# Iteration of the rational function $z - 1/z$ and a Hausdorff moment sequence ⋆


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**Abstract**

In a previous paper we considered a positive function $f$, uniquely determined for $s > 0$ by the requirements $f(1) = 1$, $\log(1/f)$ is convex and the functional equation $f(s) = \psi(f(s+1))$ with $\psi(s) = s - 1/s$. Denoting $\psi^{\circ 1}(z) = \psi(z), \psi^{\circ n}(z) = \psi(\psi^{\circ(n-1)}(z))$, $n \geq 2$, we prove that the meromorphic extension of $f$ to the whole complex plane is given by the formula $f(z) = \lim_{n\to\infty} \psi^{\circ n}(\lambda_n(\lambda_{n+1}/\lambda_n)^z)$, where the numbers $\lambda_n$ are defined by $\lambda_0 = 0$ and the recursion $\lambda_{n+1} = (1/2)(\lambda_n + \sqrt{\lambda_n^2 + 4})$. The numbers $m_n = 1/\lambda_{n+1}$ form a Hausdorff moment sequence of a probability measure $\mu$ such that $\int t^{z-1}\, d\mu(t) = 1/f(z)$.



*Key words:* Hausdorff moment sequence, iteration of rational functions
MSC 2000: 44A60, 30D05



⋆ This work was initiated during a visit of the first author to the University of Sevilla in January 2007 supported by D.G.E.S, ref. BFM2003-6335-C03-01, FQM-262 (*Junta de Andalucía*).



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## 1 Introduction and main results

Hausdorff moment sequences are sequences of the form $a_n = \int_0^1 t^n d\nu(t)$, $n \geq 0$, where $\nu$ is a positive measure on $[0,1]$. Hausdorff's characterization of moment sequences was given in [7]. See also Widder's monograph [8]. For information about moment sequences in general see [1].

In [5] we introduced a non-linear transformation $T$ of the set of Hausdorff moment sequences into itself by the formula:

$$T((a_n))_n = 1/(a_0 + a_1 + \cdots + a_n), \quad n \geq 0. \tag{1}$$

The corresponding transformation of positive measures on $[0,1]$ is denoted $\widehat{T}$. We recall from [5] that if $\nu \neq 0$, then $\widehat{T}(\nu)(\{0\}) = 0$ and

$$\int_0^1 \frac{1-t^{z+1}}{1-t}\, d\nu(t) \int_0^1 t^z \, d\widehat{T}(\nu)(t) = 1 \text{ for } \Re z \geq 0. \tag{2}$$

It is clear that if $\nu$ is a probability measure, then so is $\widehat{T}(\nu)$, and in this way we get a transformation of the convex set of normalized Hausdorff moment sequences (i.e. $a_0 = 1$) as well as a transformation of the set of probability measures on $[0,1]$. By Tychonoff's extension of Brouwer's fixed point theorem to locally convex topological vector spaces the transformation $T$ has a fixed point, and by (1) it is clear that a fixed point $(m_n)_n$ is uniquely determined by the recursive equation

$$m_0 = 1, \quad (1 + m_1 + \cdots + m_n)m_n = 1, \quad n \geq 1. \tag{3}$$

Therefore

$$m_{n+1}^2 + \frac{m_{n+1}}{m_n} - 1 = 0, \tag{4}$$

giving

$$m_1 = \frac{-1+\sqrt{5}}{2}, \quad m_2 = \frac{\sqrt{22+2\sqrt{5}} - \sqrt{5} - 1}{4}, \cdots .$$

In [6] we studied the Hausdorff moment sequence $(m_n)_n$ and its associated probability measure $\mu$, called the *fixed point measure*. It has an increasing and convex density $\mathcal{D}$ with respect to Lebesgue measure on $]0,1[$, and for $t \to 1$ we have $\mathcal{D}(t) \sim 1/\sqrt{2\pi(1-t)}$.

We studied the *Bernstein transform*

$$f(z) = \mathcal{B}(\mu)(z) = \int_0^1 \frac{1-t^z}{1-t} d\mu(t), \quad \Re z > 0 \tag{5}$$

as well as the *Mellin transform*

$$F(z) = \mathcal{M}(\mu)(z) = \int_0^1 t^z d\mu(t), \quad \Re z > 0 \tag{6}$$

of $\mu$. These functions are clearly holomorphic in the half-plane $\Re z > 0$ and continuous in $\Re z \geq 0$, the latter because $\mu(\{0\}) = 0$.

As a first result we proved:

**Theorem 1 ([6])** *The functions $f, F$ can be extended to meromorphic functions in $\mathbb{C}$ and they satisfy*

$$f(z+1)F(z) = 1, \quad z \in \mathbb{C} \tag{7}$$

$$f(z) = f(z+1) - \frac{1}{f(z+1)}, \quad z \in \mathbb{C}. \tag{8}$$

*They are holomorphic in $\Re z > -1$. Furthermore $z = -1$ is a simple pole of $f$ and $F$.*

*The fixed point measure $\mu$ has the properties*

$$\int_0^1 t^x \, d\mu(t) < \infty, \ \ x > -1; \quad \int_0^1 \frac{d\,\mu(t)}{t} = \infty. \tag{9}$$

Using the rational function

$$\psi(z) = z - \frac{1}{z}, \tag{10}$$

the functional equation (8) can be written

$$f(z) = \psi(f(z+1)), \quad z \in \mathbb{C}. \tag{11}$$

The function $f$ can be characterized in analogy with the Bohr-Mollerup theorem about the Gamma function, cf. [2]. Using the following notation for iterations of a function $\rho$

$$\rho^{\circ 1}(z) = \rho(z), \ \rho^{\circ n}(z) = \rho(\rho^{\circ(n-1)}(z)), \quad n \geq 2,$$

we proved:

**Theorem 2 ([6])** *The Bernstein transform (5) of the fixed point measure is a function $f : ]0,\infty[ \to ]0,\infty[$ with the following properties*

*(i)* $f(1) = 1$,
*(ii)* $\log(1/f)$ *is convex,*
*(iii)* $f(s) = \psi(f(s+1)), \quad s > 0.$

*Conversely, if* $\tilde{f} : ]0,\infty[ \to ]0,\infty[$ *satisfies (i)-(iii), then it is equal to* $f$ *and for* $0 < s \leq 1$ *we have*

$$\tilde{f}(s) = \lim_{n\to\infty} \psi^{\circ n}\left(\frac{1}{m_{n-1}}\left(\frac{m_{n-1}}{m_n}\right)^s\right). \tag{12}$$

*In particular (12) holds for* $f$*.*

The rational function $\psi$ is a two-to-one mapping of $\mathbb{C} \setminus \{0\}$ onto $\mathbb{C}$ with the exception that $\psi(z) = \pm 2i$ has only one solution $z = \pm i$. Moreover, $\psi(0) = \psi(\infty) = \infty$ and $\psi$ is a continuous mapping of the Riemann sphere $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ onto itself. It is strictly increasing on the half-lines $]-\infty, 0[$ and $]0,\infty[$, mapping each of them onto $\mathbb{R}$.

The sequence $(\lambda_n)_n$ is defined in terms of $(m_n)_n$ from (3) by

$$\lambda_0 = 0, \quad \lambda_{n+1} = 1/m_n, \quad n \geq 0, \tag{13}$$

i.e.

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1+\sqrt{5}}{2}, \quad \lambda_3 = \frac{\sqrt{22+2\sqrt{5}}+\sqrt{5}+1}{4}, \cdots .$$

By (6) and (7) we clearly have

$$m_n = F(n),\ \lambda_n = f(n), \quad n \geq 0, \tag{14}$$

hence by (11)

$$\lambda_n = \psi(\lambda_{n+1}), \quad n \geq 0, \tag{15}$$

which can be reformulated to

$$\lambda_{n+1} = \frac{1}{2}\left(\lambda_n + \sqrt{\lambda_n^2 + 4}\right), \quad n \geq 0. \tag{16}$$

The main purpose of this paper is to prove that equation (12) holds for $f$ in the whole complex plane.

**Theorem 3** *Let* $a_n, b_n : \mathbb{C} \to \mathbb{C}$ *be the entire functions defined by*

$$a_n(z) = \lambda_n\left(\frac{\lambda_n}{\lambda_{n-1}}\right)^z, n \geq 2, \quad b_n(z) = \lambda_n\left(\frac{\lambda_{n+1}}{\lambda_n}\right)^z, n \geq 1. \tag{17}$$

*The meromorphic function* $f$ *from Theorem 1 is given for* $z \in \mathbb{C}$ *by*

$$f(z) = \lim_{n\to\infty} \psi^{\circ n}(a_n(z)) = \lim_{n\to\infty} \psi^{\circ n}(b_n(z)), \tag{18}$$

*and the convergence is uniform on compact subsets of* $\mathbb{C}$*.*

**Remark 4** In [6] it was proved that the Julia and Fatou sets of $\psi$ are respectively $\mathbb{R}^*$ and $\mathbb{C} \setminus \mathbb{R}$. For $z \in \mathbb{C} \setminus \mathbb{R}$ we have $\psi^{\circ n}(z) \to \infty$ for $n \to \infty$. Notice that $a_n(z), b_n(z)$ are close to $\lambda_n$ when $n$ is large because $\lambda_{n+1}/\lambda_n \to 1$ according to Lemma 5 below. Also $\psi^{\circ n}(\lambda_n) = 0$ for all $n$.

## 2 Proofs

For a domain $G \subseteq \mathbb{C}$ we denote by $\mathcal{H}(G)$ the space of holomorphic functions on $G$ equipped with the topology of uniform convergence on compact subsets of $G$.

We first recall some easily established properties of the sequence $(\lambda_n)_n$, which are needed in the proof of Theorem 3.

**Lemma 5 ([6])** *(1)* $\sqrt{n} \le \lambda_n \le \sqrt{2n}$, $n \ge 0$.
*(2)* $(\lambda_n)_n$ *is an increasing divergent sequence and* $\lambda_{n+1}/\lambda_n$ *is decreasing with* $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$.
*(3)* $\lim_{n\to\infty} (\lambda_{n+1}^2 - \lambda_n^2) = 2$.
*(4)* $\lim_{n\to\infty} \frac{\lambda_n^2}{n} = 2$.

The following lemma gives monotonicity properties of the sequences in (18).

**Lemma 6** *The sequence* $\left(\psi^{\circ n}(a_n(s))\right)_n$ *is decreasing for* $s > 0$. *The sequence* $\left(\psi^{\circ n}(b_n(s))\right)_n$ *is increasing for* $0 < s < 1$ *and decreasing for* $1 < s$, *while* $\psi^{\circ n}(b_n(1)) = 1$ *for all* $n \ge 1$.

*Proof:* Define the function

$$\rho(y) = \log(2\sinh y) = \log \psi(e^y), \quad y > 0, \tag{19}$$

which is easily seen to be increasing and concave, and positive for $y > \log \lambda_2$. It satisfies

$$\rho(\log \lambda_{n+1}) = \log \lambda_n, \quad n \ge 1. \tag{20}$$

Let $s > 0$. The slope of the chord of $\rho$ over the interval $[\log \lambda_n, \log \lambda_{n+1}]$ majorizes the slope over $[\log \lambda_{n+1}, \log \lambda_{n+1} + s\log(\lambda_{n+1}/\lambda_n)]$ by concavity, i.e.

$$\frac{\rho(\log \lambda_{n+1}) - \rho(\log \lambda_n)}{\log \lambda_{n+1} - \log \lambda_n} \ge \frac{\rho(\log \lambda_{n+1} + s\log(\lambda_{n+1}/\lambda_n)) - \rho(\log \lambda_{n+1})}{s\log(\lambda_{n+1}/\lambda_n)},$$

hence using (20)

$$\rho(\log(a_{n+1}(s)) \le \log a_n(s),$$

or

$$\psi(a_{n+1}(s)) \leq a_n(s),$$

which shows that the sequence $(\psi^{\circ n}(a_n(s)))_n$ is decreasing.

For $s > 1$ the slope of the chord of $\rho$ over the interval $[\log \lambda_{n+1}, \log \lambda_{n+2}]$ majorizes the slope over $[\log \lambda_{n+1}, \log \lambda_{n+1} + s \log(\lambda_{n+2}/\lambda_{n+1})]$ by concavity, i.e.

$$\frac{\rho(\log \lambda_{n+2}) - \rho(\log \lambda_{n+1})}{\log \lambda_{n+2} - \log \lambda_{n+1}} \geq \frac{\rho(\log \lambda_{n+1} + s \log(\lambda_{n+2}/\lambda_{n+1})) - \rho(\log \lambda_{n+1})}{s \log(\lambda_{n+2}/\lambda_{n+1})},$$

hence using (20)

$$\rho(\log(b_{n+1}(s)) \leq \log b_n(s),$$

or

$$\psi(b_{n+1}(s)) \leq b_n(s),$$

which shows that the sequence $(\psi^{\circ n}(b_n(s)))_n$ is decreasing.

If $0 < s < 1$ the inequality between the slopes is reversed and we get that $(\psi^{\circ n}(b_n(s)))_n$ is increasing. $\square$

*Proof of Theorem 3.*

The proof is given in a number of steps.

$1^\circ$ : *For any* $0 < s < \infty$ *we have*

$$\lim_{n\to\infty} [\psi^{\circ n}(a_n(s)) - \psi^{\circ n}(b_n(s))] = 0.$$

Note that the quantity under the limit is positive because $a_n(s) > b_n(s)$ and $\psi$ is increasing.

By the mean value theorem we get for a certain $w \in ]b_n(s), a_n(s)[$

$$\psi^{\circ n}(a_n(s)) - \psi^{\circ n}(b_n(s)) = (a_n(s) - b_n(s))(\psi^{\circ n})'(w)$$

$$= (a_n(s) - b_n(s))\psi'(\psi^{\circ n-1}(w))\psi'(\psi^{\circ n-2}(w)) \cdots \psi'(w).$$

Since $\lambda_n < b_n(s) < w < a_n(s)$, we get $\lambda_{n-k} < \psi^{\circ k}(b_n(s)) < \psi^{\circ k}(w)$, $k = 0, 1, \ldots, n$, hence

$$
\begin{aligned}
&|\psi^{\circ n}(a_n(s)) - \psi^{\circ n}(b_n(s))| \\
&\le |a_n(s) - b_n(s)| \prod_{k=0}^{n-1} |\psi'(\psi^{\circ k}(w))| \\
&\le |a_n(s) - b_n(s)| \prod_{k=0}^{n-1} \left(1 + \frac{1}{\lambda_{n-k}^2}\right) \\
&= \lambda_n \left(\left(\frac{\lambda_n}{\lambda_{n-1}}\right)^s - \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^s\right) \prod_{k=1}^{n} \left(1 + \frac{1}{\lambda_k^2}\right) \\
&\le \lambda_n \left(\left(\frac{\lambda_n}{\lambda_{n-1}}\right)^s - \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^s\right) \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) \\
&= (n+1)\lambda_n \left(\left(\frac{\lambda_n}{\lambda_{n-1}}\right)^s - \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^s\right),
\end{aligned}
$$

where we have used $\sqrt{k} \le \lambda_k$ from Lemma 5 part 1.

For $1 < y < x$ there exists $y < \xi < x$ such that

$$
x^s - y^s = s(x-y)\xi^{s-1} \le \begin{cases} s(x-y) & \text{if } 0 < s \le 1 \\ s x^{s-1}(x-y) & \text{if } 1 < s, \end{cases} \tag{21}
$$

and using $\lambda_n/\lambda_{n-1} \le \lambda_2$ for $n \ge 2$ we get

$$
\left(\frac{\lambda_n}{\lambda_{n-1}}\right)^s - \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^s \le s \max(\lambda_2^{s-1}, 1) \left(\frac{\lambda_n}{\lambda_{n-1}} - \frac{\lambda_{n+1}}{\lambda_n}\right).
$$

By (16) we have

$$
\begin{aligned}
&\frac{\lambda_n}{\lambda_{n-1}} - \frac{\lambda_{n+1}}{\lambda_n} \\
&= \frac{1}{2}\left(\sqrt{1 + \frac{4}{\lambda_{n-1}^2}} - \sqrt{1 + \frac{4}{\lambda_n^2}}\right) = \frac{2\left(\frac{1}{\lambda_{n-1}^2} - \frac{1}{\lambda_n^2}\right)}{\sqrt{1 + \frac{4}{\lambda_{n-1}^2}} + \sqrt{1 + \frac{4}{\lambda_n^2}}} \le \frac{\lambda_n^2 - \lambda_{n-1}^2}{\lambda_n^2 \lambda_{n-1}^2},
\end{aligned}
$$

so the final estimate is

$$
|\psi^{\circ n}(a_n(s)) - \psi^{\circ n}(b_n(s))| \le \max(\lambda_2^{s-1}, 1) \frac{s(n+1)}{\lambda_n \lambda_{n-1}^2} (\lambda_n^2 - \lambda_{n-1}^2),
$$

which tends to zero by part 2, 3 and 4 of Lemma 5.

$2^\circ$ : *The two sequences* $(\psi^{\circ n}(a_n(s)))_n$ *and* $(\psi^{\circ n}(b_n(s)))_n$ *have the same limit* $R(s)$ *for* $s > 0$*, and* $R$ *is a function satisfying the properties of Theorem 2, hence* $R(s) = f(s)$*.*

By the monotonicity of Lemma 6 and the inequality

$$\psi^{\circ n}(a_n(s)) > \psi^{\circ n}(b_n(s)) > \psi^{\circ n}(\lambda_n) = 0,$$

it follows that the two sequences converge. By $1^\circ$ the limits are the same, which we denote by $R(s)$. The inequality $R(s) > 0$ for $s > 0$ follows for $0 < s \le 1$ from $R(s) \ge \psi(b_1(s)) = \psi(\lambda_2^s)$ and for $s > 1$ from $\psi^{\circ n}(b_n(s)) \ge \psi^{\circ n}(\lambda_{n+1}) = 1$. Clearly $R(1) = 1$. In particular we have

$$\psi^{\circ(n+1)}(a_{n+1}(s)) \to R(s), \quad \psi^{\circ n}(b_n(s+1)) \to R(s+1),$$

and using

$$\varphi(x) = (1/2)(x + \sqrt{x^2+4}), \quad x \in \mathbb{R} \tag{22}$$

as the continuous inverse of $\psi|\,]0,\infty[$ we get

$$\psi^{\circ n}(a_{n+1}(s)) \to \varphi(R(s)).$$

From the equation $a_{n+1}(s) = b_n(s+1)$ we finally get $\psi(R(s+1)) = R(s)$.

Iterating (19) leads to $\rho^{\circ n}(y) = \log \psi^{\circ n}(e^y)$ for all $n \ge 1$, and therefore

$$\log \psi^{\circ n}(a_n(s)) = \rho^{\circ n}(\log \lambda_n + s \log(\lambda_n/\lambda_{n-1}))$$

is a sequence of concave functions of $s > 0$. Therefore the limit $\log R(s)$ is concave. It now follows from the uniqueness part of Theorem 2 that $f(s) = R(s)$ for $s > 0$.

**Remark 7** The inequality

$$\psi^{\circ n}(b_n(s)) \le f(s) \le \psi^{\circ n}(a_n(s)), \quad 0 < s \le 1, \tag{23}$$

follows from the monotonicity of Lemma 6 because $R(s) = f(s)$. It can however be seen directly from the properties of $f$ without using the uniqueness part of Theorem 2. Combining (23) with the result of $1^\circ$, we then see that the sequences $(\psi^{\circ n}(a_n(s)))_n$ and $(\psi^{\circ n}(b_n(s)))_n$ converge to $f(s)$ for $0 < s \le 1$.

To see (23), we notice that for any convex function $h$ on $]0,\infty[$ we have

$$h(n) - h(n-1) \le \frac{h(n+s) - h(n)}{s} \le h(n+1) - h(n), \quad 0 < s \le 1, n \ge 2.$$

By taking $h = \log(1/f)$, we get using $f(n) = \lambda_n$

$$\log \frac{\lambda_{n-1}}{\lambda_n} \le \frac{1}{s} \log \frac{\lambda_n}{f(n+s)} \le \log \frac{\lambda_n}{\lambda_{n+1}},$$

hence

$$\lambda_n \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^s \le f(n+s) \le \lambda_n \left(\frac{\lambda_n}{\lambda_{n-1}}\right)^s, \quad 0 < s \le 1,$$

and (23) follows by applying $\psi^{\circ n}$.

In the following steps we need the quantity

$$\rho_{n,N} = \lambda_n - \lambda_{n-N} = \sum_{k=1}^{N} \frac{1}{\lambda_{n+1-k}}, \quad n \ge N \ge 1, \tag{24}$$

where the second equality follows by summing $\lambda_j - \lambda_{j-1} = 1/\lambda_j$ from $j = n+1-N$ to $j = n$.

We denote $D(a,r) = \{z \in \mathbb{C} \mid |z-a| < r\}$.

$3^\circ$ : *For* $N \in \mathbb{N}, 0 < c \le 1, n > N$

$$\psi(D(\lambda_n, c\rho_{n,N})) \subseteq D(\lambda_{n-1}, c\rho_{n-1,N}).$$

If $|z - \lambda_n| < c\rho_{n,N}$ we get

$$|\psi(z) - \lambda_{n-1}| = |z - \lambda_n - (\frac{1}{z} - \frac{1}{\lambda_n})| < c\rho_{n,N}(1 + \frac{1}{|z|\lambda_n})$$

and

$$|z| = |\lambda_n - (\lambda_n - z)| \ge \lambda_n - |\lambda_n - z| > \lambda_n - \rho_{n,N} = \lambda_{n-N},$$

hence

$$|\psi(z) - \lambda_{n-1}| < c\rho_{n,N}\left(1 + \frac{1}{\lambda_n \lambda_{n-N}}\right) = c\left(\sum_{k=1}^{N} \frac{1}{\lambda_{n+1-k}} + \frac{\lambda_n - \lambda_{n-N}}{\lambda_n \lambda_{n-N}}\right) = c\rho_{n-1,N}.$$

Iterating $n - N$ times using $\rho_{N,N} = \lambda_N - \lambda_0 = \lambda_N$ we get

$4^\circ$ : *For* $1 \le N \le n$ *and* $0 < c \le 1$

$$\psi^{\circ(n-N)}(D(\lambda_n, c\rho_{n,N})) \subseteq D(\lambda_N, c\lambda_N).$$

5°: *For* $0 < c \le 1, |z| \le cN, N \le n$ *we have* $b_n(z) \in D(\lambda_n, c\rho_{n,N})$.

For $a > 1$ and $z \in \mathbb{C}, |z| \le 1$ we have the elementary inequality

$$|a^z - 1| \le |z|(a-1). \tag{25}$$

Applying this with $a = (\lambda_{n+1}/\lambda_n)^N$ we get

$$|b_n(z) - \lambda_n| = \lambda_n |(\frac{\lambda_{n+1}}{\lambda_n})^z - 1| \le \lambda_n c \left( (\frac{\lambda_{n+1}}{\lambda_n})^N - 1 \right)$$

$$= \lambda_n c (\frac{\lambda_{n+1}}{\lambda_n} - 1) \sum_{k=0}^{N-1} (\frac{\lambda_{n+1}}{\lambda_n})^k = \frac{c}{\lambda_{n+1}} \sum_{k=0}^{N-1} (\frac{\lambda_{n+1}}{\lambda_n})^k$$

$$= c \left( \frac{1}{\lambda_{n+1}} + \sum_{k=1}^{N-1} \frac{\lambda_{n+1}^{k-1}}{\lambda_n^k} \right) \le c \sum_{k=0}^{N-1} \frac{1}{\lambda_{n-k+1}} < c \sum_{k=0}^{N-1} \frac{1}{\lambda_{n-k}} = c\rho_{n,N},$$

where we have used the inequalities

$$\lambda_{n+1}^{k-1} \lambda_{n-k+1} \le \lambda_n^k, \quad k = 1, \dots, N-1, \tag{26}$$

which are equivalent to

$$(k-1) \log f(n+1) + \log f(n-k+1) \le k \log f(n),$$

but they hold because $R = \log f$ is concave.

Combining 4° and 5° we get

6° : *For* $0 < c \le 1, n \ge N, |z| \le cN$

$$\psi^{\circ(n-N)}(b_n(z)) \in D(\lambda_N, c\lambda_N).$$

*In particular, the sequence* $\psi^{\circ(n-N)}(b_n(z)), n \ge N$ *of holomorphic functions in the disc* $D(0, N)$ *is bounded on compact subsets of this disc.*

7° : *For* $0 < s < \infty, n \ge N$ *we have*

$$\lim_{n \to \infty} \psi^{\circ(n-N)}(b_n(s)) = f(s+N).$$

Since $b_n(s) > \lambda_n$ we know that $\psi^{\circ(n-N)}(b_n(s)) > \lambda_N$. For each $N \ge 1$ we see that $\psi^{\circ N} |\, ]\lambda_{N-1}, \infty[ \to \mathbb{R}$ is a homeomorphism with inverse $\varphi^{\circ N}$, where $\varphi$ is given by (22). Since $\psi^{\circ n}(b_n(s)) \to f(s)$ we get

$$\varphi^{\circ N}(\psi^{\circ n}(b_n(s)) \to \varphi^{\circ N}(f(s)) = f(s+N)$$

i.e.

$$\lim_{n\to\infty} \psi^{\circ(n-N)}(b_n(s)) = f(s+N).$$

$8^\circ$ : *Let $N \in \mathbb{N}$. For $z \in D(0,N)$ we have*

$$\lim_{n\to\infty} \psi^{\circ(n-N)}(b_n(z)) = f(z+N),$$

*and the convergence is uniform on compact subsets of $D(0,N)$.*

By Montel's theorem the sequence $\psi^{\circ(n-N)}(b_n(z))$ has accumulation points $h$ in the space $\mathcal{H}(D(0,N))$ of holomorphic functions on $D(0,N)$. By $7^\circ$ we know that $h(s) = f(s+N)$ for $0 < s < N$. From the uniqueness theorem for holomorphic functions, all accumulation points then agree with $f(z+N) \in \mathcal{H}(D(0,N))$, and the result follows.

$9^\circ$: *For $z \in \mathbb{C}$ we have*

$$\lim_{n\to\infty} \psi^{\circ n}(b_n(z)) = f(z),$$

*uniformly on compact subsets of $\mathbb{C}$.*

For a compact subset $K \subset \mathbb{C}$ we choose $N \in \mathbb{N}$ such that $K \subset D(0,N)$ and know by $8^\circ$ that $\psi^{\circ(n-N)}(b_n(z))$ converges uniformly to $f(z+N)$ for $z \in K$. We next use that $\psi^{\circ N} : \mathbb{C}^* \to \mathbb{C}^*$ is continuous, hence uniformly continuous with respect to the chordal metric on $\mathbb{C}^*$, and since $\psi^{\circ N}(f(z+N)) = f(z)$, the result follows.

$10^\circ$ : *For $z \in \mathbb{C}$ we have*

$$\lim_{n\to\infty} \psi^{\circ n}(a_n(z)) = f(z),$$

*uniformly on compact subsets of $\mathbb{C}$.*

In fact,

$$\psi^{\circ n}(a_{n+1}(z)) = \psi^{\circ n}(b_n(z+1)) \to f(z+1)$$

so

$$\psi(\psi^{\circ n}(a_{n+1}(z))) \to \psi(f(z+1)) = f(z).$$

**Acknowledgment** The authors want to thank Michael Olesen, Nykøbing Katedralskole for useful comments to an early version of this manuscript.

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