Orthogonal polynomials and analytic functions associated to positive definite matrices *

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Abstract

For a positive definite infinite matrix A, we study the relationship between its associated sequence of orthonormal polynomials and the asymptotic behaviour of the smallest eigenvalue of its truncation A_n of size $n \times n$. For the particular case of A being a Hankel or a Hankel block matrix, our results lead to a characterization of positive measures with finite index of determinacy and of completely indeterminate matrix moment problems, respectively.

1 Introduction

To each positive definite infinite matrix $A = (a_{n,m})_{n,m}$ can be associated an inner product defined on the linear space of polynomials $\mathbb{P}$ as follows: if $p(t) = \sum_n \alpha_n t^n, q(t) = \sum_n \beta_n t^n$ then

$$\langle p, q \rangle = (\alpha_0, \alpha_1, \cdots) \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \bar{\beta}_0 \\ \bar{\beta}_1 \\ \vdots \end{pmatrix} = \sum_{k,n} \alpha_n a_{n,k} \bar{\beta}_k.$$

By definition we have that $a_{n,k}$ are the "moments" of this inner product, that is, $a_{n,k} = \langle t^n, t^k \rangle$. We can associate to that inner product a sequence of orthonormal polynomials $(p_n)_n$, p_n with degree n, which is unique assuming the leading coefficients of p_n to be positive; we also say that $(p_n)_n$ is the sequence of orthonormal polynomials with respect to the matrix A. In all of this paper we

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consider the linear space $\mathbb{P}$ endowed with the topology generated by this inner product.
We will consider the truncated matrices A_n, $n \geq 1$, of size $n \times n$ of the matrix A. Since A is positive definite, these matrices A_n, $n \geq 1$, are also positive definite; we can then write $0 < \lambda_{1,n} \leq \lambda_{2,n} \leq \cdots \leq \lambda_{n,n}$, for the eigenvalues of A_n.
The aim of this paper is to study the relationship between the asymptotic behaviour of the smallest eigenvalue $\lambda_{1,n}$, $n \geq 0$, of the matrix A_n, $n \geq 0$, and the sequence of orthonormal polynomials $(p_n)_n$ with respect to the matrix A.
In Section 2 we prove the following characterization of the boundedness below of the smallest eigenvalues:

Theorem 1.1. *The following conditions are equivalent*

- *There exists a constant* $c > 0$ *such that* $\lambda_{1,n} \geq c > 0$, $n \in \mathbb{N}$.
- *The linear mapping* T *defined by* $T(t^n) = p_n$, $n \in \mathbb{N}$, *is bounded, that is, there exists* $C > 0$ *such that for any* $p \in \mathbb{P}$

$$\sum_n \left| \frac{p^{(n)}(0)}{n!} \right|^2 \leq C\langle p, p \rangle.$$

Moreover, if one of these properties holds then

$$\lim_n \lambda_{1,n} = \|T\|^{-2}.$$

We complete Section 2 by studying particular but important cases of Theorem 1.1:

- For Hankel matrices or, equivalently, for inner products defined by a positive measure μ on the real line –i.e. the sequence $(p_n)_n$ satisfying a three term recurrence relation–, Theorem 1.1 leads to a characterization of indeterminate measures originally proved in [BChI].
- The boundedness of the operator T in certain subspaces of $\mathbb{P}$ of finite codimension –related to the kernel of Dirac's deltas and their derivatives at points of the complex plane– also characterizes determinate measures with finite index of determinacy (see [BD1], [BD2] and [BD3] for the definition and study of the index of determinacy). We also prove that measures with finite index of determinacy equal to k have the property that the sequence of the $(k+2)$-smallest eigenvalues $(\lambda_{k+2,n})_n$ of $(A_n)_n$ is bounded below. If this property characterizes measures with finite index of determinacy remains as an open question.

The orthonormal polynomials associated to a positive definite infinite matrix A whose sequence of smallest eigenvalues is bounded below have an important convergence property

Theorem 1.2. *If there exists $c > 0$ such that $\lambda_{1,n} \geq c > 0$, $n \in \mathbb{N}$, then $(p_n(z))_n \in \ell^2$ for $|z| < 1$ and moreover*

$$\sum_{n=0}^{\infty} |p_n(z)|^2 \leq \frac{1}{c}\frac{1}{1-|z|} \quad \textit{for } |z| < 1.$$

In the case when $\sum_{n=0}^{\infty} |p_n(z)|^2$ has an L^1-extension to $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the converse of Theorem 1.2 is also true:

Theorem 1.3. *If $(p_n(z))_n \in \ell^2$ for almost all z in $\mathbb{T}$ and*

$$f\left(e^{i\theta}\right) = \sum_{n=0}^{\infty} |p_n(e^{i\theta})|^2 \in L^1(\mathbb{T})$$

then there exists $c > 0$ such that $\lambda_{1,n} \geq c > 0$, $n \in \mathbb{N}$.

As a consequence of Theorems 1.2 and 1.3 we find a characterization of complete indeterminacy for matrix weights: the smallest eigenvalue of the truncations of the corresponding Hankel block matrix is bounded below.
Using Theorem 1.2, we finally associate to each positive definite infinite matrix A, whose sequence of smallest eigenvalues is bounded below, a linear mapping from ℓ^2 to the Bergman space $A_p(\mathbb{D})$, $0 < p < 2$, of analytic functions in $\mathbb{D}$.

2 Smallest eigenvalues of positive definite matrices

We start this Section by proving Theorem 1.1:

Proof. As in the Introduction we write $\lambda_{1,n}$ for the smallest eigenvalue of the truncation A_n of size $n \times n$ of the positive definite infinite matrix $A = (a_{n,m})_{n,m}$. We now consider the linear application $T : \mathbb{P} \to \mathbb{P}$ defined by $t^n \to p_n$, where $(p_n)_n$ is the sequence of orthonormal polynomials with respect to the inner product $\langle t^n, t^m \rangle = a_{n,m}$. The norm $\|T\|$ is then the supremum of the norms $\|T_n\|$ of the restrictions $T_n : \mathbb{P}_n \to \mathbb{P}_n$, where as usual we write $\mathbb{P}_n$ for the linear space of polynomials of degree less than or equal to n.
We then have:

$$\begin{aligned} \|T_n\|^2 &= \sup\left\{\langle Tp, Tp\rangle : \langle p, p\rangle = 1, p \in \mathbb{P}_n\right\} \\ &= \sup\left\{\frac{\langle Tp, Tp\rangle}{\langle p, p\rangle} : p \neq 0, p \in \mathbb{P}_n\right\}; \end{aligned}$$

if $p = \sum_{i=0}^{n} a_i t^i$ we write $x = (a_0, \cdots, a_n) \in \mathbb{C}^{n+1}$ and then since $\langle t^k, t^m \rangle = a_{k,m}$

we have

$$\begin{aligned}\frac{1}{\|T_n\|^2} &= \inf\left\{\frac{\langle p,p\rangle}{\langle Tp,Tp\rangle} : p\neq 0, p\in\mathbb{P}_n\right\}\\ &= \inf\left\{\frac{xA_{n+1}x^*}{xx^*} : x\neq 0, x\in\mathbb{C}^{n+1}\right\}\\ &= \inf\left\{xA_{n+1}x^* : xx^*=1, x\in\mathbb{C}^{n+1}\right\} = \lambda_{1,n+1}.\end{aligned}$$

Now it is easy to finish the proof.

□

We now study some important cases of this theorem.

Remark 2.1. *Assume that $A=(a_{n,m})_{n,m}$ is a Hankel matrix: i.e. there exists a sequence $(s_n)_n$ such that $a_{n,m}=s_{n+m}$.*

The sequence $(s_n)_n$ is then the sequence of moments of a positive measure μ on the real line, and the orthonormal polynomials $(p_n)_n$ associated to A are the orthonormal polynomials with respect to μ. From Th. 1.1 of [BChI], we know that the boundedness below of the smallest eigenvalues $(\lambda_{1,n})_n$ is equivalent to the indeterminacy of the measure μ. Therefore, the boundedness of the operator T gives another characterization of indeterminate measures.

Remark 2.2. *Determinate measures with finite index of determinacy can also be characterized in terms of the boundedness of the operator T in certain subspaces of $\mathbb{P}$ of finite codimension.*

The index of determinacy of a determinate measure μ was introduced and studied by the authors in [BD1]. This index checks the determinacy under multiplication by even powers of $|t-z|$ for z a complex number, and it is defined by

$$\operatorname{ind}_z(\mu)=\sup\{k\in\mathbb{N} : |t-z|^{2k}\mu \text{ is determinate}\}.$$

Using the index of determinacy, determinate measures can be classified as follows:
If μ is constructed from an N-extremal measure ν –i.e. ν is indeterminate and the linear space of polynomials is dense in $L^2(\nu)$– by removing the mass at $k+1$ points in the support of ν, then μ is determinate with

$$\operatorname{ind}_z(\mu)=\begin{cases} k, & \text{for } z\notin \operatorname{supp}(\mu),\\ k+1, & \text{for } z\in\operatorname{supp}(\mu).\end{cases} \tag{2.1}$$

For an arbitrary determinate measure μ the index of determinacy is either infinite for every z, or finite for every z. In the latter case the index has the form (2.1), and μ is derived from an N-extremal measures by removing the mass at $k+1$ points.

Using that the index of determinacy is constant at complex numbers outside of the support of μ, we define the index of determinacy of μ by

$$\text{ind}\,(\mu) := \text{ind}\,_z(\mu), \quad z \notin \text{supp}\,(\mu).$$

We are now ready to prove that finite index of determinacy can be characterized in terms of the boundedness of the operator T in subspaces of $\mathbb{P}$ of finite codimension related to the kernel of Dirac's deltas and their derivatives at points of the complex plane.

Theorem 2.3. *Let μ be a positive measure with ind $(\mu) = N$ and consider complex numbers $z_1, \cdots, z_m$ and nonnegative integers $k_1, \cdots, k_m$, such that $\sum_{l:\mu(\{z_l\})>0} k_l + \sum_{l:\mu(\{z_l\})=0}(k_l+1) \geq N+1$. We write R for the polynomial $R(t) = \prod_{l=1}^m (t-z_l)^{k_l+1}$. Then the operator T is bounded in the subspace*

$$X = \{p(z) = R(z)q(z) : q \in \mathbb{P}\} = \bigcap_{l=1}^{m} \bigcap_{i=0}^{k_l} \ker\left(\delta_{z_l}^{(i)}\right).$$

Proof. For $p = Rq \in X$ using Leibniz's rule we get:

$$p^{(k)}(t) = \sum_{i=0}^{k} \binom{k}{i} R^{(k-i)}(t) q^{(i)}(t).$$

Putting $M = \sum_{l=1}^m (k_l+1)$, since the degree of R is just M, we have that

$$p^{(k)}(t) = \sum_{i=\max\{0,k-M\}}^{k} \binom{k}{i} R^{(k-i)}(t) q^{(i)}(t).$$

This gives
(2.2)

$$\left|\frac{p^{(k)}(0)}{k!}\right|^2 = \left|\sum_{i=\max\{0,k-M\}}^{k} \frac{R^{(k-i)}(0)}{(k-i)!}\frac{q^{(i)}(0)}{i!}\right|^2 \leq K \sum_{i=\max\{0,k-M\}}^{k} \left|\frac{q^{(i)}(0)}{i!}\right|^2,$$

since

$$\sum_{i=\max\{0,k-M\}}^{k} \left|\frac{R^{(k-i)}(0)}{(k-i)!}\right|^2 \leq \sum_{i=0}^{M} \left|\frac{R^{(i)}(0)}{i!}\right|^2 = K$$

and K does not depend neither on the polynomials p and q nor on the nonnegative integer k.
Using (2.2), we can bound the operator T_μ in the subspace $X \subset \mathbb{P}$ endowed with the L^2-norm defined by the measure μ as follows –since we are going to consider two measures μ and σ, we denote the corresponding operator T by T_μ and T_σ respectively–:

(2.3)

$$\|T_\mu(p)\|^2_{L^2(\mu)} = \sum_k \left|\frac{p^{(k)}(0)}{k!}\right|^2 \leq K\sum_k \sum_{i=\max\{0,k-M\}}^{k} \left|\frac{q^{(i)}(0)}{i!}\right|^2 \leq K(M+1)\sum_k \left|\frac{q^{(k)}(0)}{k!}\right|^2.$$

Consider now the operator U defined from X to $\mathbb{P}$ by

$$U(p) = \sum_k \frac{q^{(k)}(0)}{k!} p_k.$$

To finish the proof, it is enough to prove that U is bounded in the norm generated by the measure μ: indeed, to see that just write (2.3) in the following way

$$\begin{aligned} \|T_\mu(p)\|^2_{L^2(\mu)} &\leq K(M+1)\sum_k \left|\frac{q^{(k)}(0)}{k!}\right|^2 \\ &= K(M+1)\left\|\sum_k \frac{q^{(k)}(0)}{k!} p_k\right\|^2_{L^2(\mu)} = K(M+1)\|U(p)\|^2_{L^2(\mu)}. \end{aligned}$$

We have to consider the measure σ defined by $\sigma = |R(t)|^2\mu$. Since

$$\text{ind}\,\mu = N < \sum_{l:\mu(\{z_l\})>0} k_l + \sum_{l:\mu(\{z_l\})=0} (k_l+1),$$

it follows from Lemma 2.1, p. 132 in [BD2] that the measure σ is indeterminate. Hence, writing $(q_n)_n$ for the sequence of orthonormal polynomials with respect to σ, we have from Remark 2.1 that the operator $T_\sigma(t^n) = q_n$ is bounded in $\mathbb{P}$ endowed with the L^2-norm defined by σ.
To prove that the operator U is bounded we factorize it as $U = H \circ T_\sigma \circ D$, where $D : X \to \mathbb{P}$ is defined by $D(p) = p/R = q$ and H is the linear isometry, from $\mathbb{P}$ endowed with the L^2-norm defined by the measure σ into $\mathbb{P}$ with the L^2-norm defined by the measure μ, defined by $H(q_n) = p_n$. It is clear that D is also an isometry from X endowed with the norm defined by the measure μ into $\mathbb{P}$ with the L^2-norm defined by the measure σ. That implies that $H \circ T_\sigma \circ D$ is bounded from X into $\mathbb{P}$ both endowed with the L^2-norm generated by the measure μ. It is straightforward to see that $U = H \circ T_\sigma \circ D$.

□

The converse of Theorem 2.3 is also true

Theorem 2.4. *Let μ be a determinate measure and consider like in Theorem 2.3 complex numbers $z_1, \cdots, z_m$, nonnegative integers $k_1, \cdots, k_m$, such that $\sum_{l:\mu(\{z_l\})>0} k_l + \sum_{l:\mu(\{z_l\})=0}(k_l+1) = N+1$ and the polynomial $R(t) = \prod_{l=1}^m (t-z_l)^{k_l+1}$. If the operator T is bounded in the subspace*

$$X = \{p(z) = R(z)q(z) : q \in \mathbb{P}\} = \bigcap_{l=1}^{m}\bigcap_{i=0}^{k_l} \ker\left(\delta_{z_l}^{(i)}\right)$$

then ind $(\mu) \leq N$.

Proof. As in the proof of Theorem 2.3, we consider the measure $\sigma = |R(t)|^2\mu$. Again from Lemma 2.1, p. 132 in [BD2] we deduce that if the measure σ is indeterminate then

$$\text{ind}\,\mu < \sum_{l:\mu(\{z_l\})>0} k_l + \sum_{l:\mu(\{z_l\})=0} (k_l+1) = N+1;$$

that is ind $\mu \leq N$. We then prove that σ is indeterminate.
To do that, we find a suitable expression for $\|T_\mu p\|_{L^2(\mu)}$. Indeed, any polynomial p can be written as $p(t) = \sum_k c_k p_k = \sum_k a_k t^k$; by the definition of T_μ we have that

$$\|T_\mu p\|^2_{L^2(\mu)} = \sum_k |a_k|^2 = \int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi}. \tag{2.4}$$

Since T_μ is bounded from X to $\mathbb{P}$, there exists a constant $c > 0$ such that $\|T_\mu p\|^2_{L^2(\mu)} \leq c \int |p(t)|^2 d\mu(t)$, for any $p \in X$; hence (2.4) gives that for any $p \in X$

$$\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} \leq c \int |p(t)|^2 d\mu(t). \tag{2.5}$$

We now take a complex number $a \notin \mathbb{R}$, $a \neq z_l$, $l = 1, \cdots, m$, and $|a| < 1$. Cauchy's formula, the Cauchy-Schwarz inequality and (2.5) give for $p \in X$ that

$$\begin{aligned} |p(a)|^2 &= \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{p\left(e^{i\theta}\right)}{e^{i\theta} - a} e^{i\theta} d\theta \right|^2 \\ &\leq \int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{1}{|e^{i\theta} - a|^2} \frac{d\theta}{2\pi} \\ &\leq K \int |p(t)|^2 d\mu(t). \end{aligned}$$

Since $X = \{R(t)q(t) : q \in \mathbb{P}\}$, we then have for any $q \in \mathbb{P}$ that

$$|R(a)q(a)|^2 \leq K \int |q(t)|^2 |R(t)|^2 d\mu(t);$$

that is

$$|q(a)|^2 \leq K' \int |q(t)|^2 d\sigma(t).$$

Taking in particular $q(t) = \sum_k \overline{q_k(a)} q_k(t)$, we deduce that

$$\sum_k |q_k(a)|^2 \leq K'.$$

From this it follows that $(q_n(a))_n \in \ell^2$, and since $a \notin \mathbb{R}$ the measure σ must be indeterminate.

□

Let us notice that as a consequence of Theorem 2.4 if ind $(\mu) = N$ then T is not bounded in any subspace of the form

$$X = \bigcap_{l=1}^{m} \bigcap_{i=0}^{k_l} \ker\left(\delta_{z_l}^{(i)}\right),$$

if $\sum_{l:\mu(\{z_l\})>0} k_l + \sum_{l:\mu(\{z_l\})=0}(k_l+1) < N+1$; i.e. Theorem 2.3 is sharp.

We complete this section proving that the boundedness of the operator T in a subspace of $\mathbb{P}$ of finite codimension implies the boundedness below of certain sequences of eigenvalues associated to the matrix A:

Theorem 2.5. *Write $0 < \lambda_{1,n} \leq \lambda_{2,n} \leq \cdots \leq \lambda_{n,n}$ for the eigenvalues of the truncated matrix A_n, $n \geq 1$, of size $n \times n$ of the positive definite infinite matrix A. If there exists a subspace $X \subset \mathbb{P}$ of codimension k such that the operator T is bounded in X then the sequence of eigenvalues $(\lambda_{k+1,n})_n$ is bounded below: i.e., there exists $c > 0$ such that $\lambda_{k+1,n} \geq c > 0$.*

Proof. Since the subspace X has codimension k, we can choose k infinite sequences $y_i = (y_{i,l})_l$, $i = 1, \cdots, k$, such that

$$X = \{p(t) = \sum_l a_l t^l : \sum_l a_l y_{i,l} = 0, i = 1, \cdots, k\}.$$

We now write $T_n\big|_X$ for the restriction of T to the subspace of polynomials in X with degree less than or equal to n. As in the proof of Theorem 1.1 we have that:

$$\begin{aligned}
\frac{1}{\|T_n\big|_X\|^2} &= \inf\left\{ xA_{n+1}x^* : xx^* = 1, \sum_{l=0}^{n} x_l y_{i,l} = 0, i = 1, \cdots, k, x \in \mathbb{C}^{n+1} \right\} \\
&\leq \sup_{z_1,\cdots,z_k \in \mathbb{C}^{n+1}} \inf\left\{ xA_{n+1}x^* : xx^* = 1, \sum_{l=0}^{n} x_l z_{i,l} = 0, i = 1, \cdots, k, x \in \mathbb{C}^{n+1} \right\} \\
&= \lambda_{k+1,n+1}.
\end{aligned}$$

(The last equality is the Courant-Fischer Theorem [HJ], p. 179). And now it is easy to finish the proof.

□

As a corollary we get that, a measure with finite index of determinacy equal to N has its associated sequence of $(N+2)$-smallest eigenvalues bounded below:

Corollary 2.6. *If ind $\mu = N$ then there exists $c > 0$ such that $\lambda_{N+2,n} \geq c > 0$, $n \in \mathbb{N}$.*

Proof. Theorem 2.3 implies that the operator T is bounded in a subspace of $\mathbb{P}$ of codimension $N+1$. It is now enough to apply the previous Theorem.

□

Remark 2.7. *The converse of Corollary 2.6 remains as an open problem: let μ be a determinate positive measure such that for some $k > 1$ the sequence of eigenvalues $(\lambda_{k,n})_n$ is bounded below, does μ have finite index of determinacy? If the answer is yes, we would have a characterization of finite index of determinacy in terms of the asymptotic behaviour of $(\lambda_{k,n})_n$, $k > 1$ (See [BD3] for another characterization of finite index of determinacy).*

3 Analytic functions associated to positive definite infinite matrices

We start by proving Theorem 1.2

Proof. (of Theorem 1.2)
According to the Theorem 1.1, we have that $\|T\|^2 \leq 1/c$, so for any $p(t) = \sum_{k=0}^{n} a_k t^k$ this gives

$$c\langle Tp, Tp\rangle \leq \langle p, p\rangle,$$

but by definition of the operator T this inequality is that

$$c\sum_{k=0}^{n} |a_k|^2 \leq \langle p, p\rangle. \tag{3.1}$$

Assume now that $|z| < 1$ and $p(z) = \sum_{k=0}^{n} a_k z^k = 1$; then, applying the Cauchy-Schwarz inequality

$$1 \leq \sum_{k=0}^{n} |a_k|^2 \sum_{k=0}^{n} |z^k|^2 \leq \frac{\sum_{k=0}^{n} |a_k|^2}{1-|z|^2}.$$

Using (3.1) we get that

$$\inf\{\langle p, p\rangle : p(t) = \sum_{k=0}^{n} a_k t^k, p(z) = 1\} \geq c(1-|z|^2).$$

The left-hand side is equal to $\dfrac{1}{\sum_{k=0}^{n} |p_k(z)|^2}$ and therefore by letting $n \to \infty$ we finally have that:

$$\sum_{k=0}^{\infty} |p_k(z)|^2 \leq \frac{1}{c}\frac{1}{1-|z|^2} \leq \frac{1}{c}\frac{1}{1-|z|}.$$

□

The convergence of $\sum_{k=0}^{\infty} |p_k(z)|^2$ can not be extended, in general, further than the open unit disc. Indeed, consider the sequence of polynomials $(z^n)_n$. They are orthonormal with respect to the inner product whose Gram-matrix is the identity. The smallest eigenvalues are then constant equal to 1 and hence bounded below, but

$$\sum_{n=0}^{\infty} |p_n(z)|^2 = \sum_{n=0}^{\infty} |z|^{2n},$$

which diverges for $|z| \geq 1$.

The converse of Theorem 1.2 is not true as the following example proves: let $A = (a_{n,k})_{n,k}$ be a diagonal infinite matrix with entries:

$$a_{k,k} = \begin{cases} 2^{-n}, & k = 2^n, n \geq 0, \\ 1, & \text{otherwise.} \end{cases}$$

It is clear that the smallest eigenvalue of A_n, $n \geq 1$, tends to 0 as n tends to infinity. The orthonormal polynomials with respect to A are

$$p_k(z) = \begin{cases} \left(\sqrt{2}\right)^n z^{2^n}, & k = 2^n, n \geq 0, \\ z^k, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{k=0}^{\infty} |p_k(z)|^2 \leq \sum_{k=0}^{\infty} |z|^{2k} + \sum_{k=0}^{\infty} 2^k |z|^{2^{k+1}}. \tag{3.2}$$

But taking into account that $|z|^i \leq |z|^j$ for $i \geq j$ and that $2^{k+1} = 2^k + 2^k$ we have

$$2^k |z|^{2^{k+1}} = |z|^{2^{k+1}} + \cdots + |z|^{2^{k+1}} \leq |z|^{2^k+1} + |z|^{2^k+2} + \cdots + |z|^{2^{k+1}},$$

which together with (3.2) give that

$$\sum_{k=0}^{\infty} |p_k(z)|^2 \leq \sum_{k=0}^{\infty} |z|^{2k} + \sum_{k=0}^{\infty} |z|^k \leq \frac{2}{1-|z|}.$$

We now prove Theorem 1.3 which establishes a converse of Theorem 1.2 assuming that $\sum_{k=0}^{\infty} |p_k(z)|^2$ has an L^1-extension to $|z| = 1$. This assumption is however not necessary for the boundedness below of the sequence of smallest eigenvalues of a positive definite matrix. Indeed, taking as before the identity matrix, we have $p_n(z) = z^n$, $n \in \mathbb{N}$, and

$$\sum_{k=0}^{\infty} |p_k(z)|^2 = \sum_{k=0}^{\infty} |z|^{2k} = \frac{1}{1-|z|^2}$$

which does not have an L^1-extension to $|z| = 1$.

Proof. (of Theorem 1.3)
According to the proof of Theorem 1.1, we have that

$$\lambda_{1,n+1} = \inf \left\{ \frac{\langle p,p\rangle}{\langle Tp,Tp\rangle} : p \neq 0, p \in \mathbb{P}_n \right\}.$$

If we write $p(t) = \sum_k c_k p_k$, using (2.4) we can write:

$$\begin{aligned} \frac{1}{\lambda_{1,n+1}} &= \sup \left\{ \frac{\langle Tp,Tp\rangle}{\langle p,p\rangle} : p \neq 0, p \in \mathbb{P}_n \right\} \\ &= \sup \left\{ \frac{\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi}}{\sum_k |c_k|^2} : p \neq 0, p = \sum_k c_k p_k \in \mathbb{P}_n \right\} \\ &= \sup \left\{ \int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} : \sum_k |c_k|^2 = 1, p = \sum_k c_k p_k \in \mathbb{P}_n \right\}. \end{aligned}$$

We now consider the matrix $(\kappa_{l,j})_{l,j=0,\cdots,n}$ defined by

$$\kappa_{l,j} = \int_0^{2\pi} p_l\left(e^{i\theta}\right) \overline{p_j\left(e^{i\theta}\right)} \frac{d\theta}{2\pi}.$$

Using it, we can write

$$\begin{aligned} \frac{1}{\lambda_{1,n+1}} &= \sup \left\{ \int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} : \sum_k |c_k|^2 = 1, p = \sum_k c_k p_k \in \mathbb{P}_n \right\} \\ &= \sup \left\{ \sum_{l,j} \kappa_{l,j} c_l \bar{c}_j : \sum_{k=0}^{n} |c_k|^2 = 1 \right\}. \end{aligned}$$

But this last supremum is the largest eigenvalue of the matrix $(\kappa_{l,j})_{l,j}$; since this matrix is positive definite, this largest eigenvalue is less than the sum of all the eigenvalues, that is, the trace of $(\kappa_{l,j})_{l,j}$. Therefore:

$$\begin{aligned} \frac{1}{\lambda_{1,n+1}} &\leq \sum_{l=0}^{n} \kappa_{l,l} = \sum_{l=0}^{n} \int_0^{2\pi} \left|p_l\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} \\ &\leq \int_0^{2\pi} \sum_{l=0}^{\infty} \left|p_l\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} = \int_0^{2\pi} f\left(e^{i\theta}\right) \frac{d\theta}{2\pi} < \infty, \end{aligned}$$

and the proof is finished. □

As a consequence of Theorems 1.2 and 1.3 we get a characterization of complete indeterminacy for matrix weights in terms of the behaviour of the smallest eigenvalues of its truncated Hankel block matrices (see Theorem 1.1 of [BChI] for the analogous result for positive measures). As usual, for a matrix weight W

–i.e. a non degenerate positive definite $N\times N$ matrix of measures with moments of any order–, we define the $N \times N$ matrix of k-moments as

$$A_k = \int t^k dW(t), k = 0, 1, \cdots .$$

The corresponding Hankel block matrix is $(A_{k+n})_{k,n=0,1,\cdots}$, which is a positive definite infinite matrix. Complete indeterminacy is defined as follows; let $(P_n)_n$ be a sequence of orthonormal matrix polynomials with respect to W. These matrix polynomials satisfy a three term recurrence relation of the form:

$$tP_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + A_n^*P_{n-1}(t), \quad n \geq 0, \tag{3.3}$$

where A_n and B_n are $N \times N$ matrices such that $\det A_n \neq 0$ and $B_n = B_n^*$. The determinacy or indeterminacy of the matrix weight W is related to the deficiency indices δ^+ and δ^- of the operator J defined by the infinite N-Jacobi matrix

$$J = \begin{pmatrix} B_0 & A_1 & \theta & \theta & \cdots \\ A_1^* & B_1 & A_2 & \theta & \cdots \\ \theta & A_2^* & B_2 & A_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

on the space ℓ^2, where A_n and B_n are the coefficients which appear in the three-term recurrence relation (3.3).
The deficiency indices of a matrix weight are by definition the deficiency indices of the operator defined on the space ℓ^2 by its associated N-Jacobi matrix. In [L], Theorem 3.1 (see also [B], Theorem 2.6, p. 570) it is proved that the rank of the limit matrix $R(\lambda)$ defined by

$$R(\lambda) = \lim_{n\to\infty} \left(\sum_{k=0}^{n} P_k^*(\bar{\lambda})P_k(\lambda) \right)^{-1}$$

is constant in every half-plane $\Im\lambda > 0$ and $\Im\lambda < 0$, and it coincides with the deficiency indices of J. Thus the deficiency indices can be any natural number from 0 to N, both being equal to 0 in the determinate case and both being equal to N is the so-called completely indeterminate case.

Corollary 3.1. *If A is an infinite $N \times N$ block Hankel matrix corresponding to the matrix weight W, then W is completely indeterminate if and only if the sequence of smallest eigenvalues $(\lambda_{1,n})_n$ is bounded below.*

Proof. We need to consider the operators $R_{N,m} : \mathbb{P} \to \mathbb{P}$, $m = 0, \cdots, N-1$, defined by

$$R_{N,m}(p)(t) = \sum_n \frac{p^{(nN+m)}(0)}{(nN+m)!}t^n,$$

i.e., the operator $R_{N,m}$ takes from p just those powers with remainder m modulo N and then removes t^m and changes t^N to t (for more details see [D], p. 92

or [DV], Sect. 2). From the polynomials $R_{N,m}(p)$, $m = 0, \cdots, N-1$, we can recover the polynomial p just writing

(3.4)

$$p(t) = R_{N,0}(p)(t^N) + tR_{N,1}(p)(t^N) + t^2 R_{N,2}(p)(t^N) + \cdots + t^{N-1} R_{N,N-1}(p)(t^N).$$

Notice that

$$(3.5) \qquad R_{N,i}(t^{kN+j}) = t^k \delta_{i,j}, \quad k \in \mathbb{N}, i, j = 0, \cdots, N-1.$$

We have $A = (A_{k+n})_{k,n=0,1,\cdots}$, where A_k, $k = 0, 1, \cdots$, is the $N \times N$ matrix of k-moments of W, that is, $A_k = \int t^k dW(t)$, $k = 0, 1, \cdots$. The entry $(kN + i, nN + j)$, $k, n = 0, 1, \cdots$ and $i, j = 0, \cdots, N-1$, of the matrix A is then equal to the entry (i, j) of the block A_{k+n} of A. Taking into account (3.5), we can then write

$$\langle t^{kN+i}, t^{nN+j} \rangle = a_{kN+i,nN+j} = (A_{k+n})_{i,j} = \left(\int t^{k+n} dW(t) \right)_{i,j}$$

$$= \int (R_{N,0}(t^{kN+i}), \cdots, R_{N,i}(t^{kN+i}), \cdots, R_{N,N-1}(t^{kN+i})) dW(t) \begin{pmatrix} \overline{R_{N,0}(t^{nN+j})} \\ \vdots \\ \overline{R_{N,j}(t^{nN+j})} \\ \vdots \\ \overline{R_{N,N-1}(t^{nN+j})} \end{pmatrix}.$$

From this equation we deduce the following expression for the inner product associated to A:

$$(3.6) \qquad \langle p, q \rangle = \int (R_{N,0}(p), \cdots, R_{N,N-1}(p)) dW(t) \begin{pmatrix} \overline{R_{N,0}(q)} \\ \vdots \\ \overline{R_{N,N-1}(q)} \end{pmatrix}.$$

This shows that if $(p_n)_n$ is the sequence of orthonormal polynomials with respect to A then the sequence of $N \times N$ matrix polynomials $(P_n)_n$ defined by

$$(3.7) \qquad P_n = \begin{pmatrix} R_{N,0}(p_{nN}) & \cdots & R_{N,N-1}(p_{nN}) \\ R_{N,0}(p_{nN+1}) & \cdots & R_{N,N-1}(p_{nN+1}) \\ \vdots & \ddots & \vdots \\ R_{N,0}(p_{nN+N-1}) & \cdots & R_{N,N-1}(p_{nN+N-1}) \end{pmatrix}$$

are orthonormal with respect to the matrix weight W.

To prove the Theorem we use the following characterization of completely indeterminate matrix weights: the matrix weight W is completely indeterminate if and only if for some $z \in \mathbb{C} \setminus \mathbb{R}$ the series $\sum_n (P_n(z))^* P_n(z)$ converges (see [L], Sect. 2). In the affirmative case the series converges for all $z \in \mathbb{C}$, even uniformly on compact subsets of $\mathbb{C}$.

Since the matrices $(P_n(z))^*P_n(z)$, $n \in \mathbb{N}$, are positive definite, it follows that $\sum_n (P_n(z))^*P_n(z)$ converges if and only if the entries on its diagonal are convergent sequences. Since the entries on its diagonal are (see (3.7))

$$\sum_n \Big(|R_{N,k}(p_{nN})(z)|^2 + \cdots + |R_{N,k}(p_{nN+N-1})(z)|^2 \Big),$$

$k = 0, \cdots, N-1$, we deduce that

$$\begin{array}{c} \sum_n (P_n(z))^*P_n(z) \text{ converges if and only if} \\ \text{for } k = 0, \cdots, N-1, \ \sum_n |R_{N,k}(p_n)(z)|^2 \text{ converges.} \end{array} \tag{3.8}$$

If we assume that W is completely indeterminate, we then have for $z \in \mathbb{C}$ and $k = 0, \cdots, N-1$, that $\sum_n |R_{N,k}(p_n)(z)|^2$ converges; from (3.4) we get that also for $z \in \mathbb{C}$, $\sum_n |p_n(z)|^2$ converges. But this is then a continuous function and hence

$$f\left(e^{i\theta}\right) = \sum_n |p_n(e^{i\theta})|^2 \in L^1(\mathbb{T}).$$

From Theorem 1.3 we then deduce that the sequence of smallest eigenvalues $(\lambda_{1,n})_n$ is bounded below.
Conversely, if $(\lambda_{1,n})_n$ is bounded below, we deduce from Theorem 1.2 that $\sum_n |p_n(z)|^2$ converges for $|z| < 1$. We now take $z = i/2$ and denote by z_k, $k = 0, \cdots, N-1$, the N different N-th roots of $i/2$. We then have that $(p_n(z_k))_n \in \ell^2$ for $k = 0, \cdots, N-1$. But (3.4) then gives for $k = 0, \cdots, N-1$:

$$\begin{aligned} p(z_k) &= R_{N,0}(p)(z_k^N) + z_k R_{N,1}(p)(z_k^N) + z_k^2 R_{N,2}(p)(z_k^N) + \cdots + z_k^{N-1} R_{N,N-1}(p)(z_k^N) \\ &= R_{N,0}(p)(i/2) + z_k R_{N,1}(p)(i/2) + z_k^2 R_{N,2}(p)(i/2) + \cdots + z_k^{N-1} R_{N,N-1}(p)(i/2). \end{aligned}$$

Since the matrix $B = (z_k^j)_{k,j=0,\cdots,N-1}$ is nonsingular and $(p_n(z_k))_n \in \ell^2$ for $k = 0, \cdots, N-1$, we can conclude that also $(R_{N,m}(p_n)(i/2))_n \in \ell^2$ for $m = 0, \cdots, N-1$. But, according to (3.8) this implies that $\sum_n (P_n(i/2))^*P_n(i/2)$ converges, and therefore that the matrix weight W is completely indeterminate. □

Corollary 3.1 has the following interesting consequence concerning again measures with finite index of determinacy. Each positive measure μ can be considered as a matrix weight as follows: for $N \in \mathbb{N}$ we define the matrix of measures $W_\mu = (\mu_{i,j})_{i,j=0}^{N-1}$ by $\mu_{i,j} = \psi_N(t^{i+j}\mu)$, the image measure of $t^{i+j}\mu$ under $\psi_N(t) = t^N$, i.e.

$$\mu_{i,j}(A) = \int_{\psi_N^{-1}(A)} t^{i+j} d\mu(t), \quad A \in \mathbb{B},$$

the Borel subsets of $\mathbb{R}$. It is easy to see that W_μ is a matrix weight. A straightforward computation now shows that

$$\langle p, q\rangle_{W_\mu} = \int (R_{N,0}(p), \cdots, R_{N,N-1}(p)) dW_\mu(t) \begin{pmatrix} \overline{R_{N,0}(q)} \\ \vdots \\ \overline{R_{N,N-1}(q)} \end{pmatrix} = \int p(t)\overline{q(t)} d\mu(t),$$

that is, the inner product associated to the matrix weight W_μ using the operators $R_{N,m}$, $m = 0, \cdots, N-1$, coincides with the usual inner product defined by the positive measure μ.
Positive measures with finite index of determinacy have the following surprising property: *if* $\operatorname{ind}(\mu) = k$ *then for* $N = 1, 2, \cdots, k+1$ *the matrix weight* W_μ *is determinate, but for* $N \geq k+2$ *the matrix weight* W_μ *is indeterminate* (Th. 2, p. 525 of [BD3]). Using Corollary 3.1, we can say more about this property:

Corollary 3.2. *If* $\operatorname{ind}(\mu) = k$*, then for* $N \geq k+2$ *the matrix weight* W_μ *is indeterminate but never completely indeterminate.*

Proof. It is straightforward to see that μ and W_μ have the same Hankel matrices. Since μ is determinate we have that the sequence $(\lambda_{1,n})_n$ tends to 0 as n tends to infinity, but from Corollary 3.1 follows that W_μ is not completely indeterminate. □

Using Theorem 1.2 a linear mapping from ℓ^2 to the space of analytic functions in $\mathbb{D}$ can be associated to any positive definite infinite matrix A with smallest eigenvalues bounded below.
Indeed, if $(p_n)_n$ is the sequence of orthonormal polynomials with respect to A then, we define $R : \ell^2 \to H(\mathbb{D})$ by $R((a_n)_n) = \sum_n a_n p_n(z)$. According to Theorem 1.2 the function $\sum_n a_n p_n(z)$ defines an analytic function in $\mathbb{D}$ satisfying:

$$\left|\sum_n a_n p_n(z)\right| \leq \left(\sum_n |a_n|^2\right)^{1/2} \left(\sum_n |p_n(z)|^2\right)^{1/2} \leq \frac{K}{\sqrt{1-|z|}}.$$

Each Fourier series $\sum_n a_n p_n$, $(a_n)_n \in \ell^2$, is then realized as an holomorphic function in the Bergman space $A_p(\mathbb{D})$, $0 < p < 2$:

$$\int_{\mathbb{D}} \left|\sum_n a_n p_n(z)\right|^p dm_2(z) \leq \int_0^1 \int_0^{2\pi} \frac{K^p}{(1-r)^{p/2}} r dr d\theta = 2\pi K^p \int_0^1 \frac{r}{(1-r)^{p/2}} dr < \infty.$$

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