Chapter 2

Differentiable manifolds

The recognition that, in classical mechanics as well as in special relativity, space-time events can be labeled, at least locally on a cosmological scale, by a continuum of three space coordinates and one time coordinate leads naturally to the concept of a 4-dimensional manifold as the basis of theories of space-time. Trajectories of moving particles in space-time are described by curves on the manifold. The requirement of being able to associate a velocity to a moving particle at each instant of time leads to the notion of tangent vectors, which in turn necessitates the presence of a differentiable structure on space-time.

In this chapter we introduce the above mentioned basic notions. Besides forming the basis of the whole area of differential geometry it turns out that these concepts have a great variety of other applications in physics, as is exemplified by the usefulness of considering the phase space of a classical mechanical system as a differentiable manifold.

2.1 Manifolds

Let M be a set and $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$ a covering of M, i.e. $U_\alpha \subseteq M$, $\alpha \in I$, and

$$\bigcup_{\alpha \in I} U_\alpha = M \; .$$

A set of mappings $\{x_\alpha \mid \alpha \in I\}$, $x_\alpha : U_\alpha \to \mathbb{R}^n$, where n is fixed, is called a C^N-*atlas* on M, $N \in \mathbb{N} \cup \{0, \infty\}$, if for all $\alpha, \beta \in I$:

i) x_α maps U_α bijectively onto $x_\alpha(U_\alpha)$,

ii) $x_\alpha(U_\alpha \cap U_\beta)$ is an open subset of $\mathbb{R}^n$. In particular, $x_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ is open.

iii) If $U_\alpha \cap U_\beta \neq \emptyset$, the *overlap function*

$$x_{\alpha\beta} \equiv x_\alpha \circ x_\beta^{-1} : x_\beta(U_\alpha \cap U_\beta) \to x_\alpha(U_\alpha \cap U_\beta)$$

is $\mathcal{C}^N$, i.e. it is N times continuously differentiable.

Definition 2.1 A $\mathcal{C}^N$*-manifold* is a pair $(M, \{x_\alpha \mid \alpha \in I\})$, where $\{x_\alpha \mid \alpha \in I\}$ is a $\mathcal{C}^N$-atlas on M. A mapping $x : U \to \mathbb{R}^n$ is called a *coordinate system* on $(M, \{x_\alpha \mid \alpha \in I\})$, or just on M, and U is called a *coordinate patch* on M, if $\{x_\alpha \mid \alpha \in I\} \cup \{x\}$ is a $\mathcal{C}^N$-atlas on M.

Remark 2.2 Two $\mathcal{C}^N$-atlases $\{x_\alpha \mid \alpha \in I\}$ and $\{x_\alpha \mid \alpha \in J\}$ on M are *equivalent* if $\{x_\alpha \mid \alpha \in I \cup J\}$ is a $\mathcal{C}^N$-atlas on M. An equivalence class of $\mathcal{C}^N$-atlases on M is called a $\mathcal{C}^N$*-structure* on M. Thus equivalent atlases determine the same $\mathcal{C}^N$-structure on M.

A C^0-manifold is called a *topological* or a *continuous manifold.* A *differentiable manifold* is a $\mathcal{C}^N$-manifold, where $N \geq 1$, and correspondingly we refer to a $\mathcal{C}^N$-structure, where $N \geq 1$, as a *differentiable structure.* A $\mathcal{C}^\infty$-manifold is also called a *smooth manifold.* In order to equip the set of tangent vectors to a differentiable manifold with a differentiable structure (see section 1.4) we need $N \geq 2$. For simplicity we assume in the following that $N = \infty$ unless otherwise explicitly stated. We shall, however, also encounter *Lipschitz manifolds*, also called $\mathcal{C}^{1-}$-manifolds, defined as above by requiring the overlap functions to satisfy a Lipschitz condition instead of the differentiability requirement in iii).

The concepts and constructions associated to a smooth manifold to be developed in the following will generally only depend on the differentiable structure defined by its atlas. We shall therefore subsequently make no distinction between an atlas and the differentiable structure which it represents.

An atlas $\{x_\alpha \mid \alpha \in I\}$ on M induces a unique topology on M by the requirement that each U_α be an open set in M and each x_α be a homeomorphism from U_α (with induced topology from M) onto $x_\alpha(U_\alpha)$.

It is easy to see that equivalent atlases yield the same topology on M, and that

$$O \subseteq M \text{ is open } \Leftrightarrow x_\alpha(O \cap U_\alpha) \subseteq \mathbb{R}^n \text{ is open } \forall \alpha \in I \, .$$

We shall always consider M as a topological space with the so defined topology, which we henceforth assume to be *Hausdorff and connected.*

Example 2.3 1) The atlas $\{x = \mathrm{id}_{\mathbb{R}^n}\}$ defines a differentiable structure on $\mathbb{R}^n$, referred to as the standard differentiable structure, and whose corresponding topology is the standard one.

2) The $(n-1)$-dimensional unit sphere is defined as $S^{n-1} \equiv \{\underline{x} \in \mathbb{R}^n \mid |\underline{x}| = 1\}$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

The atlas $\{x_1, x_2\}$ consisting of two stereo-graphic projections w.r.t. two antipodal points defines what we call the standard differentiable structure on S^{n-1} and the corresponding topology is easily seen to be the one induced from $\mathbb{R}^n$.

If the two antipodal points are $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$, we have

$$\begin{aligned} x_1(\underline{r}) = x_1(r_1, \ldots, r_n) &= \tfrac{1}{1-r_n}(r_1, \ldots, r_{n-1}), \ |\underline{r}| = 1, \ r_n \neq 1 \, , \\ x_2(\underline{r}) = x_2(r_1, \ldots, r_n) &= \tfrac{1}{1+r_n}(r_1, \ldots, r_{n-1}), \ |\underline{r}| = 1, \ r_n \neq -1 \, , \\ x_1^{-1}(u) &= \tfrac{1}{|u|^2+1}(2u, |u|^2 - 1) \, , \quad u \in \mathbb{R}^{n-1} \, , \\ x_2^{-1}(u) &= \tfrac{1}{|u|^2+1}(2u, 1 - |u|^2) \, , \quad u \in \mathbb{R}^{n-1} \end{aligned}$$

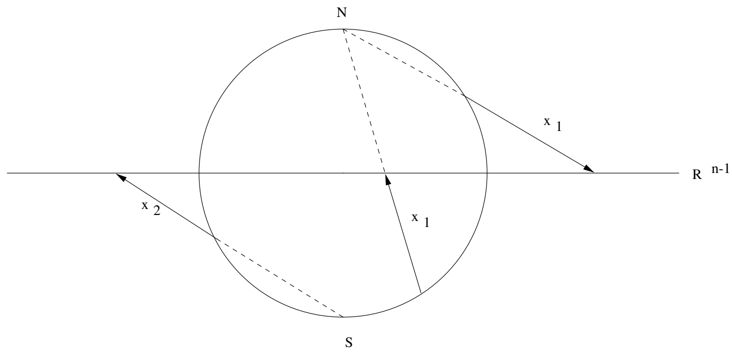

Figure 2.1: The stereo-graphic projection

and

$$x_{12}(u) = x_{21}(u) = \frac{u}{|u|^2} , \quad u \in \mathbb{R}^{n-1} \setminus \{0\} .$$

3) Given a smooth manifold M with atlas $\{x_\alpha \mid \alpha \in I\}$ and an open set $U \subseteq M$, it is clear that $\{x_\alpha\big|_{U_\alpha \cap U} \mid \alpha \in I\}$, where $x_\alpha\big|_{U_\alpha \cap U}$ denotes the restriction of x_α to $U_\alpha \cap U$, constitutes an atlas on U and that the corresponding topology on U equals the induced topology from M. We say that U with the so defined differentiable structure is an *open submanifold* of M.

4) Given, in addition to M as above, a second smooth manifold N with atlas $\{y_\beta : V_\beta \to \mathbb{R}^m \mid \beta \in J\}$ it is easy to see that $\{x_\alpha \times y_\beta \mid \alpha \in I,\ \beta \in I\}$, where $x_\alpha \times y_\beta : U_\alpha \times V_\beta \to \mathbb{R}^{n+m}$ is defined by

$$(x_\alpha \times y_\beta)(u, v) = (x_\alpha(u), y_\beta(v)) ,$$

is an atlas on $M \times N$. We call $M \times N$ with the so defined differentiable structure the product manifold of M and N.

By repeated application of this construction we obtain e.g. the *n-torus* T^n as the product of n copies of the circle, $T^n = S^1 \times \ldots \times S^1$.

2.2 Maps

Let $(M, \{x_\alpha : U_\alpha \to \mathbb{R}^n \mid \alpha \in I\})$, $(N, \{y_\beta : V_\beta \to \mathbb{R}^m \mid \beta \in J\})$ be smooth manifolds.

Definition 2.4 A continuous mapping $F : M \to N$ is called a $\mathcal{C}^r$-mapping, $r \geq 1$, if for all $\alpha \in I$, $\beta \in J$ such that $U_\alpha \cap F^{-1}(V_\beta) \neq \emptyset$ the *coordinate representation* of F with respect to x_α and y_β, given by

$$y_\beta \circ F \circ x_\alpha^{-1} : x_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \to y_\beta(V_\beta) ,$$

is $\mathcal{C}^r$. A $\mathcal{C}^1$-mapping will also be called a *differentiable mapping* and a $\mathcal{C}^\infty$-mapping is called a *smooth mapping* or simply a map.

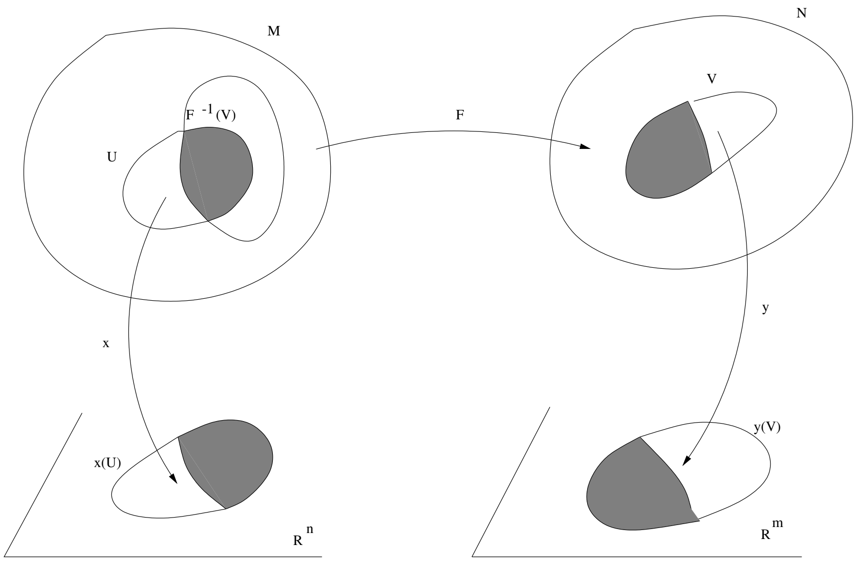

Figure 2.2: Coordinate representation of a map

It is easy to see that a continuous mapping $F : M \to N$ is $\mathcal{C}^r$ if and only if for each $p \in M$ there exist coordinate systems $x : U \to \mathbb{R}^n$ and $y : V \to \mathbb{R}^m$ around p and $F(p)$, respectively, such that $y \circ F \circ x^{-1}$ is $\mathcal{C}^r$.

Likewise, it follows that $F : M \to N$ is a $\mathcal{C}^r$-mapping if there exists an open covering $\mathcal{U}$ of M such that $F\big|_U$ is $\mathcal{C}^r$ for each U in $\mathcal{U}$. Given a third manifold O and $\mathcal{C}^r$-mappings $F : M \to N$ and $G : N \to O$ it follows immediately from Definition 2.4 and standard results on differentiable functions of several real variables that $G \circ F\colon M \to O$ is also $\mathcal{C}^r$.

Definition 2.5 A homeomorphism $F : M \to N$ is a *diffeomorphism* if both F and F^{-1} are smooth.

Definition 2.6 A curve in M is a mapping $\gamma : I \to M$, where $I \subseteq \mathbb{R}$ is an interval. In case I is not open then γ is called a $\mathcal{C}^r$-curve if it can be extended to a $\mathcal{C}^r$-mapping on an open interval containing I.

We let

$$\mathcal{C}^r(M) = \{f : M \to \mathbb{R} \mid f \text{ is } \mathcal{C}^r\} .$$

and use the notation $\mathcal{C}(M)$ instead of $\mathcal{C}^\infty(M)$.

Example 2.7 1) Let $x : U \to \mathbb{R}^n$ be a coordinate system, and let $x^i : U \to \mathbb{R}$ denote the i'th coordinate function. Then clearly x^i is a smooth function on U.

2) Let again $x : U \to \mathbb{R}^n$ be a coordinate system. For $p \in U$ let $\varphi_0 : x(U) \to \mathbb{R}$ be a smooth function which equals 1 in some neighborhood $A \subseteq x(U)$ of $x(p)$ and vanishes outside a neighborhood B of $x(p)$ such that $A \subseteq B \subseteq \overline{B} \subseteq x(U)$, say A and B are balls centered at $x(p)$ with sufficiently small radii.

Then the function $\varphi : M \to \mathbb{R}$ defined by

$$\varphi(q) = \begin{cases} \varphi_0(x(q)) & \text{for} \quad q \in U \\ 0 & \text{for} \quad q \in M \setminus U \end{cases}$$

is smooth on M since it obviously is smooth on the open sets U and $M \setminus \overline{B}$, which cover M. Such a function is called a *localization function* at p in U.

Clearly, for any neighborhood U of p one may find a coordinate system $x : U \to \mathbb{R}^n$ and an associated localization function φ at p in U. Thus, given any smooth function $f : U \to \mathbb{R}$ we obtain a smooth function $\overline{\varphi \cdot f} : M \to \mathbb{R}$ which agrees with f on a neighborhood of p and vanishes outside U by setting

$$\overline{\varphi \cdot f}(q) = \begin{cases} \varphi(q) f(q) & \text{for} \quad q \in U \\ 0 & \text{for} \quad q \notin U \ . \end{cases}$$

We shall exploit the existence of such functions frequently in the following.

2.3 Tangent vectors

Let $\gamma : I \to M$ be a $\mathcal{C}^1$-curve in M and let $t_0 \in I$. The tangent vector to γ at t_0 is the linear mapping $\dot\gamma|_{t_0} : \mathcal{C}^1(M) \to \mathbb{R}$, also denoted by $\dot\gamma(t_0)$, defined by

$$\dot\gamma|_{t_0}(f) = \frac{d\, f \circ \gamma}{dt}\Big|_{t_0} \ . \tag{2.1}$$

That is to say, we define a tangent vector in terms of its action as a directional derivative on functions as expressed by the right-hand side of the above equation. If $\gamma(t_0) = p$ we say that $\dot\gamma|_{t_0}$ is a tangent vector to γ at p or with base point p. The tangent space T_pM to M at p consists of all tangent vectors with base point p:

$$T_pM = \left\{ \dot\gamma|_{t_0} \mid \gamma : I \to M \ \ \mathcal{C}^1\text{-curve with } \ \gamma(t_0) = p \right\} \ . \tag{2.2}$$

Thus T_pM is a subset of the vector space of functions from $\mathcal{C}^1(M)$ into $\mathbb{R}$. We shall next show that, in fact, T_pM is an n-dimensional subspace.

If $x : U \to \mathbb{R}^n$ is a coordinate system around p with $p = x^{-1}(u^1, \ldots, u^n)$, let $\gamma_1, \ldots, \gamma_n$ be the *coordinate curves* through p, that is

$$\gamma_i(t) = x^{-1}(u^1, \ldots, u^{i-1}, u^i + t, u^{i+1}, \ldots, u^n) \tag{2.3}$$

for $t \in]-\varepsilon, +\varepsilon[$, ε sufficiently small (see Fig.1.3).

We set $\frac{\partial}{\partial x^i}\Big|_p = \dot\gamma_i|_0$, such that

$$\frac{\partial f}{\partial x^i}(p) \equiv \frac{\partial}{\partial x^i}\Big|_p (f) = \frac{\partial\, f \circ x^{-1}}{\partial u^i}\Big|_{x(p)} \ . \tag{2.4}$$

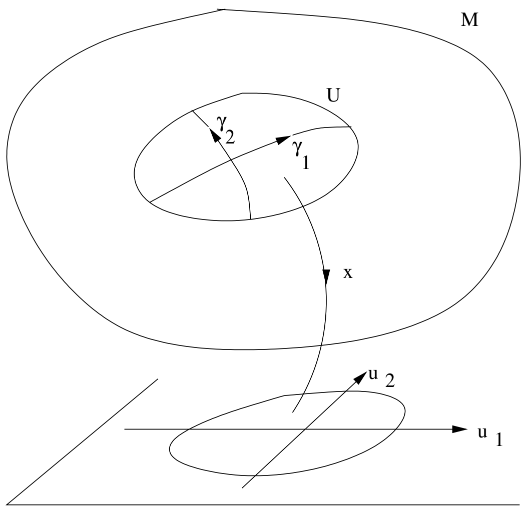

Figure 2.3: Coordinate curves

For a differentiable curve γ with $\gamma(t_0) = p$ we then have

$$\begin{aligned} \dot\gamma|_{t_0}(f) &= \frac{d\, f\circ\gamma}{dt}\Big|_{t_0} = \frac{d\, f\circ x^{-1}\circ x\circ\gamma}{dt}\Big|_{t_0} \\ &= \sum_{i=1}^n \frac{\partial\, f\circ x^{-1}}{\partial u^i}\Big|_{x(p)} \cdot \frac{d\, x^i\circ\gamma}{dt}\Big|_{t_0} \\ &= \sum_{i=1}^n \frac{d\, x^i\circ\gamma}{dt}\Big|_{t_0} \frac{\partial f}{\partial x^i}(p)\ , \quad f\in \mathcal{C}^1(M)\ , \end{aligned}$$

that is

$$\dot\gamma|_{t_0} = \sum_{i=1}^n \frac{d\, x^i\circ\gamma}{dt}\Big|_{t_0} \frac{\partial}{\partial x^i}\Big|_p\ , \tag{2.5}$$

yielding $\dot\gamma|_{t_0}$ as a linear combination of the $\frac{\partial}{\partial x^i}|_p$.

On the other hand, given $a^1,\ldots,a^n\in\mathbb{R}$, let γ be given by

$$\gamma(t) = x^{-1}(u^1 + a^1 t, \ldots, u^n + a^n t)$$

for $t\in]-\varepsilon,+\varepsilon[$, ε sufficiently small. Then

$$\begin{aligned} \dot\gamma|_0(f) &= \frac{d\, f\circ x^{-1}(u^1+ta^1,\ldots,u^n+a^n t)}{dt}\Big|_0 \\ &= \sum_i a^i \frac{\partial f\circ x^{-1}}{\partial u^i}\Big|_{x(p)} = \sum_i a^i \frac{\partial f}{\partial x^i}(p)\ , \quad f\in\mathcal{C}^1(M)\ , \end{aligned}$$

that is $\dot\gamma|_0 = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\Big|_p$. Together with (2.5) this shows that T_pM is a vector space spanned by $\left(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\right)_p \equiv \left(\frac{\partial}{\partial x^1}\Big|_p,\ldots,\frac{\partial}{\partial x^n}\Big|_p\right)$.

Clearly, a tangent vector $v_p = \dot{\gamma}|_{t_0}$ at p acts on $\mathcal{C}^1$-functions f defined on any open neighborhood U of p by formula (2.1), that is v_p can be regarded as a tangent vector at p on U. This yields a canonical isomorphism between T_pM and T_pU. In the following we shall identify T_pM and T_pU by this isomorphism. We may then rewrite (2.5) as

$$v_p = \sum_{i=1}^{n} v_p(x^i) \frac{\partial}{\partial x^i}\Big|_p . \tag{2.6}$$

Moreover, in order to show that $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)_p$ is a linearly independent set we note that, if $\sum\limits_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\Big|_p = 0$, then

$$0 = \sum_{i=1}^{n} a^i \frac{\partial x^j}{\partial x^i}(p) = a^j \ , \quad j = 1, \ldots, n \ ,$$

since

$$\frac{\partial x^j}{\partial x^i}(p) = \frac{\partial x^j \circ x^{-1}}{\partial u^i}\Big|_{x(p)} = \frac{\partial u^j}{\partial u^i}\Big|_{x(p)} = \delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} .$$

Thus we have shown

Theorem 2.8 *For each coordinate system x at $p \in M$ the set $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)_p$ is a basis of the real vector space T_pM.*

Note that T_pM actually consists of tangent vectors to smooth curves, and it would make no difference to restrict attention to smooth curves in the definition (2.2) of T_pM and to let tangent vectors act only on smooth functions.

2.4 The tangent bundle

Define

$$TM = \bigcup_{p \in M} T_pM \quad \text{(disjoint union)}$$

and $\pi : TM \to M$ by

$$\pi(v_p) = p \quad \text{for} \quad v_p \in T_pM \ .$$

Given an atlas $\{x_\alpha : U_\alpha \to \mathbb{R}^n\}$ on M there is an induced atlas $\{\widetilde{x}_\alpha : \widetilde{U}_\alpha \to \mathbb{R}^{2n}\}$ on TM given by $\widetilde{U}_\alpha = \pi^{-1}(U_\alpha)$ and

$$\widetilde{x}_\alpha(v_p) = (x_\alpha(p), v_p(x_\alpha^1), \ldots, v_p(x_\alpha^n)) \ , \tag{2.7}$$

We note that by (2.6) $v_p(x_\alpha^1), \ldots, v_p(x_\alpha^n)$ are the coordinates of the vector v_p in the basis $\left(\frac{\partial}{\partial x_\alpha^1}, \ldots, \frac{\partial}{\partial x_\alpha^n}\right)_p$. It follows, in particular, that

$$\frac{\partial}{\partial x_\alpha^i} = \sum_{j=1}^{n} \frac{\partial x_\beta^j}{\partial x_\alpha^i} \frac{\partial}{\partial x_\beta^j} \quad \text{on} \quad U_\alpha \cap U_\beta \ , \tag{2.8}$$

and therefore by (2.7) and (2.6)

$$\tilde{x}_{\beta\alpha}(u,v) = \left(x_{\beta\alpha}(u) \ , \ \frac{\partial x_\beta}{\partial x_\alpha}(x_\alpha^{-1}(u)) \cdot v \right) = \left(x_{\beta\alpha}(u) \ , \ \frac{\partial x_{\beta\alpha}}{\partial u} \cdot v \right) \tag{2.9}$$

for $(u,v) \in \tilde{x}_\alpha(\widetilde{U}_\alpha \cap \widetilde{U}_\beta) = x_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$, and where $\frac{\partial x_{\beta\alpha}}{\partial u}$ is the Jacobi matrix of $x_{\beta\alpha}$. Thus we have shown that $\{\widetilde{x}_\alpha\}$ is a $\mathcal{C}^\infty$-atlas and hence defines a $\mathcal{C}^\infty$-structure on TM.

The mapping $\pi : TM \to M$ is smooth, since

$$x_\alpha \circ \pi \circ \tilde{x}_\beta^{-1}(u,v) = x_\alpha \circ x_\beta^{-1}(u) \ , \tag{2.10}$$

and the mapping $h_\alpha : U_\alpha \times \mathbb{R}^n \to \pi^{-1}(U_\alpha)$ defined by

$$h_\alpha(p,v) = \sum_{i=1}^n v^i \frac{\partial}{\partial x_\alpha^i}\Big|_p \tag{2.11}$$

is a diffeomorphism, since

$$\tilde{x}_\alpha \circ h_\alpha \circ (x_\alpha \times \mathrm{id}_{\mathbb{R}^n})^{-1}(u,v) = (u,v) \tag{2.12}$$

for $(u,v) \in U_\alpha \times \mathbb{R}^n$. The inverse of h_α is given by

$$h_\alpha^{-1}(v_p) = (p, v_p(x_\alpha^1), \ldots, v_p(x_\alpha^n)) \tag{2.13}$$

for $v_p \in T_pM$, $p \in U_\alpha$.

Note also that

$$\pi \circ h_\alpha = \mathrm{pr}_{U_\alpha} \ , \tag{2.14}$$

where the latter denotes the projection onto U_α, and that for each $p \in U_\alpha$ the mapping $(h_\alpha)_p : \mathbb{R}^n \to T_pM$ given by

$$(h_\alpha)_p(v) = h_\alpha(p,v)$$

is an isomorphism, since $\left(\frac{\partial}{\partial x_\alpha^1}, \ldots, \frac{\partial}{\partial x_\alpha^n} \right)_p$ is a basis of T_pM.

The triple $\tau(M) = (TM, \pi, M)$ is called the *tangent bundle* over M. The diffeomorphisms h_α are called *local trivialisations* of $\tau(M)$.

Exercises

Exercise 1 Let V be a vector space of finite dimension n. Show that the set of linear isomorphisms from V onto $\mathbb{R}^n$ is an atlas on V. We call the corresponding differentiable structure on V the standard one.

For $u, v \in V$, let $\varphi_{u,v}$ be the straight line through u directed along v,

$$\varphi_{u,v}(t) = u + tv \ , \ t \in \mathbb{R} \ .$$

Show that the mapping $v \mapsto \dot{\phi}_{u,v}(0)$ is an isomorphism from V onto T_uV. We shall always identify V with T_uV by this mapping.

Exercise 2 The **real projective space** $\mathbb{R}P^n$ is by definition the set of lines through 0 in $\mathbb{R}^{n+1}$, i.e.

$$\mathbb{R}P^n = \{\tilde{v} \mid v \in \mathbb{R}^{n+1} \setminus \{0\}\} ,$$

where $\tilde{v}$ denotes the line through 0 parallel to v.

For $\alpha = 1, \ldots, n+1$ let $U_\alpha = \{\tilde{v} \mid v = (v_1, \ldots, v_{n+1}) \in \mathbb{R}^{n+1} \ , \quad v_\alpha \neq 0\}$, and define $x_\alpha : U_\alpha \mapsto \mathbb{R}^n$ by

$$x_\alpha(\tilde{v}) = v_\alpha^{-1}(v_1, \ldots, v_{\alpha-1}, v_{\alpha+1}, \ldots, v_{n+1}) \ .$$

Show that $\{x_1, \ldots, x_{n+1}\}$ is an atlas on $\mathbb{R}P^n$. This atlas defines the standard differentiable structure on $\mathbb{R}P^n$.

Prove that the mapping $F : S^1 \mapsto \mathbb{R}P^1$ defined by

$$F(\cos\theta, \sin\theta) = (\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2}))^{\sim} , \quad \theta \in [0, 2\pi] ,$$

is a diffeomorphism.

Exercise 3 a) Show that the group $\mathrm{GL}(n)$ of invertible $n \times n$-matrices is an open subset of the vector space $M(n)$ of all $n \times n$-matrices, that is $\mathrm{GL}(n)$ is an open submanifold of $M(n)$.

b) Verify that inversion $g \mapsto g^{-1}$ and multiplication $(g, h) \to gh$ are differentiable maps (from $\mathrm{GL}(n)$ to $\mathrm{GL}(n)$ and from $\mathrm{GL}(n) \times \mathrm{GL}(n)$ to $\mathrm{GL}(n)$, respectively).

This means that $\mathrm{GL}(n)$ is a Lie group according to the following

Definition A group G equipped with a differentiable structure is called a **Lie group** if inversion and multiplication are both differentiable maps, from G to G and from $G \times G$ to G, respectively. (It can be shown by use of the implicit function theorem that it is sufficient that multiplication be differentiable.)

c) Show that, if G is a Lie group, then inversion is a diffeomorphism of G, and that for fixed $g \in G$ the mappings $L_g : h \mapsto gh$ and $R_g : h \mapsto hg$, called left and right multiplication by g, respectively, are diffeomorphisms of G.

Exercise 4 Let M be a smooth manifold of dimension n and let F_pM denote the set of all (ordered) bases of T_pM, also called *frames at* p, for $p \in M$. Set

$$FM = \bigcup_{p \in M} F_pM ,$$

and define $\pi_F : FM \to M$ by $\pi_F(e) = p$ for $e \in F_pM \ , \ p \in M$.

Given an atlas $\{x_\alpha : U_\alpha \to \mathbb{R}^n \mid \alpha \in I\}$ on M, we define $k_\alpha : U_\alpha \times \mathrm{GL}(n) \to FM$ for $\alpha \in I$, such that $k_\alpha(p, b)$ is obtained by rotating the coordinate basis $(\frac{\partial}{\partial x_\alpha^1}, \ldots, \frac{\partial}{\partial x_\alpha^n})_p$ at p by the matrix b, whose matrix elements we denote by b_{ij}. That is

$$k_\alpha(p, b) = \left(\sum_{j=1}^n b_{j1} \frac{\partial}{\partial x_\alpha^j}, \ldots, \sum_{j=1}^n b_{jn} \frac{\partial}{\partial x_\alpha^j} \right)_p \equiv \left(\frac{\partial}{\partial x_\alpha^1}, \ldots, \frac{\partial}{\partial x_\alpha^n} \right)_p \cdot b \ .$$

a) Show that k_α maps $U_\alpha \times \mathrm{GL}(n)$ bijectively onto $\pi_F^{-1}(U_\alpha)$ and that $\{\hat{x}_\alpha | \alpha \in I\}$, where

$$\hat{x}_\alpha = (x \times \mathrm{id}_{\mathrm{GL}(n)}) \circ k_\alpha^{-1} ,$$

is an atlas defining a $\mathcal{C}^\infty$-structure on FM (where $\mathrm{GL}(n)$ has been identified canonically with an open subset of $\mathbb{R}^{n^2}$).

b) Show that π_F is a smooth mapping, and that the mapping

$$(e,b) \mapsto e \cdot b \equiv (\sum_{j=1}^{n} b_{j1} e_j, \ldots, \sum_{j=1}^{n} b_{jn} e_j)$$

from $FM \times \mathrm{GL}(n)$ to FM is smooth and satisfies

i) $(e \cdot b_1) \cdot b_2 = e \cdot (b_1 b_2)$, for $e \in FM$, $b_1, b_2 \in \mathrm{GL}(n)$,

ii) For fixed $e \in F_p M$ the mapping $b \to e \cdot b$ is bijective from $\mathrm{GL}(n)$ onto $F_p M$.

The properties listed above amount to showing that (FM, π_F, M) is a principal $\mathrm{GL}(n)$-bundle, called the **frame bundle** over M, according to the following

Definition Let G be a Lie group and M a smooth manifold. A **principal G-bundle** over M is a triple (P, π, M), where $\pi : P \to M$ is a smooth surjective map, together with a smooth map $(a, g) \to a \cdot g$ from $P \times G$ to P such that

i) $(a \cdot g_1) \cdot g_2 = a \cdot (g_1 g_2)$, for $a \in P,\ g_1, g_2 \in G$,

ii) For fixed $a \in \pi^{-1}(p)$ the mapping $g \to a \cdot g$ is bijective from G onto $\pi^{-1}(p)$.

In addition, it is required that there exists an open covering $\{U_\alpha \mid \alpha \in I\}$ of M, and diffeomorphisms $k_\alpha : U_\alpha \times G \to \pi^{-1}(U_\alpha)$, called *local trivialisations*, fulfilling $k_\alpha(p, g_1 g_2) = k_\alpha(p, g_1) \cdot g_2$, $p \in U_\alpha$, $g_1, g_2 \in G$.

A mapping $(a, g) \to a \cdot g$ from $P \times G$ to P fulfilling i) above is called a *G-action* on P, which is said to be *free*, resp. *transitive*, on $\pi^{-1}(p)$, if injectivity, resp. surjectivity, holds in ii), instead of bijectivity. In this language a principal G-bundle is given by a smooth surjective map $P \xrightarrow{\pi} M$ together with a smooth G-action on P, which is free and transitive on each *fiber* $\pi^{-1}(p)$, $p \in M$, subject to the requirement of local triviality. It is common to suppress the G-action from the notation and simply denote the principal bundle by $P \xrightarrow{\pi} M$.