Noncommutative End Theory

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C^*-algebra $\mathfrak{A}$, $\mathbf{1} \notin \mathfrak{A}$. Seek global properties in

$$\mathrm{C}(\mathfrak{A}) = \mathrm{M}(\mathfrak{A})/\mathfrak{A}$$

When is

- $\mathrm{C}(\mathfrak{A})$ simple?

- $\mathrm{C}(\mathfrak{A})$ prime?

  – $\mathfrak{J}_1\mathfrak{J}_2 = (0) \implies \mathfrak{J}_1 = (0)$ or $\mathfrak{J}_2 = (0)$

- $\mathrm{C}(\mathfrak{A})$ indecomposable?

  – $\mathrm{C}(\mathfrak{A}) = \mathfrak{J}_1 \oplus \mathfrak{J}_2 \Rightarrow \mathfrak{J}_1 = (0)$ or $\mathfrak{J}_2 = (0)$

  – $\mathrm{Prim}(\mathrm{C}(\mathfrak{A}))$ connected.

For certain topological spaces X:

$$\{\text{ends of } X\} \leftrightsquigarrow \{\text{components of } \beta X \backslash X\}$$

[Hans Freudenthal, 1931].

$\{\text{ends of } X\} = \{(G_n) \text{ open, connected } |(\ddagger)\}/\approx$

$$(\ddagger) = \begin{cases} G_1 \supseteq G_2 \supseteq \cdots \\ G_n \neq \emptyset \\ \bigcap_{n=1}^{\infty} \overline{G_n} = \emptyset \\ G_n \text{ has compact boundary} \end{cases}$$

$$(G_n) \approx (H_n) \Longleftrightarrow \forall n \in \mathbb{N} : G_n \cap H_n \neq \emptyset$$

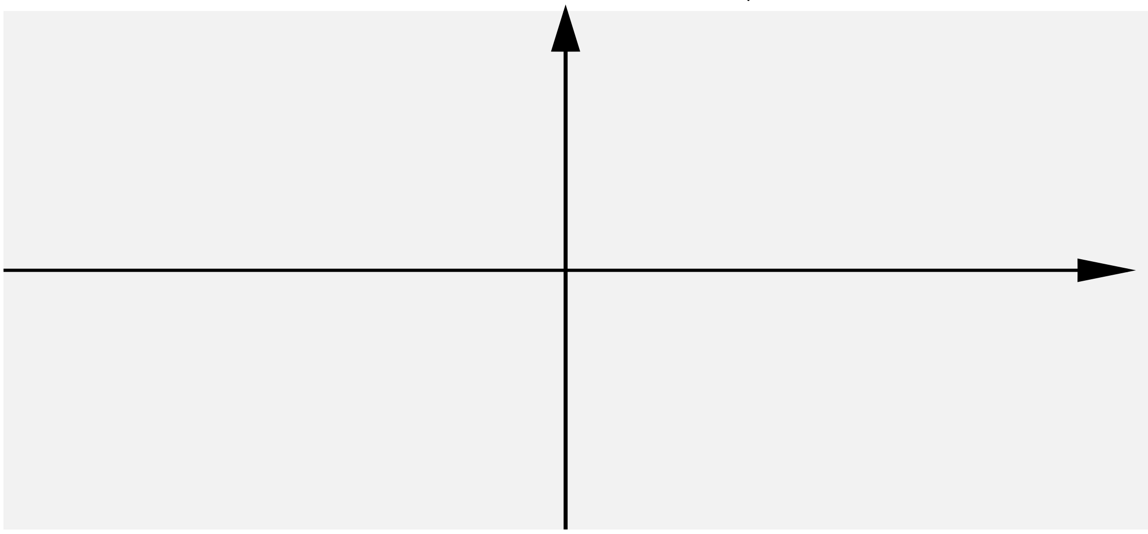
A *compact nest*

$$K_1^\circ \subseteq K_1 \subseteq K_2^\circ \subseteq \ldots, \qquad \bigcup_{i=1}^{\infty} K_i = X$$

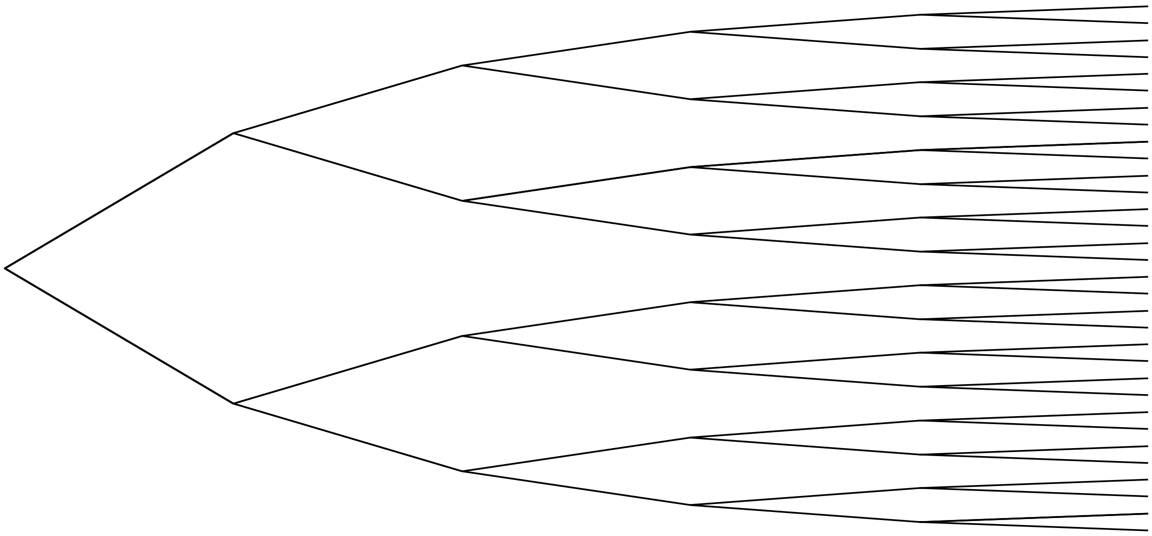
gives an algorithmic way to a full set of representatives of the ends of X!

- $\mathbb{R}$. 2 components of $\beta\mathbb{R}\backslash\mathbb{R}$.

- $\mathbb{R}^2$. 1 component of $\beta\mathbb{R}^2\backslash\mathbb{R}^2$.



- $\mathcal{T}_2$. $\mathfrak{c}$ components of $\beta\mathcal{T}_2\backslash\mathcal{T}_2$.



$\mathfrak{A}$ *Raum* algebra:

- $\mathfrak{A}$ is σ-unital.

- $\mathrm{Prim}(\mathfrak{A})$ is connected.

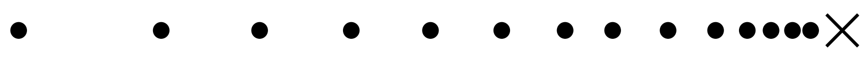
- $\mathrm{Prim}(\mathfrak{A})$ is locally connected.

- $\mathfrak{A}$ is locally connected if for every hereditary subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, there is a maximal decomposition

$$\mathfrak{B} = \sum_{j \in J} \mathfrak{I}_j$$

into (complemented) ideals of $\mathfrak{B}$.

- $C_0(\mathbb{N})$ is locally connected. $C_0(\mathbb{N})^{\sim}$ is not locally connected.



- In general, we must describe the structure of the set of complemented ideals using *components*. The components of $\mathfrak{A}$ are certain *closed* central projections of $\mathfrak{A}^{**}$.

- Define ends for a Raum algebra $\mathfrak{A}$.

  $\{\text{ends of } \mathfrak{A}\} =$
  $\{(\mathfrak{B}_n) \text{ indecomp. her. subalgs.} | (\ddagger)\}/\approx$

- $(\ddagger)$:

  - $\mathfrak{B}_1 \supseteq \mathfrak{B}_2 \supseteq \dots$
  - $\mathfrak{B}_n \neq (0)$
  - $\mathfrak{B}_n$ "go to infinity"
  - $\mathfrak{B}_n$ have "compact boundaries"

- $(\mathfrak{B}_n) \approx (\mathfrak{C}_n)$:

  $$\forall n \in \mathbb{N} : \text{ideal}(\mathfrak{B}_n)\,\text{ideal}(\mathfrak{C}_n) \neq (0)$$

- Is then

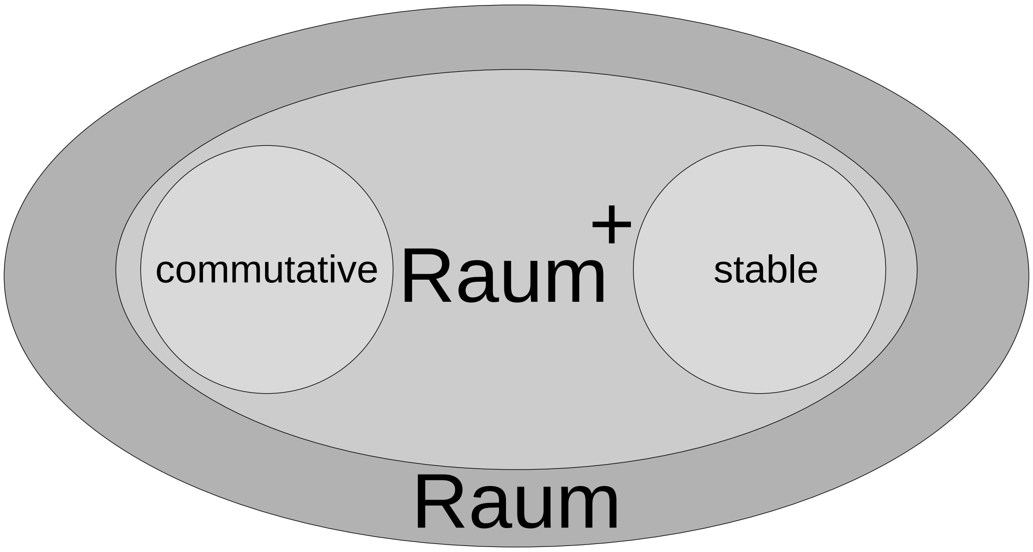
  $\{\text{ends of } \mathfrak{A}\} \leftrightsquigarrow \{\text{components of } \mathrm{C}(\mathfrak{A})\}$?

Algorithm

- Fix strictly positive element h, $\|h\| = 1$.
- Consider $\mathfrak{C}_n = \mathrm{her}(1_{[0,\frac{1}{n})}(h))$.
- Decompose $\mathfrak{C}_n = \sum_{j \in J_n} \mathfrak{I}_{nj}$.
- Arrange *unbounded* hereditary subalgebras into decreasing sequences.

There exists Raum algebra $\mathfrak{D}$:

- $\mathfrak{D}$ has no ends.
- $C(\mathfrak{D})$ has $2^{\mathfrak{c}}$ components.



$\boxed{\textbf{Proposition}}$ Let $\mathfrak{A}$ be a Raum. Then $\mathfrak{A}$ is a Raum$^+$ if and only if $C(\mathfrak{A})$ has fewer than $2^{\mathfrak{c}}$ components.

$\boxed{\textbf{Theorem}}$ Let $\mathfrak{A}$ be a Raum$^+$. Then

$$\{\text{ends of } \mathfrak{A}\} \leftrightsquigarrow \{\text{components of } \mathsf{C}(\mathfrak{A})\}.$$

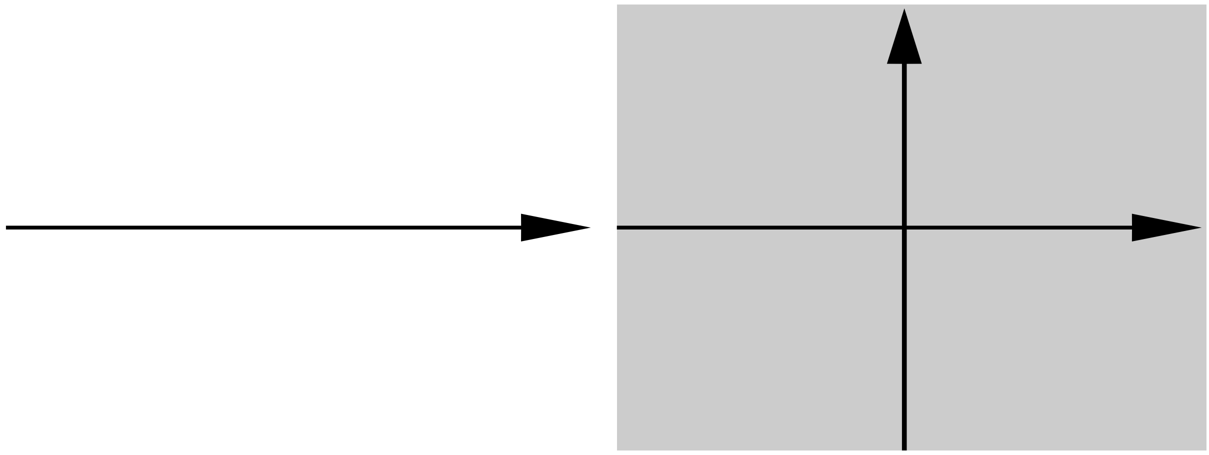
Furthermore, the algorithm outlined above yields a full set of representatives of the ends.

$\boxed{\textbf{Corollary}}$ Let $\mathfrak{A}$ be a Raum$^+$ with $N < \infty$ ends. Then there is a maximal decomposition

$$\mathsf{C}(\mathfrak{A}) = \mathfrak{I}_1 \oplus \cdots \oplus \mathfrak{I}_N.$$

$\boxed{\textbf{Corollary}}$ Let $\mathfrak{A}$ be σ-unital, primitive. Then

$$\mathsf{C}(\mathfrak{A}) \text{ is prime.}$$



$\boxed{\textbf{Theorem}}$ Let $\mathfrak{A}$ and $\mathfrak{B}$ be nuclear Raum algebras. $\mathbf{1} \notin \mathfrak{A}$, $\mathbf{1} \notin \mathfrak{B}$.

- $\mathfrak{A} \otimes \mathfrak{B}$ is a Raum$^+$.

- $\mathfrak{A} \otimes \mathfrak{B}$ has one end.

- $C(\mathfrak{A} \otimes \mathfrak{B})$ is indecomposable.

$\boxed{\textbf{Proposition}}$ Let $\mathfrak{A}$ and $\mathfrak{B}$ be nuclear Raum algebras. $\mathbf{1} \in \mathfrak{A}$, $\mathbf{1} \notin \mathfrak{B}$:

- $\mathfrak{A} \otimes \mathfrak{B}$ is a Raum$^+$ if $\mathfrak{B}$ is a Raum$^+$.

- Ends of $\mathfrak{A} \otimes \mathfrak{B}$ are in 1–1 correspondence with ends of $\mathfrak{B}$.

$\{\text{ends of } \mathfrak{A}\} = \{(p_n) \text{ connected, open } |(\ddagger)\}/\approx$

- $(\ddagger)$:
  - $p_1 \geq p_2 \geq p_3 \geq \ldots$.
  - $p_n \neq 0$.
  - $\bigwedge_{n=1}^{\infty} \overline{p_n} = 0$.
  - $\overline{\mathsf{c}(p_n)} - p_n$ is compact.
- $\approx$:

$$(p_n) \approx (q_n) \Longleftrightarrow \forall n \in \mathbb{N} : \mathsf{c}(p_n)\mathsf{c}(q_n) \neq 0$$

- The map:

$$[(p_n)] \mapsto \bigwedge_{n=1}^{\infty} \overline{p_n}^{M}.$$

- $\mathfrak{D} = C^*(\sum_1^\infty \mathbb{K}, (q_n)_1^\infty)$
  - $q_n = \mathbf{1} - r$, rank$(r) = 1$.

- Prim$(\mathfrak{D}) = \{\infty, 1, 2, \ldots\}$.
  - ∞ lies in every nonempty closed set.
  - Prim$(\mathfrak{D})$ is connected and locally connected.

- $\mathfrak{B}_n$ will be of the form $\sum_{n \in N} \mathbb{K}$.

- The minimal complemented ideals are dominated by

$$(\mathbf{1} - r, \ldots, \mathbf{1} - r, \mathbf{1}, \mathbf{1} - r, \ldots)$$

- $p \in \mathfrak{A}^{**}$ is connected when

$$p = px_0 + px_1 \Longrightarrow px_0 = 0 \text{ or } px_1 = 0,$$

  where the x_i are open and central.

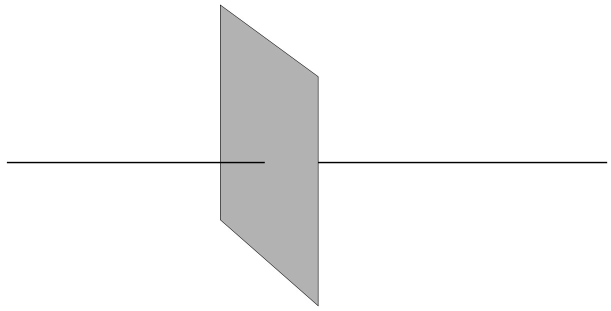
- p open: p connected $\Longleftrightarrow$ her(p) is indecomposable.

- A component is a maximal connected projection.

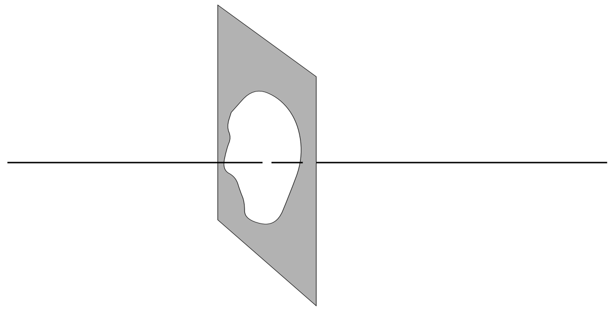
- Components are closed central projections.

- $\mathfrak{A}$ is locally connected $\Longleftrightarrow$ every component of a hereditary subalgebra is open.

- $H_3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in M_3(\mathbb{R}) \,\middle|\, x, y, z \in \mathbb{R} \right\}$.
- Prim($C^*(H_3)$):



- Prim($\mathfrak{B}_n$):



- $C^*(H_3)$ is a Raum$^+$.
- $C^*(H_3)$ has one end.
- C($C^*(H_3)$) is indecomposable.