Some transformations of Hausdorff moment sequences and harmonic numbers *

Christian Berg † and Antonio J. Durán ‡

† Institut for Matematiske Fag. Københavns Universitet

Universitetsparken 5; DK-2100 København ø, Denmark. berg@math.ku.dk

‡ Departamento de Análisis Matemático. Universidad de Sevilla

Apdo (P. O. BOX) 1160. 41080 Sevilla. Spain. duran@us.es

Abstract

We introduce some non-linear transformations from the set of Hausdorff moment sequences into itself; among them is the one defined by the formula: $T((a_n)_n) = 1/(a_0 + \cdots + a_n)$. We give some examples of Hausdorff moment sequences arising from the transformations and provide the corresponding measures: one of these sequences is the reciprocal of the harmonic numbers $(1 + 1/2 + \cdots + 1/(n+1))^{-1}$.

1 Introduction

F. Hausdorff considered in 1923 ([**H**]) the moment sequences for which the measure is concentrated on the unit interval $[0, 1]$ and characterized them by complete monotonicity. For a Hausdorff moment sequence $a_n = \int_0^1 x^n d\mu(x)$ of a probability measure μ on $[0, 1]$ we have that $a_0 = 1$ and $a_n < 1$, $n \geq 1$, unless $\mu = \delta_1$.

Moment sequences of measures supported on $[-1, 1]$ are characterized by boundedness and the positive semidefiniteness of the corresponding Hankel matrices. The main result of this paper is the following transformation from Hausdorff moment sequences to Hausdorff moment sequences:

Theorem 1.1. *Let $(a_n)_n$ be a Hausdorff moment sequence of a measure $\mu \neq 0$. Then the sequence $(b_n)_n$ defined by $b_n = 1/(a_0 + \cdots + a_n)$ is again a Hausdorff moment sequence, and its associated measure $\nu = T(\mu)$ has the properties*

*This work was done during the visit of the second author to the University of Copenhagen partially supported by D.G.E.S, ref. BFM2000-206-C04-02, FQM-262 (*Junta de Andalucía*), and by grant 51-00-0306 from the Danish Natural Science Research Council.

$\nu(\{0\}) = 0$ *and*

$$\int_0^1 \frac{1-t^{z+1}}{1-t} d\mu(t) \int_0^1 t^z d\nu(t) = 1, \quad \textit{for } \Re z \geq 0. \tag{1.1}$$

It is clear from Theorem 1.1, that if μ is a probability then so is ν, and in this way we get a transformation of the convex set of normalized Hausdorff moment sequences as well as a transformation of the set of probabilities on $[0,1]$. It is easy to see that $T(\delta_0) = \delta_1$, $T(\delta_1) = \chi_{[0,1]}(t)dt$, i.e. the Lebesgue measure on $[0,1]$. The transformation has a unique fix point $(f_n)_n$, determined by the recursive equation

$$f_0 = 1, \quad (1 + f_1 + \cdots + f_n)f_n = 1, \quad n \geq 1,$$

giving

$$f_1 = \frac{-1+\sqrt{5}}{2}, \quad f_2 = \frac{\sqrt{22+2\sqrt{5}} - \sqrt{5} - 1}{4}, \cdots$$

We hope to return to this fix point in a later paper.
The transformation $(a_n)_n \to (b_n)_n$ of Theorem 1.1 is clearly one-to-one and a sequence $(c_n)_n$ belongs to the image if and only if $c_n > 0$ for all n and $(a_n)_n$ defined by

$$a_0 = \frac{1}{c_0}, \quad a_n = \frac{1}{c_n} - \frac{1}{c_{n-1}}, n \geq 1,$$

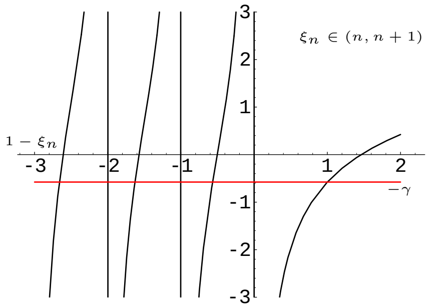
is a Hausdorff moment sequence.
The result in Theorem 1.1 is not true for measures supported on $[-1,1]$ since $\mu = (\delta_{-1} + \delta_1)/2$ is a counterexample. It can however be extended to $[-1,1]$ as follows —note that although the odd moments of the measure μ can be negative, the sums $a_0 + \cdots + a_n$ are always non-negative because they can be written as $a_0 + \cdots + a_n = \int_{-1}^1 (1 + \cdots + t^n) d\mu(t) \geq 0$; moreover, they are always positive unless $\mu = c\delta_{-1}$—:

Theorem 1.2. *Let $(a_n)_n$ be a moment sequence of a measure on $[-1,1]$. Then for any positive real number r the sequence $(b_n)_n$ defined by $b_0 = 1/r$, $b_n = 1/(r + a_0 + \cdots + a_{n-1})$ is again a moment sequence of a measure on $[-1,1]$. If $(a_n)_n$ is a Hausdorff moment sequence so is $(b_n)_n$.*

The first part of Theorem 1.1 is a consequence of Theorem 1.2: just by putting $c_n = b_{n+1} = 1/(r + a_0 + \cdots + a_n)$, $n \geq 0$, which is again a Hausdorff moment sequence if $(a_n)_n$ is so, and taking limit as r tends to 0^+, the assertion follows.
We prove these theorems in Section 2. Two different proofs are given for Theorem 1.2 —hence for Theorem 1.1— and one more for Theorem 1.1; all these proofs use the fact that analytic functions with positive Taylor coefficients preserve bounded moment sequences (see Lemma 2.1 below). Section 3 is devoted to some examples, the more interesting of which is the reciprocal of the harmonic numbers. We recall that the harmonic numbers $\mathcal{H}_n$ are the partial sums of the harmonic series, i.e.,

$$\mathcal{H}_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \quad n \geq 1,$$

cf. [**GKP**]. Take the Lebesgue measure $d\mu = \chi_{[0,1]}dx$ which gives the moments $a_n = 1/(n+1)$. According to Theorem 1.1 the sequence of the reciprocal of the harmonic numbers $1/\mathcal{H}_{n+1}$, $n \geq 0$, is again a Hausdorff moment sequence. To describe the corresponding measure ν on $[0,1]$ we consider the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$, i.e. the logarithmic derivative of the Gamma function. Then there is a sequence of numbers $(\xi_n)_{n\geq 0}$, each of them unique assuming $\xi_n \in (n, n+1)$, $n \geq 1$, $\xi_0 = 0$, satisfying that $\psi(1-\xi_n) = -\gamma$, where as usual γ stands for Euler's constant.



Write now $\alpha_n = 1/\psi'(1-\xi_n)$, $n \geq 0$, which is a sequence of positive numbers. Then $(1/\mathcal{H}_{n+1})_n$ is the moment sequence of the measure

$$d\nu = \sum_{n=0}^{\infty} \alpha_n x^{\xi_n} \chi_{(0,1]}(x)dx. \tag{1.2}$$

We also consider the one-parameter extension

$$\mathcal{H}_{n,c} = \sum_{k=1}^{n} \frac{1}{k^c}, \quad c > 0$$

of the harmonic numbers and study the representing measure of the Hausdorff moment sequence $(1/\mathcal{H}_{n+1,c})_n$, as well as the one-parameter extension

$$\left(\sum_{k=0}^{n} \frac{1}{k+a}\right)^{-1}, \quad a > 0.$$

In section 4, we relate Theorem 1.1 to the theory of convolution semigroups on the half-line or equivalently to Lévy processes. Roughly speaking one can say that T transforms the Lévy measure into the potential kernel, which also permits a characterization of the image set $T(\mathcal{M}([0,1]) \setminus \{0\})$ of the non-zero measures under T.

2 Proofs

We need to consider the Mellin transform of a measure μ with support contained in $[0,1]$: it is the function defined by

$$\mathcal{M}(\mu)(z) = \int_0^1 t^z d\mu(t), \Re z > 0.$$

For $\Re z > 0$ and $t > 0$ we define $t^z = e^{z \log t}$, and since $\lim_{t\to 0^+} t^z = 0$, we consider $t \to t^z$ as a continuous function on $[0,1]$ with the value 0 for $t = 0$. Clearly $|t^z| < 1$ for $t \in [0,1)$. The function $\mathcal{M}(\mu)$ is holomorphic in $\Re z > 0$. Suppose μ has a mass $a > 0$ at $t = 0$ and decompose $\mu = a\delta_0 + \tilde{\mu}$ with $\tilde{\mu}(\{0\}) = 0$. Then $\mathcal{M}(\mu)(z) = \mathcal{M}(\tilde{\mu})(z)$ for $\Re z > 0$ and $\lim_{x\to 0^+} \mathcal{M}(\mu)(x) = \tilde{\mu}([0,1])$. Furthermore $\mathcal{M}(\mu)$ has a continuous extension to the closed halfplane $\Re z \geq 0$ given by

$$\lim_{z\to iy;\Re z>0} \mathcal{M}(\mu)(z) = \int_0^1 e^{iy\log t} d\tilde{\mu}(t), \quad y \in \mathbb{R}.$$

All the proofs we are going to give of Theorems 1.1 and 1.2 use the following Lemma:

Lemma 2.1. *Let μ be a measure on $[-1,1]$, write $a_n = \int_{-1}^1 x^n d\mu(x)$, $n \geq 0$, and take $M > 0$ so that $|a_n| < M$ (it always exists). Let $f(z) = \sum_k b_k z^k$ be an analytic function on the disc $\{z \in \mathbb{C} : |z| < M\}$ with $b_k \geq 0$, $k \geq 0$. Then the sequence $c_n = f(a_n)$, $n \geq 0$, is again a moment sequence of some measure ν supported in $[-1,1]$. If $(a_n)_n$ is a Hausdorff moment sequence so is $(f(a_n))_n$ and the Mellin transforms of the associated measures satisfy $\mathcal{M}(\nu)(z) = f(\mathcal{M}(\mu))(z)$ for $\Re z > 0$.*

Note that if μ is a probability on $[-1,1]$ different from $\alpha\delta_1 + (1-\alpha)\delta_{-1}$, $0 \leq \alpha \leq 1$, then $|a_n| < 1$ for all $n \geq 1$.
This Lemma is already known: it is a consequence of the positive definiteness of the Schur product of positive definite matrices (see [**BCR**], Cor. 1.14, p. 70). We include here a new constructive proof using multiplicative convolution of measures on $[-1,1]$ defined by

$$\int_{-1}^1 f(x) d(\mu \diamond \nu)(x) = \int_{-1}^1 \int_{-1}^1 f(xy) d\mu(x) d\nu(y).$$

The n'th moment of $\mu \diamond \nu$ is then the product of the n'th moments of μ and ν.

Proof. We have that $f(a_n) = \sum_k b_k a_n^k$; then, by writing $\mu^{\diamond k} = \mu \diamond \cdots \diamond \mu$, ($k$ factors), $\mu^{\diamond 0} = \delta_1$, it follows straightforwardly that the measure

$$\nu = \sum_k b_k \mu^{\diamond k}$$

has the sequence $(f(a_n))_n$ as its sequence of moments. Both $f(a_n)$ and ν are well-defined because $f(z) = \sum_k b_k z^k$ converges for $|z| < M$, $|a_n| < M$, and $\|\mu^{\diamond k}\| = \|\mu\|^k$, with $\|\mu\| = a_0 < M$.

The result concerning the Mellin transforms follows easily from the previous expression taking into account that $\mathcal{M}(\mu \diamond \sigma) = \mathcal{M}(\mu)\mathcal{M}(\sigma)$.

$\square$

Corollary 2.2. *Let $(a_n)_n$ be a moment sequence of a measure on $[-1, 1]$. Then*

- *If $|a_n| < 1$, $n \geq 0$, then $(1/(1 - a_n))_n$ is also a moment sequence of a measure on $[-1, 1]$.*
- *For any positive number $t > 0$, the sequence $(e^{ta_n})_n$ is also a moment sequence of a measure on $[-1, 1]$.*

Both sequences are Hausdorff moment sequences if $(a_n)_n$ is so.

Proof of Theorem 1.1
We distinguish two cases.
(1) Suppose that $m = \int_0^1 (1 - t)^{-1} d\mu(t) < \infty$ (in particular $\mu(\{1\}) = 0$).
For $\Re z \geq 0$, we have

$$F(z) = \int_0^1 \frac{1 - t^{z+1}}{1 - t} d\mu(t) = \int_0^1 \frac{d\mu(t)}{1 - t} - \int_0^1 t^z \frac{t}{1 - t} d\mu(t),$$

and since $d\sigma(t) = t/(1 - t)d\mu(t)$ has no mass at $t = 0$, $F(z)$ is a bounded continuous function in $\Re z \geq 0$, holomorphic in $\Re z > 0$.
Note that

$$\left| \int_0^1 t^z \frac{t}{1 - t} d\mu(t) \right| \leq \int_0^1 \frac{t}{1 - t} d\mu(t) < \int_0^1 \frac{d\mu(t)}{1 - t},$$

which shows that $F(z) \neq 0$ for $\Re z \geq 0$. In particular σ has total mass $\sigma([0, 1]) < m$, so the measure

$$\nu = \sum_{k=0}^{\infty} \frac{1}{m^{k+1}} \sigma^{\diamond k} \tag{2.1}$$

is well-defined and is concentrated on $(0, 1]$.
We find

$$\mathcal{M}(\nu)(z) = \sum_{k=0}^{\infty} \frac{1}{m^{k+1}} \left(\mathcal{M}(\sigma)(z)\right)^k = \frac{1}{F(z)}, \Re z \geq 0,$$

which shows (1.1) and in particular for $z = n \in \mathbb{N}_0$,

$$b_n = \frac{1}{F(n)} = \int_0^1 t^n d\nu(t), \quad \mu([0, 1])\nu([0, 1]) = 1.$$

(2) Suppose that $\int_0^1 (1 - t)^{-1} d\mu(t) = \infty$.
For $0 < c < 1$, we define $\mu_c = \chi_{[0,c]}\mu + \mu(\{1\})\delta_c$, and for certain $c_0 < 1$ close to 1, we have $\mu_{c_0} \neq 0$. For $c_0 \leq c < 1$, we note that μ_c satisfies the conditions of

the previous part above, and $\mu_c \to \mu$ weakly for $c \to 1$. By the previous part, there exists a measure ν_c on $(0,1]$ such that

$$\int_0^1 \frac{1-t^{z+1}}{1-t} d\mu_c(t) \int_0^1 t^z d\nu_c(t) = 1, \quad \Re z \geq 0, c_0 \leq c < 1; \tag{2.2}$$

in particular $\mu_c([0,1])\nu_c([0,1]) = 1$. Since $\nu_c([0,1]) \leq 1/\mu_{c_0}([0,1])$, $c_0 \leq c < 1$, there exists a measure ν on $[0,1]$ and a sequence $c_j \to 1^-$ such that $\nu_{c_j} \to \nu$ weakly. Using that $t \to t^z$ is continuous on $[0,1]$ for $\Re z > 0$ and that $t \to (1-t^{z+1})/(1-t)$ is continuous on $[0,1]$ for $\Re z > -1$, we get from (2.2) that

$$\int_0^1 \frac{1-t^{z+1}}{1-t} d\mu(t) \int_0^1 t^z d\nu(t) = 1, \quad \Re z > 0. \tag{2.3}$$

Defining $\nu = a\delta_0 + \tilde{\nu}$ with $a = \nu(\{0\})$ and letting $z \to 0^+$, we get from (2.3) that $\mu([0,1])\tilde{\nu}([0,1]) = 1$, but since we also have $\mu([0,1])\nu([0,1]) = 1$, we conclude that $a = 0$. Finally $\nu = \tilde{\nu}$ has no mass at zero and (2.3) also holds for $\Re z = 0$. For $z = n \in \mathbb{N}_0$, we get that

$$b_n = \int_0^1 t^n d\nu(t) = \frac{1}{a_0 + \cdots + a_n}.$$

Corollary 2.3. *If μ is a non-zero measure on $[0,1]$ with $m = \int_0^1 (1-t)^{-1} d\mu(t) < \infty$. Then, the measure $\nu = T(\mu)$ of Theorem 1.1 is given by*

$$\nu = \sum_{k=0}^{\infty} \frac{1}{m^{k+1}} \sigma^{\diamond k}, \quad d\sigma(t) = \frac{t}{1-t} d\mu(t).$$

Proofs of Theorem 1.2
First proof
Any measure μ on $[-1,1]$ is weak limit of a sequence of discrete measures of the form $c_1\delta_{q_1} + \cdots + c_m\delta_{q_m}$, where $c_j > 0$, $j = 1, \cdots, m$, and $-1 < q_1 < \cdots < q_m < 1$. Hence, it is enough to prove the theorem for discrete measures of that type. Without loss of generality we can assume that the discrete measures are probabilities: $c_1 + \cdots + c_m = 1$. The moment sequence of these measures is of the form: $c_1 q_1^n + \cdots + c_m q_m^n$; hence, we have to prove that for any positive number $r > 0$ the sequence

$$b_n = \left(r + c_1 \frac{1-q_1^n}{1-q_1} + \cdots + c_m \frac{1-q_m^n}{1-q_m} \right)^{-1}, \quad n \geq 0$$

is also a moment sequence of a measure on $[-1,1]$. But we can write it in the form:

$$b_n = \frac{1}{c} \frac{1}{1-(d_1 q_1^n + \cdots + d_m q_m^n)}, \quad n \geq 0$$

where

$$c = r + \frac{c_1}{1-q_1} + \cdots + \frac{c_m}{1-q_m}, \text{ and } d_j = \frac{c_j}{c(1-q_j)}, \; j = 1, \cdots, m.$$

The sequence $d_1 q_1^n + \cdots + d_m q_m^n$ is always strictly less than one in absolute value and it is the moment sequence of the measure $d_1\delta_{q_1} + \cdots + d_m\delta_{q_m}$. It follows from the first part of Corollary 2.2 that $(b_n)_n$ is also a moment sequence of a measure on $[-1,1]$.

Second proof (cf. [**BCR**], ex. 4.6.23.)
We write σ for the measure on $[-1,1]$ with moments $(a_n)_n$; define $s_n = r + a_0 + \cdots + a_{n-1}$, $n \geq 1$, $s_0 = r > 0$. Then:

$$s_n = r + \int_{-1}^{1} \frac{1-x^n}{1-x} d\sigma(x) = r + n\sigma(\{1\}) + \int_{[-1,1)} (1-x^n) \frac{d\sigma(x)}{1-x}.$$

We claim that *for* $t \geq 0$ *there exists* $\mu_t \geq 0$ *on* $[-1,1]$ *such that* $e^{-ts_n} = \int_{-1}^{1} x^n d\mu_t(x)$. From this claim it is easy to prove the theorem: Consider $\mu = \int_0^\infty \mu_t dt$ which it is a positive measure on $[-1,1]$; then

$$\int_{-1}^{1} x^n d\mu(x) = \int_0^\infty \left(\int_{-1}^{1} x^n d\mu_t(x) \right) dt = \int_0^\infty e^{-ts_n} dt = \frac{1}{s_n}.$$

We now prove the claim. For $0 < a < 1$ let

$$\begin{aligned} s_n(a) &= r + n\sigma(\{1\}) + \int_{-1}^{a} (1-x^n) \frac{d\sigma(x)}{1-x} \\ &= r + \int_{-1}^{a} \frac{d\sigma(x)}{1-x} + n\sigma(\{1\}) - \int_{-1}^{a} x^n \frac{d\sigma(x)}{1-x} \qquad (2.4) \\ &= \alpha + \beta n - \int_{-1}^{a} x^n d\tau(x) \end{aligned}$$

where $\alpha, \beta \geq 0$, and $\tau \geq 0$ on $[-1,a]$. Then:

$$e^{-ts_n(a)} = e^{-t\alpha} e^{-t\beta n} e^{t\int_{-1}^{a} x^n d\tau(x)},$$

which is a bounded moment sequence as product of the bounded moment sequences $e^{-t\alpha}e^{-t\beta n}$ and $e^{t\int_{-1}^{a} x^n d\tau(x)}$ (this last one is a moment sequence from the second part of Corollary 2.2). Note that $s_0(a) = r$. Therefore there exists a positive measure $\mu_{t,a}$ on $[-1,1]$ so that

$$e^{-ts_n(a)} = \int_{-1}^{1} x^n d\mu_{t,a}(x),$$

and $\mu_{t,a}([-1,1]) = e^{-ts_0(a)} < 1$. As a consequence $(s_n^{-1}(a))_n$ is the moment sequence of the measure $\mu_a = \int_0^\infty \mu_{t,a} dt$ with total mass $1/r$.

By the method of moments, $\lim_{a\to 1}\mu_{t,a} = \mu_t$ is a family of positive measures on $[-1,1]$ with

$$e^{-ts_n} = \int_{-1}^{1} x^n d\mu_t(x),$$

and the claim is proved.

$\square$

Remark. We can still give another different proof of the first part of Theorem 1.1; it is based on the following result by the authors: *Let $(a_n)_n$ be a non-vanishing Hausdorff moment sequence. Then (b_n) defined by $b_0 = 1$ and $b_n = 1/(a_1\cdot\ldots\cdot a_n)$ for $n \geq 1$ is a normalized Stieltjes moment sequence* (see Theorem 1.1 in [**BD**]).
We know from the second part of Corollary 2.2 that $(e^{ta_n})_n$ is a Hausdorff moment sequence for any $t > 0$; hence $b_n = e^{t(a_n-a_0)}$ is a normalized Hausdorff moment sequence and by the previously cited result $t_n = 1/(b_0\cdot\ldots\cdot b_n)$ is a normalized Stieltjes moment sequence. But

$$t_n = \frac{1}{b_0\cdot\ldots\cdot b_n} = e^{-t(a_0+\cdots+a_n)}e^{t(n+1)a_0}.$$

Therefore $e^{-t(a_0+\cdots+a_n)}$ is a Stieltjes moment sequence, and integrating with respect to t over $(0,+\infty)$, we get that $1/(a_0+\cdots+a_n)$ is a Stieltjes moment sequence. Being bounded by $1/a_0$, it is a Hausdorff moment sequence.

$\square$

By applying the previously cited result by the authors to the first part of Corollary 2.2 we get:

Corollary 2.4. *If $(a_n)_n$ is a normalized Hausdorff moment sequence, then so is the sequence $\prod_{k=0}^{n-1}(1-a_{k+1})$, $n \geq 0$.*

3 Examples

First of all, we show some examples of moment sequences provided by the Lemma 2.1:
To do so we note that for $d\mu = t^\alpha\chi_{(0,1]}(t)dt$, $\alpha > -1$, a straightforward computation gives:

$$d(\mu^{\diamond k}) = t^\alpha \frac{(-1)^{k-1}}{(k-1)!}\log^{k-1}(t)\chi_{(0,1]}(t)dt, \quad k \geq 1. \tag{3.1}$$

1. Take now $d\mu = t\chi_{(0,1]}(t)dt$ with moments $a_n = 1/(n+2)$. The analytic functions on the unit disc $-\log(1-z)$ and $(z+(1-z)\log(1-z))/z$ provide, according to the Lemma 2.1, the following Hausdorff moment sequences

$$\begin{aligned} a_n &= \log\frac{n+2}{n+1}, \\ b_n &= 1-(n+1)\log\left(\frac{n+2}{n+1}\right). \end{aligned}$$

Taking into account the Taylor expansions:

$$\begin{aligned} -\log(1-z) &= \sum_{k=1}^{\infty} \frac{z^k}{k}, \\ \frac{z+(1-z)\log(1-z)}{z} &= \sum_{k=1}^{\infty} \frac{z^k}{(k+1)k}, \end{aligned}$$

the proof of Lemma 2.1 gives now the corresponding measures for these sequences

$$\begin{aligned} \sigma_1 &= \sum_{k=1}^{\infty} \frac{\mu^{\diamond k}}{k}, \\ \sigma_2 &= \sum_{k=1}^{\infty} \frac{\mu^{\diamond k}}{(k+1)k}, \end{aligned}$$

and using (3.1) we get:

$$\begin{aligned} d\sigma_1 &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} t \log^{k-1} t}{k!} \chi_{(0,1]}(t)dt = \frac{t-1}{\log t} \chi_{(0,1]}(t)dt; \\ d\sigma_2 &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} t \log^{k-1} t}{(k+1)!} \chi_{(0,1]}(t)dt = \frac{1-t+t\log t}{\log^2 t} \chi_{(0,1]}(t)dt. \end{aligned}$$

The first example is known (see [**BCR**], Ex. 4.6.24, p. 139), the second seems to be new.
We now show examples provided by Theorem 1.1.
2. For $\mu = \delta_q$, $0 < q < 1$, Theorem 1.1 gives the Hausdorff moment sequence

$$b_n = \frac{1-q}{1-q^{n+1}} = (1-q) \sum_{k=0}^{\infty} q^k \left(q^k\right)^n;$$

the discrete measure $\nu = (1-q)\sum_{k=0}^{\infty} q^k \delta_{q^k}$ has $(b_n)_n$ as its sequence of moments.
3. We now consider $\mu = \alpha\delta_p + \beta\delta_q$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ and $0 \leq p < q < 1$. This gives the Hausdorff moment sequence:

$$b_n = \frac{1}{\alpha \frac{1-p^{n+1}}{1-p} + \beta \frac{1-q^{n+1}}{1-q}}.$$

By Corollary 2.3, the associated measure is given by

$$\nu = \sum_{k=0}^{\infty} \frac{1}{m^{k+1}} \left(\alpha \frac{p}{1-p} \delta_p + \beta \frac{q}{1-q} \delta_q \right)^{\diamond k}, \quad m = \frac{\alpha}{1-p} + \frac{\beta}{1-q};$$

writing $m_1 = \alpha p/(m(1-p))$, $m_2 = \beta q/(m(1-q))$ and taking into account that $\delta_a \diamond \delta_b = \delta_{ab}$, we get

$$\nu = \frac{1}{m} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} m_1^j m_2^{k-j} \delta_{p^j q^{k-j}}.$$

4. For $0 < q < 1$, let $\mu = \delta_1 + (1-q)\delta_q$ be. This gives the Hausdorff moment sequence:

$$\begin{aligned} b_n &= \frac{1}{n+1+(1-q^{n+1})} = \frac{1}{n+2}\frac{1}{1-\frac{q^{n+1}}{n+2}} \\ &= \sum_{k=0}^{\infty} q^k \left(q^k\right)^n \left(\frac{1}{n+2}\right)^{k+1}; \end{aligned}$$

But the sequence $\left(\dfrac{1}{n+2}\right)^{k+1}$, $n \geq 0$, is the Hausdorff moment sequence of the measure

$$d\nu_k = \left(x\chi_{(0,1]}(x)dx\right)^{\diamond(k+1)} = \frac{(-1)^k}{k!} x \log^k x \chi_{(0,1]}(x)dx;$$

hence the measure

$$\nu = \sum_{k=0}^{\infty} q^k \delta_{q^k} \diamond \nu_k$$

has $(b_n)_n$ as it sequence of moments. It is easy to check that

$$\delta_{q^k} \diamond d\nu_k = \frac{(-1)^k}{q^{2k}k!} x \log^k(x/q^k)\chi_{(0,q^k]}(x)dx,$$

which finally gives

$$d\nu = x\sum_{k=0}^{\infty} \frac{(-1)^k}{q^k k!} \log^k(x/q^k)\chi_{(0,q^k]}(x)dx.$$

5. The probability measure on $(0,1)$ defined by

$$\mu_c = \frac{\log^{c-1}(1/t)}{\Gamma(c)}dt, \quad c > 0,$$

has Mellin transform ([**GR**, p. 551 (6)])

$$\mathcal{M}(\mu_c)(z) = \int_0^1 t^z d\mu_c(t) = \frac{1}{(z+1)^c}, \quad \Re z > -1, \tag{3.2}$$

and moments $a_n(c) = 1/(n+1)^c$. The sequence of measures $(\mu_c)_{c>0}$ is a convolution semigroup for the product convolution, i.e. a convolution semigroup on the multiplicative group $(0,+\infty)$ in the sense of [**BF**]. According to Theorem 1.1, the moments of μ_c give the Hausdorff moment sequence

$$\frac{1}{\mathcal{H}_{n+1,c}} = \frac{1}{1+\frac{1}{2^c}+\cdots+\frac{1}{(n+1)^c}}, \quad n \geq 0.$$

We now compute the representing measure ν_c for this moment sequence.

From Theorem 1.1, we know that ν_c has no mass at 0 and that its Mellin transform satisfies

$$\mathcal{M}(\nu_c)(z) f_c(z+1) = 1, \quad \Re z \geq 0,$$

where

$$f_c(z) = \int_0^1 \frac{\log^{c-1}(1/t)}{\Gamma(c)} \frac{1-t^z}{1-t} dt, \quad \Re z > 0. \tag{3.3}$$

For $z > 0$, we have by the positivity of the integrand and (3.2) that

$$\begin{aligned} f_c(z) &= \int_0^1 (1-t^z) \sum_{k=0}^{\infty} t^k \frac{\log^{c-1}(1/t)}{\Gamma(c)} dt \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{(k+1)^c} - \frac{1}{(k+z+1)^c} \right). \end{aligned} \tag{3.4}$$

For $c > 1$ (but not for $c \leq 1$), we can sum each series and get

$$f_c(z) = \zeta(c) - \zeta(c, z), \quad c > 1, \tag{3.5}$$

where $\zeta(c, z)$ is the Hurwitz's zeta function

$$\zeta(c, z) = \sum_{n=1}^{\infty} \frac{1}{(n+z)^c}, \quad c > 1, z \in \mathbb{C} \setminus (-\infty, -1].$$

Note that $\zeta(c, 0) = \zeta(c)$ is the ordinary Riemann's zeta function.
We now consider the function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz} \log \Gamma(z),$$

which satisfies, as it is well-known (see, for instance, [**GR**], p. 943), that

$$\psi(z+1) + \gamma = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z} \right) = z \sum_{n=1}^{\infty} \frac{1}{n(n+z)}. \tag{3.6}$$

This gives

$$f_1(z) = \int_0^1 \frac{1-t^z}{1-t} dt = \psi(z+1) + \gamma;$$

from where, we get

$$f_1^{(k)}(z) = (-1)^{k-1} \int_0^1 \frac{t^z \log^k(1/t)}{1-t} dt = \psi^{(k)}(z+1), \quad k = 1, 2, \cdots,$$

and so

$$\begin{aligned} f_{k+1}(z) &= \int_0^1 \frac{1-t^z}{1-t} \frac{\log^k(1/t)}{k!} dt \\ &= \zeta(k+1) - \int_0^1 \frac{t^z}{1-t} \frac{\log^k(1/t)}{k!} dt \\ &= \zeta(k+1) - \frac{(-1)^{k-1} f_1^{(k)}(z)}{k!} \\ &= \zeta(k+1) + \frac{(-1)^k \psi^{(k)}(z+1)}{k!}. \end{aligned}$$

We have then got the following table for the Mellin transform of ν_c:

$$\mathcal{M}(\nu_c)(z) = \begin{cases} \left[\sum_{k=0}^{\infty} \left(\frac{1}{(k+1)^c} - \frac{1}{(k+z+2)^c} \right) \right]^{-1} & \text{if } 0 < c < 1, \\ \frac{1}{\psi(z+2)+\gamma} & \text{if } c = 1, \\ \frac{1}{\zeta(c) - \zeta(c, z+1)} & \text{if } 1 < c, \\ \frac{1}{\zeta(c) + \frac{(-1)^{c-1}\psi^{(c-1)}(z+2)}{(c-1)!}} & \text{if } c = 2, 3, \cdots. \end{cases} \tag{3.7}$$

Formally, we can find ν_c by inversion of the Mellin transform, i.e.

$$\nu_c = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{t^{-ix-1}}{f_c(1+ix)} dx,$$

but it does not seem posible to get further in this direction.
For $0 < c \leq 1$, we choose another method based on Pick functions (cf. [**D**]) and Stieltjes functions.

Theorem 3.1. *The function $f_c(z)$ has a holomorphic continuation to $\mathbb{C} \setminus (-\infty, -1]$. It is a Pick function for $0 < c \leq 1$, i.e., $\Im f_c(z) > 0$ for $\Im z > 0$.*

Proof. The term $\dfrac{1}{n^c} - \dfrac{1}{(n+z)^c}$, $n = 1, 2, \cdots$, is holomorphic in $\mathbb{C} \setminus (-\infty, -1]$ and for $z \in \mathbb{C} \setminus (-\infty, -1]$, $|z| \leq N$, $n > N$,

$$\begin{aligned} \frac{1}{n^c} - \frac{1}{(n+z)^c} &= \frac{1}{n^c}\left(1 - \left(1 + \frac{z}{n}\right)^{-c}\right) = \frac{1}{n^c}\left(-\binom{-c}{1}\frac{z}{n} - \binom{-c}{2}\left(\frac{z}{n}\right)^2 - \cdots\right) \\ &= \frac{z}{n^{c+1}}\left(c - \binom{-c}{2}\frac{z}{n} - \cdots\right), \end{aligned}$$

in modulus behaving like $1/n^{c+1}$ which is the term of a converging series. Therefore, the right-hand side of (3.4) is holomorphic in $\mathbb{C} \setminus (-\infty, -1]$.
Using that ([**GR**, p. 288, (3.221.1)])

$$t^{-c} = \frac{\sin(\pi c)}{\pi} \int_0^\infty \frac{s^{-c}}{s+t} ds, \quad 0 < c < 1,$$

we get

$$\begin{aligned} \frac{1}{n^c} - \frac{1}{(n+z)^c} &= \frac{\sin(\pi c)}{\pi} \int_0^\infty s^{-c} \left(\frac{1}{s+n} - \frac{1}{s+n+z} \right) ds \\ &= z \frac{\sin(\pi c)}{\pi} \int_0^\infty \frac{s^{-c} ds}{(s+n)(s+n+z)}, \end{aligned} \tag{3.8}$$

showing that $\dfrac{1}{n^c} - \dfrac{1}{(n+z)^c}$ is a Pick function. This shows the assertion for $0 < c < 1$. For $c \to 1^-$, we get the assertion for $c = 1$, but it is also a consequence of (3.6).

□

We now find the measure ν_c by inverting the table (3.7).
We start by taking $c = 1$. The sequence of moments is

$$\frac{1}{\mathcal{H}_{n+1}} = \frac{1}{1 + \frac{1}{2} + \cdots + \frac{1}{n+1}}, \quad n \geq 0,$$

and we next show that the representing measure ν_1 is given by (1.2).
For $c = 1$, (3.6) and (3.7) gives that

$$\mathcal{M}(\nu_1)(z) = \frac{1}{\psi(z+2) + \gamma} = \frac{1}{f_1(z+1)}.$$

We say that a function $\phi : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is a Stieltjes function if it has the form

$$\phi(z) = a + \int_0^\infty \frac{d\sigma(x)}{x+z}$$

where $a \geq 0$ and σ is a measure on $[0, +\infty)$. The function $f_1(z)/z$ is a Stieltjes function with $a = 0$ and $\sigma = \sum_{k=1}^\infty \delta_k / k$:

$$\frac{f_1(z)}{z} = \sum_{k=1}^\infty \frac{1}{k(k+z)} = \int_0^\infty \frac{d\sigma(t)}{x+z}.$$

Since $f_1(x)/x$ is strictly decreasing on each interval $(-1, +\infty)$, $(-2, -1)$, $\cdots$ — take into account that $\dfrac{d}{dx}\left(\dfrac{f_1(x)}{x}\right) = -\displaystyle\sum_{k=1}^\infty \frac{1}{k(k+x)^2} < 0$— and has poles at $-1, -2, \cdots$, we have that the function has a simple zero in each of the intervals $(-n-1, -n)$, $n = 1, 2, \cdots$. Let $\xi_n \in (n, n+1)$ be so that $f_1(-\xi_n)/(-\xi_n) = 0$

—that is $\psi(1-\xi_n) = -\gamma$—. There is no zero at $(-1,+\infty)$ because the function is positive there. We put now $\xi_0 = 0$ so $(-\xi_n)_{n\geq 0}$ are the zeros of f_1 –a Stieltjes function has no zeros outside the real axis—.
It is known that for a Stieltjes function ϕ also $1/(z\phi(z))$ is a Stieltjes function (see [**R**], [**I**] or [**B**]). Hence $1/f_1(z)$ is a Stieltjes function with simple poles precisely at $(-\xi_n)_{n\geq 0}$:

$$\frac{1}{f_1(z)} = \sum_{n=0}^{\infty} \frac{\alpha_n}{z+\xi_n},$$

where

$$\alpha_n = \lim_{z\to -\xi_n} \frac{z+\xi_n}{f_1(z)} = \frac{1}{f_1'(-\xi_n)} = \frac{1}{\psi'(1-\xi_n)} > 0.$$

Note that the constant a is zero because $\lim_{z\to\infty} f_1(z) = \infty$. In particular,

$$\alpha_0 = \frac{1}{\psi'(1)} = \frac{6}{\pi^2}.$$

We have then found that

$$\mathcal{M}(\nu_1)(z) = \frac{1}{f_1(z+1)} = \sum_{n=0}^{\infty} \frac{\alpha_n}{z+1+\xi_n} = \int_0^1 \left(\sum_{n=0}^{\infty} \alpha_n x^{\xi_n}\right) x^z\,dx,$$

from where we deduce that the probability ν_1 has the density $\sum\limits_{n=0}^{\infty} \alpha_n x^{\xi_n}$ with respect to the Lebesgue measure $\chi_{(0,1]}(x)dx$.
We now give the asymptotic behaviour of ξ_n and α_n:

$$\xi_n = n+1-\delta_n, \quad \text{where } 0 < \delta_{n+1} < \delta_n < \tfrac{1}{2}, \quad \delta_n \sim 1/\log n \tag{3.9}$$

$$\alpha_n \sim 1/\log^2 n. \tag{3.10}$$

Indeed, since $\xi_n \in (n, n+1)$, we define δ_n so that $\xi_n = n+1-\delta_n$, $n \geq 1$. By Euler's product formula for Γ we find

$$\psi(x) - \psi(1-x) = -\pi\cot(\pi x). \tag{3.11}$$

Hence, for $x = \xi_n$, we get $\psi(\xi_n) + \gamma = \pi\cot(\pi\delta_n)$. The digamma function is strictly increasing with $\psi(1) = -\gamma$, so $\cot(\pi\delta_n) > 0$, showing that $0 < \delta_n < 1/2$ is decreasing. Since $\psi(x) \sim \log x$ for $x \to \infty$ (cf. [**GR**]), we easily get $\delta_n \sim 1/\log n$ for $n \to \infty$.
Differentiating (3.11), we get

$$\psi'(\xi_n) + \frac{1}{\alpha_n} = \frac{\pi^2}{\sin^2(\pi\delta_n)},$$

and since $\psi'(x) = \sum_{k=0}^{\infty} 1/(x+k)^2 \to 0$ for $x \to +\infty$, we get $\lim_{n\to\infty} \delta_n^2/\alpha_n = 1$, and (3.10) follows.

The first 5 of the numbers ξ_n, $n \geq 1$, are given with 5 correct digits $\xi_1 = 1.56735$, $\xi_2 = 2.62846$, $\xi_3 = 3.66038$, $\xi_4 = 4.68118$, $\xi_5 = 5.69626$.

For $0 < c < 1$, $f_c(z)/z$ is also a Stieltjes function. Indeed, from (3.8), we get

$$\frac{1}{n^c} - \frac{1}{(n+z)^c} = z\frac{\sin(\pi c)}{\pi} \int_n^\infty \frac{1}{t(t-n)^c} \frac{dt}{t+z},$$

and so

$$f_c(z) = z\frac{\sin(\pi c)}{\pi} \int_0^\infty \left(\sum_{1 \leq n < t} \frac{1}{(t-n)^c} \right) \frac{dt}{t(t+z)} = z\frac{\sin(\pi c)}{\pi} \int_0^\infty \frac{\phi_c(t)}{t} \frac{dt}{t+z},$$

where

$$\phi_c(t) = \sum_{1 \leq n < t} \frac{1}{(t-n)^c} = \begin{cases} 0, & \text{if } 0 < t \leq 1, \\ \frac{1}{(t-1)^c}, & \text{if } 1 < t \leq 2, \\ \frac{1}{(t-1)^c} + \frac{1}{(t-2)^c}, & \text{if } 2 < t \leq 3, \\ \quad \vdots & \\ \frac{1}{(t-1)^c} + \frac{1}{(t-2)^c} + \cdots + \frac{1}{(t-n)^c}, & \text{if } n < t \leq n+1. \end{cases}$$

Since $f_c(z)/z$ is a Stieltjes transform, so is $1/f_c(z)$, i.e.

$$\frac{1}{f_c(z)} = \int_0^\infty \frac{d\kappa_c(t)}{t+z}.$$

Note that the constant a is zero because $\lim_{z \to \infty} f_c(z) = \infty$ by (3.3). Hence

$$\begin{aligned} \mathcal{M}(\nu_c)(z) &= \int_0^\infty \frac{d\kappa_c(t)}{t+z+1} = \int_0^\infty \left(\int_0^1 x^{t+z} dx \right) d\kappa_c(t) \\ &= \int_0^1 x^z \left(\int_0^\infty x^t d\kappa_c(t) \right) dx, \end{aligned}$$

so

$$d\nu_c(x) = \int_0^\infty x^t d\kappa_c(t).$$

In principle, one can find κ_c as the limit for $y \to 0^+$ of the densities

$$\frac{1}{\pi} \Im \left[\frac{1}{f_c(-x+iy)} \right], \quad x > 0.$$

Since f_c increases on $(-1, +\infty)$ from $-\infty$ to $+\infty$ with a simple zero for $z = 0$ with $f_c'(0) = c\zeta(c+1)$,
we see that

$$\kappa_c = (c\zeta(c+1))^{-1}\delta_0 + \tilde{\kappa}_c,$$

where $\tilde{\kappa}_c$ is concentrated at $[1, +\infty)$. The determination of $\tilde{\kappa}_c$ involves calculation of the Hilbert transform of the density $\phi_c(t)/t$, but an explicit expression does not seem possible.
For $c > 1$, $f_c(z)/z$ is not a Stieltjes function, but $m = \int_0^1 (1-t)^{-1} d\mu_c(t) = \zeta(c) < \infty$. According to Corollary 2.3, we define

$$d\sigma_c = \frac{t}{1-t} \frac{\log^{c-1}(1/t)}{\Gamma(c)} \chi_{(0,1]}(t) dt.$$

and get

$$\nu_c = \sum_{k=0}^{\infty} \frac{\sigma_c^{\diamond k}}{\zeta^{k+1}(c)}.$$

It seems impossible to find a closed expression for the convolution $\sigma_c^{\diamond k}$. Note that ν_c has the mass $1/\zeta(c)$ at 1 and it is absolutely continuous on $(0,1)$ with respect to the Lebesgue measure.

In the case $c = 1$, we have the following one-parameter extension
6. For $a > 0$, let $\mu_a = x^{a-1} \chi_{(0,1]}(x) dx$ with moments $s_n(\mu_a) = 1/(a+n)$, $n \geq 0$. The case $a = 1$ corresponds to $c = 1$ in Example 3.5. We get

$$\int_0^1 \frac{1-t^{z+1}}{1-t} d\mu_a(t) = \sum_{n=0}^{\infty} \left(\frac{1}{n+a} - \frac{1}{n+a+z+1} \right) = \frac{1}{f(z+1)}, \quad \Re z > -1,$$

where

$$f(z) = z \sum_{n=0}^{\infty} \frac{1}{(n+a)(n+a+z)},$$

It follows that $f(z)/z$ and $1/f(z)$ are Stieltjes functions

$$\frac{1}{f(z)} = \sum_{n=0}^{\infty} \frac{\alpha n_(a)}{z + \xi_n(a)}, \quad \alpha_n(a) = \frac{1}{f'(-\xi_n(a))},$$

where $\xi_0(a) = 0$, $\xi_n(a) \in (a-1+n, a+n)$, $n \geq 1$, are such that $f(-\xi_n(a)) = 0$. The measure ν_a with moments

$$s_n(\nu_a) = \left(\sum_{k=0}^{n} \frac{1}{a+k} \right)^{-1}, \quad n \geq 0$$

is given as

$$\nu_a = \left(\alpha_0(a) + \sum_{n=1}^{\infty} \alpha_n(a) t^{\xi_n(a)} \right) \chi_{(0,1]}(t) dt.$$

7. As we mentioned in the introduction, the measure $\dfrac{\delta_{-1} + \delta_1}{2}$ shows that Theorem 1.1 is no longer true for measures on $[-1,1]$. However, Theorem 1.2

gives for any $r > 0$ that $b_n = \dfrac{1}{r + [(n+1)/2]}$ is again a moment sequence on $[-1,1]$ —as usual $[x]$ denotes the integer part of the real number x—. Its associated measure μ can be computed using the second proof of Theorem 1.2. Indeed, with the notation of that proof, we straightforwardly have that the sequence (2.4) is given by:

$$s_n = r + \frac{1}{4} + \frac{n}{2} - \frac{1}{4}(-1)^n;$$

(note that it is independent of a). The measures μ_t with moments e^{-ts_n}, $t > 0$, are given by the formula

$$\mu_t = e^{-t(r+1/4)}\delta_{e^{-t/2}} \diamond \sigma_t,$$

where the measure σ_t has moments $e^{(-1)^n t/4}$. This measure can be calculated using the proof of Lemma 2.1:

$$\begin{aligned}
\sigma_t &= \sum_{l=0}^{\infty} \frac{(t/4)^l}{l!}\delta_{-1}^{\diamond l} \\
&= \left(\sum_{l=0}^{\infty} \frac{(t/4)^{2l}}{(2l)!}\right)\delta_1 + \left(\sum_{l=0}^{\infty} \frac{(t/4)^{2l+1}}{(2l+1)!}\right)\delta_{-1} \\
&= \frac{e^{t/4} + e^{-t/4}}{2}\delta_1 + \frac{e^{t/4} - e^{-t/4}}{2}\delta_{-1}.
\end{aligned}$$

This gives for the measure μ_t the formula

$$\mu_t = e^{-t(r+1/4)}\left(\cosh(t/4)\delta_{e^{-t/2}} + \sinh(t/4)\delta_{-e^{-t/2}}\right).$$

The measure μ is then $\int_0^\infty \mu_t dt$; this can be computed easily to show that the measure μ has density

$$|x|^{2r}\left(\frac{1}{|x|} + \operatorname{sgn}(x)\right)$$

with respect to the Lebesgue measure on $[-1,1]$.

4 Relation to convolution semigroup

Let $(\nu_c)_{c>0}$ be a convolution semigroup of sub-probabilities on $[0,\infty)$ with Laplace exponent or Bernstein function f given by

$$\int_0^\infty e^{-sx}d\nu_c(x) = e^{-cf(s)}, \quad s \geq 0 \tag{4.1}$$

cf. [**BF**], [**Bt**]. We recall that f has the integral representation

$$f(x) = a + bs + \int_0^\infty \left(1 - e^{-sx}\right)d\lambda(x), \tag{4.2}$$

where $a, b \geq 0$ and the Lévy measure λ on $(0,\infty)$ satisfies the integrability condition $\int x/(1+x)d\lambda(x) < \infty$. Note that $\nu_c([0,\infty)) = e^{-ac}$, so that $(\nu_c)_{c\geq 0}$ consists of probabilities if and only if $a = 0$. In the following we shall exclude the Bernstein function identically equal to zero, which correspond to the convolution semigroup $\nu_c = \delta_0$, $c > 0$.
The potential kernel κ of the convolution semigroup ($f \not\equiv 0$) is given as

$$\kappa = \int_0^\infty \nu_c dc, \quad \int_0^\infty e^{-sx} d\kappa(x) = \frac{1}{f(s)}. \tag{4.3}$$

From (4.2), we find

$$f(s+1) - f(s) = b + \int_0^\infty e^{-sx}\left(1 - e^{-x}\right) d\lambda(x),$$

which shows that $f(s+1) - f(s)$ is the Laplace transform of the finite measure

$$b\delta_0 + \left(1 - e^{-x}\right) d\lambda(x). \tag{4.4}$$

In particular, $s \to f(s+1) - f(s)$ is completely monotonic. Note that any finite measure on $[0,\infty)$ has the form (4.4) for a uniquely determined pair (b,λ), where $b \geq 0$ and λ is a Lévy measure.
Let us now introduce the group isomorphism ρ of the additive group $\mathbb{R}$ onto the multiplicative group $(0,\infty)$ given as $\rho(x) = e^{-x}$ with inverse $\rho^{-1}(y) = \log(1/y)$. The image measures $\tilde{\nu}_c = \rho(\nu_c)$ of the convolution semigroup $(\nu_c)_{c>0}$ form a convolution semigroup on $(0,1]$ with respect to the product convolution $\diamond$ and formula (4.1) reads

$$\int_0^1 x^s d\tilde{\nu}_c(x) = e^{-cf(s)}. \tag{4.5}$$

We now introduce the potential kernel $\tilde{\kappa}$ and the Lévy measure $\tilde{\lambda}$ of $(\tilde{\nu}_c)_{c>0}$ as

$$\tilde{\kappa} = \rho(\kappa) = \int_0^\infty \tilde{\nu}_c dc, \quad \tilde{\lambda} = \rho\left(\left(1 - e^{-x}\right)\lambda\right),$$

so, (4.2) and (4.3) are transformed to

$$f(s) = a + bs + \int_0^1 \frac{1-t^s}{1-t} d\tilde{\lambda}(t) \tag{4.6}$$

$$\int_0^1 t^s d\tilde{\kappa}(t) = \frac{1}{f(s)} \tag{4.7}$$

Note that the formulas (4.1), (4.2), (4.5), (4.6) remain valid when the variable s takes values in the half-plane $\Re z \geq 0$, while in (4.3), (4.7), we can let s assume values in $\Re z > 0$.
The transformation T of Theorem 1.1 defined on the set of measures $\mu \neq 0$ on $[0,1]$ can now be described in terms of convolution semigroups-Bernstein functions in the following way.

Theorem 4.1. *Let μ be a non-zero measure on $[0,1]$ and write $\mu = a\delta_0 + b\delta_1 + \tilde{\lambda}$ with $a = \mu(\{0\})$, $b = \mu(\{1\})$, $\tilde{\lambda} = \mu|_{(0,1)}$. Let f be the non-zero Bernstein function given by (4.6). Then, the measure $\nu = T(\mu)$ is given as $\nu = xd\tilde{\kappa}(x)$, where $\tilde{\kappa}$ is the potential kernel of the corresponding product convolution semigroup. The moment sequence $b_n = 1/(a_0 + \cdots + a_n)$ of ν is given as $b_n = 1/f(n+1)$, $n \geq 0$.*

Proof. From (4.6) and (4.7), we get

$$f(s+1) = \int_0^1 \frac{1 - x^{s+1}}{1-x} d\mu(x), \qquad \int_0^1 x^s x d\tilde{\kappa}(x) = \frac{1}{f(s+1)},$$

and comparing with (1.1), we get $\nu = T(\mu) = xd\tilde{\kappa}(x)$, and clearly $b_n = 1/f(n+1)$.

□

Examples

1. For the Bernstein function $f(s) = \log(1+s)$, we have $a = b = 0$ and $\lambda = (e^{-x}/x)\,dx$. This gives $\mu = (1-t)/\log(1/t)dt$. The corresponding product convolution semigroup is

$$\tilde{\nu}_c = \frac{1}{\Gamma(c)} \log^{c-1}(1/t)\chi_{(0,1)}(t)dt,$$

cf. Example 5 in Section 3. The potential kernel $\tilde{\kappa}$ has the density on $(0,1)$

$$\int_0^\infty \frac{\log^{c-1}(1/t)}{\Gamma(c)} dc.$$

Note that the function (of x)

$$\int_0^\infty \frac{x^{c-1}}{\Gamma(c)} dc$$

can be considered as an "integral version" of the exponential function.
We find

$$T(\mu) = t\int_0^\infty \frac{\log^{c-1}(1/t)}{\Gamma(c)} dc\chi_{(0,1)}(t)dt.$$

2. For the Bernstein function $f(s) = s^\alpha$, $0 < \alpha < 1$, we have $a = b = 0$ and $\lambda = (\alpha/\Gamma(1-\alpha))x^{-\alpha-1}dx$ since

$$s^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-xs}\right) \frac{dx}{x^{\alpha+1}},$$

cf. [**BF**, p. 71]. Therefore

$$\mu = \frac{\alpha}{\Gamma(1-\alpha)} \frac{1-t}{t} \log^{-\alpha-1}(1/t)\chi_{(0,1)}(t)dt.$$

The corresponding convolution semigroup $(\nu_c)_{c>0}$ on $(0,\infty)$ is called the one-sided stable semigroup of order α. The potential kernel is

$$\kappa = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \chi_{(0,\infty)}(x) dx$$

and we have

$$\nu = T(\mu) = \frac{1}{\Gamma(\alpha)} \log^{\alpha-1}(1/t) \chi_{(0,1)}(t) dt.$$

3. The Bernstein function $f(s) = c\,(1 - e^{-as})$ corresponds to the Poisson semigroup on $[0,\infty)$ and it leads to the measures μ, ν studied in Example 3.2 with $q = e^{-a}$, $c = 1/(1-q)$.

4. The study of Example 3.5 corresponds to the Bernstein function (3.3) with $a = b = 0$ and $\tilde{\lambda} = (1/\Gamma(c)) \log^{c-1}(1/t) \chi_{(0,1)}(t) dt$. The corresponding potential kernel $\tilde{\kappa}_c$ is related to ν_c by $\nu_c = t\tilde{\kappa}_c$.

In terms of potential kernels it is clear that we can characterize the image of $\mathcal{M}_+([0,1]) \setminus \{0\}$ under the transformation T as the set of measures of the form $td\tilde{\kappa}(t)$, where $\tilde{\kappa}$ is a potential kernel of a product convolution semigroup $(\tilde{\nu}_c)_{c>0}$ on $(0,1]$. Similarly, the set of moment sequences $b_n = 1/(a_0 + \cdots + a_n)$, where $(a_n)_n$ is a Hausdorff moment sequence with $a_0 > 0$, is the same as the sequences $(1/f(n+1))_n$ where f is a non-zero Bernstein function.

References

[B] C. Berg, *Quelques remarques sur le cône de Stieltjes.* In: Séminaire de Théorie du potentiel, Paris n.5. Lecture Notes in Mathematics **814**. Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[Bt] J. Bertoin, *Lévy processes.* Cambridge University Press, Cambridge, 1996.

[BCR] C. Berg, J. P. R. Christensen and P. Ressel, Harmonic analysis on semigroups. Theory of positive definite and related functions. Graduate Texts in Mathematics vol. **100**. Springer-Verlag, Berlin-Heidelberg-New York, 1984.

[BD] C. Berg and A. J. Durán, *A transformation from Hausdorff to Stieltjes moment sequences.* University of Copenhagen, Preprint Series 2002, n. 9.

[BF] C. Berg and G. Forst, Potential theory on locally compact abelian groups. Ergebnisse der Mathematik und ihrer Grenzgebiete Band 87. Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[D] W.F. Donoghue, Jr., Monotone Matrix Functions and Analytic Continuation. Springer-Verlag, Berlin-Heidelberg-New York, 1974.

[GR] I.S. Gradshteyn and I.M. Ryzhik, Table of integrals, series and products. Academic Press, New York, 1980.

[GKP] R.L. Graham, D.E. Knuth and O. Patashnik, Concrete Mathematics: A foundation for computer science, Reading, MA: Addison Wesley, 1994.

[H] F. Hausdorff, *Momentprobleme für ein endliches Intervall,* Math. Z. **16** (1923) 220–248.

[I] M. Itô, *Sur les cônes convexes de Riesz et les noyaux complètement sous-harmoniques.* Nagoya Math. J. **55** (1974), 111–144.

[R] G.E.H. Reuter, *Über eine Volterrasche Integralgleichung mit totalmonotonem Kern.* Arch. Math. **7** (1956), 59–66.