

# Asymptotic behaviour of the Urbanik semigroup

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## Abstract

We revisit the product convolution semigroup of probability densities  $e_c(t)$ ,  $c > 0$  on the positive half-line with moments  $(n!)^c$  and determine the asymptotic behaviour of  $e_c$  for large and small  $t > 0$ . This shows that  $(n!)^c$  is indeterminate as Stieltjes moment sequence if and only if  $c > 2$ .

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## 1 Introduction

We consider a family of probability densities  $e_c(t)$ ,  $c > 0$  on the half-line given by

$$e_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{ix-1} \Gamma(1-ix)^c dx, \quad t > 0. \quad (1)$$

In this formula we use that  $\Gamma(z)$  is a non-vanishing holomorphic function in the cut plane

$$\mathcal{A} = \mathbb{C} \setminus (-\infty, 0], \quad (2)$$

so we can define

$$\Gamma(z)^c = \exp(c \log \Gamma(z)), \quad z \in \mathcal{A}$$

using the holomorphic branch of  $\log \Gamma$  which is 0 for  $z = 1$ .

As far as we know it was proved first by Urbanik in [10, Section 4] that  $e_c$  is a probability density, and that the following product convolution equation holds

$$e_{c+d}(t) = \int_0^\infty e_c(t/x) e_d(x) \frac{dx}{x}, \quad c, d > 0. \quad (3)$$

Furthermore, it was noticed that

$$\int_0^\infty t^n e_c(t) dt = (n!)^c, \quad c > 0, n = 0, 1, \dots \quad (4)$$

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Defining the probability measure  $\tau_c$  on  $(0, \infty)$  by

$$d\tau_c = e_c(t) dt = te_c(t) dm(t), \quad c > 0, \quad (5)$$

where  $dm(t) = (1/t) dt$  is the Haar measure on the locally compact abelian group  $G = (0, \infty)$  under multiplication, we can write (3) as  $\tau_c \diamond \tau_d = \tau_{c+d}$ , where  $\diamond$  denotes the (product) convolution of measures on the multiplicative group  $G$ . The family  $(\tau_c)_{c>0}$  is a convolution semigroup in the sense of [4]. We propose to call this semigroup the Urbanik semigroup because of [10].

The continuous characters of the group  $G$  can be given as  $t \rightarrow t^{ix}$ , where  $x \in \mathbb{R}$  is arbitrary, and in this way the dual group  $\widehat{G}$  of  $G$  can be identified with the additive group of real numbers, and by the inversion theorem of Fourier analysis for LCA-groups, (1) is equivalent to

$$\widehat{\tau}_c(x) = \int_0^\infty t^{-ix} d\tau_c(x) = \exp(c \log(\Gamma(1 - ix))), \quad x \in \mathbb{R}. \quad (6)$$

To establish the existence of a product convolution semigroup  $(\tau_c)$  satisfying (6) is therefore equivalent to proving that

$$\rho(x) := -\log \Gamma(1 - ix), \quad x \in \mathbb{R} \quad (7)$$

is a continuous negative definite function on  $\mathbb{R}$  in the terminology of [4] or [8].

This was done in [10] by giving the Lévy-Khinchin representation of  $\rho$ , using Malmsten's formula, cf. [5, 8.341(3)]:

$$\log \Gamma(z) = \int_0^\infty \left[ \frac{e^{-zt} - e^{-t}}{1 - e^{-t}} + (z - 1)e^{-t} \right] \frac{dt}{t}, \quad \operatorname{Re}(z) > 0. \quad (8)$$

In fact this formula can be written

$$-\log \Gamma(1 - ix) = \int_0^\infty \left[ 1 - e^{ixt} + \frac{itx}{1 + t^2} \right] \frac{e^{-t}}{t(1 - e^{-t})} dt - iax, \quad (9)$$

where

$$a = \int_0^\infty \left[ \frac{1}{(1 + t^2)(1 - e^{-t})} - \frac{1}{t} \right] e^{-t} dt,$$

showing that  $\rho(x) = -\log \Gamma(1 - ix)$  is negative definite with the Lévy measure

$$d\mu = \frac{e^{-t}}{t(1 - e^{-t})} dt$$

concentrated on  $(0, \infty)$ .

Another proof of the negative definiteness of  $\rho$  was given in [3] based on the Weierstrass product for  $\Gamma$ , where  $\operatorname{Log}$  denotes the principal logarithm in the cut plane  $\mathcal{A}$ , cf. (2):

$$-\log \Gamma(z) = \gamma z + \operatorname{Log} z + \sum_{k=1}^{\infty} (\operatorname{Log}(1 + z/k) - z/k), \quad z \in \mathcal{A}.$$

Clearly,

$$\rho_n(z) := \gamma z + \operatorname{Log} z + \sum_{k=1}^n (\operatorname{Log}(1 + z/k) - z/k)$$

converges locally uniformly to  $-\log \Gamma(z)$  for  $z \in \mathcal{A}$ , and since

$$\rho_n(1 - ix) = \rho_n(1) - i \left( \gamma - \sum_{k=1}^n \frac{1}{k} \right) x + \sum_{k=1}^{n+1} \operatorname{Log}(1 - ix/k)$$

is negative definite, because  $\operatorname{Log}(1 + iax)$  is so for  $a \in \mathbb{R}$  and

$$\rho_n(1) = \gamma + \log(n+1) - \sum_{k=1}^n \frac{1}{k} > 0,$$

we conclude that the limit function  $\rho(x) = -\log \Gamma(1 - ix)$  is negative definite.

As noticed in [3, Lemma 2.1], (4) is a special case of

$$\int_0^\infty t^z e_c(t) dt = \Gamma(1+z)^c, \quad \operatorname{Re}(z) > -1, \quad (10)$$

and letting  $z$  tend to  $-1$  along the real axis, we get

$$\int_0^\infty e_c(t) \frac{dt}{t} = \int_0^\infty e_c(1/t) \frac{dt}{t} = \infty, \quad c > 0. \quad (11)$$

It follows from (4) that  $(n!)^c$  is a Stieltjes moment sequence for any  $c > 0$ , and while it is easy to see that it is S-determinate for  $c \leq 2$  in the sense, that there is only one measure on the half-line with these moments, namely  $\tau_c$ , it is rather delicate to see that it is S-indeterminate for  $c > 2$ . This was proved in Theorem 2.5 in [3]. The proof was based on a relationship between  $\tau_c$  and stable distributions, and it used heavily asymptotic results of Skorokhod from [9] and exposed in [12]. Further details are given at the end of this section.

The purpose of the present paper is to establish the asymptotic behaviour of the densities  $e_c(t)$  for  $t \rightarrow \infty$  and  $t \rightarrow 0$ . The behaviour for  $t \rightarrow \infty$  will lead to a direct proof of the S-indeterminacy for  $c > 2$ .

We mention that the product convolution semigroup  $(\tau_c)_{c>0}$  corresponds to the Bernstein function  $f(s) = s$  in the following result from [3, Theorem 1.8].

**Theorem 1.1** *Let  $f$  be a non-zero Bernstein function. The uniquely determined measure  $\kappa = \kappa(f)$  with moments  $s_n = f(1) \cdots f(n)$  is infinitely divisible with respect to the product convolution. The unique product convolution semigroup  $(\kappa_c)_{c>0}$  with  $\kappa_1 = \kappa$  has the moments*

$$\int_0^\infty x^n d\kappa_c(x) = (f(1) \cdots f(n))^c, \quad c > 0, n = 0, 1, \dots \quad (12)$$

It is an easy consequence of Carleman's criterion that the measures  $\kappa_c$  are S-determinate for  $c \leq 2$ , cf. [3, Theorem 1.6].

In [3] we consider three Bernstein functions  $f_\alpha, f_\beta, f_\gamma$  with corresponding product convolution semigroups  $(\alpha_c)_{c>0}, (\beta_c)_{c>0}, (\gamma_c)_{c>0}$ :

$$f_\alpha(s) = (1 + 1/s)^s, \quad f_\beta(s) = (1 + 1/s)^{-s-1}, \quad f_\gamma(s) = s(1 + 1/s)^{s+1}.$$

It is proved that the measures  $\alpha_c, \beta_c$  have compact support, so they are clearly S-determinate for all  $c > 0$ , but  $\gamma_c$  is S-indeterminate for  $c > 2$ . Using that  $\tau_c = \beta_c \diamond \gamma_c$ , it is possible to infer that also  $\tau_c$  is S-indeterminate, see [3] for details.

As noticed in [10], the measures  $\tau_c$ ,  $c \geq 1$  are also infinitely divisible for the additive structure, because  $e_c(t)$  is completely monotonic. To see this, notice that the convolution equation (3) with  $d = 1$  can be written

$$e_{c+1}(t) = \int_0^\infty e^{-tx} e_c(1/x) \frac{dx}{x}, \quad c > 0, \quad (13)$$

showing that  $e_c(t)$  is completely monotonic for  $c > 1$ , and it tends to infinity for  $t \rightarrow 0$  because of (11).

It is well-known that the exponential distribution  $\tau_1$  is infinitely divisible for the additive structure and with a completely monotonic density  $e_1(t)$ .

Urbanik also showed that  $\tau_c$  is not infinitely divisible for the additive structure when  $0 < c < 1$ .

Formula (1) states roughly speaking that  $te_c(t)$  is the Fourier transform of the Schwartz function  $\Gamma(1 - ix)^c$  evaluated at  $\log t$ , thus showing that  $e_c$  is  $C^\infty$  on  $(0, \infty)$ . By Riemann-Lebesgue's Lemma we also see that  $te_c(t)$  tends to zero for  $t$  tending to zero and to infinity. Much more will be obtained in the main results below.

## 2 Main results

Our main results are

**Theorem 2.1** *For  $c > 0$  we have*

$$e_c(t) = \frac{(2\pi)^{(c-1)/2} \exp(-ct^{1/c})}{\sqrt{c} t^{(c-1)/(2c)}} \left[ 1 + \mathcal{O}\left(\frac{1}{t^{1/c}}\right) \right], \quad t \rightarrow \infty. \quad (14)$$

**Remark 2.2** The densities  $e_c$  are not explicitly known except for  $c = 1, 2$ , where

$$e_1(t) = e^{-t}, \quad e_2(t) = \int_0^\infty \exp(-x - t/x) \frac{dx}{x} = 2K_0(2\sqrt{t}).$$

In the last formula  $K_0$  is a modified Bessel function, see [7, Chap. 10, Sec. 25].

**Corollary 2.3** *The measure  $\tau_c = e_c(t) dt$  is S-indeterminate for  $c > 2$ .*

**Theorem 2.4** *For  $c > 0$  we have*

$$e_c(t) = \frac{(\log(1/t))^{c-1}}{\Gamma(c)} + \mathcal{O}((\log(1/t))^{c-2}), \quad t \rightarrow 0. \quad (15)$$

**Remark 2.5** Formula (15) shows that  $e_c(t)$  tends to infinity as a power of  $\log(1/t)$  when  $c > 1$ , but so slowly that multiplication with  $t$  forces the density to tend to zero. When  $0 < c < 1$  the density  $e_c(t)$  tends to zero.

### 3 Proofs

We will first give a proof of Theorem 2.1 in the case, where  $c$  is a natural number. Note that the asymptotic expression in (14) for  $c = 1$  reduces to  $e_1(t) = e^{-t}$ . When  $c = n + 1$ , where  $n$  is a natural number, we know that  $e_{n+1}(t)$  is the  $n$ 'th product convolution power of  $e_1$ , hence

$$e_{n+1}(t) = \int_0^\infty \dots \int_0^\infty e^{-\frac{t}{u_1 \dots u_n}} e^{-u_1} \dots e^{-u_n} \frac{du_1}{u_1} \dots \frac{du_n}{u_n}.$$

For  $t > 0$  fixed, the change of variables  $u_j = t^{1/(n+1)} v_j, j = 1, \dots, n$  leads to

$$e_{n+1}(t) = \int_0^\infty \dots \int_0^\infty g(v_1, \dots, v_n) e^{-t^{1/(n+1)} f(v_1, \dots, v_n)} dv_1 \dots dv_n, \quad (16)$$

with

$$g(v_1, \dots, v_n) := \frac{1}{v_1 \dots v_n}, \quad f(v_1, \dots, v_n) := v_1 + \dots + v_n + g(v_1, \dots, v_n).$$

The phase function  $f(v_1, \dots, v_n)$  is convex in  $C = \{v_1 > 0, \dots, v_n > 0\}$  because the Hessian matrix of second derivatives is

$$Hf(v_1, \dots, v_n) = g(v_1, \dots, v_n) \begin{pmatrix} \frac{2}{v_1^2} & \frac{1}{v_1 v_2} & \dots & \frac{1}{v_1 v_n} \\ \frac{1}{v_2 v_1} & \frac{2}{v_2^2} & \dots & \frac{1}{v_2 v_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{v_n v_1} & \frac{1}{v_n v_2} & \dots & \frac{2}{v_n^2} \end{pmatrix},$$

which is easily seen to be positive definite. The phase function therefore has a global minimum at the unique stationary point  $\vec{v}_0$  such that  $\vec{\nabla} f(\vec{v}_0) = \vec{0}$ , that is, at  $\vec{v}_0 = (1, \dots, 1)$ . At that point, the Hessian matrix of  $f(\vec{v})$  is

$$A := Hf(1, \dots, 1) = \begin{pmatrix} 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 2 \end{pmatrix},$$

with determinant  $\det(A) = n + 1$ .

By Laplace's asymptotic method for multiple dimensional Laplace transforms, cf. [11, Theorem 3, p. 495], we know that for  $t \rightarrow \infty$ ,

$$e_{n+1}(t) = \left( \frac{2\pi}{t^{1/(n+1)}} \right)^{n/2} g(\vec{v}_0) (\det(A))^{-1/2} e^{-t^{1/(n+1)} f(\vec{v}_0)} \left[ 1 + \mathcal{O} \left( \frac{1}{t^{1/(n+1)}} \right) \right].$$

We have that  $g(\vec{v}_0) = 1$  and  $f(\vec{v}_0) = n + 1$ , hence

$$e_{n+1}(t) = \frac{(2\pi)^{n/2}}{\sqrt{n+1}} \frac{e^{-(n+1)t^{1/(n+1)}}}{t^{n/(2(n+1))}} \left[ 1 + \mathcal{O} \left( \frac{1}{t^{1/(n+1)}} \right) \right], \quad (17)$$

which agrees with (14) for  $c = n + 1$ .

The proof of Theorem 2.1 for arbitrary  $c > 0$  is more delicate. We first apply Cauchy's integral theorem to move the integration in (1) to an arbitrary horizontal line

$$L_a := \{z = x + ia \mid x \in \mathbb{R}\}, \quad a > 0. \quad (18)$$

**Lemma 3.1** *With  $L_a$  as in (18) we have*

$$e_c(t) = \frac{1}{2\pi} \int_{L_a} t^{iz-1} \Gamma(1-iz)^c dz, \quad t > 0. \quad (19)$$

*Proof:* For  $t, c > 0$  fixed,  $f(z) = t^{iz-1} \Gamma(1-iz)^c$  is holomorphic in the simply connected domain  $\mathbb{C} \setminus i(-\infty, -1]$ , so the Lemma follows from Cauchy's integral theorem provided the integral

$$\int_0^a f(x+iy) dy$$

tends to 0 for  $x \rightarrow \pm\infty$ . We have

$$|f(x+iy)| = t^{-y-1} |\Gamma(1+y-ix)|^c$$

and since

$$|\Gamma(u+iv)| \sim \sqrt{2\pi} e^{-\pi/2|v|} |v|^{u-1/2}, \quad |v| \rightarrow \infty, \text{ uniformly for bounded real } u,$$

cf. [1, p.141, eq. 5.11.9], [5, 8.328(1)], the result follows.  $\square$

In the following we will use Lemma 3.1 with the line of integration  $L = L_a$ , where  $a = t^{1/c} - 1$  for  $t > 1$ . Therefore, using the parametrization  $z = x + i(t^{1/c} - 1)$  we get

$$e_c(t) = t^{-t^{1/c}} \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{ix} \Gamma(t^{1/c} - ix)^c dx,$$

and after the change of variable  $x = t^{1/c} u$

$$e_c(t) = t^{1/c-t^{1/c}} \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{iut^{1/c}} \Gamma(t^{1/c}(1-iu))^c du. \quad (20)$$

Binet's formula for  $\Gamma$  is ([5, 8.341(1)])

$$\Gamma(z) = \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z+\mu(z)}, \quad \operatorname{Re}(z) > 0, \quad (21)$$

where

$$\mu(z) = \int_0^{\infty} \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-zt}}{t} dt, \quad \operatorname{Re}(z) > 0. \quad (22)$$

Notice that  $\mu(z)$  is the Laplace transform of a positive function, so we have the estimates for  $z = r + is, r > 0$

$$|\mu(z)| \leq \mu(r) \leq \frac{1}{12r}, \quad (23)$$

where the last inequality is a classical version of Stirling's formula, thus showing that the estimate is uniform in  $s \in \mathbb{R}$ .

Inserting this in (20), we get after some simplification

$$e_c(t) = (2\pi)^{c/2-1} t^{1/c-1/2} e^{-ct^{1/c}} \int_{-\infty}^{\infty} e^{ct^{1/c}f(u)} g_c(u) M(u, t) du, \quad (24)$$

where

$$f(u) := iu + (1 - iu) \operatorname{Log}(1 - iu), \quad g_c(u) := (1 - iu)^{-c/2} \quad (25)$$

and

$$M(u, t) := \exp[c\mu(t^{1/c}(1 - iu))]. \quad (26)$$

From (23) we get  $M(u, t) = 1 + \mathcal{O}(t^{-1/c})$  for  $t \rightarrow \infty$ , uniformly in  $u$ . We shall therefore consider the behaviour of

$$\int_{-\infty}^{\infty} e^{ct^{1/c}f(u)} g_c(u) du. \quad (27)$$

From here we need to apply the saddle point method to obtain the approximation of (27) for large positive  $t$ . For convenience, we use Theorem 1 in [6]. We have that the only saddle point of the phase function  $f(u)$  is  $u = 0$  and  $f(0) = f'(0) = 0$ ,  $f''(0) = -1$ ,  $f'''(0) \neq 0$ ; also  $g_c(0) = 1$ . Then, the parameters used in that theorem are  $m = 2$ ,  $p = 3$ ,  $\phi = \pi$ ,  $N = 0$ ,  $M = 1$  and the large variable used in the theorem is  $x \equiv ct^{1/c}$ . We have that the steepest descendent path used in the theorem is  $\Gamma = \Gamma_0 \cup \Gamma_1 = (-\infty, 0) \cup (0, \infty)$ , that is, it is just the original integration path in the above integral, and therefore does not need any deformation. From [6, Theorem 1] with the notation used there, we read that the integral (27) has an expansion of the form

$$e^{xf(0)} [c_0 \Psi_0(x) + c_1 \Psi_1(x) + c_2 \Psi_2(x) + \dots],$$

with  $\Psi_n(x) = \mathcal{O}(x^{-(n+1)/2})$  and  $c_n$  is independent of  $x$ . Because the factors  $c_{2n+1}$  vanish we find

$$c_0 \Psi_0(x) + c_1 \Psi_1(x) + c_2 \Psi_2(x) + \dots = c_0 \Psi_0(x) [1 + \mathcal{O}(x^{-1})]$$

with  $c_0 = 1$  and

$$\Psi_0(x) = a_0(x) \Gamma\left(\frac{1}{2}\right) \left| \frac{2}{xf''(0)} \right|^{1/2}$$

with

$$a_0(x) = e^{-xf(0)} A_0(x) B_0, \quad A_0(x) = e^{xf(0)}, \quad B_0 = g_c(0),$$

hence  $a_0(x) = B_0 = 1$ . Using all these data we finally obtain

$$\int_{-\infty}^{\infty} e^{ct^{1/c}f(u)} g_c(u) du = \frac{\sqrt{2\pi}}{\sqrt{ct^{1/(2c)}}} [1 + \mathcal{O}(t^{-1/c})],$$

and

$$e_c(t) = \frac{(2\pi)^{(c-1)/2}}{\sqrt{c}} \frac{e^{-ct^{1/c}}}{t^{(c-1)/(2c)}} [1 + \mathcal{O}(t^{-1/c})].$$

□

*Proof of Corollary 2.3.* We apply the Krein criterion for S-indeterminacy of probability densities concentrated on the half-line, using a version given in [2, Theorem 5.1]. It states that if

$$\int_0^\infty \frac{\log e_c(t) dt}{\sqrt{t}(1+t)} > -\infty, \quad (28)$$

then  $\tau_c = e_c(t) dt$  is S-indeterminate. We shall see that (28) holds for  $c > 2$ .

From Theorem 2.1 combined with the fact that  $e_c(t)$  is decreasing when  $c > 1$ , we see that the inequality in (28) holds if and only if

$$\int_0^\infty \frac{\log((2\pi)^{(c-1)/2}/\sqrt{c}) - ct^{1/c} - ((c-1)/(2c)) \log t}{\sqrt{t}(1+t)} dt > -\infty,$$

and the latter holds precisely for  $c > 2$ . This shows that  $\tau_c$  is S-indeterminate for  $c > 2$ . □

*Proof of Theorem 2.4.*

Since we are studying the behaviour for  $t \rightarrow 0$ , we assume that  $0 < t < 1$  so that  $\Lambda := \log(1/t) > 0$ .

We will need integration along the vertical lines

$$V_a := \{a + iy \mid y = -\infty \dots \infty\}, \quad a \in \mathbb{R}, \quad (29)$$

and we can therefore express (1) as

$$e_c(t) = \frac{1}{2\pi i} \int_{V_{-1}} t^z \Gamma(-z)^c dz. \quad (30)$$

By the functional equation for  $\Gamma$  we get

$$e_c(t) = \frac{1}{2\pi i} \int_{V_{-1}} (-z)^{-c} t^z \Gamma(1-z)^c dz. \quad (31)$$

To ease the writing we define

$$\varphi(z) := t^z \Gamma(1-z)^c, \quad g(z) := (-z)^{-c} = \exp(-c \operatorname{Log}(-z)),$$

and note that  $\varphi$  is holomorphic in  $\mathbb{C} \setminus [1, \infty)$ , while  $g$  is holomorphic in  $\mathbb{C} \setminus [0, \infty)$ . Here  $\operatorname{Log}$  is the principal logarithm in the cut plane  $\mathcal{A}$ , cf. (2).

Note that for  $x > 0$

$$g_\pm(x) := \lim_{\varepsilon \rightarrow 0} g(x \pm i\varepsilon) = x^{-c} e^{\pm i\pi c}.$$

Formula (31) can now be written

$$e_c(t) = \frac{1}{2\pi i} \int_{V_{-1}} g(z) \varphi(z) dz. \quad (32)$$

**Case 1.** We will first treat the case  $0 < c < 1$ .



We fix  $0 < s < 1$  and choose  $0 < \varepsilon < s$  and integrate  $g(z)\varphi(z)$  over the contour  $\mathcal{C}$

$$\{-1+iy \mid y = \infty \dots 0\} \cup [-1, -\varepsilon] \cup \{\varepsilon e^{i\theta} \mid \theta = \pi \dots 0\} \cup [\varepsilon, s] \cup \{s+iy \mid y = 0 \dots \infty\}$$

and get 0 by the integral theorem of Cauchy. On the interval  $[\varepsilon, s]$  we use the values of  $g_+$ .

Similarly we get 0 by integrating  $g(z)\varphi(z)$  over the complex conjugate contour  $\bar{\mathcal{C}}$ , and now we use the values of  $g_-$  on the interval  $[\varepsilon, s]$ .

Subtracting the second contour integral from the first leads to

$$\int_{V_s} - \int_{V_{-1}} - \int_{|z|=\varepsilon} g(z)\varphi(z) dz + \int_{\varepsilon}^s \varphi(x)(g_+(x) - g_-(x)) dx = 0,$$

where the integral over the circle is with positive orientation. Note that the two integrals over  $[-1, -\varepsilon]$  cancel. Using that  $0 < c < 1$  it is easy to see that the just mentioned integral converges to 0 for  $\varepsilon \rightarrow 0$ , and we finally get for  $\varepsilon \rightarrow 0$

$$e_c(t) = \frac{1}{2\pi i} \int_{V_s} g(z)\varphi(z) dz + \frac{\sin(\pi c)}{\pi} \int_0^s x^{-c} \varphi(x) dx := I_1 + I_2.$$

We claim that the first integral  $I_1$  is  $o(t^s)$  for  $t \rightarrow 0$ . To see this we insert the parametrization of  $V_s$  and get

$$I_1 = \frac{t^s}{2\pi} \int_{-\infty}^{\infty} (-s - iy)^{-c} t^{iy} \Gamma(1 - s - iy)^c dy$$

and the integral is  $o(1)$  by Riemann-Lebesgue's Lemma, so  $I_1 = o(t^s)$ .

The substitution  $u = x \log(1/t) = x\Lambda$  in the integral  $I_2$  leads to

$$I_2 = \frac{\sin(\pi c)}{\pi} \Lambda^{c-1} \int_0^{s\Lambda} u^{-c} e^{-u} \Gamma(1 - u/\Lambda)^c du. \quad (33)$$

We split the integral in (33) as

$$\Gamma(1 - c) + \int_0^{s\Lambda} u^{-c} e^{-u} [\Gamma(1 - u/\Lambda)^c - 1] du - \int_{s\Lambda}^{\infty} u^{-c} e^{-u} du, \quad (34)$$

and by the mean-value theorem and  $\Psi = \Gamma'/\Gamma$  we have

$$\Gamma(1 - u/\Lambda)^c - 1 = -\frac{u}{\Lambda} c \Gamma(1 - \theta u/\Lambda)^c \Psi(1 - \theta u/\Lambda)$$

for some  $0 < \theta < 1$ , but this implies that

$$|\Gamma(1 - u/\Lambda)^c - 1| \leq \frac{cu}{\Lambda} M(s), \quad 0 < u < s\Lambda,$$

where

$$M(s) := \max\{\Gamma(x)^c |\Psi(x)| \mid 1 - s \leq x \leq 1\},$$

so the first integral in (34) is  $\mathcal{O}(\Lambda^{-1})$ . The second integral is an incomplete Gamma function, and by known asymptotics for this, see [5], we get that the

second integral is  $\mathcal{O}(\Lambda^{-c}t^s)$ . Putting things together and using Euler's reflection formula for  $\Gamma$ , we see that

$$e_c(t) = \frac{\Lambda^{c-1}}{\Gamma(c)} + \mathcal{O}(\Lambda^{c-2}),$$

which is (15).

**Case 2.** We now assume  $1 < c < 2$ .

The Gamma function decays so rapidly when  $z = -1 + iy \in V_{-1}, y \rightarrow \pm\infty$ , that we can integrate by parts in (31) to get

$$e_c(t) = -\frac{1}{2\pi i} \int_{V_{-1}} \frac{(-z)^{-(c-1)}}{c-1} \frac{d}{dz} (t^z \Gamma(1-z)^c) dz. \quad (35)$$

Defining

$$\varphi_1(z) := \frac{d}{dz} (t^z \Gamma(1-z)^c) = t^z \Gamma(1-z)^c (\log t - c\Psi(1-z)),$$

and using the same contour technique as in case 1 to the integral in (35), where now  $0 < c-1 < 1$ , we get for  $0 < s < 1$  fixed

$$e_c(t) = -\frac{1}{c-1} \frac{1}{2\pi i} \int_{V_s} (-z)^{-(c-1)} \varphi_1(z) dz - \frac{\sin(\pi(c-1))}{(c-1)\pi} \int_0^s x^{-(c-1)} \varphi_1(x) dx.$$

The first integral is  $o(t^s\Lambda)$  by Riemann-Lebesgue's Lemma, and the substitution  $u = x\Lambda$  in the second integral leads to

$$\begin{aligned} & \int_0^s x^{-(c-1)} \varphi_1(x) dx \\ &= \Lambda^{c-2} \int_0^{s\Lambda} u^{-(c-1)} \varphi_1(u/\Lambda) du \\ &= -\Lambda^{c-1} \int_0^{s\Lambda} u^{-(c-1)} e^{-u} du - \Lambda^{c-1} \int_0^{s\Lambda} u^{-(c-1)} e^{-u} (\Gamma(1-u/\Lambda)^c - 1) du \\ &\quad - c\Lambda^{c-2} \int_0^{s\Lambda} u^{-(c-1)} e^{-u} \Gamma(1-u/\Lambda)^c \Psi(1-u/\Lambda) du \\ &= -\Lambda^{c-1} \Gamma(2-c) + \mathcal{O}(\Lambda^{c-2}). \end{aligned}$$

Using that

$$\left( -\frac{\sin(\pi(c-1))}{(c-1)\pi} \right) (-\Lambda^{c-1} \Gamma(2-c)) = \frac{\Lambda^{c-1}}{\Gamma(c)}$$

by Euler's reflection formula, we see that (15) holds.

**Case 3.** We now assume  $c > 2$ .

We perform the change of variable  $w = \Lambda z$  in (31) and obtain

$$e_c(t) = \frac{\Lambda^{c-1}}{2\pi i} \int_{V_{-\Lambda}} (-w)^{-c} e^{-w} \Gamma(1-w/\Lambda)^c dw.$$

Using Cauchy's integral theorem, we can shift the contour  $V_{-\Lambda}$  to  $V_{-1}$  as the integrand is holomorphic in the vertical strip between both paths and exponentially small at both extremes of that vertical strip. Then,

$$e_c(t) = \frac{\Lambda^{c-1}}{2\pi i} \int_{V_{-1}} (-w)^{-c} e^{-w} \Gamma(1 - w/\Lambda)^c dw.$$

For any holomorphic function  $h$  in a domain  $G$  which is star-shaped with respect to 0 we have

$$h(z) = h(0) + z \int_0^1 h'(uz) du, \quad z \in G.$$

If this is applied to  $G = \mathbb{C} \setminus [1, \infty)$  and  $h(z) = \Gamma(1 - z)^c$  we find

$$\Gamma(1 - w/\Lambda)^c = 1 - \frac{cw}{\Lambda} \int_0^1 \Gamma(1 - uw/\Lambda)^c \Psi(1 - uw/\Lambda) du. \quad (36)$$

Defining

$$R(w) = \int_0^1 \Gamma(1 - uw/\Lambda)^c \Psi(1 - uw/\Lambda) du,$$

we get

$$e_c(t) = \Lambda^{c-1} 2\pi i \int_{V_{-1}} (-w)^{-c} e^{-w} dw + \frac{c\Lambda^{c-2}}{2\pi i} \int_{V_{-1}} (-w)^{1-c} e^{-w} R(w) dw. \quad (37)$$

For any  $w \in V_{-1}$ ,  $0 \leq u \leq 1$  and for  $\Lambda \geq 1$  we have that  $1 - uw/\Lambda \in \Omega$ , where  $\Omega$  is the closed vertical strip located between the vertical lines  $V_1$  and  $V_2$ . Because  $\Gamma(z)^c \Psi(z)$  is continuous in  $\Omega$  and exponentially small at the upper and lower limits of  $\Omega$ , the function  $R(w)$  is bounded for  $w \in V_{-1}$  by a constant independent of  $\Lambda \geq 1$ . Therefore,

$$\frac{c\Lambda^{c-2}}{2\pi i} \int_{V_{-1}} (-w)^{1-c} e^{-w} R(w) dw = \mathcal{O}(\Lambda^{c-2}),$$

where we use that  $(-w)^{1-c} e^{-w}$  is integrable over  $V_{-1}$  because  $c > 2$ .

On the other hand, in the first integral of (37), the contour  $V_{-1}$  may be deformed to a Hankel contour

$$\mathcal{H} := \{x - i \mid x = \infty \dots 0\} \cup \{e^{i\theta} \mid \theta = -\pi/2 \dots -3\pi/2\} \cup \{x + i \mid x = 0 \dots \infty\}$$

surrounding  $[0, \infty)$ , and the integral over  $\mathcal{H}$  is Hankel's integral representation of the inverse of the Gamma function:

$$\frac{1}{2\pi i} \int_{\mathcal{H}} (-w)^{-c} e^{-w} dw = \frac{1}{\Gamma(c)}.$$

Therefore, when we join everything, we obtain that for  $c > 2$ :

$$e_c(t) = \frac{(\log(1/t))^{c-1}}{\Gamma(c)} + \mathcal{O}((\log(1/t))^{c-2}), \quad t \rightarrow 0.$$

**Case 4.**  $c = 1, c = 2$ .

These cases are easy since  $e_1(t) = e^{-t}$  and  $e_2(t) = 2K_0(2\sqrt{t})$ .  $\square$

**Remark 3.2** The behaviour of  $e_c(t)$  for  $t \rightarrow 0$  can be obtained from (30) using the residue theorem when  $c$  is a natural number. In fact, in this case  $\Gamma(-z)^c$  has a pole of order  $c$  at  $z = 0$ , and a shift of the contour  $V_{-1}$  to  $V_s$ , where  $0 < s < 1$ , has to be compensated by a residue, which will give the behaviour for  $t \rightarrow 0$ .

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## References

- [1] R. A. Askey, R. Roy, Chapter 5, Gamma Function, NIST Handbook of Mathematical Functions, NIST and Cambridge Univ. Press, 2010.
- [2] C. Berg, Indeterminate moment problems and the theory of entire functions, *J. Comput. Appl. Math.* **65** (1995), 27–55.
- [3] C. Berg, On powers of Stieltjes moment sequences, I, *J. Theor. Prob.* **18** (2005), 871–889.
- [4] C. Berg, G. Forst, Potential Theory on Locally Compact Abelian Groups, Springer-Verlag, Berlin-Heidelberg-New York, 1975.
- [5] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals, Series and Products. Sixth Edition, Academic Press, San Diego, 2000.
- [6] J. L. López, P. Pagola and E. Pérez Sinusía, A systematization of the saddle point method. Application to the Airy and Hankel functions. *J. Math. Anal. Appl.* **354** (2009), 347–359.
- [7] F. W. J. Olver, L. C. Maximon, Chapter 10, Bessel Functions, NIST Handbook of Mathematical Functions, NIST and Cambridge Univ. Press, 2010.
- [8] R. L. Schilling, R. Song and Z. Vondraček, Bernstein functions. Theory and applications. De Gruyter Studies in Mathematics **37**, de Gruyter, Berlin 2010.
- [9] Skorokhod, A.V., Asymptotic formulas for stable distribution laws, *Dokl. Akad. Nauk SSSR* **98** (1954), 731–734; English transl., Selected Transl. Math. Statist. and Probab., Vol. **1** (1961), 157–161, Amer. Math. Soc., Providence, R. I.
- [10] K. Urbanik, Functionals on transient stochastic processes with independent increments. *Studia Math.* **103** (1992), 299–315.
- [11] R. Wong, Asymptotic Approximations of Integrals, Classics in applied mathematics **34**, SIAM, Philadelphia, USA, 2001.
- [12] V. M. Zolotarev, One-dimensional Stable Distributions, Translations of Mathematical Monographs **65**, Amer. Math. Soc., Providence, R. I., 1986.

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