int_{\Omega_0^+} \overline{\mathcal{F}\varphi(\omega)} \ \overline{\mathcal{F}\psi(\omega)} \, d\omega \ \ , \ \varphi , \psi \in \mathcal{K}(G)^\natural, \qquad (5)$$

and $\mathcal{F}$ can be extended in a unique way to an isometry of $L^2(G)^\natural$ onto $L^2(\Omega_0^+ , d\omega)$.

For $\mu \in M^1(G)^\natural$ we define the transform $\mathcal{F}\mu : \Omega^+ \to C$ as the function

$$\mathcal{F}\mu(\omega) = \int \overline{\omega(g)} \, d\mu(g) \ .$$

This extends $\mathcal{F}$ from $L^1(G)^\natural$ to a homomorphism of the algebra $M^1(G)^\natural$ into the algebra of bounded continuous functions on $\Omega^+$.

2.1. LEMMA. – *Let* $\mu \in M^1(G)^\natural$. *If* $\mathcal{F}\mu(\omega) = 0$ *for all* $\omega \in \Omega_0^+$, *then* $\mu = 0$.

*Proof.* – If $\mu \in L^2(G)^\natural$ this is an immediate consequence of (5), and the general case is reduced to this situation by convolution with an approximative identity $(\varphi_V^\natural)$ in $L^1(G)^\natural$, which is obtained in the following way. For any neighbourhood V of the origin in G we choose a positive function $\varphi_V \in \mathcal{K}(G)$ with support in V and with integral 1. We next form the bi-invariant function $\varphi_V^\natural$ . ‖

In particular it follows that $L^1(G)^\natural$ and $M^1(G)^\natural$ are semi-simple algebras.

2.2. *The inversion* THEOREM. – *Let* $\mu \in M^1(G)^\natural$ *and suppose that* $\mathfrak{F}\mu \in \mathcal{L}^1(\Omega_0^+, d\omega)$. *Then* $\mu$ *has a continuous density* $\varphi$ *with respect to* $dg$ *and*

$$\varphi(g) = \int_{\Omega_0^+} \omega(g)\, \mathfrak{F}\mu(\omega)\, d\omega\, .$$

*Proof.* – Suppose first that $f \in L^1(G)^\natural$ is continuous and bounded, and that $\mathfrak{F}f \in \mathcal{L}^1(\Omega_0^+, d\omega)$. The function

$$\varphi(g) = \int_{\Omega_0^+} \omega(g)\, \mathfrak{F}f(\omega)\, d\omega$$

is continuous, bi-invariant and bounded. Since also $f \in L^2(G)^\natural$ (5) gives

$$\int_G \psi(g)\, \overline{\varphi(g)}\, dg = \int_{\Omega_0^+} \mathfrak{F}\psi(\omega)\, \overline{\mathfrak{F}f(\omega)}\, d\omega = \int_G \psi(g)\, \overline{f(g)}\, dg$$

for all $\psi \in \mathcal{K}(G)^\natural$, and we get $\varphi = f$.

This result applies to $\mu * \varphi_V^\natural$, where as before $(\varphi_V^\natural)$ is an approximative identity, and we get

$$\mu * \varphi_V^\natural(g) = \int_{\Omega_0^+} \omega(g)\, \mathfrak{F}\mu(\omega)\, \mathfrak{F}\varphi_V^\natural(\omega)\, d\omega\, .$$

This implies the desired result, because $\mathfrak{F}\varphi_V^\natural \mapsto 1$ uniformly over compact subsets of $\Omega^+$ as V shrinks to $e$. ‖

2.3. COROLLARY. – *Suppose that* $\mu \in M^1(G)^\natural$. *Then* $\mu$ *is positive definite if and only if* $\mathfrak{F}\mu$ *is positive on* $\Omega_0^+$.

*If this is the case, the measure* $\hat{\mu}$ *in the Godement-Plancherel theorem has* $\mathfrak{F}\mu$ *as density with respect to the Plancherel measure* $d\omega$, *i.e.* $\hat{\mu} = \mathfrak{F}\mu(\omega)\, d\omega$.

*Proof.* – Suppose first that $\mu$ is positive definite. We will show that the measure $\mathcal{F}\mu(\omega)\,d\omega$ satisfies (3).

For $\varphi, \psi \in \mathcal{K}(G)^{\natural}$, $h = \varphi * \psi^*$, the inversion theorem gives

$$h(g) = \int_{\Omega_0^+} \overline{\omega(g)}\, \mathcal{F}\check{h}(\omega)\, d\omega \,,$$

and then we have

$$\int_{\Omega_0^+} \overline{\mathcal{F}} h(\omega)\, \mathcal{F}\mu(\omega)\, d\omega = \int_{\Omega_0^+} \left(\mathcal{F}\check{h}(\omega) \int_G \overline{\omega(g)}\, d\mu(g)\right) d\omega =$$

$$= \int_G h(g)\, d\mu(g) \,,$$

which shows that $\hat{\mu} = \mathcal{F}\mu(\omega)\,d\omega$. Since $\hat{\mu}$ is a positive measure, it follows that $\mathcal{F}\mu(\omega) \geqslant 0$ for all $\omega \in \Omega_0^+$.

Suppose next that $\mathcal{F}\mu$ is positive on $\Omega_0^+$. If $\mathcal{F}\mu$ is integrable with respect to the Plancherel measure, the inversion theorem implies that

$$\mu(f * f^*) = \int_G \left(f * f^*(g) \int_{\Omega_0^+} \omega(g)\, \mathcal{F}\mu(\omega)\, d\omega\right) dg \geqslant 0 \,,$$

for all $f \in \mathcal{K}(G)$, i.e. $\mu$ is positive definite.

If as before $(\varphi_V^{\natural})$ denotes an approximative identity, we find that $\mu * \varphi_V^{\natural} * (\varphi_V^{\natural})^*$ is positive definite, because $\mathcal{F}\mu\,|\mathcal{F}\varphi_V^{\natural}|^2$ is integrable and positive on $\Omega_0^+$. If we let V shrink to $e$, we get the desired result. ‖

For each $\omega \in \Omega^+$ we have a canonically defined Hilbert space $H_\omega$ with a unit vector $e_\omega$, and a continuous irreducible representation $\pi_\omega$ of G in $H_\omega$ such that $e_\omega$ is a cyclic vector. Moreover

$$\omega(g) = (e_\omega \,, \pi_\omega(g)\, e_\omega) \,,$$

and $\pi_\omega(k)\, e_\omega = e_\omega$ for all $k \in K$, i.e. $\pi_\omega$ is of class one.

The representation $\pi_\omega$ can in the usual way be extended to a representation of $M^1(G)$.

2.4. Lemma. – *For $\mu \in M^1(G)^{\natural}$ the operator $\pi_\omega(\mu)$ is given by*

$$\pi_\omega(\mu)\, a = \mathcal{F}\mu(\omega)\,(a \,, e_\omega)\, e_\omega \;, \quad a \in H_\omega \,, \omega \in \Omega^+ \,.$$

*In particular $\pi_\omega(\mu)$ is in the trace class and $tr\,\pi_\omega(\mu) = \mathcal{F}\mu(\omega)$.*

*Proof.* – Let $g_1, g_2 \in G$, $a = \pi_\omega(g_1)e_\omega$ ; $b = \pi_\omega(g_2^{-1})e_\omega$. It suffices to show

$$(\pi_\omega(\mu)\, a\, , b) = \mathcal{F}\mu(\omega)\, (a, e_\omega)\, (e_\omega\, , b) = \mathcal{F}\mu(\omega)\, \overline{\omega(g_1)}\, \overline{\omega(g_2)}\ .$$

We have

$$(\pi_\omega(\mu)\, a\, , b) = \int_G (\pi_\omega(g)\, a\, , b)\, d\mu(g) = \int_G \overline{\omega(g_2 g g_1)}\, d\mu(g)\ ,$$

but since $\mu$ is bi-invariant, we also have

$$\int_G \overline{\omega(g_2 g g_1)}\, d\mu(g) = \int_G \bar{\omega}(g_2\, kg\, lg_1)\, d\mu(g)$$

for all $k$, $l \in K$. Thus, integrating over K with respect to $k$ and $l$, we obtain

$$(\pi_\omega(\mu)\, a\, , b) = \overline{\omega(g_1)}\, \overline{\omega(g_2)} \int_G \overline{\omega(g)}\, d\mu(g)\ .\ \|$$

2.5. THEOREM. – *For any function* $F \in L^1 \cap L^2(X\, , \xi)$ *and any* $\omega \in \Omega^+$, $\pi_\omega(F)$ *is a Hilbert-Schmidt operator in* $H_\omega$. *The function* $\omega \to \| \pi_\omega(F) \|_{H.S.}$ *is continuous and square integrable with respect to the Plancherel measure, and satisfies*

$$\int_G |\, F(g)\, |^2\, dg = \int_{\Omega_0^+} \| \pi_\omega(F) \|_{H.S.}^2\, d\omega\ . \qquad (6)$$

*The mapping* $F \to (\pi_\omega(F))_\omega$ *can be extended uniquely to an isometry of* $L^2(X\, , \xi)$ *into the Hilbert space of square integrable vector fields of Hilbert-Schmidt operators.*

*Proof.* – The function $h = F^* * F$ is bi-invariant and positive definite. The measure $\mathcal{F}h(\omega)\, d\omega$ in the Godement-Plancherel theorem is bounded, so $\mathcal{F}h$ is integrable. By the inversion theorem we then have

$$\int_G |\, F(g)\, |^2\, dg = h(e) = \int_{\Omega_0^+} \mathcal{F}h(\omega)\, d\omega\ ,$$

but since $\pi_\omega(h) = \pi_\omega(F)^*\, \pi_\omega(F)$ is in the trace class, $\pi_\omega(F)$ is a Hilbert-Schmidt operator and

$$\mathcal{F}h(\omega) = tr\, \pi_\omega(h) = \| \pi_\omega(\mathrm{F}) \|^2_{\mathrm{H.S.}} \; .$$

The extension to $L^2(X, \xi)$ is classical, cf. [8]. ‖

## 3. Characterization of G-invariant semigroups.

We shall now deal with the question of determining the strongly continuous contraction-semigroups of submarkovian operators in $L^2(X, \xi)$, which commute with the action of G in X.

3.1. LEMMA. – *In order that a bounded operator* T *in* $L^2(X, \xi)$ *satisfies*

i) $T(\lambda(s) F) = \lambda(s) T(F)$ *for all* $s \in G$, $F \in L^2(X, \xi)$.

ii) *If* $0 \leqslant F \leqslant 1$ $\xi$-a.e., *then* $0 \leqslant TF \leqslant 1$ $\xi$-a.e. (T *is submarkovian*)

*it is necessary and sufficient that there is a positive bi-invariant measure* $\mu$ *on* G *with* $\|\mu\| \leqslant 1$, *such that* $TF = F * \mu$ *for all* $F \in L^2(X, \xi)$.

*The measure* $\mu$ *is uniquely determined.*

*Proof.* – It is immediate to verify that $TF = F * \mu$ defines a bounded operator with the properties i) and ii).

For the converse we proceed as follows :

*a*) We first assume that TF is continuous for every $F \in L^2(X, \xi)$. It is easy to see that i) implies that $T(F)^\natural = T(F^\natural)$ for all $F \in \mathcal{K}(X)$. The mapping $F \mapsto T((F^\natural)^\vee)(e)$ is a positive linear form on $\mathcal{K}(G)$, so it defines a positive Radon measure $\mu$ on G, which is clearly bi-invariant and of total mass $\leqslant 1$.

For $F \in \mathcal{K}(X)$ we have

$$F * \mu(e) = \int \check{F}\, d\mu = T(F^\natural)(e) = T(F)^\natural(e) = T(F)(e) \; ,$$

and finally for any $g \in G$

$$F * \mu(g^{-1}) = [\lambda(g) F] * \mu(e) = T(\lambda(g) F)(e) = \lambda(g) T(F)(e) = \\ = T(F)(g^{-1}) \; .$$

By the density of $\mathcal{K}(X)$ in $L^2(X, \xi)$ it follows that $TF = F * \mu$ for all $F \in L^2(X, \xi)$.

b) In the general case we consider the operators

$$T_V F = TF * \varphi_V^\natural \quad , \quad F \in L^2(X, \xi) ,$$

where $(\varphi_V^\natural)$ is an approximative identity. Now $T_V$ satisfy i) and ii) and the condition of case a), so there is a positive bi-invariant measure $\mu_V$ with $\|\mu_V\| \leqslant 1$ such that $T_V F = F * \mu_V$. If $\mu$ is a vague accumulation point for the net $(\mu_V)$, it is easy to see that $TF = F * \mu$ (in $L^2(X, \xi)$) for all $F \in \mathcal{K}(X)$, and then for all $F \in L^2(X, \xi)$. ∎

If in the above correspondence $\mu$ is associated with T, the adjoint operator $T^*$ is associated with $\check{\mu}$, and if S is another operator of the same type associated with $\nu$, then ST is associated with $\mu * \nu$. This proves in particular that two bounded operators in $L^2(X, \xi)$ commute, if they satisfy the conditions of the lemma. The identity operator is associated with the normalized Haar measure $\omega_K$ of K (considered as a measure on G).

DEFINITION. – A *vaguely continuous convolution semigroup on* G *is a family* $(\mu_t)_{t \geqslant 0}$ *of positive bi-invariant measures on* G *satisfying*

i) $\mu_s * \mu_t = \mu_{s+t}$ , $s, t \geqslant 0$ , $\mu_0 = \omega_K$ .

ii) $\|\mu_t\| \leqslant 1$ .

iii) $\mu_t \to \omega_K$ vaguely as $t \to 0$.

Under these conditions we even have $\mu_t(f) \to \omega_K(f)$ for $t \to 0$ for all continuous bounded functions $f$. It is then easy to obtain the following result :

3.2. THEOREM. – *There is a one to one correspondence between strongly continuous contraction-semigroups* $(T_t)_{t \geqslant 0}$ *of operators in* $L^2(X, \xi)$ *satisfying* i) *and* ii) *of lemma 3.1, and vaguely continuous convolution semigroups* $(\mu_t)_{t \geqslant 0}$ *on* G. *The correspondence is given by*

$$T_t F = F * \mu_t \quad , \quad t \geqslant 0 \quad , \quad F \in L^2(X, \xi) .$$

DEFINITION. – *A continuous function* $p : \Omega^+ \to \mathbf{C}$ *is called positive definite, if* $p = \mathcal{F}\mu$ *for some (necessarily unique) positive measure* $\mu \in M^1(G)^\natural$.

Notice that $|p(\omega)| \leqslant p(1) = \|\mu\|$ for all $\omega \in \Omega^+$.

A continuous function $q : \Omega^+ \to \mathbf{C}$ is called *negative definite,* if $q(1) \geqslant 0$ and if $\exp(-tq)$ is positive definite for all $t > 0$.

Notice that

$$|e^{-tq(\omega)}| \leqslant e^{-tq(1)} \leqslant 1 \quad , \quad \omega \in \Omega^+ \ ,$$

for all $t \geqslant 0$. This implies that $\mathrm{Re}\, q \geqslant q(1) \geqslant 0$.

The sets $\mathcal{P}$ and $\mathcal{Q}$ of positive and negative definite functions are convex cones containing the positive constant functions. The cone $\mathcal{P}$ is even stable under multiplication.

In § 5 we are concerned with an intrinsic characterization of these cones.

3.3. THEOREM. – *There is a one to one correspondence between negative definite functions $q$ on $\Omega^+$ and vaguely continuous convolution semigroups $(\mu_t)_{t \geqslant 0}$ on* G. *The correspondence is given by*

$$\mathcal{F}\mu_t(\omega) = e^{-tq(\omega)} \quad , \quad t \geqslant 0 \quad , \quad \omega \in \Omega^+ \ . \tag{7}$$

*Proof.* – If $(\mu_t)_{t \geqslant 0}$ is given, and $\omega \in \Omega^+$ is fixed, there is a uniquely determined complex number $q(\omega)$ such that (7) is fullfilled for all $t \geqslant 0$. Since all the functions $\exp(-tq)$, $t \geqslant 0$ are continuous on the locally compact space $\Omega^+$, the next lemma shows that $q$ is continuous. Since $\exp(-tq(1)) = \|\mu_t\| \leqslant 1$ for all $t \geqslant 0$, we have $q(1) \geqslant 0$.

Conversely, if $q$ is negative definite, we have by definition a family $(\mu_t)_{t \geqslant 0}$ of positive bi-invariant measures satisfying (7). Consequently we have

$$\mathcal{F}(\mu_s * \mu_t) = \mathcal{F}(\mu_s)\,\mathcal{F}(\mu_t) = e^{-(s+t)q} = \mathcal{F}(\mu_{s+t}) \ ,$$

which shows the semigroup property. The formula

$$\mu_t(\varphi * \psi^*) = \int_{\Omega_0^+} \mathcal{F}\varphi(\omega)\, \overline{\mathcal{F}\psi(\omega)}\, e^{-tq(\omega)}\, d\omega \quad , \quad \varphi, \psi \in \mathcal{K}(G)^{\natural},$$

implies that $\mu_t(\varphi * \psi^*) \to \omega_K(\varphi * \psi^*)$ for $t \to 0$, because

$$|e^{-tq(\omega)}| = |\mathcal{F}\mu_t(\omega)| \leqslant \|\mu_t\| = e^{-tq(1)} \leqslant 1 \ .$$

This is sufficient to ensure the continuity property of the semigroup. ‖

3.4. Lemma. – *Let* $f : Y \to \mathbf{R}$ *be a real function on a compact Hausdorff space* Y. *If* $f_t : Y \to \mathbf{C}$ *given as*

$$f_t(y) = \exp(itf(y))$$

*is continuous for every* $t \in \mathbf{R}$, *then f is continuous.*

*Proof.* – Let T be the group of complex numbers with absolute value 1 and define

$$\beta : \mathbf{R} \to \mathbf{T}^{\mathbf{R}} \quad \text{as} \quad \beta(x)(t) = \exp(itx) .$$

By definition $\beta \circ f$ is continuous, so $\beta(f(Y))$ is compact. A theorem of Glicksberg [7] then shows, that $f(Y)$ is compact in R, and now it is easy to prove that $f$ is continuous. ‖

*Remarks.* – The lemma extends to $k$-spaces, in particular to locally compact spaces. On the other hand, if we put $Y = \beta(\mathbf{R})$, $f = \beta^{-1}$ we get an example which shows, that the lemma is false for topological spaces in general.

Note that a positive definite function $p = \mathcal{F}\mu$ on $\Omega^+$ is real if and only if $\mu$ is symmetric ($\mu = \check{\mu}$), and consequently a negative definite function $q$ on $\Omega^+$ is real (and then positive) if and only if the corresponding convolution semigroup $(\mu_t)_{t>0}$ consists of symmetric measures. It follows from corollary 2.3, that the measures $\mu_t$ are positive definite in this case.

## 4. Characterization of G-invariant Dirichlet forms.

4.1. Theorem. – *There is a one to one correspondence between G-invariant Dirichlet forms* (Q , V) *on* $L^2(X, \xi)$ *and real negative definite functions q on* $\Omega^+$. *The correspondence is given by*

$$Q(F) = \int_{\Omega_0^+} \| \pi_\omega(F) \|_{H.S.}^2 \, q(\omega) \, d\omega \quad \textit{for} \quad F \in V , \tag{8}$$

*and* V *is the set of functions* $F \in L^2(X, \xi)$ *for which the integral in* (8) *is finite.*

*Proof.* – By the general theorem of Beurling and Deny (cf. [3]), there is a one to one correspondence between the Dirichlet forms $(Q, V)$ on $L^2(X, \xi)$ and the strongly continuous contraction-semigroups $(T_t)_{t>0}$ of hermitian and submarkovian operators in $L^2(X, \xi)$. The correspondence is given by

$$\lim_{t \to 0} \left(\frac{1}{t}(F - T_t F), F\right) = \begin{cases} Q(F) & \text{for } F \in V \\ \infty & \text{for } F \in L^2(X, \xi)\backslash V . \end{cases}$$

The Dirichlet form $(Q, V)$ is G-invariant if and only if each of the operators $T_t$ satisfy i) of lemma 3.1. (For the "only if" part one considers the semigroup $T_t^{(s)} F = \lambda(s^{-1}) T_t(\lambda(s) F)$, $s \in G$).

The correspondence is now proved by theorem 3.2 and 3.3. In order to prove (8), we use the expressions

$$T_t F = F * \mu_t \quad , \quad \mathcal{F}\mu_t(\omega) = e^{-tq(\omega)} \tag{9}$$

For $F \in L^1 \cap L^2(X, \xi)$ we get by theorem 2.5 and lemma 2.4

$$\left(\frac{1}{t}(F - T_t F), F\right) = \left(F * \frac{1}{t}(\omega_K - \mu_t), F\right) =$$

$$= \int_{\Omega_0^+} tr\left(\pi_\omega(F)^* \pi_\omega\left(F * \frac{1}{t}(\omega_K - \mu_t)\right)\right) d\omega$$

$$= \int_{\Omega_0^+} tr\, \pi_\omega\left((F^* * F) * \frac{1}{t}(\omega_K - \mu_t)\right) d\omega =$$

$$= \int_{\Omega_0^+} \frac{1}{t}(1 - e^{-tq(\omega)}) \| \pi_\omega(F) \|_{H.S.}^2 \, d\omega .$$

By continuity, this formula holds for all $F \in L^2(X, \xi)$. When $t$ decreases to zero, $t^{-1}(1 - \exp(-tq(\omega)))$ increases to $q(\omega)$, and the proof is finished. ∎

If G is compact, $H_\omega$ is of finite dimension $N_\omega$ and can be taken to be the subspace of $L^2(X, \xi)$ spanned by the functions $\lambda(s)\,\omega$, $s \in G$. Furthermore $L^2(X, \xi)$ is the Hilbert sum of the spaces $H_\omega$, $\omega \in \Omega^+$, so $F \in L^2(X, \xi)$ has a $L^2$-expansion

$$F = \sum_{\omega \in \Omega^+} P_\omega F ,$$

where $P_\omega$ is the orthogonal projection on $H_\omega$. It turns out that

$$P_\omega F = N_\omega F * \omega \quad , \quad \| P_\omega F \|^2 = N_\omega \| \pi_\omega(F) \|^2_{H.S.} \; .$$

Formula (8) is now reduced to

$$Q(F) = \sum_{\omega \in \Omega^+} \| P_\omega F \|^2 \, q(\omega) \; , \tag{10}$$

because $\Omega^+$ is discrete, and the Plancherel measure has the mass $N_\omega$ in $\omega$. (Note that $\Omega = \Omega^+ = \Omega_0^+$). This generalizes results for the sphere [1].

In the case $G = \mathcal{U} \times \mathcal{U}$ where $\mathcal{U}$ is a compact group and K is the diagonal in G, we obtain a characterization of the Dirichlet forms on the compact group $\mathcal{U}$, which are invariant under the inner automorphisms of $\mathcal{U}$.

Finally, in the case where G is abelian, $K = \{e\}$, theorem 4.1 reduces to the theorem of Beurling and Deny [3 p. 190].

## 5. Positive and negative definite functions on $\Omega^+$.

We now introduce intrinsic definitions of positive and negative definite functions.

DEFINITION. – *A continuous function* $p : \Omega^+ \to C$ *is called a* PD-*function, if the following property holds :*

$\forall a_1 , \ldots , a_n \in C , \forall \omega_1 , \ldots , \omega_n \in \Omega^+$

$$\left( \mathrm{Re}\left( \sum_{i=1}^n a_i \omega_i \right) \geq 0 \;\text{ on }\; G \Rightarrow \mathrm{Re}\left( \sum_{i=1}^n a_i p(\omega_i) \right) \geq 0 \right) .$$

A continuous function $q : \Omega^+ \to C$ is called a ND-*function* if $q(1) \geq 0$, and if the following property holds :

$\forall a_1, \ldots, a_n \in \mathbf{C}, \forall \omega_1, \ldots, \omega_n \in \Omega^+$

$$\left(\sum_{i=1}^{n} a_i = 0, \operatorname{Re}\left(\sum_{i=1}^{n} a_i \omega_i\right) \geqslant 0 \text{ on } G \Rightarrow \operatorname{Re}\left(\sum_{i=1}^{n} a_i q(\omega_i)\right) \leqslant 0\right).$$

5.1. Theorem. – *Every positive (resp. negative) definite function on $\Omega^+$ is a* PD- *(resp.* ND-*) function.*

*Proof.* – If $p = \mathcal{F}\mu$ for $\mu \in M_+^1(G)^\natural$, and if

$$\operatorname{Re}\left(\sum_{i=1}^{n} a_i \omega_i\right) \geqslant 0 \quad \text{on} \quad G,$$

we have

$$\operatorname{Re}\left(\sum_{i=1}^{n} a_i p(\omega_i)\right) = \int \operatorname{Re}\left(\sum_{i=1}^{n} a_i \omega_i(g)\right) d\check{\mu}(g) \geqslant 0 .$$

Suppose next that $q$ is negative definite and that

$$\operatorname{Re}\left(\sum_{i=1}^{n} a_i \omega_i\right) \geqslant 0 \quad \text{on } G \quad \text{with } \sum_{i=1}^{n} a_i = 0 .$$

This implies that for all $t > 0$

$$\operatorname{Re}\left(\sum_{i=1}^{n} a_i \frac{1}{t}\left(1 - e^{-tq(\omega_i)}\right)\right) \leqslant 0 ,$$

and for $t \to 0$ we get $\operatorname{Re}\left(\sum_{i=1}^{n} a_i q(\omega_i)\right) \leqslant 0$. ‖

The converse to theorem 5.1 is not true in general, because there are symmetric spaces of rank one, for which 1 is an isolated point in $\Omega^+$, [12](1). On the other hand it is simple to check that PD- and ND-functions are usual continuous positive and negative definite functions on $\Omega^+$ in the case of G abelian, $K = \{e\}$, in the case of which $\Omega^+$ is the character group of G.

We shall now prove that the converse of theorem 5.1 is true, whenever G is compact.

(1) In these cases the function $p(\omega) = 0$ for $\omega \neq 1$, $p(1) = 1$ is a PD-function, but not a positive definite function.

5.2. THEOREM – ([1]) *If* G *is compact, then every* PD- (*resp.* ND-) *function is a positive (resp. negative) definite function.*

*Proof.* – a) The subspace E of $\mathcal{K}(G)^{\natural}$ spanned by the spherical functions $\omega \in \Omega^+$ is dense in $\mathcal{K}(G)^{\natural}$ under the uniform norm (lemma 2.1). If $p$ is given to be a PD-function, we can in a unique way extend $p$ to a linear form $L_p : E \to C$, namely

$$L_p\left(\sum_{i=1}^{n} a_i \omega_i\right) = \sum_{i=1}^{n} a_i p(\omega_i) .$$

The definition of a PD-function implies that $L_p$ is real and positive, and consequently continuous. This shows that there is a uniquely determined positive bi-invariant measure $\mu$ on G such that

$$L_p(f) = \int f d\mu \quad \text{for all } f \in E ,$$

in particular $\mathcal{F}\check{\mu}(\omega) = p(\omega)$ for all $\omega \in \Omega^+$.

b) Let $q : \Omega^+ \to C$ be a ND-function. We define the operator A in E by

$$A\left(\sum_{i=1}^{n} a_i \omega_i\right) = -\sum_{i=1}^{n} a_i q(\omega_i) \omega_i$$

and shall prove that A satisfies *the maximum principle* :

If $f = \sum_{i=1}^{n} a_i \omega_i$ satisfies $\operatorname{Re} f(g_0) = \sup_G \operatorname{Re} f \geq 0$ for some $g_0 \in G$, then $\operatorname{Re} Af(g_0) \leq 0$.

To see this put $h = \sum_{i=1}^{n} a_i \omega_i(g_0) \omega_i$, so that

$$\sup \operatorname{Re} h \geq \operatorname{Re} h(e) = \operatorname{Re} f(g_0) \geq 0 .$$

Next, note that $p_g(\omega) = \omega(g_0) \omega(g)$ is a positive definite function on $\Omega^+$ for fixed $g$, so since

([1]) A mild modification of the proof gives that the theorem is valid in the non-compact case if all the functions $\omega \in \Omega^+ \backslash \{1\}$ tend to 0 at infinity, and if 1 is not an isolated point in $\Omega^+$. These conditions are satisfied for instance for the symmetric spaces of euclidean type and the real and complex hyperbolic spaces.

$$\operatorname{Re}\Big( f(g_0)\, 1 - \sum_{i=1}^{n} a_i\, \omega_i \Big) \geqslant 0 \quad \text{on G} ,$$

we get

$$\operatorname{Re}\ \Big( f(g_0) - \sum_{i=1}^{n} a_i\, p_g(\omega_i) \Big) \geqslant 0 ,$$

that is $\operatorname{Re} h \leqslant \operatorname{Re} f(g_0)$.

This shows that $\sup \operatorname{Re} h = \operatorname{Re} h(e) \geqslant 0$, so we have

$$\operatorname{Re}\ \Big( h(e) \cdot 1 - \sum_{i=1}^{n} a_i\, \omega_i(g_0)\, \omega_i \Big) \geqslant 0 \quad \text{on G} ,$$

which implies that

$$\operatorname{Re}\ \Big( h(e)\, q(1) - \sum_{i=1}^{n} a_i\, \omega_i(g_0)\, q(\omega_i) \Big) \leqslant 0 ,$$

i.e. $\operatorname{Re} \mathrm{A} f(g_0) \leqslant - q(1) \operatorname{Re} h(e) \leqslant 0$ .

c) We next show that $(\lambda \mathrm{I} - \mathrm{A})\, \mathrm{E} = \mathrm{E}$ for $\lambda > 0$, where I is the identity operator on E.

For any $\omega \in \Omega^+$ we have $\operatorname{Re}(1 - \omega) \geqslant 0$ on G, which implies that $\operatorname{Re} q(\omega) \geqslant \operatorname{Re} q(1) = q(1) \geqslant 0$. Since

$$(\lambda \mathrm{I} - \mathrm{A}) \Big( \sum_{i=1}^{n} a_i\, \omega_i \Big) = \sum_{i=1}^{n} (\lambda + q(\omega_i))\, a_i\, \omega_i ,$$

it is clear that $(\lambda \mathrm{I} - \mathrm{A})\, \mathrm{E} = \mathrm{E}$ for $\lambda > 0$ and that

$$(\lambda \mathrm{I} - \mathrm{A})^{-1} \Big( \sum_{i=1}^{n} a_i\, \omega_i \Big) = \sum_{i=1}^{n} \frac{a_i}{\lambda + q(\omega_i)}\, \omega_i .$$

In particular $(\lambda \mathrm{I} - \mathrm{A})\, \mathrm{E}$ is dense in $\mathcal{K}(\mathrm{G})^{\natural}$ for $\lambda > 0$, and then one knows (cf. [2] or [13]), that the closure of A is the infinitesimal generator of a Feller semigroup $(\mathrm{P}_t)_{t \geqslant 0}$ on $\mathcal{K}(\mathrm{G})^{\natural}$. (One should think of $\mathcal{K}(\mathrm{G})^{\natural}$ as the space of continuous functions on the double coset space $\mathrm{K} \backslash \mathrm{G} / \mathrm{K}$).

The resolvent $(\mathrm{V}_\lambda)_{\lambda > 0}$ of the semigroup is given as $\mathrm{V}_\lambda = (\lambda \mathrm{I} - \mathrm{A})^{-1}$ on E. We therefore obtain

$$\exp(t\lambda(\lambda \mathrm{V}_\lambda - \mathrm{I})) \Big( \sum_{i=1}^{n} a_i\, \omega_i \Big) = \sum_{i=1}^{n} \exp\Big( - \frac{t\lambda\, q(\omega_i)}{\lambda + q(\omega_i)} \Big)\, a_i\, \omega_i .$$

For $\lambda$ tending to $\infty$ we get

$$P_t\left(\sum_{i=1}^{n} a_i\,\omega_i\right) = \sum_{i=1}^{n} \exp(-\,tq(\omega_i))\,a_i\,\omega_i\ ,$$

which shows that $\exp(-\,tq)$ is a PD-function for all $t > 0$. ‖

In the important case where G is a noncompact connected semisimple Lie group with finite center, and K a maximal compact subgroup of G, the negative definite functions on $\Omega^+$ can be expressed by a Levi-Khinchine formula due to Gangolli [6].

In this case we have

5.3. LEMMA. – *Let* $q : \Omega^+ \to \mathrm{R}$ *be a real negative definite function. If* $q$ *is not identical zero, then* $q(\omega) > 0$ *for all* $\omega \in \Omega_0^+$.

*Proof.* – For $\omega \in \Omega_0^+$ it is well known that $\omega(x) \to 0$ for $x \to \infty$ in G (Theorem 2 p. 585 in [10]). Consequently $\{x \in \mathrm{G} \mid \omega(x) = 1\}$ is a compact set, but just by the fact that $\omega$ is positive definite, it follows that $\{x \in \mathrm{G} \mid \omega(x) = 1\}$ is a subgroup. Thus, by the maximality of K, we conclude that $\{x \in \mathrm{G} \mid \omega(x) = 1\} = \mathrm{K}$.

If we suppose that $q(\omega) = 0$ for some $\omega \in \Omega_0^+$, we have by (7) that

$$1 = \mathcal{F}\mu_t(\omega) = \int_{\mathrm{G}} \mathrm{Re}\,\omega(x)\,d\mu_t(x) \quad \text{for all } t > 0\ ,$$

which implies that $\operatorname{supp}\mu_t \subseteq \mathrm{K}$, $\|\mu_t\| = 1$ for all $t > 0$. Since $\mu_t$ is bi-invariant, we must have $\mu_t = \omega_{\mathrm{K}}$ for all $t > 0$ i.e. $q \equiv 0$. ‖

If we furthermore suppose that X is of rank 1 the following holds :

*For* $x \in \mathrm{G}\backslash\mathrm{K}$, *the function* $\omega \mapsto \omega(x)$ *tends to zero at infinity on the locally compact space* $\Omega_0^+$.

For a proof, see f. ex. [5, th. 2]. It seems to be unknown whether the same property holds for higher rank.

This has the following consequence :

5.4. COROLLARY. – *Suppose that* X *is a symmetric space of noncompact type and of rank* 1, *and that* $q : \Omega^+ \to \mathrm{R}$ *is a real negative* *definite function not identical zero. Then there is a constant* $a > 0$ *such that* $q(\omega) \geqslant a$ *for all* $\omega \in \Omega_0^+$.

*Proof.* – Put $a = \inf\{q(\omega) \mid \omega \in \Omega_0^+\}$ and suppose that $a = 0$. We then have a sequence $\omega_n \in \Omega_0^+$ such that $\lim q(\omega_n) = 0$. Suppose that some subsequence $\omega_{n_p}$ is contained in a compact subset of $\Omega_0^+$. For a cluster point $\omega_0$ of $\omega_{n_p}$ we would then have $q(\omega_0) = 0$, and then $q$ is identical zero. Thus $\omega_n \to \infty$ in $\Omega_0^+$. By (7) we have

$$\int_G \omega_n(x)\, d\mu_t(x) = e^{-tq(\omega_n)} \quad \text{for all } t > 0 .$$

The above property of the spherical functions implies that the integral tends to $\mu_t(K)$, so we get $\mu_t(K) = 1$ for all $t > 0$. Consequently we have $\mu_t = \omega_K$ for all $t > 0$ and thus $q \equiv 0$, which is a contradiction. ‖

## 6. Characterization of G-invariant Dirichlet spaces.

Let $(Q, V)$ be a G-invariant Dirichlet form on $L^2(X, \xi)$. It is known (cf. [3]) that Q is positive definite and that the completion $\hat{V}$ of V under the norm $Q^{1/2}$ is a Dirichlet space if and only if

$$\int_0^\infty (T_t F, F)\, dt < \infty \quad \text{for all } F \in \mathcal{K}_+(X) , \tag{11}$$

$(T_t)_{t>0}$ being the corresponding semigroup.

Using the expressions (9) this condition amounts to the requirement that

$$\int_{\Omega_0^+} \| \pi_\omega(F) \|_{H.S.}^2 \, \frac{1}{q(\omega)}\, d\omega < \infty \tag{12}$$

for all $F \in \mathcal{K}_+(X)$. ($\| \pi_\omega(F) \|_{H.S.}^2$ is a function in $C_0(\Omega^+) \cap \mathcal{L}^1(\Omega_0^+, d\omega)$).

For any $\omega \in \Omega_0^+$ there is $\varphi \in \mathcal{K}_+(G)^\natural$ such that $|\mathcal{F}\varphi(\omega)|^2 > 0$, and (12) then implies that $1/q$ is integrable over some neighbourhood of $\omega$ with respect to the Plancherel measure. We have thus proved :

6.1. Lemma. – *Let* $(Q, V)$ *be a G-invariant Dirichlet form and suppose that Q is positive definite and that the completion* $\hat{V}$ *of V under* $Q^{1/2}$ *is a Dirichlet space. Then* $1/q \in \mathcal{L}^1_{loc}(\Omega_0^+, d\omega)$.

We do not know whether the converse of lemma 6.1 is valid in general, i.e. if $1/q \in \mathscr{L}^1_{\text{loc}}(\Omega_0^+, d\omega)$, is it then true that Q is positive definite and that $\hat{V}$ is a (necessarily G-invariant) Dirichlet space ?

This is known however in the abelian case (see [3]), and we shall now prove it in the compact case and when X is a symmetric space of noncompact type of rank 1.

6.2. THEOREM. – *Suppose that* G *is compact and let* (Q , V) *be a* G-*invariant Dirichlet form on* $L^2(X, \xi)$ *with associated negative definite function* $q$. *Then* $1/q \in \mathscr{L}^1_{\text{loc}}(\Omega_0^+, d\omega)$ *if and only if* $q(1) > 0$, *and in this case* Q *is positive definite. Moreover,* V *is a regular,* G-*invariant Dirichlet space under the norm* $Q^{1/2}$.

*Proof.* – Recall that $\Omega^+$ is discrete and that $q(\omega) \geqslant q(1)$ for all $\omega \in \Omega^+$, which by (10) implies that

$$Q(F) \geqslant q(1) \, \| F \|^2 \quad \text{for all } F \in V \ .$$

This shows that Q is positive definite, and since V is complete under the norm $(\| F \|^2 + Q(F))^{1/2}$, V is also complete under $Q^{1/2}$, and this proves that V is a Dirichlet space. It is obvious that V is regular, because all the functions $\lambda(s)\,\omega$, $s \in G$, $\omega \in \Omega^+$ are contained in V. ▯

6.3. THEOREM. – *Suppose that* $X = G/K$ *is a symmetric space of noncompact type of rank* 1, *and let* (Q , V) *be a* G-*invariant Dirichlet form on* $L^2(X, \xi)$ *with associated negative definite function* $q$ *not identical zero.*

*Then* Q *is positive definite and* V *is a regular* G-*invariant Dirichlet space under the norm* $Q^{1/2}$.

*Proof.* – By corollary 5.4 there is some constant $a > 0$ such that $q(\omega) \geqslant a$ for all $\omega \in \Omega_0^+$, and from (8) we then get the inequality

$$Q(F) \geqslant a \, \| F \|^2 \quad \text{for all } F \in V .$$

This proves as above that Q is positive definite and that V is a Dirichlet space under $Q^{1/2}$. The regularity of V is proved like in [3]. ▯

Let $(\mu_t)_{t \geqslant 0}$ be the vaguely continuous convolution semigroup corresponding to $q$ satisfying the conditions in one of the two theorems. We have

$$\mu_t(\varphi * \psi^*) = \int_{\Omega_0^+} \bar{\mathfrak{F}}\varphi(\omega)\, \overline{\bar{\mathfrak{F}}\psi(\omega)}\, e^{-tq(\omega)}\, d\omega$$

for all $\varphi, \psi \in \mathcal{K}(G)^\natural$, and since $1/q$ is bounded over $\Omega_0^+$, we get

$$\int_0^\infty \mu_t(\varphi * \psi^*)\, dt = \int_{\Omega_0^+} \bar{\mathfrak{F}}\varphi(\omega)\, \overline{\bar{\mathfrak{F}}\psi(\omega)}\, \frac{1}{q(\omega)}\, d\omega$$

for all $\varphi, \psi \in \mathcal{K}(G)^\natural$.

Now, since any function $f \in \mathcal{K}_+(G)$ can be dominated by a function of the form $\varphi * \psi^*$, where $\varphi, \psi \in \mathcal{K}_+(G)^\natural$, the formula

$$\nu = \int_0^\infty \mu_t\, dt$$

defines a positive definite, positive and bi-invariant measure $\nu$ on G satisfying

$$\nu(\varphi * \psi^*) = \int_{\Omega_0^+} \bar{\mathfrak{F}}\varphi(\omega)\, \overline{\bar{\mathfrak{F}}\psi(\omega)}\, \frac{1}{q(\omega)}\, d\omega \quad , \quad \varphi, \psi \in \mathcal{K}(G)^\natural .$$

The measure $\nu$ is the potential kernel of the Dirichlet space V : The potential generated by $F \in \mathcal{K}(X)$ is (represented by) the function $F * \nu$.

Since $V \subseteq L^2(X, \xi)$, we have proved that the potentials of finite energy are square integrable with respect to $\xi$ under the hypothesis of theorem 6.2 or 6.3. This in turn implies, that any function in $L^2(X, \xi)$ is a measure of finite energy.

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Manuscrit reçu le 16 mai 1972
accepté par G. Choquet

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