MARKOV'S THEOREM REVISITED

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Abstract. The fact that Markov's Theorem holds for determinate measures is often overlooked and the theorem is stated for measures with compact support as did Markov. We shall give a brief survey of the history of the theorem as well as a proof in the determinate case. We also prove a version of Markov's theorem in the indeterminate case. The results are applied to the shifted moment problem.

0. Introduction. The classical theorem of Markov [11] states that

$$\lim_{n\to\infty}\frac{Q_n(z)}{P_n(z)}=\int\frac{d\mu(x)}{z-x}\quad \text{for } z\in\mathbb{C}\setminus[a,b]\,, \tag{1}$$

where μ is a (positive) measure on the finite interval $[a,b]$. Here and in the following P_n are the orthonormal polynomials associated with μ, and (Q_n) are the corresponding polynomials of the second kind

$$Q_n(x)=\int\frac{P_n(x)-P_n(y)}{x-y}\,d\mu(y)\;. \tag{2}$$

Markov considered a measure with a density, but this reflects the period and is not essential in his proof.

In this paper we shall look at the various extensions of Markov's Theorem which have appeared since [11],[12]. The theorem holds in fact for any determinate measure μ, and that was proved by Hamburger in the fundamental paper [10], Theorem 14 p.292. In the monographs by Akhiezer [1] and Shohat-Tamarkin [21] Markov's Theorem is not stated explicitely (but one can find equivalent statements without Markov's name), and in Szegö [24] and Chihara [7] the theorem is stated only for measures on a finite interval, and this may lead to the erroneous conclusion that the extension to more general classes of measures is not known.

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Hamburger's extension of Markov's Theorem is connected to complete convergence of the associated continued fraction, a concept which was introduced by Hamburger, who also proved that it is equivalent to determinacy of the moment problem. In the first third of this century the moment problem was intimately connected with the theory of continued fractions, and in Perron's influential monograph on the subject, which appeared in 3 editions in the period 1913 to 1957, cf. [15],[17],[18], the moment problem is treated from the continued fractions point of view. Markov's Theorem is treated in all three editions and the extension by Hamburger is contained in [17] and [18]. We shall give more details below. In later treatments of the moment problem functional analysis has replaced continued fractions as the main tool, and in e.g. Akhiezer [1] continued fractions only enter marginally.

In the sequel $s = (s_n)_{n\geq 0}$ denotes a Hamburger moment sequence, normalized ($s_0 = 1$) and assumed positive definite, i.e. $\Delta_n = \det \mathcal{H}_n > 0$ for $n \geq 0$, where $\mathcal{H}_n$ is the Hankel matrix $(s_{i+j})_{0\leq i,j\leq n}$. Any solution μ having s as sequence of moments is a probability measure with infinite support. The polynomials (P_n) and (Q_n) are uniquely determined by s with the convention that P_n has positive leading coefficient.

For each $n \geq 1$ let Λ_n denote the set of zeros of P_n and consider the discrete probability τ_n with mass

$$m_\lambda = \left(\sum_{i=0}^{n-1} P_i(\lambda)^2\right)^{-1} \quad \text{in } \lambda \in \Lambda_n \ .$$

It is well-known that

$$\frac{Q_n(z)}{P_n(z)} = \int \frac{d\tau_n(x)}{z-x} \quad \text{for} \ \ z \in \mathbb{C} \setminus \Lambda_n \ , \tag{3}$$

and

$$\int x^k \, d\mu(x) = \int x^k \, d\tau_n(x) \ , \ k = 0, 1, \cdots, 2n-1 \ , \tag{4}$$

cf. Akhiezer [1] p. 22, 31.

The basic notion of convergence for probability measures is weak convergence: A sequence (μ_n) of probabilities on a metric space X converges weakly to μ if

$$\lim_{n\to\infty} \int f \, d\mu_n = \int f \, d\mu \tag{5}$$

for any continuous and bounded function $f : X \to \mathbb{C}$. For a treatment of this classical concept see Billingsley [5].

Defining

$$\Lambda = \bigcap_{N=1}^{\infty} M_N \ , \text{ where } \ M_N = \overline{\bigcup_{n=N}^{\infty} \Lambda_n} \ , \tag{6}$$

we get a closed subset of $\mathbb{R}$, and it is clear that any *natural solution* μ of the moment problem, i.e. any weak accumulation point of the sequence $(\tau_n)_{n\geq 1}$, cf. [7] p. 60, has $\mathrm{supp}(\mu) \subseteq \Lambda$.

Furthermore, if for any solution μ of the moment problem we define $a_\mu = \inf \mathrm{supp}(\mu)$, $b_\mu = \sup \mathrm{supp}(\mu)$, then $\Lambda \subseteq M_N \subseteq [a_\mu, b_\mu]$.

1. The determinate case.

We shall prove Hamburger's extension of Markov's Theorem using the following result.

Theorem 1.1. Method of moments. *Suppose that* (μ_n) *and* μ *are probabilities on* $\mathbb{R}$ *with moments of every order and that* μ *is* $\det(H)$.

If

$$\lim_{n\to\infty} \int x^k d\mu_n(x) = \int x^k d\mu(x) \quad \textit{for } \ k = 0, 1, \cdots$$

then $\mu_n \to \mu$ *weakly.*

For a proof see Feller [9]. A very general version of the method of moments, including measures on $\mathbb{R}^k$ can be found in [4].

Theorem 1.2. *Assume that* μ *is* $\det(H)$. *Then*

$$\lim_{n\to\infty} \frac{Q_n(z)}{P_n(z)} = \int \frac{d\mu(x)}{z-x} \quad \textit{for } \ z \in \mathbb{C} \setminus \Lambda, \tag{7}$$

and the convergence is uniform for z *in compact subsets of* $\mathbb{C} \setminus \Lambda$.

Proof. By (4) the k'th moment of τ_n converges for $n \to \infty$ to the k'th moment of μ for any k (they are in fact equal for n sufficiently big). By the method of moments $\tau_n \to \mu$ weakly on $\mathbb{R}$ and a fortiori on the closed subset M_N for any $N \in \mathbb{N}$, since it contains $\mathrm{supp}(\mu)$ and $\mathrm{supp}(\tau_n)$ for $n \geq N$.

It follows in particular that

$$\lim_{n\to\infty} \int \frac{d\tau_n(x)}{z-x} = \int \frac{d\mu(x)}{z-x}$$

for any $z \in \mathbb{C} \setminus \Lambda$. To see that the convergence is uniform for $z \in K$, where $K \subseteq \mathbb{C} \setminus \Lambda$ is compact, we notice that $K \cap M_N = \emptyset$ for N sufficiently big, and then there exists $C > 0$ such that

$$|z - x| \geq C \ \text{ for } \ z \in K \ , \, x \in M_N \ .$$

For given $\varepsilon > 0$ there exist $z_1, \cdots, z_p \in K$ such that the discs $D(z_i, \varepsilon)$ cover K. For $z \in K$ we choose $i \in \{1, \cdots, p\}$ such that $|z - z_i| < \varepsilon$, and hence for $x \in M_N$

$$\left| \frac{1}{z-x} - \frac{1}{z_i - x} \right| \le \frac{\varepsilon}{C^2} .$$

For $n \ge N$ we finally get

$$\left| \int \frac{d\mu(x)}{z-x} - \int \frac{d\tau_n(x)}{z-x} \right| \le \frac{2\varepsilon}{C^2} + \left| \int \frac{d\mu(x)}{z_i - x} - \int \frac{d\tau_n(x)}{z_i - x} \right| ,$$

from which the uniform convergence follows. $\square$

Remark 1.3. One cannot replace Λ by $\mathrm{supp}(\mu)$ in (7). If μ is a symmetric measure then

$$\frac{Q_n(0)}{P_n(0)} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \infty & \text{if } n \text{ is odd ,} \end{cases}$$

so if $\mathrm{supp}(\mu)$ has a hole containing 0, e.g. $\mathrm{supp}(\mu) = \mathbb{R} \setminus]-1, 1[$, then (7) cannot hold for $z = 0$.

Historical remarks.

Already Markov [11] noticed that his theorem holds for some measures with unbounded support including the densities leading to the Laguerre polynomials. In [16] Perron extended the theorem to measures μ on a half-line $[a, \infty[$ satisfying

$$\liminf_{n \to \infty} \frac{\sqrt[n]{s_n}}{n} < \infty, \tag{8}$$

(noticing that $s_n > 0$ for n sufficiently big), but he could only prove the convergence in (7) for $\mathrm{Re}\, z < a$ unless $a \ge 0$. This restriction in the convergence was removed by Szász [23] who also removed the restriction about support. Without restriction on the support Szász replaced condition (8) by

$$\liminf_{n \to \infty} \frac{\sqrt[2n]{s_{2n}}}{\sqrt{n}} < \infty. \tag{9}$$

Riesz showed in [19] that the following weaker condition is sufficient

$$\liminf_{n \to \infty} \frac{\sqrt[2n]{s_{2n}}}{n} < \infty, \tag{10}$$

which was later improved by Carleman [6] to

$$\sum_0^\infty \frac{1}{\sqrt[2n]{s_{2n}}} = \infty. \tag{11}$$

The conditions (8)-(11) are in fact conditions which ensure determinacy of the moment sequence. In the second edition of Perron's monograph [17] it is shown that (10) implies determinacy (Satz 14 p.413) and that (7) holds for all $z \in \mathbb{C} \setminus \mathbb{R}$ (Satz 16 p. 418). It is clear that the proof uses only the determinacy of the moment sequence , but apparently Perron has not considered determinacy to be so important a concept that he would use it as an assumption in a theorem.

In [17] Perron does not discuss the complete convergence (introduced in [10]) of the associated continued fraction , but this is done in [18] p.220. The associated continued fraction is of Grommer type ([18] p.192) and is given as

$$\cfrac{1}{z - a_0 - \cfrac{b_0^2}{z - a_1 - \cfrac{b_1^2}{z - a_2 - \ddots}}}, \tag{12}$$

where a_n, b_n are the coefficients of the recurrence relation

$$zP_n(z) = b_n P_{n+1}(z) + a_n P_n(z) + b_{n-1} P_{n-1}(z). \tag{13}$$

The approximating fractions of (12) are $Q_n(z)/P_n(z)$, cf.[1] p.24. The continued fraction (12) is called *completely convergent* with limit a at the point $z \in \mathbb{C}$ if

$$\lim_{n\to\infty} \frac{Q_n(z)t + Q_{n-1}(z)}{P_n(z)t + P_{n-1}(z)} = a$$

uniformly for $t \in \mathbb{R}$. Hamburger proved that the associated continued fraction is completely convergent for all $z \in \mathbb{C} \setminus \mathbb{R}$ if and only if the moment sequence is determinate, and in the affirmative case the limit is $\int \frac{d\mu(x)}{z-x}$. In [18] this follows by combination of Theorems 4.11 and 4.15.

We finally note that Theorem 1.2 follows from Theorem 4.1 in [21] and from Theorem 1.3.3 in [1].

The paper by Van Assche [26] contains a far reaching generalization of Markov's Theorem in the determinate case.

2. The indeterminate case.

In this case the set of measures admitting s as sequence of moments is described via four entire functions A, B, C, D, cf. [1] p. 98. The Nevanlinna extremal solutions $(\mu_t)_{t\in\mathbb{R}\cup\{\infty\}}$ are given by the formula

$$\int \frac{d\mu_t(x)}{z-x} = \frac{A(z)t - C(z)}{B(z)t - D(z)} \quad , \; z \in \mathbb{C} \setminus \mathrm{supp}(\mu_t) \; . \tag{14}$$

Note that $\mathbb{R}^* = \mathbb{R}\cup\{\infty\}$ shall be considered topologically as the one-point compactification of $\mathbb{R}$. To say that $\alpha_n \in \mathbb{R}$ converges to ∞ therefore means that $|\alpha_n| \to \infty$ in the ordinary sense.

Theorem 2.1. *Assume that μ is indeterminate.*

If

$$\lim_{n\to\infty}\frac{P_n(0)}{Q_n(0)}=\alpha \quad \text{in } \mathbb{R}^*\ ,$$

then

$$\lim_{n\to\infty}\frac{Q_n(z)}{P_n(z)}=\int\frac{d\mu_\alpha(x)}{z-x} \quad \text{for } z\in\mathbb{C}\setminus\operatorname{supp}(\mu_\alpha)\ ,$$

and the convergence is uniform for z in compact subsets of $\mathbb{C}\setminus\operatorname{supp}(\mu_\alpha)$.

Proof. Since P_n and Q_n have no common zeros the quotient $P_n(0)/Q_n(0)$ is well--defined in $\mathbb{R}^*$. Putting $\alpha_n=P_n(0)/Q_n(0)$ we have by [1] p. 14

$$\frac{Q_n(z)}{P_n(z)}=\frac{A_n(z)\alpha_n-C_n(z)}{B_n(z)\alpha_n-D_n(z)} \tag{15}$$

for $z\in\mathbb{C}$ with the obvious interpretations if $P_n(z)=0$ or $\alpha_n=\infty$. The polynomials A_n,B_n,C_n,D_n converge to the entire functions A,B,C,D uniformly for z in compact subsets of $\mathbb{C}$. Therefore, if $\alpha_n\to\alpha$ in $\mathbb{R}^*$ then

$$\frac{Q_n(z)}{P_n(z)}\to\frac{A(z)\alpha-C(z)}{B(z)\alpha-D(z)} \tag{16}$$

uniformly for z in compact subsets of $\mathbb{C}\setminus N_\alpha$, where

$$\begin{aligned} N_\alpha&=\{z\in\mathbb{C}\mid B(z)\alpha-D(z)=0\}\ ,\ \alpha\neq\infty\ ,\\ N_\infty&=\{z\in\mathbb{C}\mid B(z)=0\}\ .\end{aligned}$$

We recall that the Nevanlinna extremal measure μ_α is discrete with $\operatorname{supp}(\mu_\alpha)=N_\alpha$. The assertion of the theorem now follows from (14). $\square$

Remark 2.2. It follows easily from (15) that the convergence of $P_n(0)/Q_n(0)$ in $\mathbb{R}^*$ is also a necessary condition for the convergence of $Q_n(z)/P_n(z)$ in $\mathbb{C}\setminus\mathbb{R}$ or even in just one point $z_0\in\mathbb{C}\setminus\mathbb{R}$. Note that μ_α is a natural solution so that $\operatorname{supp}(\mu_\alpha)\subseteq\Lambda$.

Remark 2.3. If μ is a symmetric indeterminate measure on $\mathbb{R}$ then we see as in Remark 1.3 that $P_n(0)/Q_n(0)$ is divergent in $\mathbb{R}^*$, so in this case $Q_n(z)/P_n(z)$ does not converge. However we get

$$\begin{aligned}\lim_{n\to\infty}\frac{Q_{2n}(z)}{P_{2n}(z)}&=\int\frac{d\mu_\infty(x)}{z-x} \quad \text{for } z\in\mathbb{C}\setminus\operatorname{supp}(\mu_\infty)\\ \lim_{n\to\infty}\frac{Q_{2n+1}(z)}{P_{2n+1}(z)}&=\int\frac{d\mu_0(x)}{z-x} \quad \text{for } z\in\mathbb{C}\setminus\operatorname{supp}(\mu_0)\ ,\end{aligned}$$

and the convergence is again uniform for z in compact subsets of the domains in question.

3. The Stieltjes case.

We shall now consider the case where s is a Stieltjes moment sequence, i.e. there exists at least one solution μ of the moment problem for which $\mathrm{supp}(\mu) \subset [0,\infty[$. Equivalently both s and the shifted sequence $\tilde{s} = (s_{n+1})_{n\geq 0}$ have positive Hankel determinants. A Stieltjes moment sequence can be determinate in the sense of Stieltjes meaning that there is precisely one solution supported by $[0,\infty[$. To have a short notation we write det(S) in this case, and the oppositie case is denoted indet(S). Similarly we write det(H) or indet(H) if the moment sequence is determinate or indeterminate considered as a Hamburger moment sequence. We recall that a Stieltjes moment sequence can be det(S) and yet indet(H), cf. [1] p.240, [21] p.76.

To a Stieltjes moment sequence there is a so-called corresponding continued fraction ([18] p.191) which is of Stieltjes type. We shall write it in the terminology of [1] p.232-233:

$$\frac{1}{m_1 z + \frac{1}{l_1 + \frac{1}{m_2 z + \ddots}}}, \tag{17}$$

where $m_i, l_i > 0$ are related to the coefficients a_n, b_n of the three term recurrence relation (13) by

$$a_0 = \frac{1}{m_1 l_1}, \quad a_n = \frac{1}{m_{n+1}}\left(\frac{1}{l_n} + \frac{1}{l_{n+1}}\right), \quad n \geq 1 \tag{18}$$

$$b_n = \frac{1}{l_{n+1}\sqrt{m_{n+1}m_{n+2}}}, \quad n \geq 0. \tag{19}$$

The approximating fractions $S_n(z)/T_n(z), n \geq 0$ are given by the equations

$$\begin{pmatrix} S_{2n+1}(z) \\ T_{2n+1}(z) \end{pmatrix} = \begin{pmatrix} S_{2n}(z) & S_{2n-1}(z) \\ T_{2n}(z) & T_{2n-1}(z) \end{pmatrix} \begin{pmatrix} m_{n+1} z \\ 1 \end{pmatrix}, \quad n \geq 0 \tag{20}$$

$$\begin{pmatrix} S_{2n+2}(z) \\ T_{2n+2}(z) \end{pmatrix} = \begin{pmatrix} S_{2n+1}(z) & S_{2n}(z) \\ T_{2n+1}(z) & T_{2n}(z) \end{pmatrix} \begin{pmatrix} l_{n+1} \\ 1 \end{pmatrix}, n \geq 0 \tag{21}$$

with

$$\begin{pmatrix} S_0(z) & S_{-1}(z) \\ T_0(z) & T_{-1}(z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{22}$$

cf. [17] p.5. Eliminating $S_{2n+1}(z), T_{2n+1}(z)$ from these equations we see that $\sqrt{m_{n+1}}S_{2n}(-z), \sqrt{m_{n+1}}T_{2n}(-z)$ satisfy the recurrence relation (13). Using that $P_n(z), Q_n(z)$ are uniquely determined by (13) and the initial conditions

$$\begin{pmatrix} Q_1(z) & Q_0(z) \\ P_1(z) & P_0(z) \end{pmatrix} = \begin{pmatrix} \frac{1}{b_0} & 0 \\ \frac{1}{b_0}(z - a_0) & 1 \end{pmatrix},$$

we see that

$$Q_n(z) = (-1)^{n-1}\sqrt{m_{n+1}m_1}S_{2n}(-z), \quad P_n(z) = (-1)^n\sqrt{\frac{m_{n+1}}{m_1}}T_{2n}(-z). \tag{23}$$

By (20),(21) and (23) we then get

$$-m_1\frac{S_{2n}(-z)}{T_{2n}(-z)} = \frac{Q_n(z)}{P_n(z)}, \tag{24}$$

$$-m_1\frac{S_{2n-1}(-z)}{T_{2n-1}(-z)} = \frac{\sqrt{\frac{m_n}{m_{n+1}}}Q_n(z)+Q_{n-1}(z)}{\sqrt{\frac{m_n}{m_{n+1}}}P_n(z)+P_{n-1}(z)}. \tag{25}$$

From (20)-(22) we get

$$\begin{pmatrix} S_{2n}(0) & S_{2n-1}(0) \\ T_{2n}(0) & T_{2n-1}(0) \end{pmatrix} = \begin{pmatrix} l_1+\cdots+l_n & 1 \\ 1 & 0 \end{pmatrix}, n \geq 0$$

and hence by (23)

$$P_n(0) = (-1)^n\sqrt{\frac{m_{n+1}}{m_1}}, \quad Q_n(0) = (-1)^{n-1}(l_1+\cdots+l_n)\sqrt{m_1 m_{n+1}}, \tag{26}$$

so that

$$\alpha_n = \frac{P_n(0)}{Q_n(0)} = -\frac{1}{m_1}(l_1+\cdots+l_n)^{-1} \tag{27}$$

which converges to

$$\alpha = -\frac{1}{m_1}(\sum_1^\infty l_n)^{-1}. \tag{28}$$

Using [1] p.14 (25) can be rewritten

$$-m_1\frac{S_{2n-1}(-z)}{T_{2n-1}(-z)} = \frac{C_n(z)}{D_n(z)}. \tag{29}$$

Stieltjes proved in [22] that

$$\lim_{n\to\infty}\frac{S_{2n+i}(z)}{T_{2n+i}(z)} = \frac{1}{m_1}\int_0^\infty \frac{d\mu^{(i)}(x)}{z+x}, \quad i=0,-1,\ z\in\mathbb{C}\setminus]-\infty,0],$$

where $\mu^{(i)}, i=0,-1$ are solutions to the Stieltjes moment problem, and he furthermore showed that the problem is det(S) if and only if $\sum(l_n+m_n)=\infty$, cf. [18], Satz 4.9, 4.10.

In particular already Stieltjes knew that

$$\lim_{n\to\infty} \frac{Q_n(z)}{P_n(z)} = \int_0^\infty \frac{d\mu^{(0)}(x)}{z-x}, \quad z \in \mathbb{C} \setminus [0,\infty[, \tag{30}$$

which can be rephrased as "Markov's Theorem holds for an arbitrary Stieltjes moment problem." If the problem is det(S) then $\mu^{(0)}$ is of course the unique solution supported by $[0,\infty[$. That Markov's Theorem holds in this form for a sequence which is det(S) was noticed by Askey and Wimp [2].

In case the problem is indet(S) or more generally indet(H) we shall next identify the solutions $\mu^{(i)}$ as Nevanlinna extremal measures and determine the corresponding parameters t.

Theorem 3.1. *Consider a Stieltjes moment sequence which is indet(H). Then $\mu^{(0)} = \mu_\alpha$, where α is given by (28) and $\mu^{(-1)} = \mu_0$.*

Proof. The assertions follow from Theorem 2.1 and the equations (28) and (29). □

Remark 3.2. It is worth noticing that the Stieltjes problem in Theorem 3.1 is det(S) if and only if $\alpha = 0$. This is easily derived from the criteria in [1] p. 237, 240. In this case $\mu_\alpha = \mu_0$ is the unique solution concentrated on $[0,\infty[$.

For $\alpha < 0$ the problem is indet(S) and the Nevanlinna extremal solutions $(\mu_t)_{t\in\mathbb{R}^*}$ for which $\mathrm{supp}(\mu_t) \subseteq [0,\infty[$ are characterized by $t \in [\alpha, 0]$, cf. [8] p. 340.

4. Applications to the shifted moment problem.

Let μ be a probability with infinite support and moments of any order. The polynomial sequences $y_n = P_n(z)$ and $y_n = Q_n(z), n \geq 0$ satisfy the second order difference equation

$$zy_n = b_n y_{n+1} + a_n y_n + b_{n-1} y_{n-1} \quad , \; n \geq 1. \tag{31}$$

The sequence $(P_n(z))$ resp. $(Q_n(z))$ is uniquely determined by (31) and the initial conditions

$$y_0 = 1 \;,\; y_1 = \frac{1}{b_0}(z - a_0) \quad , \quad \text{resp.} \;\; y_0 = 0 \;,\; y_1 = \frac{1}{b_0} \;. \tag{32}$$

Replacing (a_n) and (b_n) in (31) and (32) by the shifted sequences $\tilde{a}_n = a_{n+1}$, $\tilde{b}_n = b_{n+1}$, the corresponding unique solutions $(\tilde{P}_n(z))$ and $(\tilde{Q}_n(z))$ are given by

$$\tilde{P}_n(z) = b_0 Q_{n+1}(z) \tag{33}$$

$$\tilde{Q}_n(z) = P_1(z) Q_{n+1}(z) - \frac{1}{b_0} P_{n+1}(z) \;, \tag{34}$$

These equations are not new. Equation (33) can be found in Sherman [20], and both equations are derived in Belmehdi [3] and Pedersen [14]. By Favard's theorem $(\tilde{P}_n)$ are the orthonormal polynomials associated with some probability $\tilde{\mu}$, and $(\tilde{Q}_n)$ are the corresponding polynomials of the second kind. This new moment problem will be called the *shifted moment problem*. The Jacobi matrix $\tilde{J}$ for this problem is obtained from the Jacobi matrix J for the original problem by deleting the first row and column. Let (s_n) resp. $(\tilde{s}_n)$ denote the corresponding moment sequences. Then

$$s_n = J_{11}^n \quad , \; \tilde{s}_n = \tilde{J}_{11}^n$$

meaning that s_n is the element in the first row and first column of the n'th power of the matrix J and similarly with $\tilde{s}_n$. This shows how $\tilde{s}_n$ can be expressed in terms of (a_n) and (b_n). By (33) we immediately get that the two moment problems are determinate simultaneously, and we shall now relate the Stieltjes transforms of the measures μ and $\tilde{\mu}$ in the determinate case. The result is due to Sherman [20] p. 68. See also Nevai [13].

Theorem 4.1. *Suppose that μ and hence $\tilde{\mu}$ are* $\det(H)$. *Then*

$$b_0^2 \int \frac{d\tilde{\mu}(x)}{z-x} = z - a_0 - \left(\int \frac{d\mu(x)}{z-x} \right)^{-1} \qquad \text{for } \; z \in \mathbb{C} \setminus \mathbb{R} \,. \tag{35}$$

Proof. By (33) and (34) we get for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\frac{\tilde{Q}_n(z)}{\tilde{P}_n(z)} = \frac{z - a_0}{b_0^2} - \frac{1}{b_0^2} \frac{P_{n+1}(z)}{Q_{n+1}(z)} \,, \tag{36}$$

and the result follows from Theorem 1.2. □

Example 4.2. Sherman [20]. If μ is the Arcsin-distribution with density $(1/\pi)(1-x^2)^{-\frac{1}{2}}$ on the interval $]-1,1[$, we find that $\tilde{\mu}$ has the density $(2/\pi)(1-x^2)^{1/2}$. This can be verified by inserting the expressions for μ and $\tilde{\mu}$ in (35), but follows also from the fact that the corresponding orthonormal polynomials are the Čebyčev polynomials of the first and second kind. Note that $a_n = 0$, $n \geq 0$ and $b_0 = 1/\sqrt{2}$, $b_n = 1/2$ for $n \geq 1$. The shifted sequences are constant, $\tilde{a}_n = 0$, $\tilde{b}_n = 1/2$, $n \geq 0$ which shows that $\tilde{\mu} = \tilde{\tilde{\mu}}$, i.e. $\tilde{\mu}$ is fixpoint under the operation $\sim$. All the fixpoints under $\sim$ are the image measures of $\tilde{\mu}$ under affine transformations $x \mapsto \alpha x + \beta$, $\alpha > 0, \beta \in \mathbb{R}$ for which the (a_n) and (b_n) sequences are the constant sequences (β) and $(\alpha/2)$.

In the Stieltjes case, which is characterized by $b_k > 0$ and the positivity of the quadratic forms

$$\sum_{k=0}^{n} a_k \xi_k^2 + 2 \sum_{k=0}^{n-1} b_k \xi_k \xi_{k+1}, \quad \xi \in \mathbb{R}^{n+1}, \quad n \geq 0$$

cf. [1] p.233, the shifted moment problem is again a Stieltjes problem. If the original Stieltjes problem is indet(H) so is the shifted problem, and we can use Theorem 3.1 to obtain the following:

Theorem 4.3. *Consider a Stieltjes problem which is indet (H), let μ_α be the Nevanlinna extremal solution of the Stieltjes problem given by (28) and let $\tilde{\mu}_{\tilde{\alpha}}$ be the corresponding solution of the shifted problem.*

Then we have for $z \in \mathbb{C} \setminus \mathbb{R}$

$$b_0^2 \int \frac{d\tilde{\mu}_{\tilde{\alpha}}(x)}{z-x} = z - a_0 - \left(\int \frac{d\mu_\alpha(x)}{z-x} \right)^{-1} , \tag{37}$$

and the parameters α and $\tilde{\alpha}$ are related by the equation

$$\tilde{\alpha} = -\frac{b_0^2}{a_0 + \alpha} . \tag{38}$$

proof. We know from (27) that $(P_n(0)/Q_n(0))$ is strictly increasing with limit α given by (28). Since $P_1(0)/Q_1(0) = -a_0$ we have $a_0 + \alpha > 0$.

By (33), (34) we get

$$\frac{\tilde{P}_n(0)}{\tilde{Q}_n(0)} = -\frac{b_0^2 Q_{n+1}(0)}{a_0 Q_{n+1}(0) + P_{n+1}(0)} \to -\frac{b_0^2}{a_0 + \alpha} , \tag{39}$$

so $\tilde{\alpha} = -b_0^2/(a_0+\alpha)$. The formula (37) follows as in the proof of Theorem 4.1. □

Remark 4.4. Formula (37) is a special case of a formula in [14] which establishes a one-to-one correspondence between the convex sets of solutions to an indeterminate Hamburger problem and its shifted counterpart.

Remark 4.5. Formula (38) shows that $\tilde{\alpha} < 0$ even if $\alpha = 0$. Thus, the shifted Stieltjes problem is always indet (S) although the original problem can be det(S) $(\alpha = 0)$ or indet(S) $(\alpha < 0)$.

The technique above can be used to give a formula for the moment $\tilde{s}_n$ in terms of the moments (s_n). A similar formula appears in Sherman [20] p. 79, but it seems justified only in the determinate case, and the sign in front of the determinant is incorrect.

Proposition 4.6. *Let (s_n) be a normalized Hamburger moment sequence and $(\tilde{s}_n)$ the shifted counterpart. Then*

$$b_0^2 \tilde{s}_n = -\beta_{n+2} \quad \textit{for} \quad n \geq 0 , \quad \textit{where}$$

$$\beta_n = (-1)^{\frac{1}{2}n(n+1)} \begin{vmatrix} 0 & 0 & \dots & s_0 & s_1 \\ 0 & 0 & \dots & s_1 & s_2 \\ \vdots & \vdots & & \vdots & \vdots \\ s_0 & s_1 & \dots & s_{n-2} & s_{n-1} \\ s_1 & s_2 & \dots & s_{n-1} & s_n \end{vmatrix} . \tag{40}$$

Proof. If μ is any positive measure with moment sequence (s_n) then the Stieltjes transform

$$F(z) = \int \frac{d\mu(x)}{z-x}$$

has the asymptotic series

$$F(z) \sim \sum_{n=0}^{\infty} \frac{s_n}{z^{n+1}}$$

for $|z| \to \infty$ in any sector $\arg(z) \in \,]\varepsilon, \pi - \varepsilon[$ in the upper half-plane.

In the determinate case (35) shows that

$$b_0^2 \int \frac{d\tilde{\mu}(x)}{z-x}$$

has an asymptotic series given by the right-hand side of (35), i.e. by

$$z - a_0 - z\sum_{n=0}^{\infty} \frac{\beta_n}{z^n} = -\sum_{n=0}^{\infty} \frac{\beta_{n+2}}{z^{n+1}} , \tag{41}$$

where (β_n) is uniquely determined such that

$$\sum_{j=0}^{n} s_{n-j}\beta_j = \delta_{n0} \ , \ n \geq 0 \ .$$

By Cramer's rule β_n is given as

$$\beta_n = \begin{vmatrix} s_0 & 0 & \dots & 0 & 1 \\ s_1 & s_0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ s_{n-1} & s_{n-2} & \dots & s_0 & 0 \\ s_n & s_{n-1} & \dots & s_1 & 0 \end{vmatrix} = (-1)^{\frac{1}{2}n(n+1)} \begin{vmatrix} 0 & 0 & \dots & s_0 & s_1 \\ 0 & 0 & \dots & s_1 & s_2 \\ \vdots & \vdots & & \vdots & \vdots \\ s_0 & s_1 & \dots & s_{n-2} & s_{n-1} \\ s_1 & s_2 & \dots & s_{n-1} & s_n \end{vmatrix} ,$$

and hence $b_0^2 \tilde{s}_n = -\beta_{n+2}$.

In the indeterminate case we choose an increasing sequence (n_j) of positive integers such that

$$\lim_{j\to\infty} \frac{P_{n_j}(0)}{Q_{n_j}(0)} = t \;\text{ in }\; \mathbb{R}^* \,.$$

By the first equality sign in (39) we get

$$\lim_{j\to\infty} \frac{\tilde{P}_{n_j-1}0)}{\tilde{Q}_{n_j-1}(0)} = -\frac{b_0^2}{a_0+t} =: \tilde{t} \,.$$

By the same reasoning as in Theorem 2.1 we obtain

$$\lim_{j\to\infty} \frac{Q_{n_j}(z)}{P_{n_j}(z)} = \int \frac{d\mu_t(x)}{z-x} \qquad \text{for } \; z \in \mathbb{C}\setminus\mathbb{R}$$

$$\lim_{j\to\infty} \frac{\tilde{Q}_{n_j-1}(z)}{\tilde{P}_{n_j-1}(z)} = \int \frac{d\tilde{\mu}_{\tilde{t}}(x)}{z-x} \qquad \text{for } \; z \in \mathbb{C}\setminus\mathbb{R} \,,$$

so by (36) we find

$$b_0^2 \int \frac{d\tilde{\mu}_{\tilde{t}}(x)}{z-x} = z - a_0 - \left(\int \frac{d\mu_t(x)}{z-x} \right)^{-1} \,. \tag{42}$$

By the same reasoning as in the determinate case this formula yields the asymptotic series (41) for the left-hand side of (42), and this shows again (40). □

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