On powers of Stieltjes moment sequences, I

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Abstract

For a Bernstein function f the sequence $s_n = f(1) \cdot \ldots \cdot f(n)$ is a Stieltjes moment sequence with the property that all powers $s_n^c, c > 0$ are again Stieltjes moment sequences. We prove that s_n^c is Stieltjes determinate for $c \leq 2$, but it can be indeterminate for $c > 2$ as is shown by the moment sequence $(n!)^c$, corresponding to the Bernstein function $f(s) = s$. Nevertheless there always exists a unique product convolution semigroup $(\rho_c)_{c>0}$ such that ρ_c has moments s_n^c. We apply the indeterminacy of $(n!)^c$ for $c > 2$ to prove that the distribution of the product of p independent identically distributed normal random variables is indeterminate if and only if $p \geq 3$.

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1 Introduction and main results

The starting point of this paper is Theorem 1.6 below, proved in [6]. We shall prove that the border $c = 2$ in this theorem is best possible by proving that the sequence $(n!)^c$ becomes indeterminate as a Stieltjes moment sequence when $c > 2$. The proof is given in section 2. It uses that $(n + 1)^{c(n+1)}$ is indeterminate for $c > 2$. Since this sequence is derived from the Bernstein function $f(s) = s(1 + 1/s)^{s+1}$, also this more complicated Bernstein function illustrates the sharpness of Theorem 1.6.

The result about $(n!)^c$ implies the following: If $X_1, \ldots, X_p$ are independent identically distributed random variables with exponential distribution, then the distribution of the product $X_1 \cdot \ldots \cdot X_p$ is determinate for $p \leq 2$ but indeterminate for $p \geq 3$. In section 3 we establish a similar result about the product of independent identically distributed normal random variables, thereby answering a question by A. DasGupta.

In his fundamental memoir [26] Stieltjes characterized sequences of the form

$$s_n = \int_0^\infty x^n \, d\mu(x), \tag{1}$$

where μ is a non-negative measure on $[0,\infty[$, by certain quadratic forms being non-negative. These sequences are now called Stieltjes moment sequences. They are called normalized if $s_0 = 1$. A Stieltjes moment sequence is called *S-determinate*, if there is only one measure μ on $[0,\infty[$ such that (1) holds; otherwise it is called *S-indeterminate.* In the latter case there are also solutions μ to (1), which are not supported by $[0,\infty[$, i.e. solutions to the corresponding Hamburger moment problem.

Later Hausdorff, cf. [18], characterized the Stieltjes moment sequences for which the measure is concentrated on the unit interval $[0,1]$ by complete monotonicity. Both results can be found in [1],[23], [30] or in [4]. A Hausdorff moment sequence

$$a_n = \int_0^1 x^n \, d\mu(x), \; n = 0,1,2,\ldots, \tag{2}$$

is either non-vanishing (i.e. $a_n \neq 0$ for all n) or of the form $a_n = c\delta_{0n}$ with $c \geq 0$, where (δ_{0n}) is the sequence $(1,0,0,\ldots)$. The latter corresponds to the Dirac measure δ_0 with mass 1 concentrated at 0.

The main result in [6] is the following construction of Stieltjes moment sequences from Hausdorff moment sequences:

Theorem 1.1 *Let (a_n) be a non-vanishing Hausdorff moment sequence. Then (s_n) defined by $s_0 = 1$ and $s_n = 1/(a_1 \cdot \ldots \cdot a_n)$ for $n \geq 1$ is a normalized Stieltjes moment sequence.*

This result generalizes and unifies results of various authors, cf. [12],[13],[28].

A statement similar to Theorem 1.1 is contained in [20, Theorem 3.3], but the proof given is erroneous. The inequality used to prove Theorem 3.2 in [20] is not valid, and therefore the result of Boas cannot be applied. In fact, the result of Boas hinted to cannot yield the desired result because it says that if a sequence (s_n) increases so rapidly that $s_n \geq (ns_{n-1})^n$, then (s_n) is a Stieltjes moment sequence. A measure having these moments must have unbounded support, but at the same time Theorem 3.1 in [20] states that it has bounded support.

Remark 1.2 A variant of Theorem 1.1 is that also (t_n) defined by $t_0 = 1, t_n = 1/(a_0 \cdot \ldots \cdot a_{n-1}), n \geq 1$, is a normalized Stieltjes moment sequence.

This follows from the proof in [6].

Since non-vanishing Hausdorff moment sequences roughly speaking can be interpolated by completely monotonic functions we also have the following variant of Theorem 1.1, cf. [6]:

Theorem 1.3 *Let φ be a non-zero completely monotonic function. Then (s_n) defined by $s_0 = 1$ and $s_n = 1/(\varphi(1) \cdot \ldots \cdot \varphi(n))$ for $n \geq 1$ is a normalized Stieltjes moment sequence.*

We recall that a function $\varphi :]0,\infty[\mapsto [0,\infty[$ is called *completely monotonic*, if it is C^∞ and $(-1)^k\varphi^{(k)}(s) \geq 0$ for $s > 0, k = 0, 1, \ldots$. By the Theorem of Bernstein we have

$$\varphi(s) = \int_0^\infty e^{-sx}\, d\alpha(x) = \mathcal{L}\alpha(s), \tag{3}$$

where α a non-negative measure on $[0,\infty[$ and $\mathcal{L}$ denotes Laplace transformation.

As an example of Theorem 1.3 the completely monotonic function $\varphi(s) = \exp(-cs), c > 0$ leads to the Stieltjes moment sequence

$$s_n = \exp(c\binom{n+1}{2}), \tag{4}$$

which is S-indeterminate. The moments are closely related to the moments of the log-normal distribution. A recent treatment of this moment problem can be found in [14].

We are going to discuss a special case of Theorem 1.3 related to convolution semigroups. Let $(\eta_t)_{t>0}$ be a convolution semigroup of sub-probabilities on $[0,\infty[$ with *Laplace exponent* or *Bernstein function* f given by

$$\int_0^\infty e^{-sx}\, d\eta_t(x) = e^{-tf(s)}, \quad s > 0,$$

cf. [7],[10]. We recall that f has the integral representation

$$f(s) = a + bs + \int_0^\infty (1 - e^{-sx})\, d\nu(x), \tag{5}$$

where $a, b \geq 0$ and the Lévy measure ν on $]0,\infty[$ satisfies the integrability condition $\int x/(1+x)\, d\nu(x) < \infty$. Note that $\eta_t([0,\infty[) = \exp(-at)$, so that $(\eta_t)_{t>0}$ consists of probabilities if and only if $a = 0$.

In the following we shall exclude the Bernstein function identically equal to zero, which corresponds to the convolution semigroup $\eta_t = \delta_0, t > 0$.

It is well-known and easy to see that $f(s)/s$ and $1/f(s)$ are completely monotonic functions, when f is a non-zero Bernstein function, viz. the Laplace transforms of the following measures

$$\lambda = b\delta_0 + (a + \nu(]x,\infty[))\, dY(x), \quad \kappa = \int_0^\infty \eta_t\, dt, \tag{6}$$

where Y denotes Lebesgue measure on $]0,\infty[$.

These two completely monotonic functions lead to the following known results as special cases of Theorem 1.3:

Corollary 1.4 ([12],[13],[28]). *Let f be a non-zero Bernstein function. Then $s_0 = 1, s_n = n!/(f(1)\cdot \ldots \cdot f(n))$ for $n \geq 1$ is a Stieltjes moment sequence of a uniquely determined probability measure I_f on $[0,\infty[$.*

Corollary 1.5 ([11]). *Let f be a non-zero Bernstein function. Then $s_0 = 1, s_n = f(1) \cdot \ldots \cdot f(n)$ for $n \geq 1$ is a Stieltjes moment sequence of a uniquely determined probability measure ρ_f on $[0, \infty[$.*

Notice that the determinacy statements in the two corollaries cannot be deduced from Theorem 1.3 because of example (4). The S-determinacy is an easy consequence of Carleman's Criterion, cf. [6].

In [11] Bertoin and Yor remarked that the S-determinacy leads to a factorization of moments and distributions

$$n! = [n!/(f(1) \cdot \ldots \cdot f(n))][f(1) \cdot \ldots \cdot f(n)], \quad \exp(-x)\, dY(x) = I_f \diamond \rho_f. \qquad (7)$$

Here $\diamond$ denotes product convolution of measures on $[0, \infty[$. The product convolution $\mu \diamond \nu$ of two measures μ and ν on $[0, \infty[$ is defined as the image measure of $\mu \otimes \nu$ under the product mapping $s, t \mapsto st$. The second equation follows from the first since the n'th moment of the product convolution is the product of the n'th moments of the factors. Therefore the product convolution has the same moments as the exponential distribution, which is determinate.

Note that the second equation in (7) implies that neither I_f nor ρ_f has mass at zero.

The following result is an extension of Corollary 1.5.

Theorem 1.6 *Let f be a non-zero Bernstein function and let $c > 0$ be arbitrary. Then $s_0 = 1, s_n = (f(1) \cdot \ldots \cdot f(n))^c$ for $n \geq 1$ is a Stieltjes moment sequence, which is S-determinate for $c \leq 2$.*

Proof: It suffices to show that $1/f^c$ is completely monotonic, which follows since more generally $\varphi(f(s))$ is completely monotonic when φ is so, cf. [7]. Here we use the completely monotonic function $\varphi(s) = s^{-c}$. (One can also see that $1/f^c$ is the Laplace transform of the measure

$$\frac{1}{\Gamma(c)} \int_0^\infty t^{c-1} \eta_t\, dt,$$

which is the c'th convolution power of the potential kernel κ of the semigroup $(\eta_t)_{t>0}$ defined in (6).)

Let us define $s_{n,c} = (f(1) \cdot \ldots \cdot f(n))^c$ for $c > 0, n > 0$ while $s_{0,c} = 1$.

To see the S-determinacy for $c \leq 2$ we notice that $f(s) \leq f(1)s, s \geq 1$ because $f(s)/s$ is decreasing. Therefore $s_{n,c} \leq (f(1)^n n!)^c$, so by Stirling's formula

$$\sqrt[2n]{s_{n,c}} \leq f(1)^{c/2} (n!)^{c/(2n)} \sim (f(1)n/e)^{c/2},$$

which shows the the divergence of the series

$$\sum_{n=0}^{\infty} 1/\sqrt[2n]{s_{n,c}}$$

for $c \leq 2$. The Criterion of Carleman implies the S-determinacy of $s_{n,c}$. $\square$

Remark 1.7 M. Yor asked the question if $c = 2$ is best possible as border for S-determinacy in Theorem 1.6. In this paper we show that this is true. In fact, for the moment sequence $s_n = n!$ of the exponential distribution, corresponding to the Bernstein function $f(s) = s$, we show in Section 2 that $(n!)^c$ is S-indeterminate for $c > 2$.

The notion of an infinitely divisible probability measure has been studied for arbitrary locally compact groups, cf. [19]. In accordance with the general definition we say that a probability ρ on $]0, \infty[$ is infinitely divisible on the multiplicative group of positive real numbers, if it has p'th product convolution roots for any natural number p, i.e. if there exists a probability $\tau(p)$ on $]0, \infty[$ such that $(\tau(p))^{\diamond p} = \rho$. This condition implies the existence of a unique family $(\rho_c)_{c>0}$ of probabilities on $]0, \infty[$ such that $\rho_c \diamond \rho_d = \rho_{c+d}$, $\rho_1 = \rho$ and $\lim_{c\to 0} \rho_c = \delta_1$ (weakly), which we call a product convolution semigroup. It is a (continuous) convolution semigroup in the abstract sense of [7] or [19]. A p'th root $\tau(p)$ is unique and one defines $\rho_{1/p} = \tau(p), \rho_{m/p} = (\tau(p))^{\diamond m}, m = 1, 2, \ldots$. Finally ρ_c is defined by continuity when $c > 0$ is irrational.

The family of image measures $(\log(\rho_c))$ under the log-function is a convolution semigroup of infinitely divisible measures in the ordinary sense on the real line considered as an additive group.

Theorem 1.6 leads to the following result.

Theorem 1.8 *Let f be a non-zero Bernstein function. The uniquely determined measure $\rho = \rho_f$ with moments $s_n = f(1) \cdot \ldots \cdot f(n)$ is infinitely divisible with respect to the product convolution. The unique product convolution semigroup $(\rho_c)_{c>0}$ with $\rho_1 = \rho$ has the moments*

$$\int_0^\infty x^n \, d\rho_c(x) = (f(1) \cdot \ldots \cdot f(n))^c, \quad c > 0, n = 0, 1, \ldots. \tag{8}$$

Proof:

Let ρ denote the unique measure on $[0, \infty[$, the moment sequence of which is $s_n = f(1) \cdot \ldots \cdot f(n)$. It follows by (7) that $\rho(\{0\}) = 0$, so ρ is indeed a measure on $]0, \infty[$. It is enough to prove that for each $p \in \mathbb{N}$ there exists a probability $\tau(p)$ on $]0, \infty[$ such that the p'th convolution power $(\tau(p))^{\diamond p}$ is equal to ρ, cf. [19]. By Theorem 1.6 there exists a unique probability $\tau(p)$ on $[0, \infty[$ with moment sequence $({s_n}^{1/p})$. It follows that the moment sequence of $(\tau(p))^{\diamond p}$ is (s_n). By S-determinacy of (s_n) we get $\rho = (\tau(p))^{\diamond p}$, and by this equation $\tau(p)$ can have no mass at 0, since ρ has no mass at 0. Let $(\rho_c)_{c>0}$ be the unique product convolution semigroup with $\rho_1 = \rho$. For a rational number $c = m/p > 0$ it is known that $\rho_c = (\tau(p))^{\diamond m}$ and it follows that (8) holds for such c. The equation (8) for arbitrary $c > 0$ follows by continuity. □

It is interesting to examine whether also the first factor I_f of (7) is infinitely divisible with respect to the product convolution. This is true by Theorem 1.8

if $s/f(s)$ is a Bernstein function. In [29, Theorem 3.3] Urbanik has given a necessary and sufficient condition, which can be rephrased as condition (ii) in the next result.

Theorem 1.9 *Let f be a non-zero Bernstein function with $f(0) = 0$. The following conditions are equivalent*

(i) I_f *given by* (7) *is infinitely divisible for the product convolution,*

(ii) $\log(s+1) - \log f(s+1) + \log f(1)$ *is a Bernstein function,*

(iii) $1/s - f'(s)/f(s)$ *is completely monotonic,*

(iv) $[\nu(]x,\infty[)Y - x\nu] * \kappa \geq 0$.

Proof: The equivalence of (i) and (ii) is a combination of the Theorems 2.3 and 3.3 in [29].

Note that $f(s+1)/(s+1) \leq f(1)$ so the function in (ii) is nonnegative. Therefore it is a Bernstein function if and only if its derivative

$$\frac{1}{s+1} - \frac{f'(s+1)}{f(s+1)} \tag{9}$$

is completely monotonic. By (5) and (6) with $a = 0$ we have the following Laplace transforms

$$f'(s) = \mathcal{L}(b\delta_0 + x\nu)(s), \quad \frac{1}{f(s)} = \mathcal{L}\kappa(s),$$

so (9) is the Laplace transform of

$$e^{-x}Y - [e^{-x}(b\delta_0 + x\nu)] * [e^{-x}\kappa] = e^{-x}[Y - (b\delta_0 + x\nu) * \kappa].$$

It follows that (9) is completely monotonic if and only if

$$Y - (b\delta_0 + x\nu) * \kappa \geq 0,$$

which is equivalent to (iii). Finally, since $Y = \lambda * \kappa$ by (6), we get that (iii) and (iv) are equivalent. □

Define $\mathcal{B}^*$ as the set of non-zero Bernstein functions f with $f(0) = 0$ for which I_f is infinitely divisible for the product convolution. We shall use condition (iii) to prove some sufficient conditions for f to belong to $\mathcal{B}^*$.

Theorem 1.10 *(a) If φ is a Stieltjes transform*

$$\varphi(s) = a + \int_0^\infty \frac{d\sigma(x)}{x+s}$$

satisfying $\varphi(0) = \infty$ ($a \geq 0, \sigma \geq 0$), then $f = 1/\varphi \in \mathcal{B}^*$.

(b) If $f, g \in \mathcal{B}^*$ then $f \circ g \in \mathcal{B}^*$.

Proof: (a) The reciprocal of a Stieltjes transform is always a Bernstein function, cf. [7, Prop. 14.11] or [8, Prop. 1.3]. We get

$$\frac{1}{s} - \frac{f'(s)}{f(s)} = \frac{1}{s} + \frac{\varphi'(s)}{\varphi(s)} = \left(a + \int_0^\infty \frac{x\,d\sigma(x)}{(x+s)^2}\right)\frac{1}{s\varphi(s)},$$

which is completely monotonic as product of two completely monotonic functions. The second factor $1/(s\varphi(s))$ is completely monotonic because it is again a Stieltjes transform, cf.[22].

(b) The result is due to Urbanik [29, Theorem 3.5] and follows from the identity

$$\frac{1}{s} - \frac{(f\circ g)'}{f\circ g} = \frac{1}{s} - \frac{g'}{g} + \left(\frac{1}{g} - \frac{f'\circ g}{f\circ g}\right) g',$$

because $\Phi\circ g$ is completely monotonic when Φ if is so. □

2 The Stieltjes moment sequence $(n!)^c$

In this section we shall prove that the constant $c = 2$ of Theorem 1.6 is best possible, cf. Remark 1.7. The Bernstein function $f(s) = s$ leads to the moment sequence of the heading, and since the exponential distribution with density $\exp(-x)$ on the half-line has the moments $n!$, Theorem 1.8 yields the existence of a uniquely determined product convolution semigroup $(e_c)_{c>0}$ such that $de_1(x) = \exp(-x)\,dY(x)$.

From (8) we know that

$$\int_0^\infty x^n\,de_c(x) = (n!)^c, \quad c>0, n=0,1,\ldots. \tag{10}$$

The S-indeterminacy of e_c for $c > 2$ is formulated in Theorem 2.5 below. The proof consists of a series of preliminary results, some of which are of independent interest. We shall start by calculating the Mellin transform of the measures e_c.

Lemma 2.1 *For $z \in \mathbb{C}$, $Re\, z > -1$ we have*

$$\int_0^\infty t^z\,de_c(t) = \Gamma(1+z)^c, \quad c>0. \tag{11}$$

Proof: We define $\log\Gamma(z)$ in the cut plane $\mathcal{A} = \mathbb{C}\setminus]-\infty,0]$ as the branch which vanishes at $z = 1$, and we use it to define $\Gamma(z)^c$. We next define the functions $\psi_n(z)$ for $z \in \mathcal{A}$ and $n = 1,2,\ldots,\infty$ by

$$\psi_n(z) = \mathrm{Log}(z) + \gamma z + \sum_{k=1}^{n}(\mathrm{Log}(1+z/k) - z/k),$$

where γ is Euler's constant, and Log is the principal logarithm in $\mathcal{A}$. From Weierstrass' product expansion we have

$$\Gamma(z)^c = \exp(-c\psi_\infty(z)), \quad c > 0, z \in \mathcal{A}.$$

We first prove that $\psi_n(1+ix)$ is a continuous negative definite function for $x \in \mathbb{R}$ and for each $n \geq 1$, where we use a terminology from potential theory and harmonic analysis, cf. [7],[4]. Letting $n \to \infty$ we get that $\psi_\infty(1+ix) = -\operatorname{Log}\Gamma(1+ix)$ is a continuous negative definite function. We have

$$\psi_n(1) = \gamma + \log(n+1) - \sum_{k=1}^{n} \frac{1}{k} > 0,$$

(note that $n \to \sum_{k=1}^{n} \frac{1}{k} - \log(n+1)$ is completely alternating with limit γ, cf. [4, p. 139]), and

$$\psi_n(1+ix) = \psi_n(1) + i(\gamma - \sum_{k=1}^{n} \frac{1}{k})x + \sum_{k=1}^{n+1} \operatorname{Log}(1+ix/k),$$

which shows that ψ_n is negative definite since iax is negative definite for $a \in \mathbb{R}$ and $\log(1+s)$ is a Bernstein function, cf. [7, p.69]. This shows that there exists a product convolution semigroup $(\tau_c)_{c>0}$ of probabilities on $]0, \infty[$ such that

$$\int_0^\infty t^{ix}\, d\tau_c(t) = \exp(-c\psi_\infty(1+ix)) = \Gamma(1+ix)^c, \quad x \in \mathbb{R}.$$

For $c = 1$ this equation shows that $\tau_1 = e_1$, since their Mellin transforms are equal. By the uniqueness of a convolution semigroup, for which the measure for $c = 1$ is given, we conclude that $e_c = \tau_c$ for all $c > 0$. We have now proved (11) for $z \in i\mathbb{R}$. Since both sides of the equation are continuous functions in the closed right half-plane and holomorphic in the open half-plane, we see that (11) holds for $\operatorname{Re} z \geq 0$. However, $\Gamma(1+z)^c$ is holomorphic for $\operatorname{Re} z > -1$ and e_c is a positive measure, so by a classical result (going back to Landau for Dirichlet series, see [30, p.58]) the integral must converge for $\operatorname{Re} z > -1$ and (11) holds for these values of z. □

Remark 2.2 By the inversion formula for the Fourier or Mellin transformation it follows that e_c has a smooth nonnegative density (with respect to Y, Lebesgue measure on $]0, \infty[$ and also denoted e_c). It is given as

$$e_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{ix-1}\Gamma(1-ix)^c\, dx.$$

By the Riemann-Lebesgue Lemma $te_c(t) \to 0$ for $t \to 0$ and for $t \to \infty$. For $c = 2$ it can be given in a closed form as

$$e_2(t) = \int_0^\infty \exp(-x - t/x)\frac{dx}{x} = 2K_0(\sqrt{t}),$$

where K_0 is the modified Bessel function. As far as we know the densities $e_c(t)$ for $c \neq 1, 2$ can not be expressed in a closed form by means of classical special functions.

The formula

$$e_{c+1}(t) = \int_0^\infty \exp(-t/x) e_c(x) \frac{dx}{x}, \quad c > 0$$

shows that e_c is completely monotonic for $c \geq 1$. In particular e_c is infinitely divisible for the additive structure for $c \geq 1$ by the Goldie-Steutel Theorem.

The densities e_c have also been studied in [28], and it is proved that they are not infinitely divisible for the additive structure if $c < 1$.

To prove that e_c is S-indeterminate for $c > 2$ one can in principle apply the Krein integral criterion (see below), but this will require knowledge about the asymptotic behaviour of the densities e_c, which seems to be a difficult task. We shall instead exploit knowledge about the *stable distribution of index* 1, thus avoiding the asymptotic analysis.

In a recent paper Alzer and Berg proved the following, cf. [2, Theorem 3]:

The functions $p(s) = e - (1 + 1/s)^s$ and $q(s) = (1 + 1/s)^{s+1} - e$ are Stieltjes transforms with the representations $(s > 0)$

$$p(s) = \frac{1}{s+1} + \int_0^1 \frac{g(x)}{1-x} \frac{dx}{x+s}, \quad q(s) = \frac{1}{s} + \int_0^1 \frac{g(x)}{x} \frac{dx}{x+s}, \tag{12}$$

and in particular they are completely monotonic. Here $g(x)$ is given by

$$g(x) = \frac{1}{\pi} x^x (1-x)^{1-x} \sin(\pi x) \quad (0 \leq x \leq 1). \tag{13}$$

From these results we shall construct 3 Bernstein functions $f_\alpha, f_\beta, f_\gamma$. The corresponding product convolution semigroups according to Theorem 1.8 will be denoted $(\alpha_c)_{c>0}, (\beta_c)_{c>0}, (\gamma_c)_{c>0}$.

The function $f_\alpha(s) = (1+1/s)^s$ is a Bernstein function because $f'_\alpha(s) = -p'(s)$ is completely monotonic. We have

$$f_\alpha(1) \cdot \ldots \cdot f_\alpha(n) = (n+1)^n / n!$$

and

$$\int_0^\infty t^n \, d\alpha_c(t) = \left((n+1)^n / n!\right)^c, \quad c > 0, n = 0, 1, \ldots. \tag{14}$$

The function $(1 + 1/s)^{s+1} = e + q(s)$ is a Stieltjes transform by (12). Therefore $f_\beta(s) = (1+1/s)^{-s-1}$ is a Bernstein function, because the reciprocal of a Stieltjes transform is always a Bernstein function, cf. [7, Prop. 14.11] or [8, Prop. 1.3]. We have

$$f_\beta(1) \cdot \ldots \cdot f_\beta(n) = n!/(n+1)^{n+1}$$

and

$$\int_0^\infty t^n \, d\beta_c(t) = \left((n!)/(n+1)^{n+1}\right)^c, \quad c>0, n=0,1,\ldots. \tag{15}$$

From (12) we get that

$$f_\gamma(s) = s(1+1/s)^{s+1} = 1 + es + \int_0^1 \frac{s}{x+s}\frac{g(x)}{x}\, dx, \tag{16}$$

showing that f_γ is a Bernstein function. We have

$$f_\gamma(1)\cdot\ldots\cdot f_\gamma(n) = (n+1)^{n+1}$$

and

$$\int_0^\infty t^n \, d\gamma_c(t) = (n+1)^{c(n+1)}, \quad c>0, n=0,1,\ldots. \tag{17}$$

Lemma 2.3 *For $c>0$ the measures α_c and β_c are concentrated on $]0,e^c[$ and $]0,e^{-c}[$ respectively. Their Mellin transforms are*

$$\int_0^{e^c} t^z \, d\alpha_c(t) = \left((z+1)^z/\Gamma(z+1)\right)^c, \quad Re\, z>-1, c>0, \tag{18}$$

$$\int_0^{e^{-c}} t^z \, d\beta_c(t) = \left(\Gamma(z+1)/(z+1)^{z+1}\right)^c, \quad Re\, z>-1, c>0. \tag{19}$$

Proof: The digamma function $\Psi(s)=\Gamma'(s)/\Gamma(s)$ satisfies [17, 8.361,8]

$$\Psi(s) - \log(s) = \int_0^\infty e^{-sx}\left(\frac{1}{x} - \frac{1}{1-e^{-x}}\right) dx, \quad s>0. \tag{20}$$

Adding $1/s = \int_0^\infty e^{-sx}\, dx$, replacing s by $s+1$ and integrating with respect to s yields $(s>0)$

$$f_0(s) = \log\Gamma(s+1) - s\log(s+1) + s = \int_0^\infty (1-e^{-sx})\left(\frac{1}{x} - \frac{1}{e^x-1}\right)\frac{e^{-x}}{x}\, dx. \tag{21}$$

It follows that f_0 is a Bernstein function so there exists a uniquely determined convolution semigroup $(\eta_c)_{c>0}$ of probabilities on $[0,\infty[$ such that

$$\int_0^\infty e^{-sy}\, d\eta_c(y) = e^{-cf_0(s)}, \quad s,c>0. \tag{22}$$

Since $f_0(s)\to\infty$ for $s\to\infty$ we have $\eta_c(\{0\})=0$ for each $c>0$. The shifted family $(\eta_c * \delta_{-c})_{c>0}$ is a convolution semigroup of probabilities on $\mathbb{R}$ such that $\eta_c*\delta_{-c}$ is concentrated on $]-c,\infty[$ with Fourier transform

$$\int e^{-ixy}\, d\eta_c*\delta_{-c}(y) = \exp(-c(f_0(ix)-ix)) = \left((ix+1)^{ix}/\Gamma(ix+1)\right)^c, \quad x\in\mathbb{R}. \tag{23}$$

The image measures of $(\eta_c * \delta_{-c})_{c>0}$ under $\exp(-y)$ form a product convolution semigroup $(\tilde{\alpha}_c)_{c>0}$ such that $\tilde{\alpha}_c$ is concentrated on $]0, e^c[$ and

$$\int t^{ix}\, d\tilde{\alpha}_c(t) = \left((ix+1)^{ix}/\Gamma(ix+1)\right)^c, \quad x \in \mathbb{R}, c > 0. \tag{24}$$

Since the function $((z+1)^z/\Gamma(z+1))^c$ is holomorphic in $\mathrm{Re}\, z > -1$, it follows from [30, p.58] that

$$\int_0^{e^c} t^\sigma\, d\tilde{\alpha}_c(t) < \infty \text{ for } \sigma > -1$$

and that

$$\int_0^{e^c} t^z\, d\tilde{\alpha}_c(t) = ((z+1)^z/\Gamma(z+1))^c, \quad \mathrm{Re}\, z > -1.$$

In particular $\tilde{\alpha}_c$ has the moments (14) of α_c, hence $\tilde{\alpha}_c = \alpha_c$. This shows the first part of the lemma.

Replacing s by $s+1$ in (20) and integrating leads to

$$f_1(s) = (s+1)\log(s+1) - s - \log\Gamma(s+1) = \int_0^\infty (1-e^{-sx}) \left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right) \frac{e^{-x}}{x}\, dx,$$

which is a Bernstein function such that $f_1(s) \to \infty$ for $s \to \infty$. Like above this gives a product convolution semigroup $(\tilde{\beta}_c)_{c>0}$ such that $\tilde{\beta}_c$ is concentrated on $]0, e^{-c}[$ and

$$\int_0^{e^{-c}} t^z\, d\tilde{\beta}_c(t) = \left(\Gamma(z+1)/(z+1)^{z+1}\right)^c, \quad \mathrm{Re}\, z > -1.$$

Finally $\tilde{\beta}_c = \beta_c$ because they have the same moments. $\square$

For the last of the three Bernstein functions f_γ we have the following result.

Theorem 2.4 *The Mellin transform of the measure γ_c is given by*

$$\int_0^\infty t^z\, d\gamma_c(t) = (z+1)^{c(z+1)}, \quad Re\, z > -1, \tag{25}$$

and $\gamma_c = g_c(\log t)\, dt$, where $(g_c)_{c>0}$ are the densities of the stable semigroup of index 1 given by the Fourier integral

$$g_c(x) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ix\xi - c\psi(\xi)}\, d\xi, \quad x \in \mathbb{R},$$

and

$$\psi(\xi) = \frac{\pi}{2}|\xi| + i\xi \log|\xi|, \quad \xi \in \mathbb{R}$$

is the corresponding negative definite function. The moment sequence

$$s_n = (n+1)^{c(n+1)} \tag{26}$$

is S-determinate for $c \le 2$, S-indeterminate for $c > 2$.

Proof: The study of stable distributions is a classical subject, and we refer to Zolotarev's monograph [31] for detailed information.

It is well-known that

$$e^{cz\,\mathrm{Log}\,z} = \int_{-\infty}^{\infty} e^{zx} g_c(x)\,dx, \quad \mathrm{Re}\,z \geq 0$$

(for details see e.g. [4, p. 218]), in particular

$$n^{nc} = \int_0^{\infty} t^n g_c(\log t)/t\,dt, \quad n = 0, 1, \ldots.$$

Since the measures $(g_c(\log t)\,dt)_{c>0}$ form a product convolution semigroup with the same moments as $(\gamma_c)_{c>0}$, and since the moments are S-determinate for $c \leq 2$ by Theorem 1.6, the measures are equal for $c \leq 2$ hence for all c by Theorem 1.8.

To see that the moment sequence (26) is indeterminate for $c > 2$ we make use of the criterion of Krein, according to which the density $g_c(\log t)$ is indeterminate provided

$$\int_0^{\infty} \frac{\log g_c(\log t^2)}{1+t^2}\,dt > -\infty. \tag{27}$$

Using the asymptotic behaviour of g_c at $\pm\infty$ one can prove that (27) holds precisely if $c > 2$. This has been carried out by Pakes in [20]. The criterion of Krein is discussed e.g. in [27].

For the convenience of the reader we shall outline the main points in the proof that (27) holds precisely when $c > 2$.

We have to examine for which $c > 0$

$$\int_{-\infty}^{\infty} \frac{\log g_c(x)}{\cosh(x/2)}\,dx > -\infty. \tag{28}$$

A simple computation yields

$$g_c(x) = \frac{1}{c} g_1(\frac{x}{c} + \log c),$$

and with the notation used in Zolotarev [31, Theorem C.3, p. 12] we have

$$g_1(x) = g_B(x, 1, -1),$$

i.e. the parameters in [31] are $(\alpha, \beta, \gamma, \lambda) = (1, -1, 0, 1)$. According to [31, p. 99] we have

$$g_B(x, 1, -1) \sim \frac{1}{\sqrt{2\pi}} \exp(\frac{x-1}{2} - e^{x-1}), \quad x \to \infty, \tag{29}$$

a result due to Skorokhod [24].

Using that $g_B(-x, \alpha, \beta) = g_B(x, \alpha, -\beta)$, cf. [31, p. 65], we find

$$g_c(-x) = \frac{1}{c} g_B(\frac{x}{c} - \log c, 1, 1),$$

and the asymptotic behaviour of $g_B(x,1,1)$ is

$$g_B(x,1,1) \sim x^{-2}, \quad x \to \infty, \tag{30}$$

cf. [31, p. 101]. (Note that the polynomial $P_1(\log x) = \pi$.) From (29) and (30) we find

$$g_c(x) \sim \frac{1}{\sqrt{2\pi c}} \exp\left\{\frac{1}{2}(x/c + \log c - 1) - \exp(x/c + \log c - 1)\right\}, \quad x \to \infty$$

$$g_c(-x) \sim \frac{1}{c(x/c - \log c)^2}, \quad x \to \infty.$$

From this we see that integrability at $-\infty$ in (28) holds for all $c > 0$, while integrability at ∞ holds if and only if $c > 2$. □

Combining the previous results we get:

Theorem 2.5 *The Stieltjes moment sequence $(n!)^c$ is S-determinate for $0 < c \leq 2$ and S-indeterminate for $c > 2$. It admits the factorization as Stieltjes moment sequences for each $c > 0$*

$$(n!)^c = [(n+1)^{c(n+1)}][(n!/(n+1)^{n+1})^c], \tag{31}$$

which corresponds to the factorization of measures $e_c = \gamma_c \diamond \beta_c$.

Proof: The equation $e_c = \gamma_c \diamond \beta_c$ holds because the two sides of the equation have the same Mellin transform. Since the moment sequence (26) is S-indeterminate for $c > 2$, it follows by Lemma 2.2 in [6] that (31) is S-indeterminate. □

Remark 2.6 There is a factorization similar to (31) namely

$$(n+1)^{cn} = [((n+1)^n/n!)^c][(n!)^c]. \tag{32}$$

To find the corresponding factorization of measures, we consider the Bernstein function $f_\delta(s) = s/(s+1)$, which by Theorem 1.6 and Theorem 1.8 leads to the moments $(n+1)^{-c}$ and the product convolution semigroup $(\mu_c)_{c>0}$ given by

$$\mu_c = \frac{\log^{c-1}(1/t)}{\Gamma(c)} 1_{]0,1[}(t)\, dt. \tag{33}$$

We have $\gamma_c \diamond \mu_c = \alpha_c \diamond e_c$, which is verified by calculating the Mellin transforms.

Remark 2.7 Let X be a random variable with exponential distribution. Then the distribution of X^c has the moment sequence $\Gamma(nc+1)$ which is determinate for $c \leq 2$ and indeterminate for $c > 2$, cf.[21],[27]. This result follows easily from the theorems of Carleman and Krein. Note that $c = 2$ is the border in this case as well as in Theorem 2.5.

Remark 2.8 It is interesting to note that the closely related moment sequence $n^n/(n+1)!$ appears in recent work by Dykema and Haagerup, cf. [15],[16]. Formula (18) can be written

$$\int_0^{e^c} t^z \, d\alpha_c(t) = ((z+1)^{z+1}/\Gamma(z+2))^c,$$

so we may let $z \to -1^+$ to get

$$\int_0^{e^c} \frac{1}{t} \, d\alpha_c(t) = 1.$$

Therefore $((1/t)\alpha_c)_{c>0}$ is a product convolution semigroup with moments

$$s_n((1/t)\alpha_c) = (n^n/n!)^c,$$

and using (33) we see that

$$s_n((1/t)\alpha_c \diamond \mu_c) = (n^n/(n+1)!)^c.$$

Dykema and Haagerup proves that the density $\varphi :]0, e[\to [0, \infty[$ of $(1/t)\alpha_1 \diamond \mu_1$ is given by

$$\varphi\left((\sin(v)/v)\exp(v \cot v)\right) = (1/\pi)\sin(v)\exp(-v \cot v), \quad 0 < v < \pi.$$

Example 2.9 It follows also from [2] that $f(s) = (1+s)^{1+1/s}$ and $f(s) = (1+s)^{-1/s}$ are Bernstein functions. This leads to the following Stieltjes moment sequences via Theorem 1.6:

$$\left((n+1)! \prod_{k=1}^{n} (1+k)^{1/k}\right)^c \quad \text{and} \quad \left(\prod_{k=1}^{n} (1+k)^{1/k}\right)^{-c},$$

where $c > 0$.

3 Indeterminacy of products of normal random variables

Let X be a normal random variable with density $(1/\sqrt{\pi})\exp(-x^2)$. In [3] it was proved that the distribution of X^3 is indeterminate, by exhibiting a family of distributions with the same moments as X^3. Since one has an explicit formula for the density of X^3, it is also possible to establish the result using the criterion of Krein, cf. [27]. For an extension of the result in [3] see the paper by Slud [25].

We shall use Theorem 2.5 to answer a question by A. DasGupta about indeterminacy of the distribution of the product of 3 independent identically distributed normal random variables. The density d of the distribution is given as the integral

$$d(x) = (1/\sqrt{\pi})^3 \int_0^\infty \int_0^\infty \exp\left(-u - \frac{v}{u} - \frac{x^2}{v}\right) \frac{du\, dv}{uv}. \tag{34}$$

In principle one can prove the indeterminacy of d using Krein's criterion provided one knows the asymptotic behaviour of d. We avoid this method and obtain the following result using what we have proved about the semigroup $(e_c)_{c>0}$.

Theorem 3.1 *Let $X_1, X_2, \ldots, X_p$ be p independent identically distributed normal random variables as above. The distribution of the product $X_1 X_2 \cdot \ldots \cdot X_p$ is determinate for $p \leq 2$ and indeterminate for $p \geq 3$.*

Proof: For a symmetric measure μ on $\mathbb{R}$ we let $\psi(\mu)$ denote the image measure on $[0, \infty[$ of μ under $\psi(x) = x^2$, i.e.

$$\int_0^\infty f\, d\psi(\mu) = \int_{-\infty}^\infty f(x^2)\, d\mu(x).$$

It is well-known that μ is determinate for the corresponding Hamburger moment problem if and only if $\psi(\mu)$ is S-determinate. For a generalization to rotation invariant measures on $\mathbb{R}^n$ see [9].

For measures $\mu_1, \ldots, \mu_p$ on $\mathbb{R}$ the product convolution $\mu_1 \diamond \cdots \diamond \mu_p$ is defined as the image measure of the product measure $\mu_1 \otimes \cdots \otimes \mu_p$ on $\mathbb{R}^p$ under $(x_1, \ldots, x_p) \to x_1 \cdot \ldots \cdot x_p$ of $\mathbb{R}^p$ to $\mathbb{R}$. If $\mu_1, \ldots, \mu_p$ are the distributions of independent random variables $X_1, \ldots, X_p$, then $\mu_1 \diamond \cdots \diamond \mu_p$ is the distribution of the random variable $X_1 \cdot \ldots \cdot X_p$.

If $|x|\mu$ denotes the measure with density $|x|$ with respect to μ we have

$$(|x|\mu_1) \diamond \cdots \diamond (|x|\mu_p) = |x|(\mu_1 \diamond \cdots \diamond \mu_p). \tag{35}$$

Note that if $\mu_1, \ldots, \mu_p$ are symmetric, then $\mu_1 \diamond \cdots \diamond \mu_p$ is symmetric and

$$\psi(\mu_1 \diamond \cdots \diamond \mu_p) = \psi(\mu_1) \diamond \cdots \diamond \psi(\mu_p). \tag{36}$$

The distribution of $X_1 \cdot \ldots \cdot X_p$ is the product convolution

$$\tau_p = \left((1/\sqrt{\pi}) e^{-x^2}\, dx\right)^{\diamond p}$$

with the moment sequence

$$s_{2n}(\tau_p) = \left(\frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}}\right)^p, \quad s_{2n+1}(\tau_p) = 0.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}(\tau_p)}} = \infty \tag{37}$$

if and only if $p \leq 2$. By Carleman's Criterion τ_p is determinate for $p \leq 2$, but since the divergence of the series in (37) is only sufficient for determinacy, we cannot conclude the indeterminacy for $p \geq 3$. We shall obtain the indeterminacy by comparison with the product convolution semigroup $(e_c)_{c>0}$.

It is clear that τ_p is absolutely continuous with respect to Lebesgue measure, and its density is given by a $(p-1)$-fold integral generalizing the density (34). Since $\psi(|x|e^{-x^2}\,dx) = e_1$ we get by (35) and (36)

$$e_p = e_1^{\diamond p} = \psi\left(|x|(e^{-x^2}\,dx)^{\diamond p}\right) = \psi(\pi^{p/2}|x|\tau_p),$$

which by Theorem 2.5 shows that $\pi^{p/2}|x|\tau_p$ is indeterminate. Since $2|x| \leq 1 + x^2$ we get that $(1+x^2)\tau_p$ is indeterminate. We claim that this implies that τ_p itself is indeterminate for otherwise $\mathrm{ind}_i(\tau_p) = 0$, i.e. the index of determinacy of τ_p at $z = i$ is zero, cf. [5]. However measures with finite index of determinacy are discrete by the main theorem of [5], and this contradicts the absolute continuity of τ_p. □

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