On an iteration leading to a q-analogue of the Digamma function

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Abstract

We show that the q-Digamma function ψ_q for $0 < q < 1$ appears in an iteration studied by Berg and Durán. This is connected with the determination of the probability measure ν_q on the unit interval with moments $1/\sum_{k=1}^{n+1}(1-q)/(1-q^k)$, which are q-analogues of the reciprocals of the harmonic numbers. The Mellin transform of the measure ν_q can be expressed in terms of the q-Digamma function. It is shown that ν_q has a continuous density on $]0,1]$, which is piecewise C^∞ with kinks at the powers of q. Furthermore, $(1-q)e^{-x}\nu_q(e^{-x})$ is a standard p-function from the theory of regenerative phenomena.

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1 Introduction

For a measure μ on the unit interval $[0,1]$ we consider its *Bernstein transform*

$$\mathcal{B}(\mu)(z) = \int_0^1 \frac{1-t^z}{1-t}\,d\mu(t), \quad \Re z > 0, \tag{1}$$

as well as its *Mellin transform*

$$\mathcal{M}(\mu)(z) = \int_0^1 t^z\,d\mu(t), \quad \Re z > 0. \tag{2}$$

These functions are clearly holomorphic in the right half-plane $\Re z > 0$.

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The two integral transformations are combined in the following theorem from [4] about Hausdorff moment sequences, i.e., sequences $(a_n)_{n\geq 0}$ of the form

$$a_n = \int_0^1 t^n \, d\mu(t), \tag{3}$$

for a positive measure μ on the unit interval.

Theorem 1.1 *Let $(a_n)_{n\geq 0}$ be a Hausdorff moment sequence as in (3) with $\mu \neq 0$. Then the sequence $(T(a_n))_{n\geq 0}$ defined by $T(a_n)_n = 1/(a_0 + \ldots + a_n)$ is again a Hausdorff moment sequence, and its associated measure $\widehat{T}(\mu)$ has the properties $\widehat{T}(\mu)(\{0\}) = 0$ and*

$$\mathcal{B}(\mu)(z+1)\mathcal{M}(\widehat{T}(\mu))(z) = 1 \quad \textit{for} \quad \Re z > 0. \tag{4}$$

This means that the measure $\widehat{T}(\mu)$ is determined as the inverse Mellin transform of the function $1/\mathcal{B}(\mu)(z+1)$.

It follows by Theorem 1.1 that T maps the set of normalized Hausdorff moment sequences (i.e., $a_0 = 1$) into itself. By Tychonoff's extension of Brouwer's fixed point theorem, T has a fixed point (m_n). Furthermore, it is clear that a fixed point (m_n) is uniquely determined by the equations

$$(1 + m_1 + \ldots + m_n)m_n = 1, \quad n \geq 1. \tag{5}$$

Therefore

$$m_{n+1}^2 + \frac{m_{n+1}}{m_n} - 1 = 0, \tag{6}$$

giving

$$m_1 = \frac{-1+\sqrt{5}}{2}, \quad m_2 = \frac{\sqrt{22+2\sqrt{5}} - \sqrt{5} - 1}{4}, \ldots .$$

Similarly, $\widehat{T}$ maps the set $M_+^1([0,1])$ of probability measures on $[0,1]$ into itself. It has a uniquely determined fixed point ω and

$$m_n = \int_0^1 t^n \, d\omega(t), \quad n = 0, 1, \ldots . \tag{7}$$

Berg and Durán studied this fixed point in [5],[6], and it was proved that the Bernstein transform $f = \mathcal{B}(\omega)$ is meromorphic in the whole complex plane and characterized by a functional equation and a log-convexity property in analogy with Bohr-Mollerup's characterization of the Gamma function, cf. [2]. Let us also mention that ω has an increasing and convex density with respect to Lebesgue measure m on the unit interval.

An important step in the proof of these results is to establish that ω is an attractive fixed point so that in particular the iterates $\widehat{T}^{\circ n}(\delta_1)$ converge weakly to ω. Here and in the following δ_a denotes the Dirac measure with mass 1 concentrated in $a \in \mathbb{R}$.

It is easy to see that $\widehat{T}(\delta_1) = m$, because

$$T(1,1,\ldots)_n = \frac{1}{n+1} = \int_0^1 t^n\,dt.$$

It is well-known that the Bernstein transform of Lebesgue measure m on $[0,1]$ is related to the Digamma function ψ, i.e., the logarithmic derivative of the Gamma function, since

$$\int_0^1 \frac{1-t^z}{1-t}dt = \psi(z+1)+\gamma = \sum_{n=1}^{\infty}\frac{z}{n(n+z)}, \quad \Re z > 0, \tag{8}$$

cf. [10, 8.36]. Here $\gamma = -\psi(1)$ is Euler's constant.

Therefore $\nu_1 := \widehat{T}(m) = \widehat{T}^{\circ 2}(\delta_1)$ is determined by

$$\mathcal{M}(\nu_1)(z) = \frac{1}{\mathcal{B}(m)(z+1)} = \frac{1}{\psi(z+2)+\gamma}.$$

The measure $\nu_1 = \widehat{T}(m)$ is given explicitly in [4] as

$$\nu_1 = (\sum_{n=0}^{\infty}\alpha_n t^{\xi_n})\,dt, \tag{9}$$

where $\xi_0 = 0$, $\xi_n \in (n, n+1), n = 1, 2, \ldots$ is the solution to $\psi(1-\xi_n) = -\gamma$ and $\alpha_n = 1/\psi'(1-\xi_n)$. The moments of the measure ν_1 are the reciprocals of the harmonic numbers, i.e.,

$$\int_0^1 t^n\,d\nu_1(t) = \frac{1}{\mathcal{H}_{n+1}} = \left(\sum_{k=1}^{n+1}\frac{1}{k}\right)^{-1}. \tag{10}$$

The purpose of the present paper is to study the first two elements of the sequence $\widehat{T}^{\circ n}(\delta_q)$, where $0 < q < 1$ is fixed. The reason for excluding $q = 0$ is that $\widehat{T}(\delta_0) = \delta_1$. Since ω is an attractive fixed point, we know that the sequence converges weakly to ω.

The first step in the iteration is easy:

$$\widehat{T}(\delta_q) = (1-q)\sum_{k=0}^{\infty} q^k \delta_{q^k}, \tag{11}$$

because

$$\int_0^1 t^z\,d\widehat{T}(\delta_q)(t) = \frac{1-q}{1-q^{z+1}} = (1-q)\sum_{k=0}^{\infty} q^k q^{kz}. \tag{12}$$

This shows that $\widehat{T}(\delta_q)$ is the Jackson d_qt-measure on $[0,1]$ used in the theory of q-integrals, cf. [9]. It is a q-analogue of Lebesgue measure in the sense that $d_qt \to m$ weakly for $q \to 1$.

It is therefore to be expected that $\nu_q := \widehat{T}(d_qt) = \widehat{T}^{\circ 2}(\delta_q)$ is a q-analogue of the measure ν_1, and we are going to determine ν_q. We have

$$\mathcal{M}(\nu_q)(z) = \frac{1}{f_q(z+1)}, \tag{13}$$

where f_q is defined as the Bernstein transform of d_qt:

$$f_q(z) = \int_0^1 \frac{1-t^z}{1-t}\, d_qt = (1-q)\left(z + \sum_{k=1}^{\infty} q^k \frac{1-q^{kz}}{1-q^k}\right). \tag{14}$$

This formula is a q-analogue of (8) and the relationship between f_q and q-versions of Euler's constant and the Digamma function is given in (20).

The moments of ν_q are q-analogues of (10)

$$\int_0^1 t^n\, d\nu_q(t) = \left(\sum_{k=0}^{n} \frac{1-q}{1-q^{k+1}}\right)^{-1}. \tag{15}$$

It is easily seen that $q \to \int_0^1 t^n\, d\nu_q(t)$ is an increasing function mapping $[0,1]$ onto $[(n+1)^{-1}, \mathcal{H}_{n+1}^{-1}]$.

Formally ν_q is the inverse Mellin transform of $1/f_q(z+1)$, i.e.,

$$\nu_q = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{t^{-z}}{f_q(z+1)}\, dz,$$

and we shall exploit that in section 3, where we transfer the harmonic analysis on the multiplicative group $]0,\infty[$ to the additive group of real numbers, hence replacing the Mellin transformation by the ordinary Fourier transformation. If we denote by τ_q the image measure of ν_q under the transformation $\log(1/t)$, we formally get

$$\tau_q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iyx} \frac{dy}{f_q(1+iy)}, \tag{16}$$

but since $y \to 1/f_q(1+iy)$ is a square integrable and non-integrable function as shown in Section 3, this only yields that τ_q is a square integrable function. In Theorem 3.1 we prove that τ_q is the restriction to $[0,\infty[$ of a continuous symmetric positive definite function.

The main tool to obtain further regularity properties of ν_q will be a direct approach using convolution. In fact, we realize $(1-q)\tau_q$ as the convolution of an exponential density and an elementary kernel $\sum_0^\infty N^{*n}$, see (26), which makes it possible to prove that

$$\tau_q(x) = e^{-(1+c_q)x} p_n(x), \quad x \in [n\log(1/q), (n+1)\log(1/q)], \quad n = 0, 1, \ldots$$

for a certain constant c_q, cf. (22), and a polynomial p_n of degree n the coefficients of which are given explicitly as functions of q, see (31). In Remark 2.2 we point out that $(1-q)\tau_q(x)$ is a standard p-function in the sense of [11].

Going back to the interval $]0,1]$ we can state our main result that ν_q is a C^∞-spline, i.e. it has a piecewise C^∞-density with respect to Lebesgue measure on $[0,1]$:

Theorem 1.2 *The measure ν_q has a continuous and piecewice C^∞-density denoted $\nu_q(t)$ on $]0,1]$. We have*

$$\nu_q(t) = t^{c_q} p_n(\log(1/t)), \quad t \in [q^{n+1}, q^n],\ n = 0, 1, \dots, \tag{17}$$

where c_q is given by (22) and p_n is a polynomial of degree n given by (31). The derivative of ν_q has a jump of size $1/(1-q^n)(1-q)$ at the point $q^n, n = 1, 2, \dots$.

Remark 1.3 It follows that the behaviour of $\nu_q(t)$ is oscillatory, and therefore quite different from that of $\nu_1(t)$ given by (9), which is increasing and convex. See Figure 1 and 2 which shows the graph of $(1-q)\nu_q$ for $q = 0.5$ and $q = 0.9$.

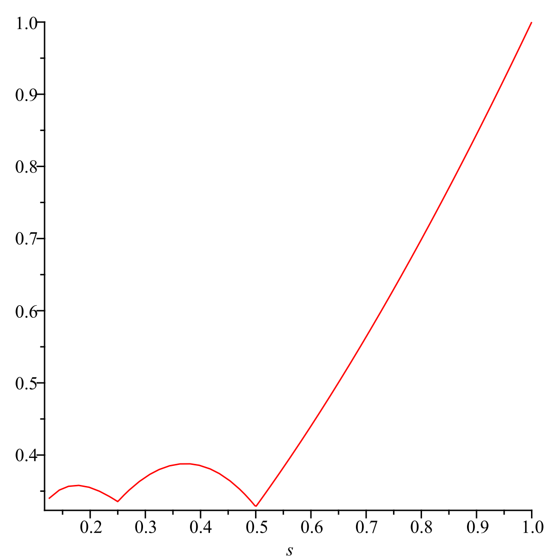

Figure 1: The graph of $(1-q)\nu_q(t)$ on $[q^3, 1]$ for $q = 0.5$

2 Proofs

Jackson's q-analogue of the Gamma function is defined as

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z},$$

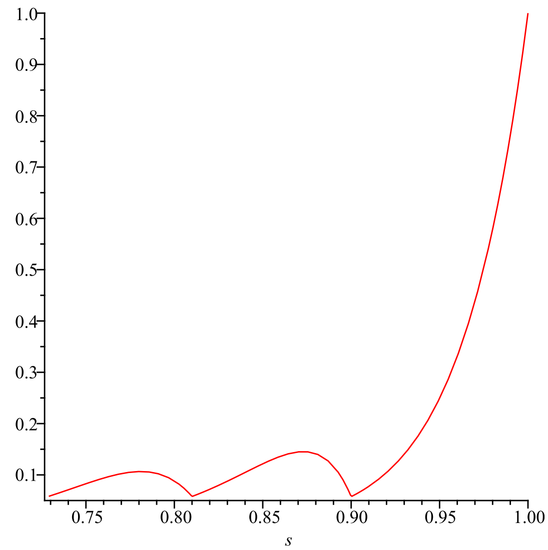

Figure 2: The graph of $(1-q)\nu_q(t)$ on $[q^3, 1]$ for $q = 0.9$

cf. [9], and its logarithmic derivative

$$\psi_q(z) = \frac{d}{dz} \log \Gamma_q(z) = -\log(1-q) + \log q \sum_{k=0}^{\infty} \frac{q^{k+z}}{1-q^{k+z}} \tag{18}$$

has been proposed in [12] as a q-analogue of the Digamma function ψ. See also the recent paper [13]. We define the q-analogue of Euler's constant as

$$\gamma_q = -\psi_q(1) = \log(1-q) - \log q \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}. \tag{19}$$

The Bernstein transform f_q of $d_q t$ is given in (14), hence

$$\begin{aligned} \frac{f_q(z)}{1-q} &= z + \sum_{k=1}^{\infty} q^k \sum_{n=0}^{\infty} q^{kn}(1-q^{kz}) \\ &= z + \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} (q^{k(n+1)} - q^{k(n+1+z)}) \right) \\ &= z + \frac{1}{\log(1/q)} \left(\gamma_q + \psi_q(z+1)\right), \end{aligned}$$

showing that

$$f_q(z) = (1-q)z + \frac{1-q}{\log(1/q)} \left(\gamma_q + \psi_q(z+1)\right), \tag{20}$$

so f_q has a close relationship with the q-Digamma function and the q-version of Euler's constant.

We will be using another expression for $f_q(z)/(1-q)$ derived from (14), namely

$$\frac{f_q(z)}{1-q} = z + c_q - \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} q^{kz}, \tag{21}$$

with

$$c_q = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}. \tag{22}$$

Clearly, $q/(1-q) < c_q < q/(1-q)^2$ for $0 < q < 1$ and $q \mapsto c_q$ is a strictly increasing map of $]0,1[$ onto $]0,\infty[$. We mention two other expressions

$$c_q = \sum_{n=1}^{\infty} d(n) q^n = \sum_{n=1}^{\infty} \left(1 - (q^n;q)_\infty\right),$$

where $d(n)$ is the number of divisors in n, see [8, p. 14].

In order to replace the Mellin transformation by the Laplace transformation we introduce the probability measure τ_q on $[0,\infty[$ which has ν_q as image measure under $t \to e^{-t}$, hence

$$\mathcal{L}(\tau_q)(z) = \int_0^\infty e^{-tz}\, d\tau_q(t) = \frac{1}{f_q(z+1)}.$$

The analogue of Theorem 1.2 about the measure τ_q is given in the next theorem, which we shall prove first.

Theorem 2.1 *The measure τ_q has a continuous density also denoted τ_q with respect to Lebesgue measure on $[0,\infty[$. It is C^∞ in each of the open intervals $]n\log(1/q),(n+1)\log(1/q)[, n = 0,1,\ldots$ with jump of the derivative of size*

$$J_n = \frac{q^{2n}}{(1-q^n)(1-q)} \tag{23}$$

at the point $n\log(1/q), n = 1,2,\ldots$.

Furthermore, $\tau_q(0) = 1/(1-q)$ and $\lim_{t\to\infty} \tau_q(t) = 0$.

Proof of Theorem 2.1. Introducing the discrete measure

$$\mu = \sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^k} \delta_{k\log(1/q)}$$

of finite total mass

$$||\mu||_1 = c_q - q/(1-q) < c_q, \tag{24}$$

we can write

$$\frac{f_q(z+1)}{1-q} = 1 + c_q + z - \mathcal{L}(\mu)(z),$$

hence

$$\frac{1-q}{f_q(z+1)} = \left((1+c_q+z)(1-\frac{\mathcal{L}(\mu)(z)}{1+c_q+z})\right)^{-1} = \sum_{n=0}^{\infty} \frac{(\mathcal{L}(\mu)(z))^n}{(1+c_q+z)^{n+1}}. \qquad (25)$$

Let ρ_q denote the following exponential density restricted to the positive half-line

$$\rho_q(t) = \exp(-(1+c_q)t)Y(t),$$

where Y is the usual Heaviside function equal to 1 for $t \geq 0$ and equal to zero for $t < 0$. Its Laplace transform is given as

$$\int_0^\infty e^{-tz}\rho_q(t)\,dt = (1+c_q+z)^{-1},$$

but this shows that (25) is equivalent to the following convolution equation

$$(1-q)\tau_q = \rho_q * \sum_{n=0}^{\infty}(\mu * \rho_q)^{*n} = \sum_{n=0}^{\infty} \rho_q^{*(n+1)} * \mu^{*n}. \qquad (26)$$

This equation expresses a factorization of $(1-q)\tau_q$ as the convolution of the exponential density ρ_q and an elementary kernel $\sum_0^\infty N^{*n}$ with $N = \mu * \rho_q$. For information about the basic notion of elementary kernels in potential theory, see [7, p.100]. All three measures in question τ_q, ρ_q and $\sum_0^\infty(\mu * \rho_q)^{*n}$ are potential kernels on $\mathbb{R}$ in the sense of [7].

The measure $\mu^{*n}, n \geq 1$ is a discrete measure concentrated in the points $k\log(1/q), k = n, n+1, \ldots$. The convolution powers of ρ_q are Gamma densities

$$\rho_q^{*(n+1)}(t) = \frac{t^n}{n!}e^{-(1+c_q)t}Y(t),$$

as is easily seen by Laplace transformation.

Clearly, $\rho_q * \mu$ is a bounded integrable function with

$$||\rho_q * \mu||_\infty \leq ||\rho_q||_\infty ||\mu||_1 < c_q, \quad ||\rho_q * \mu||_1 = ||\rho_q||_1 ||\mu||_1 < \frac{c_q}{1+c_q}, \qquad (27)$$

and then $\rho_q * (\rho_q * \mu)^{*n}, n \geq 1$ is a continuous integrable function on $\mathbb{R}$, vanishing for $t \leq n\log(1/q)$ and for $t \to \infty$. Furthermore,

$$||\rho_q * (\rho_q * \mu)^{*n}||_\infty < (c_q/(1+c_q))^n,$$

and this shows that the right-hand side of (26) converges uniformly on $[0,\infty[$, so $(1-q)\tau_q$ has a continuous density on $[0,\infty[$ tending to 0 at infinity. Note that $(1-q)\tau_q(0) = \rho_q(0) = 1$.

For $n \geq 1$ and $x \in [n\log(1/q), \infty[$ we get

$$\begin{aligned}
&\rho_q^{*(n+1)} * \mu^{*n}(x) \\
&= \int_0^x \frac{(x-t)^n}{n!} e^{-(1+c_q)(x-t)} Y(x-t)\, d\mu^{*n}(t) \\
&= e^{-(1+c_q)x} \sum_{k=n}^{\infty} \frac{(x-k\log(1/q))^n}{n!} q^{-k(1+c_q)} Y(x-k\log(1/q)) \mu^{*n}(k\log(1/q))
\end{aligned}$$

which is a finite sum, and

$$\mu^{*n}(k\log(1/q)) = \sum_{p_1+\ldots+p_n=k} \prod_{j=1}^{n} \frac{q^{2p_j}}{1-q^{p_j}},\ k = n, n+1, \ldots.$$

In particular,

$$\mu^{*n}(n\log(1/q)) = \left(\frac{q^2}{1-q}\right)^n, \quad \mu^{*n}((n+1)\log(1/q)) = \frac{nq^{2n+2}}{(1-q)^n(1+q)}.$$

For $n \geq 0$ and $0 \leq x < (n+1)\log(1/q)$ we then get

$$(1-q)\tau_q(x) = \tag{28}$$
$$e^{-(1+c_q)x} \sum_{j=0}^{n} q^{-j(1+c_q)} Y(x-j\log(1/q)) \sum_{k=0}^{j} \frac{(x-j\log(1/q))^k}{k!} \mu^{*k}(j\log(1/q)).$$

On $[0, \log(1/q)]$ it is equal to $\exp(-(1+c_q)x)$, on $[\log(1/q), 2\log(1/q)]$ it is equal to

$$\exp(-(1+c_q)x)\left(1 + \frac{q^{1-c_q}}{1-q}(x-\log(1/q))\right),$$

on $[2\log(1/q), 3\log(1/q)[$ it is equal to

$$\begin{aligned}
&\exp(-(1+c_q)x)\left(1 + \frac{q^{1-c_q}}{1-q}(x-\log(1/q)) + \right. \\
&\left. \frac{q^{2(1-c_q)}}{1-q^2}(x-2\log(1/q)) + \frac{q^{2(1-c_q)}}{2(1-q)^2}(x-2\log(1/q))^2\right).
\end{aligned}$$

On $[n\log(1/q), (n+1)\log(1/q)], n \geq 1$ we can write

$$\begin{aligned}
&(1-q)\tau_q(x) \\
&= e^{-(1+c_q)x}\left(1 + \sum_{j=1}^{n} q^{-j(1+c_q)} \sum_{k=1}^{j} \frac{(x-j\log(1/q))^k}{k!} \mu^{*k}(j\log(1/q))\right),
\end{aligned} \tag{29}$$

because $\mu^{*0}(j\log(1/q)) = 0$ for $j \geq 1$.

This shows that

$$\tau_q(x) = e^{-(1+c_q)x} p_n(x), \quad x \in [n \log(1/q), (n+1)\log(1/q)], n = 0, 1, \ldots, \tag{30}$$

where p_n is the polynomial of degree n given by

$$p_n(x) = \frac{1}{1-q}\left(1 + \sum_{j=1}^{n} q^{-j(1+c_q)} \sum_{k=1}^{j} \frac{(x - j\log(1/q))^k}{k!} \mu^{*k}(j \log(1/q))\right). \tag{31}$$

The derivative of the expression (29) is

$$\begin{aligned} & -(1+c_q)(1-q)\tau_q(x) \\ + \quad & e^{-(1+c_q)x} \sum_{j=1}^{n} q^{-j(1+c_q)} \sum_{k=1}^{j} \frac{(x - j\log(1/q))^{k-1}}{(k-1)!} \mu^{*k}(j\log(1/q)), \end{aligned} \tag{32}$$

and the value R_n at the point $x = n\log(1/q), n \geq 1$ is

$$\begin{aligned} & R_n \\ = \quad & q^{n(1+c_q)} \left[-(1+c_q)\left(1 + \sum_{j=1}^{n} q^{-j(1+c_q)} \sum_{k=1}^{j} \frac{((n-j)\log(1/q))^k}{k!} \mu^{*k}(j\log(1/q))\right)\right. \\ + \quad & \left. \sum_{j=1}^{n} q^{-j(1+c_q)} \sum_{k=1}^{j} \frac{((n-j)\log(1/q))^{k-1}}{(k-1)!} \mu^{*k}(j\log(1/q))\right] \\ = \quad & q^{n(1+c_q)} \left[-(1+c_q)\left(1 + \sum_{j=1}^{n-1} q^{-j(1+c_q)} \sum_{k=1}^{j} \frac{((n-j)\log(1/q))^k}{k!} \mu^{*k}(j\log(1/q))\right)\right. \\ + \quad & \sum_{j=1}^{n-1} q^{-j(1+c_q)} \sum_{k=1}^{j} \frac{((n-j)\log(1/q))^{k-1}}{(k-1)!} \mu^{*k}(j\log(1/q)) \\ + \quad & \left. q^{-n(1+c_q)} \mu(n\log(1/q)) \right]. \end{aligned}$$

The value L_{n+1} of (32) at $x = (n+1)\log(1/q)$ is

$$\begin{aligned} L_{n+1} = \; & q^{(n+1)(1+c_q)} \times \\ & \left[-(1+c_q)\left(1 + \sum_{j=1}^{n} q^{-j(1+c_q)} \sum_{k=1}^{j} \frac{((n+1-j)\log(1/q))^k}{k!} \mu^{*k}(j\log(1/q))\right)\right. \\ + \quad & \left. \sum_{j=1}^{n} q^{-j(1+c_q)} \sum_{k=1}^{j} \frac{((n+1-j)\log(1/q))^{k-1}}{(k-1)!} \mu^{*k}(j\log(1/q))\right]. \end{aligned}$$

The difference

$$R_n - L_n = q^{2n}/(1-q^n)$$

is the jump of $(1-q)\tau_q$ at $x = n\log(1/q)$, and this gives the jump J_n of (23). $\square$

To transfer the results of Theorem 2.1 to give Theorem 1.2, we use that ν_q is the image measure of τ_q under $t \mapsto e^{-t}$, hence

$$\nu_q(t) = (1/t)\tau_q(\log(1/t)),\ 0 < t \leq 1.$$

We then get

$$\begin{aligned} & D_+\nu_q(t)|_{t=q^n} \\ &= -\frac{1}{t^2}\left[D_-\tau_q(\log(1/t)) + \tau_q(\log(1/t))\right]_{t=q^n} \\ &= -\frac{1}{q^{2n}}\left[D_-\tau_q(n\log(1/q)) + \tau_q(n\log(1/q))\right], \end{aligned}$$

and similarly

$$D_-\nu_q(t)|_{t=q^n} = -\frac{1}{q^{2n}}\left[D_+\tau_q(n\log(1/q)) + \tau_q(n\log(1/q))\right],$$

hence

$$\left[D_+\nu_q(t) - D_-\nu_q(t)\right]_{t=q^n} = \frac{1}{q^{2n}}\left(D_+\tau_q(n\log(1/q)) - D_-\tau_q(n\log(1/q))\right),$$

and the assertion follows.

Remark 2.2 The representation (25) and Theorem 2.1 show that $(1-q)\tau_q$ is a standard p-function in the terminology from the theory of regenerative phenomena. This follows from [11, Theorem 3.1], and the measure $\mu + (1/(1-q))\delta_\infty$ plays the role of the measure "μ"in [11]. The size J_n of the jump of the derivative at $n\log(1/q)$ equals

$$\mu(\{n\log(1/q)\}) = \frac{q^{2n}}{1-q^n}.$$

This is in accordance with [11, Theorem 3.4].

Figure 3 and 4 show the graph of $(1-q)\tau_q$ for $q = 0.5$ and $q = 0.9$.

3 Further properties of τ_q

Formally, by Fourier inversion we get that

$$\tau_q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iyx}\frac{dy}{f_q(1+iy)}.$$

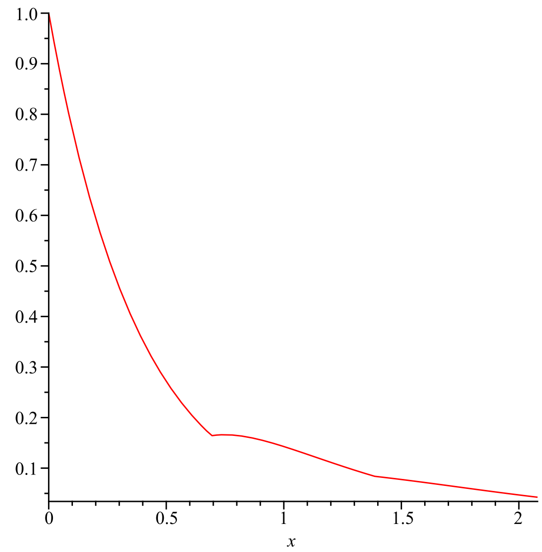

Figure 3: The graph of $(1-q)\tau_q(x)$ on $[0, 3\log(1/q)]$ for $q = 0.5$

The function $1/f_q(1+iy)$ is a non-integrable L^2-function, so the formula holds in the L^2-sense. To see this we notice that

$$\frac{f_q(z)}{z} = 1 - q + \int_0^\infty e^{-tz} h_q(t)\, dt, \quad \Re z > 0,$$

where

$$h_q(t) = (1-q) \sum_{k > t/\log(1/q)} \frac{q^k}{1-q^k}. \tag{33}$$

In particular

$$\frac{f_q(1+iy)}{1+iy} = 1 - q + \int_0^\infty e^{-ity} e^{-t} h_q(t)\, dt,$$

and since $e^{-t}h_q(t)$ is integrable, it follows from the Riemann-Lebesgue Lemma that we get the asymptotic behaviour

$$f_q(1+iy) \sim (1-q)(1+iy), \quad |y| \to \infty. \tag{34}$$

Furthermore, from (21) we get

$$\Re f_q(1+iy) = 1 - q + (1-q) \sum_{k=1}^\infty \frac{q^k}{1-q^k} \left(1 - q^k \cos(ky \log(q))\right),$$

hence

$$1 \le \Re f_q(1+iy) \le 1 - q + \sum_{k=1}^\infty q^k (1 + q^k),$$

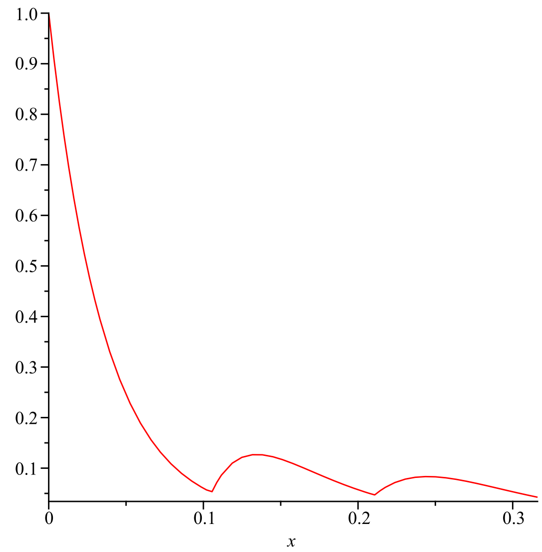

Figure 4: The graph of $(1-q)\tau_q(x)$ on $[0, 3\log(1/q)]$ for $q = 0.9$

showing that $\Re f_q(1+iy)$ is bounded below and above. It follows that the symmetrized density

$$\varphi_q(x) = \begin{cases} \tau_q(x) & \text{if } x \geq 0, \\ \\ \tau_q(-x) & \text{if } x < 0, \end{cases} \tag{35}$$

is the Fourier transform of the non-negative integrable function

$$\frac{2\Re f_q(1+iy)}{|f_q(1+iy)|^2},$$

and therefore $\varphi_q(x)$ is continuous and positive definite, so τ_q is the restriction to $[0,\infty[$ of such a function. Summing up we have proved

Theorem 3.1 *For* $x \geq 0$

$$\tau_q(x) = 4\int_0^\infty \cos(xy)\frac{\Re f_q(1+iy)}{|f_q(1+iy)|^2}\,dy$$

is the restriction of a continuous symmetric positive definite function (35)*. For* $0 < t \leq 1$

$$t\nu_q(t) = 4\int_0^\infty \cos(y\log t)\frac{\Re f_q(1+iy)}{|f_q(1+iy)|^2}\,dy.$$

Remark 3.2 The function f_q defined in (14) is a Bernstein function in the sense of [7], but not a complete Bernstein function in the sense of [14], because $f_q(z)/z$ is not a Stieltjes function as shown by formula (33). This is in contrast to

$$\lim_{q\to 1} f_q(z) = \psi(z+1) + \gamma,$$

which is a complete Bernstein function, cf. [4].

4 Relation to other work

The transformation T can be extended from normalized Hausdorff moment sequences to the set $\mathcal{K} = [0,1]^{\mathbb{N}}$ of sequences $(x_n) = (x_n)_{n\geq 1}$ of numbers from the unit interval $[0,1]$. This was done in [3], where $T : \mathcal{K} \to \mathcal{K}$ is defined by

$$(T(x_n))_n = \frac{1}{1 + x_1 + \ldots + x_n}, \quad n \geq 1. \tag{36}$$

The connection is that a normalized Hausdorff moment sequence $(a_n)_{n\geq 0}$ is considered as the element $(a_n)_{n\geq 1} \in \mathcal{K}$.

Since T is a continuous transformation of the compact convex set $\mathcal{K}$ in the space $\mathbb{R}^{\mathbb{N}}$ of real sequences equipped with the product topology, it has a fixed point by Tychonoff's theorem, and this is $(m_n)_{n\geq 1}$.

There is no reason a priori that the fixed point (m_n) of (36) should be a Hausdorff moment sequence, but as we have seen above, the motivation for the study of T comes from the theory of Hausdorff moment sequences.

Although T is not a contraction on $\mathcal{K}$ in the natural metric

$$d((a_n),(b_n)) = \sum_{n=1}^{\infty} 2^{-n}|a_n - b_n|, \quad (a_n),(b_n) \in \mathcal{K},$$

it was proved in [3] that T maps $\mathcal{K}$ into the compact convex subset

$$\mathcal{C} = \left\{(a_n) \in \mathcal{K} \mid a_1 \geq \tfrac{1}{2}\right\},$$

and the restriction of T to $\mathcal{C}$ is a contraction. It is therefore possible to infer that (m_n) is an attractive fixed point from the fixed point theorem of Banach.

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References

[1] Akhiezer, N. I., *The classical moment problem.* Oliver and Boyd, Edinburgh, 1965.

[2] Artin, E., *The Gamma Function.* Holt, Rinehart and Winston, New York 1964.

[3] Berg, C., Beygmohammadi, M., *On a fixed point in the metric space of normalized Hausdorff moment sequences*, Rend. Circ. Mat. Palermo, Serie II, Suppl. **82** (2010), 251–257.

[4] Berg, C., Durán, A. J., *Some transformations of Hausdorff moment sequences and Harmonic numbers*, Canad. J. Math. **57** (2005), 941–960.

[5] Berg, C., Durán, A. J., *The fixed point for a transformation of Hausdorff moment sequences and iteration of a rational function*, Math. Scand. **103** (2008), 11–39.

[6] Berg, C., Durán, A. J., *Iteration of the rational function* $z - 1/z$ *and a Hausdorff moment sequence*, Expo. Math. **26** (2008), 375–385.

[7] Berg, C., Forst, G., *Potential theory on locally compact abelian groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete Band 87. Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[8] Fine, N. J., *Basic hypergeometric series and applications.* American Mathematical Society, Providence, RI, 1988.

[9] Gasper, G., Rahman, M, *Basic hypergeometric series.* Cambridge University Press, Cambridge 1990, second edition 2004.

[10] Gradshteyn, I. S., Ryzhik, I. M., *Table of integrals, series and products*, Sixth Edition, Academic Press, New York, 2000.

[11] Kingman, J. F. C., *Regenerative Phenomena.* John Wiley & Sons Ltd., London 1972.

[12] Krattenthaler, C., Srivastava, H. M., *Summations for Basic Hypergeometric Series Involving a q-analogue of the Digamma Function*, Computers Math. Applic. **32** (1996), 73–91.

[13] Mansour, T., Shabani, A. S., *Some inequalities for the q-Digamma function*, J. Inequal. Pure and Appl. Math. **10** (2009), Art. 12, 8 pp.

[14] Schilling, R., Song, R. and Vondraček, Z., *Bernstein functions: Theory and Applications*, de Gruyter, Berlin 2010.

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