Convexity of the median in the gamma distribution

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Abstract

We show that the median $m(x)$ in the gamma distribution with parameter x is a strictly convex function on the positive half-line.

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1 Introduction

The median of the gamma distribution with (positive) parameter x is defined implicitly by the formula

$$\int_0^{m(x)} e^{-t}t^{x-1}\,dt = \frac{1}{2}\int_0^{\infty} e^{-t}t^{x-1}\,dt. \tag{1}$$

In a recent paper (see [5]) we showed the $0 < m'(x) < 1$ for all $x > 0$. Consequently, $m(x) - x$ is a decreasing function, which for $x = 1, 2, \ldots$ yields a positive answer to the Chen-Rubin conjecture. Other authors have solved this conjecture in its discrete setting (see [2], [1], [3]).

In [4] convexity of the sequence $m(n+1)$ has been established, and the natural question arises if $m(x)$ is a convex function. The main result of this paper is the following.

Theorem 1.1 *The median $m(x)$ defined in (1) satisfies $m''(x) > 0$. In particular it is a strictly convex function for $x > 0$.*

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2 Proofs

The proof is based on some results in [5], which we briefly describe. Convexity of m is studied through the function

$$\varphi(x) \equiv \log \frac{x}{m(x)}, \quad x > 0. \tag{2}$$

This function played a key role in [5], and we recall its crucial properties in the proposition below.

Proposition 2.1 *The function $x \to x\varphi(x)$ is strictly decreasing for $x > 0$ and*

$$\begin{aligned}
\lim_{x\to 0_+} x\varphi(x) &= \log 2, \\
\lim_{x\to\infty} x\varphi(x) &= \frac{1}{3}.
\end{aligned}$$

Remark 2.2 *Proposition 2.1 is established by showing $(x\varphi(x))' < 0$. It follows that the function $\varphi(x)$ is itself strictly decreasing and $\varphi(x) < -x\varphi'(x)$.*

The starting point for proving Theorem 1.1 is the relation [5, (10)]

$$(x\varphi(x))' = -e^{g(x)}(A(x) + B(x)), \tag{3}$$

where

$$\begin{aligned}
g(x) &= x(\varphi(x) - 1 + e^{-\varphi(x)}), \\
A(x) &= \int_0^{x\varphi(x)} e^{-s} e^{x(1-e^{-s/x})} \left(1 - \left(1 + \frac{s}{x}\right) e^{-s/x}\right) ds, \\
B(x) &= \frac{1}{2}\int_0^\infty t e^{-xt}\xi'(t+1)\, dt,
\end{aligned}$$

and where ξ is a certain positive, increasing and concave function on $[1, \infty)$ satisfying $\xi'(t+1) < 8/135$ for $t > 0$. To establish these properties of ξ is quite involved, and we refer to [5, Section 5] for details.

Before proving the theorem we state the following lemmas, whose proofs are given later.

Lemma 2.3 *For the function g we have*

$$\begin{aligned}
g(x) &< x\varphi(x), \\
-g'(x) &< -x\varphi'(x)\varphi(x) \quad \text{and} \\
-g'(x) &< -x\varphi'(x)
\end{aligned}$$

for all $x > 0$.

Lemma 2.4 *We have for $x > 0$*

$$\begin{aligned} A(x) &< \frac{x\varphi(x)^3}{6}, \\ -A'(x) &< -\frac{1}{6}\varphi(x)^3 - \frac{1}{2}x\varphi'(x)\varphi(x)^2. \end{aligned}$$

Lemma 2.5 *We have for $x > 0$*

$$\begin{aligned} B(x) &< \frac{4}{135x^2}, \\ -B'(x) &< \frac{8}{135x^3}. \end{aligned}$$

Proof of Theorem 1.1. From equation (2) we get

$$m''(x) = -e^{-\varphi(x)}\left(2\varphi'(x) + x\varphi''(x) - x\varphi'(x)^2\right),$$

so that $m''(x) > 0$ is equivalent to the inequality

$$(x\varphi(x))'' < x\varphi'(x)^2.$$

Differentiation of (3) yields

$$(x\varphi(x))'' = e^{g(x)}(-g'(x))(A(x) + B(x)) + e^{g(x)}(-A'(x) - B'(x)).$$

By using Lemma 2.4 and 2.5 it follows that

$$\begin{aligned} -A'(x) - B'(x) &< -\frac{1}{2}x\varphi'(x)\varphi(x)^2 + \frac{8}{135x^3} - \frac{1}{6}\varphi(x)^3 \\ &= -\frac{1}{2}x\varphi'(x)\varphi(x)^2 + \frac{\varphi(x)^2}{x}\left(\frac{8}{135(x\varphi(x))^2} - \frac{1}{6}x\varphi(x)\right). \end{aligned}$$

Here the expression in the brackets is positive, since $(x\varphi(x))^3 < (\log 2)^3 < 48/135$. Therefore, and because $\varphi(x) < -x\varphi'(x)$,

$$\begin{aligned} -A'(x) - B'(x) &< \frac{1}{2}x\varphi'(x)^2\, x\varphi(x) + x\varphi'(x)^2\left(\frac{8}{135(x\varphi(x))^2} - \frac{1}{6}x\varphi(x)\right) \\ &= x\varphi'(x)^2\left(\frac{8}{135(x\varphi(x))^2} + \frac{1}{3}x\varphi(x)\right). \end{aligned}$$

We also have from Lemma 2.3, 2.4 and 2.5,

$$\begin{aligned} -g'(x)(A(x) + B(x)) &< -x\varphi'(x)\varphi(x)\left(\frac{x\varphi(x)^3}{6} + \frac{4}{135x^2}\right) \\ &< x^2\varphi'(x)^2\varphi(x)^2\left(\frac{x\varphi(x)}{6} + \frac{4}{135(x\varphi(x))^2}\right). \end{aligned}$$

Combination of these inequalities yields

$$\begin{aligned}(x\varphi(x))'' &< x\varphi'(x)^2 e^{x\varphi(x)}\left(x\varphi(x)^2\left(\frac{x\varphi(x)}{6}+\frac{4}{135(x\varphi(x))^2}\right)\right.\\ &\qquad\left.+\frac{8}{135(x\varphi(x))^2}+\frac{1}{3}x\varphi(x)\right).\end{aligned}$$

Supposing that $x \geq 1$, it follows that

$$\begin{aligned}(x\varphi(x))'' &< x\varphi'(x)^2 e^{x\varphi(x)}\left((x\varphi(x))^2\left(\frac{x\varphi(x)}{6}+\frac{4}{135(x\varphi(x))^2}\right)\right.\\ &\qquad\left.+\frac{8}{135(x\varphi(x))^2}+\frac{1}{3}x\varphi(x)\right)\\ &= x\varphi'(x)^2 e^{x\varphi(x)}\left(\frac{(x\varphi(x))^3}{6}+\frac{4}{135}+\frac{8}{135(x\varphi(x))^2}+\frac{1}{3}x\varphi(x)\right)\\ &= x\varphi'(x)^2 h_1(x\varphi(x)),\end{aligned}$$

where h_1 is given by

$$h_1(t) = e^t\left(\frac{t^3}{6}+\frac{4}{135}+\frac{8}{135t^2}+\frac{t}{3}\right).$$

One can show that h_1 attains its maximum on the interval $[1/3, \log 2]$ at the left end point and that $h_1(1/3) = (551/810)\sqrt[3]{e} \approx 0.9494$. Therefore it follows that $(x\varphi(x))'' < x\varphi'(x)^2$ for $x \geq 1$.

For $0 < x < 1$ the estimate $-g'(x) < -x\varphi'(x)$ from Lemma 2.3 is used and in this way we get

$$(x\varphi(x))'' < x\varphi'(x)^2 h_2(x\varphi(x)),$$

where

$$h_2(t) = e^t\left(\frac{t^2}{6}+\frac{4}{135t}+\frac{8}{135t^2}+\frac{t}{3}\right).$$

Since $x < 1$ and $x\varphi(x)$ decreases we have $x\varphi(x) > \varphi(1) = -\log\log 2$. One can show that h_2 attains its maximum on the interval $[-\log\log 2, \log 2]$ for $t = -\log\log 2$ and that $h_2(-\log\log 2) \approx 0.9616$. Therefore $(x\varphi(x))'' < x\varphi'(x)^2$ for $x < 1$. □

Remark 2.6 *The function h_2 becomes larger than 1 on the interval $[1/3, \log 2]$, so h_2 cannot be used to obtain the inequality $(x\varphi(x))'' < x\varphi'(x)^2$ for all $x > 0$.*

Proof of Lemma 2.3. It is clear that $g(x) < x\varphi(x)$. Differentiation yields

$$\begin{aligned} -g'(x) &= -\varphi(x) + (1 - x\varphi'(x))(1 - e^{-\varphi(x)}) \\ &< -\varphi(x) + (1 - x\varphi'(x))\varphi(x) \;=\; -x\varphi'(x)\varphi(x), \end{aligned}$$

where we have used $1 - e^{-a} < a$ for $a > 0$.

To find an estimate that is more accurate for x near 0 we use

$$\begin{aligned} -g'(x) &= -x\varphi'(x)(1 - e^{-\varphi(x)}) - \varphi(x) + 1 - e^{-\varphi(x)} \\ &< -x\varphi'(x) - \varphi(x) + 1 - e^{-\varphi(x)} \;<\; -x\varphi'(x). \end{aligned}$$

□

Proof of Lemma 2.4. Using that $1 - e^{-a} < a$ and $1 - (1 + a)e^{-a} < a^2/2$ for $a > 0$, we can estimate $A(x)$ by

$$A(x) < \int_0^{x\varphi(x)} e^{-s} e^{x(s/x)} \left(\frac{s^2}{2x^2}\right) ds = \frac{x\varphi(x)^3}{6}.$$

A computation shows that

$$\begin{aligned} -A'(x) &= -(\varphi(x) + x\varphi'(x))e^{-x\varphi(x)} e^{x(1-e^{-\varphi(x)})}(1 - (1 + \varphi(x))e^{-\varphi(x)}) \\ &\quad - \int_0^{x\varphi(x)} e^{-s} e^{x(1-e^{-s/x})} \left(1 - \left(1 + \frac{s}{x}\right) e^{-s/x}\right)^2 ds \\ &\quad + \int_0^{x\varphi(x)} e^{-s} e^{x(1-e^{-s/x})} \frac{s^2}{x^3} e^{-s/x}\, ds \\ &< -(\varphi(x) + x\varphi'(x))\frac{1}{2}\varphi(x)^2 + \int_0^{x\varphi(x)} s^2 e^{-s/x}\, ds \frac{1}{x^3} \\ &< -\frac{1}{2}\varphi(x)^3 - \frac{1}{2}x\varphi'(x)\varphi(x)^2 + \frac{1}{3}\varphi(x)^3 \\ &= -\frac{1}{6}\varphi(x)^3 - \frac{1}{2}x\varphi'(x)\varphi(x)^2. \end{aligned}$$

□

Proof of Lemma 2.5. These estimates follow directly from the inequality $\xi'(t + 1) < 8/135$. □

References

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