Fibonacci numbers and orthogonal polynomials

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Abstract

We prove that the sequence $(1/F_{n+2})_{n\geq 0}$ of reciprocals of the Fibonacci numbers is a moment sequence of a certain discrete probability, and we identify the orthogonal polynomials as little q-Jacobi polynomials with $q = (1-\sqrt{5})/(1+\sqrt{5})$. We prove that the corresponding kernel polynomials have integer coefficients, and from this we deduce that the inverse of the corresponding Hankel matrices $(1/F_{i+j+2})$ have integer entries. We prove analogous results for the Hilbert matrices.

Key words: Fibonacci numbers, orthogonal polynomials

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1 Introduction

In [14] Richardson noticed that the *Filbert matrices*

$$\mathcal{F}_n = (1/F_{i+j+1}), \quad 0 \leq i, j \leq n, \quad n = 0, 1, \ldots, \tag{1}$$

where $F_n, n \geq 0$ is the sequence of Fibonacci numbers, have the property that all elements of the inverse matrices are integers. The corresponding property for the *Hilbert matrices* $(1/(i+j+1))$ has been known for a long time, see [7],[6],[15]. The last reference contains a table of the inverse Hilbert matrices up to $n = 9$.

Richardson gave an explicit formula for the elements of the inverse Filbert matrices and proved it using computer algebra. The formula shows a remarkable analogy with the corresponding formula for the elements of the inverse Hilbert matrices in the sense that one shall replace some binomial coefficients $\binom{n}{k}$ by the analogous *Fibonomial coefficients*

$$\binom{n}{k}_{\mathbb{F}} = \prod_{i=1}^{k} \frac{F_{n-i+1}}{F_i}, \quad 0 \le k \le n, \tag{2}$$

with the usual convention that empty products are defined as 1. These coefficients are defined and studied in [12] and are integers. The sequence of Fibonacci numbers is $F_0 = 0, F_1 = 1, \ldots,$ with the recursion formula $F_{n+1} = F_n + F_{n-1},\ n \ge 1$.

The Hilbert matrices are the Hankel matrices (s_{i+j}) corresponding to the moment sequence

$$s_n = 1/(n+1) = \int_0^1 x^n \, dx,$$

and that the reciprocal matrices have integer entries can easily be explained by the fact the corresponding orthogonal polynomials, namely the Legendre polynomials, have integer coefficients. See section 4 for details.

The purpose of the present paper is to show that $(1/F_{n+2})_{n\ge 0}$ is the moment sequence of a certain discrete probability. Although this is a simple consequence of Binet's formula for F_n, it does not seem to have been noticed in the literature, cf. [13]. We find the corresponding probability measure to be

$$\mu = (1-q^2)\sum_{k=0}^{\infty} q^{2k}\delta_{q^k/\phi}, \tag{3}$$

where we use the notation

$$\phi = \frac{1+\sqrt{5}}{2}, \quad q = \frac{1-\sqrt{5}}{1+\sqrt{5}} = \frac{1}{\phi} - 1, \tag{4}$$

and δ_a denotes the probability measure with mass 1 at the point a. The number ϕ is called the golden ratio.

The corresponding orthogonal polynomials are little q-Jacobi polynomials

$$p_n(x;a,b;q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, xq\right), \tag{5}$$

see [8], specialized to the parameters $a = q, b = 1$, with q taking the value from (4).

To be precise we define

$$p_n(x) := F_{n+1} p_n(x\phi; q, 1; q), \tag{6}$$

and these polynomials have integer coefficients, since they can be written

$$p_n(x) = \sum_{k=0}^{n} (-1)^{kn - \binom{k}{2}} \binom{n}{k}_{\mathbb{F}} \binom{n+k+1}{n}_{\mathbb{F}} x^k. \tag{7}$$

The orthonormal polynomials with respect to μ and having positive leading coefficients are given as

$$P_n(x) = (-1)^{\binom{n+1}{2}} \sqrt{F_{2n+2}}\, p_n(x), \tag{8}$$

so the kernel polynomial

$$K_n(x,y) = \sum_{k=0}^{n} P_k(x) P_k(y),$$

is a polynomial in x, y with integer coefficients. If we denote $a_{i,j}^{(n)}$ the coefficient to $x^i y^j$ in the kernel polynomial, then it is a general fact that the matrix

$$A_n = (a_{i,j}^{(n)}), \quad 0 \le i, j \le n \tag{9}$$

is the inverse of the Hankel matrix of the problem $(s_{i+j})_0^n$, see Theorem 2.1 below.

This explains that the elements of the inverse of the matrix $(1/F_{i+j+2})_0^n$ are integers, and we derive a formula for the entries from the orthogonal polynomials.

The Filbert matrices (1) are not positive definite but non-singular, and they are the Hankel matrices of the moments of a (real-valued) signed measure with total mass 1. The orthogonal polynomials for this signed measure are the little q-Jacobi polynomials

$$p_n(x\phi; 1, 1; q) = \sum_{k=0}^{n} (-1)^{kn - \binom{k}{2}} \binom{n}{k}_{\mathbb{F}} \binom{n+k}{n}_{\mathbb{F}} x^k, \tag{10}$$

and a simple modification of the positive definite case leads to Richardson's formula for the entries of the inverse of the Filbert matrices.

The two results can be unified in the statement that for each $\alpha \in \mathbb{N} = \{1, 2, \ldots\}$ the sequence $(F_\alpha/F_{\alpha+n})_{n\geq 0}$ is a moment sequence of a real-valued measure μ_α with total mass 1. It is a positive measure when α is even, but a signed measure when α is odd. The orthogonal polynomials are little q-Jacobi polynomials $p_n(x\phi; q^{\alpha-1}, 1; q)$. This is proved in section 3.

In section 2 we recall some basic things about orthogonal polynomials both in the positive definite and in the quasi-definite case, and Theorem 2.1 about the inverse of the Hankel matrices is proved.

In section 4 we briefly discuss the matrices $(1/(\alpha+i+j))_0^n$, where $\alpha > 0$. They are related to Jacobi polynomials transfered to the interval $]0, 1[$ and belonging to the parameters $(0, \alpha-1)$. This leads to the formula (36), which for $\alpha = 1$ is the formula for the elements of the inverse Hilbert matrices.

After the circulation of a preliminary version of this paper (dated April 10, 2006), Ismail has extended the results of section 3 to a one parameter generalization of the Fibonacci numbers, cf. [11].

2 Orthogonal Polynomials

We start by recalling some simple facts from the theory of orthogonal polynomials, cf. [1] or [10] and in particular [5] for the quasi-definite case.

The positive definite case.

We consider the set $\mathcal{M}^*$ of probability measures on $\mathbb{R}$ with moments of any order and with infinite support. The moment sequence of $\mu \in \mathcal{M}^*$ is

$$s_n = s_n(\mu) = \int x^n \, d\mu(x), \quad n = 0, 1, \ldots, \tag{11}$$

and the corresponding Hankel matrices are given by

$$H_n = \begin{pmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{pmatrix}, \quad n = 0, 1, \ldots. \tag{12}$$

The orthonormal polynomials (P_n) for μ are uniquely determined by the equations

$$\int P_n(x) P_m(x) \, d\mu(x) = \delta_{n,m}, \quad n, m \geq 0, \tag{13}$$

and the requirement that P_n is a polynomial of degree n with positive leading coefficient. This coefficient is equal to

$$\sqrt{D_{n-1}/D_n}, \tag{14}$$

where $D_n = \det H_n$. The reproducing kernel for the polynomials of degree $\le n$ is defined as

$$K_n(x, y) = \sum_{k=0}^{n} P_k(x)P_k(y), \tag{15}$$

and is called the kernel polynomial. It is clear that we can write

$$K_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{i,j}^{(n)} x^i y^j, \tag{16}$$

where the numbers $a_{i,j}^{(n)}$ are uniquely determined and satisfy $a_{i,j}^{(n)} = a_{j,i}^{(n)}$. If we collect these numbers in an $(n+1) \times (n+1)$-matrix $A_n = (a_{i,j}^{(n)})$, then it is the inverse of the Hankel matrix H_n:

Theorem 2.1

$$A_n H_n = H_n A_n = I_n,$$

where I_n is the unit matrix of order $n+1$.

Proof. For $0 \le k \le n$ we have by the reproducing property

$$\int x^k K_n(x, y)\, d\mu(x) = y^k. \tag{17}$$

On the other hand we have

$$\int x^k K_n(x, y)\, d\mu(x) = \sum_{j=0}^{n} (\sum_{i=0}^{n} s_{k+i} a_{i,j}^{(n)}) y^j,$$

and therefore

$$\sum_{i=0}^{n} s_{k+i} a_{i,j}^{(n)} = \delta_{k,j}.$$

□

The quasi-definite case.

If μ is a real-valued signed measure on $\mathbb{R}$ with total mass 1 and moments of any order, one can still define the moments (11) and the corresponding Hankel matrices (12). To define orthogonal polynomials one has to assume that (12) is a non-singular matrix for any n, i.e. that the determinants satisfy $D_n = \det H_n \neq 0$. On the other hand, if orthogonal polynomials exist with respect to a signed measure, then the Hankel determinants are non-zero. See [5, Theorem 3.1] for details. In this case the orthonormal polynomial P_n is uniquely determined by the requirement that the leading coefficient $\sqrt{D_{n-1}/D_n}$ is either positive or purely imaginary with positive imaginary part. The corresponding kernel polynomial K_n has real coefficients, and Theorem 2.1 remains valid.

3 Fibonacci numbers

The Fibonacci numbers can be given by the formula

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n), \quad n \geq 0 \tag{18}$$

usually called Binet's formula, but it is actually older, see [12],[13]. Here

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \hat{\phi} = \frac{1-\sqrt{5}}{2} = 1 - \phi.$$

Using the number $q = \hat{\phi}/\phi$, satisfying $-1 < q < 0$ and already defined in (4), leads to

$$F_n = \frac{1}{\sqrt{5}}\phi^n(1-q^n), \quad q\phi^2 = -1, \tag{19}$$

and for $\alpha \in \mathbb{N}$ and $n \geq 0$

$$\frac{F_\alpha}{F_{\alpha+n}} = \frac{\sqrt{5}F_\alpha}{\phi^{\alpha+n}} \frac{1}{1-q^{\alpha+n}} = (1-q^\alpha)\sum_{k=0}^{\infty}(q^k/\phi)^n q^{\alpha k},$$

which is the n'th moment of the real-valued measure

$$\mu_\alpha = (1-q^\alpha)\sum_{k=0}^{\infty} q^{\alpha k}\delta_{q^k/\phi} \tag{20}$$

with total mass 1. When α is even then μ_α is a probability measure, but when α is odd the masses $q^{\alpha k}$ change sign with the parity of k. Note that μ_2 is the measure considered in (3).

For the Fibonomial coefficients defined in (2) one has

$$\binom{n}{k}_{\mathbb{F}} = 1,\ 0 \le k \le n \le 2,$$

and they satisfy a recursion formula

$$\binom{n}{k}_{\mathbb{F}} = F_{k-1}\binom{n-1}{k}_{\mathbb{F}} + F_{n-k+1}\binom{n-1}{k-1}_{\mathbb{F}},\ n > k \ge 1, \tag{21}$$

see [12], which shows that the Fibonomial coefficients are integers. From (2) it is also clear that

$$\binom{n}{k}_{\mathbb{F}} = \binom{n}{n-k}_{\mathbb{F}}, \quad 0 \le k \le n.$$

In [8, Section 7.3] one finds a discussion of the little q-Jacobi polynomials defined in (5), and it is proved that

$$\sum_{k=0}^{\infty} p_n(q^k; a, b; q) p_m(q^k; a, b; q) \frac{(bq;q)_k}{(q;q)_k}(aq)^k = \frac{\delta_{n,m}}{h_n(a,b;q)}, \tag{22}$$

where

$$h_n(a,b;q) = \frac{(abq;q)_n(1-abq^{2n+1})(aq;q)_n(aq;q)_\infty}{(q;q)_n(1-abq)(bq;q)_n(abq^2;q)_\infty}(aq)^{-n}. \tag{23}$$

In [8] it is assumed that $0 < q, aq < 1$, but the derivation shows that it holds for $|q| < 1, |a| \le 1, |b| \le 1$, in particular in the case of interest here: $-1 < q < 0, a = q^{\alpha-1}, b = 1$, in the case of which we get

$$\sum_{k=0}^{\infty} p_n(q^k; q^{\alpha-1}, 1; q) p_m(q^k; q^{\alpha-1}, 1; q) q^{\alpha k} = \delta_{n,m} \frac{q^{\alpha n}(q;q)_n^2}{(q^\alpha;q)_n^2(1-q^{\alpha+2n})}. \tag{24}$$

This shows that the polynomials

$$p_n(x\phi; q^{\alpha-1}, 1; q)$$

are orthogonal with respect to μ_α and that

$$\int p_n(x\phi; q^{\alpha-1}, 1; q) p_m(x\phi; q^{\alpha-1}, 1; q)\, d\mu_\alpha(x) = \delta_{n,m} \frac{(1-q^\alpha)q^{\alpha n}(q;q)_n^2}{(q^\alpha;q)_n^2(1-q^{\alpha+2n})}.$$

To simplify this apply (19) to get

$$\frac{(1-q^\alpha)q^{\alpha n}(q;q)_n^2}{(q^\alpha;q)_n^2(1-q^{\alpha+2n})} = (-1)^{\alpha n}\tfrac{F_\alpha}{F_{\alpha+2n}}\left(\prod_{j=0}^{n-1}\tfrac{F_{1+j}}{F_{\alpha+j}}\right)^2 = (-1)^{\alpha n}\tfrac{F_\alpha}{F_{\alpha+2n}}\tbinom{\alpha+n-1}{n}_{\mathbb{F}}^{-2}.$$

Theorem 3.1 *Let $\alpha \in \mathbb{N}$. The polynomials $p_n^{(\alpha)}(x)$ defined by*

$$p_n^{(\alpha)}(x) = \tbinom{\alpha+n-1}{n}_{\mathbb{F}} p_n(x\phi;q^{\alpha-1},1;q) \tag{25}$$

can be written

$$p_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^{kn-\binom{k}{2}}\tbinom{n}{k}_{\mathbb{F}}\tbinom{\alpha+n+k-1}{n}_{\mathbb{F}}x^k, \tag{26}$$

and they satisfy

$$\int p_n^{(\alpha)}(x)p_m^{(\alpha)}(x)\,d\mu_\alpha(x) = \delta_{n,m}(-1)^{\alpha n}\tfrac{F_\alpha}{F_{\alpha+2n}}, \tag{27}$$

so the corresponding orthonormal polynomials are

$$P_n^{(\alpha)}(x) = \sqrt{(-1)^{\alpha n}F_{\alpha+2n}/F_\alpha}\,p_n^{(\alpha)}(x). \tag{28}$$

Proof. By definition, see (5)

$$\begin{aligned} p_n^{(\alpha)}(x) &= \tbinom{\alpha+n-1}{n}_{\mathbb{F}}\sum_{k=0}^{n}\frac{(q^{-n},q^{\alpha+n};q)_k}{(q,q^\alpha;q)_k}(q\phi x)^k \\ &= \tbinom{\alpha+n-1}{n}_{\mathbb{F}}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q\frac{(q^{\alpha+n};q)_k}{(q^\alpha;q)_k}(-1)^k q^{\binom{k}{2}-nk}(q\phi x)^k, \end{aligned}$$

where

$$\begin{bmatrix} n \\ k\end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

is the q-binomial coefficient. Using (19) leads to

$$\begin{bmatrix} n \\ k\end{bmatrix}_q = \binom{n}{k}_{\mathbb{F}}\phi^{k(k-n)},$$

hence

$$p_n^{(\alpha)}(x) = \tbinom{\alpha+n-1}{n}_{\mathbb{F}}\sum_{k=0}^{n}(-1)^k\tbinom{n}{k}_{\mathbb{F}}(\phi^2 q)^{\binom{k+1}{2}-nk}\prod_{j=0}^{k-1}\tfrac{F_{\alpha+n+j}}{F_{\alpha+j}}x^k,$$

which by (19) can be reduced to (26). □

Remark 3.2 The polynomials $p_n^{(\alpha)}(x)$ for $\alpha = 1$ and $\alpha = 2$ are the polynomials in (10) and in (7) respectively.

Corollary 3.3 *For* $\alpha \in \mathbb{N}$

$$\det(1/F_{\alpha+i+j})_0^n = \left((-1)^{\alpha\binom{n+1}{2}} F_\alpha \prod_{k=1}^{n} F_{\alpha+2k} \binom{\alpha+2k-1}{k}_{\mathbb{F}}^2\right)^{-1},$$

which is the reciprocal of an integer.

Proof. From the general theory it is known that the leading coefficient of the orthonormal polynomial $P_n^{(\alpha)}$ is $\sqrt{D_{n-1}/D_n}$, where

$$D_n = \det(F_\alpha/F_{\alpha+i+j})_0^n.$$

From (26) and (28) we then get

$$D_{n-1}/D_n = (-1)^{\alpha n} \tfrac{F_{\alpha+2n}}{F_\alpha} \binom{\alpha+2n-1}{n}_{\mathbb{F}}^2,$$

hence

$$\frac{1}{D_n} = \prod_{k=1}^{n} \frac{D_{k-1}}{D_k} = (-1)^{\alpha\binom{n+1}{2}} \tfrac{1}{F_\alpha^n} \prod_{k=1}^{n} F_{\alpha+2k} \binom{\alpha+2k-1}{k}_{\mathbb{F}}^2$$

and the formula follows. □

Theorem 3.4 *The* i, j*'th entry of the inverse of the matrix* $(1/F_{\alpha+i+j})_0^n$ *is given as*

$$(-1)^{n(\alpha+i+j)-\binom{i}{2}-\binom{j}{2}} F_{\alpha+i+j} \binom{\alpha+n+i}{n-j}_{\mathbb{F}} \binom{\alpha+n+j}{n-i}_{\mathbb{F}} \binom{\alpha+i+j-1}{i}_{\mathbb{F}} \binom{\alpha+i+j-1}{j}_{\mathbb{F}}. \qquad (29)$$

Proof. From Theorem 2.1 we get

$$\left((F_\alpha/F_{\alpha+i+j})_0^n\right)^{-1} = \left(a_{i,j}^{(n)}(\alpha)\right)_0^n,$$

where $a_{i,j}^{(n)}(\alpha)$ is the coefficient to $x^i y^j$ in the kernel polynomial $K_n(x, y)$ for the orthonormal polynomials $P_n^{(\alpha)}$. Inserting the expressions (26) and (28) in the kernel polynomial and changing the order of summation gives

$$F_\alpha a_{i,j}^{(n)}(\alpha) = \sum_{k=\max(i,j)}^{n} C^{(\alpha)}(k; i, j),$$

where we for $k \geq i, j$ have defined

$$C^{(\alpha)}(k; i, j) := (-1)^{k(\alpha+i+j)-\binom{i}{2}-\binom{j}{2}} F_{\alpha+2k} \binom{k}{i}_{\mathbb{F}} \binom{k}{j}_{\mathbb{F}} \binom{\alpha+k+i-1}{k}_{\mathbb{F}} \binom{\alpha+k+j-1}{k}_{\mathbb{F}}. \tag{30}$$

To prove that this expression can be summed to give (29), we use induction in n. By symmetry we can always assume $i \geq j$. The starting step $n = k = i \geq j$ is easy and is left to the reader. For the induction step let $R^{(\alpha)}(n; i, j)$ denote the expression (29). It has to be established that

$$R^{(\alpha)}(n+1; i, j) - R^{(\alpha)}(n; i, j) = C^{(\alpha)}(n+1; i, j).$$

The left-hand side of this expression can be written

$$(-1)^{(n+1)(\alpha+i+j)-\binom{i}{2}-\binom{j}{2}} F_{\alpha+i+j} \binom{\alpha+i+j-1}{i}_{\mathbb{F}} \binom{\alpha+i+j-1}{j}_{\mathbb{F}} T,$$

where

$$\begin{aligned} T &= \binom{\alpha+n+1+i}{n+1-j}_{\mathbb{F}} \binom{\alpha+n+1+j}{n+1-i}_{\mathbb{F}} - (-1)^{\alpha+i+j} \binom{\alpha+n+i}{n-j}_{\mathbb{F}} \binom{\alpha+n+j}{n-i}_{\mathbb{F}} \\ &= \frac{(F_{\alpha+n+i} \cdots F_{\alpha+i+j+1})(F_{\alpha+n+j} \cdots F_{\alpha+i+j+1})}{(F_1 \cdots F_{n+1-j})(F_1 \cdots F_{n+1-i})} \cdot \\ &\quad \left[F_{\alpha+n+i+1} F_{\alpha+n+j+1} - (-1)^{\alpha+i+j} F_{n+1-i} F_{n+1-j}\right]. \end{aligned}$$

By Lemma 3.5 below (with n replaced by $n+1$), the expression in brackets equals $F_{\alpha+2n+2} F_{\alpha+i+j}$, and now it is easy to complete the proof. $\square$

Lemma 3.5 *For $n \geq i, j \geq 0$ and $\alpha \geq 0$ the following formula holds*

$$F_{\alpha+2n} F_{\alpha+i+j} = F_{\alpha+n+i} F_{\alpha+n+j} - (-1)^{\alpha+i+j} F_{n-i} F_{n-j}. \tag{31}$$

Proof. Using Binet's formula, the right-hand side of (31) multiplied with 5 equals

$$(\phi^{\alpha+n+i} - \hat{\phi}^{\alpha+n+i})(\phi^{\alpha+n+j} - \hat{\phi}^{\alpha+n+j}) - (-1)^{\alpha+i+j}(\phi^{n-i} - \hat{\phi}^{n-i})(\phi^{n-j} - \hat{\phi}^{n-j}).$$

Using $\phi\hat{\phi} = -1$ one gets after some simplification

$$(\phi^{\alpha+2n} - \hat{\phi}^{\alpha+2n})(\phi^{\alpha+i+j} - \hat{\phi}^{\alpha+i+j}),$$

which establishes the formula. $\square$

Remark 3.6 For $\alpha = 1$ the expression (29) reduces to

$$(-1)^{n(i+j+1)-\binom{i}{2}-\binom{j}{2}} F_{i+j+1} \binom{n+i+1}{n-j}_{\mathbb{F}} \binom{n+j+1}{n-i}_{\mathbb{F}} \binom{i+j}{i}_{\mathbb{F}}^2,$$

which is the expression found by Richardson [14], except that he expressed the sign in a different but equivalent manner.

4 The Hilbert matrices

For $\alpha > 0$ the matrices

$$\mathcal{H}_n^{(\alpha)} = (\alpha/(\alpha+i+j))_0^n\,, \quad n = 0, 1, \ldots, \tag{32}$$

are the Hankel matrices for the moment sequence

$$s_n^{(\alpha)} = \alpha \int_0^1 x^n x^{\alpha-1}\,dx = \frac{\alpha}{\alpha+n}, \quad n = 0, 1, \ldots$$

of the measure $\sigma_\alpha = \alpha x^{\alpha-1} 1_{]0,1[}(x)\,dx$. The corresponding orthogonal polynomials are easily seen to be

$$r_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha+1} D^n \left[x^{\alpha-1+n}(1-x)^n\right] = (-1)^n \sum_{k=0}^n \binom{n}{k}\binom{\alpha-1+n}{k}(x-1)^k x^{n-k}, \tag{33}$$

since they are Jacobi polynomials transfered to $]0,1[$, cf. [3]. Using the binomial formula for $(x-1)^k$ we find

$$r_n^{(\alpha)}(x) = (-1)^n \sum_{j=0}^n (-1)^j x^{n-j} c_j,$$

where

$$\begin{aligned} c_j &= \sum_{k=j}^n \binom{k}{j}\binom{n}{k}\binom{\alpha-1+n}{k} = \sum_{l=0}^{n-j} \binom{j+l}{j}\binom{n}{j+l}\binom{\alpha-1+n}{j+l} \\ &= \binom{n}{j}\binom{\alpha-1+n}{j} {}_2F_1\left(\begin{matrix}-n+j, -n-\alpha+j+1\\ j+1\end{matrix}; 1\right) = \binom{n}{j}\binom{2n+\alpha-j-1}{n}, \end{aligned}$$

where the ${}_2F_1$ is summed by the Chu-Vandermonde formula, cf. [3, p. 67]. This gives

$$r_n^{(\alpha)}(x) = \sum_{j=0}^n (-1)^j \binom{n}{j}\binom{\alpha+n+j-1}{n} x^j. \tag{34}$$

The orthonormal polynomials with positive leading coefficients are given as

$$R_n^{(\alpha)}(x) = (-1)^n \sqrt{\frac{\alpha+2n}{\alpha}}\, r_n^{(\alpha)}(x),$$

so the corresponding kernel polynomials have coefficients $a_{i,j}^{(n)}(\alpha)$ which by Theorem 2.1 satisfy

$$\alpha a_{i,j}^{(n)}(\alpha) = (-1)^{i+j} \sum_{k=\max(i,j)}^{n} (\alpha+2k)\binom{k}{i}\binom{k}{j}\binom{\alpha+k+i-1}{k}\binom{\alpha+k+j-1}{k}. \tag{35}$$

Theorem 4.1 *The i,j'th element of the inverse matrix of $(1/(\alpha+i+j))_0^n$ is given as*

$$(-1)^{i+j}(\alpha+i+j)\binom{\alpha+n+i}{n-j}\binom{\alpha+n+j}{n-i}\binom{\alpha+i+j-1}{i}\binom{\alpha+i+j-1}{j}. \tag{36}$$

In particular they are integers for $\alpha \in \mathbb{N}$. Furthermore,

$$\det\,(1/(\alpha+i+j))_0^n = \left(\alpha \prod_{k=1}^{n}(\alpha+2k)\binom{\alpha+2k-1}{k}^2\right)^{-1}. \tag{37}$$

Proof. Let $R(n;i,j)$ denote the number given in (36), and define

$$C(k;i,j) = (-1)^{i+j}(\alpha+2k)\binom{k}{i}\binom{k}{j}\binom{\alpha+k+i-1}{k}\binom{\alpha+k+j-1}{k}, \quad k \geq i,j.$$

We shall prove that

$$R(n;i,j) = \sum_{k=\max(i,j)}^{n} C(k;i,j)$$

by induction in n and can assume $i \geq j$. This is easy for $n = k = i$ and we shall establish

$$R(n+1;i,j) - R(n;i,j) = C(n+1;i,j). \tag{38}$$

The left-hand side of this expression can be written

$$(-1)^{i+j}(\alpha+i+j)\binom{\alpha+i+j-1}{i}\binom{\alpha+i+j-1}{j}T,$$

where

$$\begin{aligned} T &= \binom{\alpha+n+1+i}{n+1-j}\binom{\alpha+n+1+j}{n+1-i} - \binom{\alpha+n+i}{n-j}\binom{\alpha+n+j}{n-i} \\ &= \tfrac{((\alpha+n+i)\cdots(\alpha+i+j+1))((\alpha+n+j)\cdots(\alpha+i+j+1))}{(n+1-j)!(n+1-i)!} \cdot \\ &\quad [(\alpha+n+1+i)(\alpha+n+1+j) - (n+1-j)(n+1-i)]. \end{aligned}$$

The quantity in brackets equals $(\alpha+2n+2)(\alpha+i+j)$, and now it is easy to complete the proof of (38).

The leading coefficient of $R_n^{(\alpha)}(x)$ is

$$\sqrt{\frac{D_{n-1}}{D_n}} = \sqrt{\frac{\alpha+2n}{\alpha}}\binom{\alpha+2n-1}{n},$$

where

$$D_n = \det\left(\alpha/(\alpha+i+j)\right)_0^n = \alpha^{n+1}\det\left(1/(\alpha+i+j)\right)_0^n .$$

Therefore

$$\frac{1}{D_n} = \prod_{k=1}^{n}\frac{D_{k-1}}{D_k} = \frac{1}{\alpha^n}\prod_{k=1}^{n}(\alpha+2k)\binom{\alpha+2k-1}{k}^2,$$

which proves (37). □

Replacing x by $1-x$, we see that $r_n^{(\alpha)}(1-x)$ are orthogonal polynomials with respect to the probability measure $\alpha(1-x)^{\alpha-1}1_{]0,1[}(x)\,dx$. The corresponding moment sequence is

$$s_n = \frac{1}{\binom{\alpha+n}{n}}, \tag{39}$$

and the corresponding orthonormal polynomials are $\sqrt{(\alpha+2n)/\alpha}\; r_n^{(\alpha)}(1-x)$. Therefore

$$K_n(x,y) = \sum_{k=0}^{n}\frac{\alpha+2k}{\alpha}r_k^{(\alpha)}(1-x)r_k^{(\alpha)}(1-y), \tag{40}$$

showing that the coefficient to x^iy^j in $\alpha K_n(x,y)$ is an integer when $\alpha\in\mathbb{N}$. This yields

Theorem 4.2 *Let $\alpha\in\mathbb{N}$. The inverse of the matrix*

$$\left(\frac{1}{\alpha\binom{\alpha+i+j}{\alpha}}\right)_0^n \tag{41}$$

has integer entries.

It is not difficult to prove that

$$r_n^{(\alpha)}(1-x) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{\alpha+n+k-1}{k}x^k,$$

and it follows that the entries of the inverse of (41) are given as

$$(-1)^{i+j}\sum_{k=\max(i,j)}^{n}(\alpha+2k)\tbinom{k}{i}\tbinom{k}{j}\tbinom{\alpha+k+i-1}{i}\tbinom{\alpha+k+j-1}{j}.$$

This formula holds of course for any $\alpha > 0$.

The results of this section for $\alpha = 1, 2$ have been treated in the survey paper [4], written in Danish. For $\alpha = 1$ the formula for the elements of the inverse of $\mathcal{H}_n^{(\alpha)}$ was given in [6], but goes at least back to Collar [7], while the formula for its determinant goes back to Hilbert in [9]. In this case the polynomials $r_n^{(1)}(x)$ are the Legendre polynomials for the interval $[0, 1]$, cf. [3, Section 7.7]. These polynomials have succesfully been used in the proof of the irrationality of $\zeta(3)$. For $\alpha = 2$ we have $(\alpha + 2k)/\alpha = 1 + k$, so the coefficient to $x^i y^j$ in (40) is an integer. In this case Theorem 4.2 can be sharpened: The inverse of the matrix $\left(1/\tbinom{2+i+j}{2}\right)_0^n$ has integer coefficients. This result is also given in [14].

Historical comments This paper was originally prepared for the proceedings of the International Conference on Mathematical Analysis and its Applications, Assiut, Egypt, January 3-6, 2006 and a copy was uploaded to ArXiv:math/0609283 in September 2006. The author wants to thank the organizers for the invitation to participate. Unfortunately, the proceedings of the meeting has not been published.

In June 2007 the following comments have been added because of the manuscript [2].

A result equivalent to Theorem 2.1 is given by Collar in [7]. Denoting by

$$M_n = (p_{ij}), \quad 0 \le i, j \le n$$

the matrix of coefficients of the orthonormal polynomials, i.e.

$$P_i(x) = \sum_{j=0}^{n} p_{ij} x^j, \quad i = 0, 1, \ldots, n,$$

where $p_{ij} = 0$ for $i < j$, then the orthonormality can be expressed as the matrix equation $M_n H_n M_n^t = I_n$, hence

$$H_n^{-1} = M_n^t M_n. \qquad (42)$$

Collar uses (42) to obtain formula (36) and states: "Equation (42), which provides an elegant method for the computation of the reciprocal of a moment matrix, is due to Dr A. C. Aitken. The author is grateful to Dr. Aitken for permission to describe the method and for many helpful suggestions."

The paper by Collar is not mentioned in Choi's paper [6] and was not included in the list of references in the first version of this paper.

In [2] the authors have defined a q-analogue of the Hilbert matrix for any complex q different from the roots of unity and have proved a q-analogue of (36). When $q = (1-\sqrt{5})/(1+\sqrt{5})$ one can recover the results about the Filbert matrices and for $q = -e^{-2\theta}, \theta > 0$ results of Ismail ([11]) about Hankel matrices of generalized Fibonacci numbers.

References

[1] N. I. Akhiezer, *The classical moment problem.* Oliver and Boyd, Edinburgh, 1965.

[2] J. E. Andersen and C. Berg, *Quantum Hilbert matrices and orthogonal polynomials.* J. Comput. Appl. Math. **233** (2009), 723–729. (ArXiv:math.CA/0703546)

[3] G. E. Andrews, R. Askey and R. Roy, *Special functions.* Cambridge University Press, Cambridge 1999.

[4] C. Berg, *Ortogonale polynomier og Hilbert matricen.* (In Danish) NORMAT **54** (2006), 116–133.

[5] T. S. Chihara, *An introduction to orthogonal polynomials.* Gordon and Breach, New York-London-Paris, 1978.

[6] Man-Duen Choi, *Tricks or Treats with the Hilbert Matrix*, Amer. Math. Monthly **90** (1983), 301–312.

[7] A. R. Collar, *On the Reciprocation of Certain Matrices*, Proc. Roy. Soc. Edinburgh **59** (1939), 195–206.

[8] G. Gasper and M. Rahman, *Basic hypergeometric series.* Cambridge University Press, Cambridge 1990, second edition 2004.

[9] D. Hilbert, *Ein Beitrag zur Theorie des Legendreschen Polynoms*, Acta Math. **18** (1894), 155–159. (367–370 in "Gesammelte Abhandlungen II", Berlin 1933.)

[10] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*. Cambridge University Press, Cambridge 2005.

[11] M. E. H. Ismail, *One Parameter Generalizations of the Fibonacci and Lucas Numbers*. Fibonacci Quarterly **46/47** no. 2 (2008/2009), 167–180. (ArXiv:math.CA/0606743)

[12] D. E. Knuth, *The Art of Computer Programming*. Vol. 1, 2nd Ed., Addison-Wesley, 1973

[13] T. Koshy, *Fibonacci and Lucas Numbers With Applications*. John Wiley, New York, 2001.

[14] T. M. Richardson, *The Filbert matrix*, Fibonacci Quart. **39** no. 3 (2001), 268–275.

[15] R. Savage, E. Lukacs, *Tables of Inverses of Finite Segments of the Hilbert Matrix*, National Bureau of Standards Applied Mathematics Series **39** (1954), 105–108.

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