Quantum Hilbert matrices and orthogonal polynomials

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Abstract

Using the notion of quantum integers associated with a complex number $q \neq 0$, we define the quantum Hilbert matrix and various extensions. They are Hankel matrices corresponding to certain little q-Jacobi polynomials when $|q| < 1$, and for the special value $q = (1-\sqrt{5})/(1+\sqrt{5})$ they are closely related to Hankel matrices of reciprocal Fibonacci numbers called Filbert matrices. We find a formula for the entries of the inverse quantum Hilbert matrix.

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1 Introduction

In [9] Hilbert introduced the *Hilbert matrices*

$$\mathcal{H}_n = (1/(\ell + j + 1)), \quad 0 \le \ell, j \le n, \quad n = 0, 1, \ldots, \tag{1}$$

and found the following expression for their determinants

$$\det \mathcal{H}_n = \left(\prod_{k=1}^{n} (2k+1) \binom{2k}{k}^2 \right)^{-1}, \tag{2}$$

showing that they are the reciprocal of integers. This fact is also a consequence of the observation that the inverse matrices $\mathcal{H}_n^{-1}$ have integer entries and Collar [7] found the following integer expression for them

$$(\mathcal{H}_n^{-1})_{\ell,j} = (-1)^{\ell+j} (\ell + j + 1) \binom{n+\ell+1}{n-j} \binom{n+j+1}{n-\ell} \binom{\ell+j}{\ell} \binom{\ell+j}{j}. \tag{3}$$

Collar [7] actually found the inverse of a one parameter extension $\mathcal{H}_n^{(\alpha)} = (\alpha/(\ell+j+\alpha))$ of the Hilbert matrices, where $\alpha > 0$,

$$(\mathcal{H}_n^{(\alpha)})^{-1}_{\ell,j} = (-1)^{\ell+j}\tfrac{\ell+j+\alpha}{\alpha}\tbinom{n+\ell+\alpha}{n-j}\tbinom{n+j+\alpha}{n-\ell}\tbinom{\ell+j+\alpha-1}{\ell}\tbinom{\ell+j+\alpha-1}{j}. \tag{4}$$

This formula was rederived by the second author in [4, Theorem 4.1] without know-ledge of Collar's paper, which was not mentioned in the survey by Choi [6]. The idea of proof of (4) is to observe that $\mathcal{H}_n^{(\alpha)}, n \ge 0$ are the Hankel matrices of a moment problem. After having determined the corresponding orthonormal polynomials (P_n) and their kernel polynomials

$$K_n(x,y) = \sum_{k=0}^{n} P_k(x)P_k(y) = \sum_{\ell,j=0}^{n} a_{\ell,j}^{(n)} x^\ell y^j \tag{5}$$

one uses the result that the matrix

$$(a_{\ell,j}^{(n)}), \quad 0 \le \ell, j \le n$$

of coefficients of the kernel polynomial is the inverse of the Hankel matrix. This result is called the ABC theorem in the survey paper [15] by Simon. It is equivalent to formula (1) in [7], and it was rediscovered as [4, Theorem 2.1] with the remark that it also holds for orthonormal polynomials with respect to signed measures. We add that the determinant formula for the kernel polynomials given in [1, p.9] is equivalent to the ABC theorem.

In [14] Richardson noticed that the *Filbert matrices*

$$\mathcal{F}_n = (1/F_{\ell+j+1}), \quad 0 \le \ell, j \le n, \quad n = 0, 1, \ldots, \tag{6}$$

where $F_n, n \ge 0$ is the sequence of Fibonacci numbers, have the property that all elements of the inverse matrices are integers. Richardson gave an explicit formula for the elements of the inverse matrices and proved it using computer algebra. It is the special case $\alpha = 1, \sinh\theta = \frac{1}{2}$ of (44) given below. The formula shows a remarkable analogy with formula (3) in the sense that one shall replace the binomial coefficients $\binom{n}{k}$ by the analogous *Fibonomial coefficients*

$$\binom{n}{k}_{\mathbb{F}} = \prod_{j=1}^{k} \frac{F_{n-j+1}}{F_j}, \quad 0 \le k \le n, \tag{7}$$

with the usual convention that empty products are defined as 1. These coefficients are defined and studied in [12] and are integers. We recall that the sequence of Fibonacci numbers is $F_0 = 0, F_1 = 1, \ldots,$ with the recursion formula $F_{n+1} = F_n + F_{n-1},\ n \ge 1$.

The Hilbert matrices are the Hankel matrices $(s_{\ell+j})$ corresponding to the moment sequence

$$s_n = 1/(n+1) = \int_0^1 x^n\, dx, \quad n \ge 0, \tag{8}$$

and the corresponding orthogonal polynomials are the Legendre polynomials for the interval $[0,1]$, see [2, Section 7.7]. This was used in [7],[4] and in the survey paper [3] to prove Collar's formula (4).

In [4] it was noticed that for $\alpha \in \mathbb{N}$ the sequence $(F_\alpha/F_{n+\alpha})_{n\geq 0}$ is the moment sequence of a signed measure of total mass one. The corresponding orthogonal polynomials were identified as special little q-Jacobi polynomials for the value $q = (1-\sqrt{5})/(1+\sqrt{5})$, and from this Richardson's formula for the elements of the inverse Filbert matrices was derived as the case $\alpha = 1$ of [4, (29)].

The results about Fibonacci numbers have been extended by Ismail in [11] to a one-parameter family of sequences $(F_n(\theta))_{n\geq 0}, \theta > 0$, determined by the recursion

$$F_{n+1}(\theta) = 2\sinh\theta F_n(\theta) + F_{n-1}(\theta),\ n \geq 1, \quad F_0(\theta) = 0, F_1(\theta) = 1. \tag{9}$$

When $\sinh\theta = \frac{1}{2}$ we have $F_n(\theta) = F_n$, and when $2\sinh\theta$ is a positive integer then all $F_n(\theta)$ are integers.

The purpose of this paper is to give a common generalization of all these results by the use of *quantum integers* defined by

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}, \quad n = 0, 1, \ldots \tag{10}$$

where $q \in \mathbb{C} \setminus \{0\}$. To make the definition precise, we consider $q^{1/2}$ as the holomorphic branch in the cut plane $\mathbb{C}\setminus]-\infty, 0]$ which is positive for $q > 0$, and extend it to the cut by $(-q)^{1/2} = i\sqrt{q}$ for $q > 0$. In this way $q \to [n]_q$ is holomorphic in the cut plane and for $q \in]-\infty, 0[$ we have

$$[n]_q = \lim_{\varepsilon\to 0^+} [n]_{q+i\varepsilon}. \tag{11}$$

Clearly $[n]_q = n$ for $n = 0, 1$ and $[n]_1 = n$ for all $n \in \mathbb{N}$. The quantum integer $[n]_q$ vanishes for $n \geq 1$ precisely if q is an n'th root of unity different from 1. Note also that $[n]_q = [n]_{1/q} = \overline{[n]_{\bar{q}}}$ for q in the cut plane. For $q = e^{i\theta}, \theta \in]-\pi, \pi[$ we find

$$[n]_{e^{i\theta}} = U_{n-1}(\theta/2),\ n \geq 1, \tag{12}$$

where $U_n(\theta) = \sin((n+1)\theta)/\sin\theta$ is the Chebyshev polynomial of the second kind. Inserting $\theta = \pm\pi$ in (12) yields the following value, which agrees with $[n]_{-1}$ defined according to (11):

$$[n]_{-1} = \begin{cases} 0 & \text{if } n = 2k \\ (-1)^k & \text{if } n = 2k+1. \end{cases} \tag{13}$$

We see in (38) below that the quantum integers for $q = (1-\sqrt{5})/(1+\sqrt{5})$ are closely related to the Fibonacci numbers.

It should be mentioned that some authors define quantum integers differently, namely as $(1-q^n)/(1-q)$.

Assume now $0<|q|<1$. We consider the complex measure in the complex plane depending on $\alpha\in\mathbb{N}$ and q, defined by

$$\mu^{(\alpha)}(q)=(1-q^\alpha)\sum_{k=0}^{\infty}q^{k\alpha}\delta_{q^{k+1/2}}, \tag{14}$$

where δ_c is the Dirac measure concentrated at $c\in\mathbb{C}$. It is easy to see the following q-analogue of (8)

$$\frac{1}{[n+1]_q}=\int x^n\,d\mu^{(1)}(q)(x),\quad n\geq 0 \tag{15}$$

as well as

$$\frac{[\alpha]_q}{[n+\alpha]_q}=\int x^n\,d\mu^{(\alpha)}(q)(x),\quad n\geq 0. \tag{16}$$

The Hankel matrices corresponding to $\mu^{(1)}(q)$ are defined as

$$\mathcal{H}_n(q)=\left(\frac{1}{[\ell+j+1]_q}\right),\quad 0\leq \ell,j\leq n,\quad n=0,1,\ldots, \tag{17}$$

and are called the *quantum Hilbert matrices*. We will also consider the *generalized quantum Hilbert matrices* ($\alpha\in\mathbb{N}$)

$$\mathcal{H}_n^{(\alpha)}(q)=\left(\frac{[\alpha]_q}{[\ell+j+\alpha]_q}\right),\quad 0\leq \ell,j\leq n,\quad n=0,1,\ldots. \tag{18}$$

These matrices are well-defined for non-zero complex numbers q which are not roots of unity of order $\leq 2n+\alpha$. When $0<q<1$ the measure $\mu^{(1)}(q)$ is a probability measure on $[0,1]$ which for $q\to 1$ converges weakly to Lebesgue measure on $[0,1]$. The quantum Hilbert matrices $\mathcal{H}_n(q)$ converge to the ordinary Hilbert matrices when $q\to 1$.

We prove in section 2 that the generalized quantum Hilbert matrices are regular and find a formula for the elements of the inverse matrix, see Theorem 2.1. This is a q-analogue of Collar's formula (4). The proof uses that the orthogonal polynomials with respect to $\mu^{(\alpha)}(q)$ are little q-Jacobi polynomials.

In section 3 we consider the special values $q=-e^{-2\theta}$, $\theta>0$, which for $\sinh\theta=\frac{1}{2}$ gives $q=(1-\sqrt{5})/(1+\sqrt{5})$. We prove that $\mathcal{H}_n^{(\alpha)}(q)$ is unitarily related to the *generalized Filbert matrix*

$$\mathcal{F}_n^{(\alpha)}(\theta)=(F_\alpha(\theta)/F_{\ell+j+\alpha}(\theta)),\quad 0\leq \ell,j\leq n,\quad n=0,1,\ldots \tag{19}$$

in the sense that

$$\mathcal{H}_n^{(\alpha)}(q)=U_n\mathcal{F}_n^{(\alpha)}(\theta)U_n, \tag{20}$$

where U_n is a unitary diagonal matrix with diagonal elements $i^\ell,\ell=0,1,\ldots,n$. This makes it possible to deduce Richardson's formula for the elements of the inverse of $\mathcal{F}_n$ and its generalizations $\mathcal{F}_n^{(\alpha)}(\theta)$ from the q-analogue of Collar's formula.

2 Quantum Hilbert matrices

Let us recall the definition of the basic hypergeometric function ${}_2\phi_1$. For details see [8]. We have

$$ {}_2\phi_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};q,z\right)=\sum_{n=0}^{\infty}\frac{(\alpha;q)_n(\beta;q)_n}{(\gamma;q)_n(q;q)_n}z^n, \tag{21} $$

where the q-shifted factorial is defined as

$$ (w;q)_n=\prod_{k=1}^{n}(1-wq^{k-1}) $$

for a complex number w and $n=0,1,\ldots$. If $|q|<1$ we can consider the infinite product $(w;q)_\infty$.

The little q-Jacobi polynomials are given by

$$ p_n(x;a,b;q)={}_2\phi_1\left(\begin{matrix}q^{-n},abq^{n+1}\\ aq\end{matrix};q,xq\right), \tag{22} $$

where a,b are complex parameters satisfying $|a|,|b|\le 1$. In [8, Section 7.3] one finds a discussion of the little q-Jacobi polynomials, and it is proved that

$$ \sum_{k=0}^{\infty}p_n(q^k;a,b;q)p_m(q^k;a,b;q)\frac{(bq;q)_k}{(q;q)_k}(aq)^k=\frac{\delta_{n,m}}{h_n(a,b;q)}, \tag{23} $$

where

$$ h_n(a,b;q)=\frac{(abq;q)_n(1-abq^{2n+1})(aq;q)_n(aq;q)_\infty}{(q;q)_n(1-abq)(bq;q)_n(abq^2;q)_\infty}(aq)^{-n}. \tag{24} $$

In [8] it is assumed that $0<q,aq<1$, but the derivation shows that it holds for $0<|q|<1,|a|\le 1,|b|\le 1$, in particular in the case of interest here: $a=q^{\alpha-1},\alpha\in\mathbb{N},b=1$, in the case of which we get

$$ \sum_{k=0}^{\infty}p_n(q^k;q^{\alpha-1},1;q)p_m(q^k;q^{\alpha-1},1;q)q^{\alpha k}=\delta_{n,m}\frac{q^{\alpha n}(q;q)_n^2}{(q^\alpha;q)_n^2(1-q^{2n+\alpha})}. \tag{25} $$

The Gaussian q-binomial coefficients

$$ \begin{bmatrix}n\\ k\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} $$

are polynomials in q. Using the quantum integers (10) we get

$$ \begin{bmatrix}n\\ k\end{bmatrix}_q=\prod_{j=1}^{k}\frac{[n-j+1]_q q^{(n-j)/2}}{[j]_q q^{(j-1)/2}}=q^{k(n-k)/2}\prod_{j=1}^{k}\frac{[n-j+1]_q}{[j]_q}, $$

and introducing the notation

$$[n]_q! = \prod_{k=1}^{n} [k]_q, \quad [0]_q! = 1,$$

we define the *quantum binomial coefficients* by

$$\binom{n}{k}_q = \prod_{j=1}^{k} \frac{[n-j+1]_q}{[j]_q} = \frac{[n]_q!}{[k]_q![n-k]_q!}, \tag{26}$$

hence

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{k(n-k)/2} \binom{n}{k}_q . \tag{27}$$

This formula shows that $\binom{n}{k}_q$ is a holomorphic function of q in the cut plane $\mathbb{C}\setminus]-\infty,0]$ with a continuous extension to the upper part of the cut.

Defining

$$p_n^{(\alpha)}(q;x) := \binom{n+\alpha-1}{n}_q p_n(x/q^{1/2}; q^{\alpha-1}, 1; q), \tag{28}$$

some calculation leads to the simple expression

$$p_n^{(\alpha)}(q;x) = \sum_{j=0}^{n} \binom{n}{j}_q \binom{n+j+\alpha-1}{n}_q (-1)^j x^j, \tag{29}$$

which is a q-analogue of the polynomials $r_n^{(\alpha)}$ of [4].

The equation (25) can be written

$$\int p_n^{(\alpha)}(q;x) p_m^{(\alpha)}(q;x)\, d\mu^{(\alpha)}(q)(x) = \delta_{n,m} \frac{[\alpha]_q}{[2n+\alpha]_q} \tag{30}$$

showing that

$$P_n^{(\alpha)}(q;x) = ([2n+\alpha]_q/[\alpha]_q)^{1/2} p_n^{(\alpha)}(q;x),$$

are orthonormal polynomials. The corresponding kernel polynomials are defined by

$$K_n^{(\alpha)}(q;x,y) = \sum_{k=0}^{n} P_k^{(\alpha)}(q;x) P_k^{(\alpha)}(q;y) = \sum_{k=0}^{n} ([2k+\alpha]_q/[\alpha]_q)\, p_k^{(\alpha)}(q;x) p_k^{(\alpha)}(q;y). \tag{31}$$

While $P_n^{(\alpha)}(q;x)$ depends on the choice of a square root, the kernel polynomials $K_n^{(\alpha)}(q;x,y)$ are independent of this choice. Since the orthogonal polynomials exist with respect to $\mu^{(\alpha)}(q)$ and the moments are given by (16), it follows by [5, Theorem 3.1] that $\mathcal{H}_n^{(\alpha)}(q)$ is regular for each n. For $q \to 1$ and $\alpha = 1$

the expressions (29)–(31) tend to classical formulas for Legendre polynomials for $[0,1]$.

Writing

$$K_n^{(\alpha)}(q;x,y)=\sum_{\ell,j=0}^{n} a_{\ell,j}^{(n)}(q;\alpha)x^\ell y^j,$$

it follows by (29) that the coefficients $a_{\ell,j}^{(n)}(q;\alpha)$ are given by

$$a_{\ell,j}^{(n)}(q;\alpha)=(-1)^{\ell+j}\sum_{k=\max(\ell,j)}^{n}\tfrac{[2k+\alpha]_q}{[\alpha]_q}\tbinom{k}{\ell}_q\tbinom{k}{j}_q\tbinom{k+\ell+\alpha-1}{k}_q\tbinom{k+j+\alpha-1}{k}_q. \tag{32}$$

Theorem 2.1 *The ℓ,j'th element of the inverse matrix of the generalized quantum Hilbert matrix $\mathcal{H}_n^{(\alpha)}(q)$ defined in (18) is given as*

$$(-1)^{\ell+j}\tfrac{[\ell+j+\alpha]_q}{[\alpha]_q}\tbinom{n+\ell+\alpha}{n-j}_q\tbinom{n+j+\alpha}{n-\ell}_q\tbinom{\ell+j+\alpha-1}{\ell}_q\tbinom{\ell+j+\alpha-1}{j}_q. \tag{33}$$

Furthermore,

$$\det\mathcal{H}_n^{(\alpha)}(q)=[\alpha]_q^n\left(\prod_{k=1}^{n}[2k+\alpha]_q\tbinom{2k+\alpha-1}{k}_q^2\right)^{-1}. \tag{34}$$

Proof. As explained in the introduction the coefficients $a_{\ell,j}^{(n)}(q)$ of the kernel polynomial are the entries of the inverse of the Hankel matrix.

Let $R(n;\ell,j)$ denote the expression in (33), and define

$$C(k;\ell,j)=(-1)^{\ell+j}\tfrac{[2k+\alpha]_q}{[\alpha]_q}\tbinom{k}{\ell}_q\tbinom{k}{j}_q\tbinom{k+\ell+\alpha-1}{k}_q\tbinom{k+j+\alpha-1}{k}_q,\quad k\ge\ell,j.$$

We shall prove that

$$R(n;\ell,j)=\sum_{k=\max(\ell,j)}^{n}C(k;\ell,j) \tag{35}$$

by induction in n, and can assume $\ell\ge j$ without loss of generality. The equation (35) is easy for $n=k=\ell$ and is left to the reader. We shall establish the induction step

$$R(n+1;\ell,j)-R(n;\ell,j)=C(n+1;\ell,j). \tag{36}$$

The left-hand side of this expression can be written

$$(-1)^{\ell+j}\tfrac{[\ell+j+\alpha]_q}{[\alpha]_q}\tbinom{\ell+j+\alpha-1}{\ell}_q\tbinom{\ell+j+\alpha-1}{j}_q T,$$

where

$$\begin{aligned}T&=\tbinom{n+\ell+\alpha+1}{n+1-j}_q\tbinom{n+j+\alpha+1}{n+1-\ell}_q-\tbinom{n+\ell+\alpha}{n-j}_q\tbinom{n+j+\alpha}{n-\ell}_q\\&=\tfrac{([n+\ell+\alpha]_q\cdots[\ell+j+\alpha+1]_q)([n+j+\alpha]_q\cdots[\ell+j+\alpha+1]_q)}{[n+1-j]_q![n+1-\ell]_q!}\,.\end{aligned}$$

$$\{[n+\ell+\alpha+1]_q[n+j+\alpha+1]_q - [n+1-j]_q[n+1-\ell]_q\}\,.$$

The quantity in braces equals $[2n+2+\alpha]_q[\ell+j+\alpha]_q$, and now it is easy to complete the proof of (36).

From the general theory of orthogonal polynomials, cf. [1],[5],[10], it is known that the leading coefficient of the orthonormal polynomial $P_n^{(\alpha)}(q;x)$ is $\sqrt{D_{n-1}/D_n}$, where

$$D_n = \det \mathcal{H}_n^{(\alpha)}(q).$$

From (29) and (30) we then get

$$D_{n-1}/D_n = ([2n+\alpha]_q/[\alpha]_q)\binom{2n+\alpha-1}{n}_q^2,$$

hence

$$\frac{1}{D_n} = \prod_{k=1}^{n} \frac{D_{k-1}}{D_k} = \prod_{k=1}^{n} ([2k+\alpha]_q/[\alpha]_q)\binom{2k+\alpha-1}{k}_q^2,$$

and (34) follows. □

Remark 2.2 By analytic continuation, the formulas (33) and (34) of Theorem 2.1 are valid for $q \neq 0$ which is not a root of unity of order $\leq 2n+\alpha$.

We have not been able to derive (35) from known summation formulas for q-series.

3 Filbert matrices

In this section we specialize to $q = -e^{-2\theta}$ for $\theta > 0$. It is easy to see that the unique solution $F_n(\theta), n \geq 0$ to (9) is given by

$$F_n(\theta) = \frac{e^{n\theta} - (-1)^n e^{-n\theta}}{e^{\theta} + e^{-\theta}}, \tag{37}$$

which is Ismail's definition in [11], hence

$$[n]_q = (-i)^{n-1} F_n(\theta). \tag{38}$$

For $\theta = \theta_0 > 0$ such that $\sinh \theta_0 = \frac{1}{2}$ we get $q = (1-\sqrt{5})/(1+\sqrt{5})$ and $F_n = F_n(\theta_0)$. For information about Fibonacci numbers, see [12],[13],[16]. Ismail [11] also considered the generalized Fibonomial coefficients

$$\binom{n}{k}_{\mathbb{F}(\theta)} = \prod_{j=1}^{k} \frac{F_{n-j+1}(\theta)}{F_j(\theta)}, \quad 0 \leq k \leq n, \tag{39}$$

with the usual convention that empty products are 1, and gave the following recursion

$$\binom{n}{k}_{\mathbb{F}(\theta)} = F_{k-1}(\theta)\binom{n-1}{k}_{\mathbb{F}(\theta)} + F_{n-k+1}(\theta)\binom{n-1}{k-1}_{\mathbb{F}(\theta)}, \quad n > k \geq 1, \tag{40}$$

which shows that they are integers when $2\sinh\theta$ is an integer. Using (38) we next get that the quantum binomial coefficients can be expressed by the Fibonomial coefficients of (39) as

$$\binom{n}{k}_q = (-i)^{k(n-k)}\binom{n}{k}_{\mathbb{F}(\theta)}, \tag{41}$$

hence for $0 \le \ell, j \le n, \alpha \in \mathbb{N}$

$$\frac{[\alpha]_q}{[\ell+j+\alpha]_q} = i^{\ell+j}\frac{F_\alpha(\theta)}{F_{\ell+j+\alpha}(\theta)}. \tag{42}$$

Letting U_n denote the unitary diagonal $(n+1)\times(n+1)$-matrix with ℓ'th diagonal element equal to $i^\ell, \ell = 0, 1, \ldots, n$, then formula (42) implies

$$\mathcal{H}_n^{(\alpha)}(q) = U_n\mathcal{F}_n^{(\alpha)}(\theta)U_n, \quad \mathcal{F}_n^{(\alpha)}(\theta)^{-1} = U_n\mathcal{H}_n^{(\alpha)}(q)^{-1}U_n. \tag{43}$$

This leads to Berg's and Ismail's generalizations of Richardson's formula, cf. [4], [11].

Theorem 3.1 *Let A be the matrix $(1/F_{\ell+j+\alpha}(\theta)), 0 \le \ell, j \le n$. Then A^{-1} has the entries*

$$(-1)^{n(\ell+j+\alpha)-\binom{\ell}{2}-\binom{j}{2}}F_{\ell+j+\alpha}(\theta)\tbinom{n+\ell+\alpha}{n-j}_{\mathbb{F}(\theta)}\tbinom{n+j+\alpha}{n-\ell}_{\mathbb{F}(\theta)}\tbinom{\ell+j+\alpha-1}{\ell}_{\mathbb{F}(\theta)}\tbinom{\ell+j+\alpha-1}{j}_{\mathbb{F}(\theta)}, \tag{44}$$

and

$$\det A = (-1)^{\alpha\binom{n+1}{2}}\left(F_\alpha(\theta)\prod_{k=1}^n F_{2k+\alpha}(\theta)\tbinom{2k+\alpha-1}{k}_{\mathbb{F}(\theta)}^2\right)^{-1}. \tag{45}$$

Proof. We use

$$\det U_n = i^{\binom{n+1}{2}}, \quad \det\mathcal{H}_n^{(\alpha)}(q) = (-1)^{\binom{n+1}{2}}\det\mathcal{F}_n^{(\alpha)}(\theta)$$

and the formulas (38) and (41) to make the calculation, the only non-obvious thing being the sign in the two formulas. In the proof of (44) we get the following sign for the ℓj'th element of A^{-1}:

$$\begin{aligned} &i^{\ell+j}(-1)^{\ell+j}(-i)^{\ell+j+(n-j)(\ell+j+\alpha)+(n-l)(\ell+j+\alpha)+\ell(j+\alpha-1)+j(\ell+\alpha-1)} \\ =\ &(-1)^{\ell+j}(-i)^{2n(\ell+j+\alpha)-\ell(\ell+1)-j(j+1)} = (-1)^{n(\ell+j+\alpha)-\binom{\ell}{2}-\binom{j}{2}}. \end{aligned}$$

In the proof of (45) we get the sign

$$\begin{aligned} &(-1)^{\binom{n+1}{2}}(-i)^{(\alpha-1)n}\left(\prod_{k=1}^n(-i)^{2k+\alpha-1+2k(k+\alpha-1)}\right)^{-1} \\ =\ &(-1)^{\binom{n+1}{2}}\left(\prod_{k=1}^n(-i)^{2k(k+1)+2k(\alpha-1)}\right)^{-1} = (-1)^{\binom{n+1}{2}}\left(\prod_{k=1}^n(-1)^{k(\alpha-1)}\right)^{-1} \\ =\ &(-1)^{\alpha\binom{n+1}{2}}. \end{aligned}$$

□

Remark 3.2 For $q = -e^{2\theta}, \theta > 0$ we get $[n]_q = i^{n-1}F_n(\theta)$ with $F_n(\theta)$ given by (37). Thus, for $q < 0$ we have $[n]_{1/q} = \overline{[n]_q}$.

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