$\mathbb{I}^\omega$- and $\mathbb{R}^\omega$-manifolds

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- If X is Polish, then $C(Y,X)$ has the Baire property.

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  - ($\star$) For each $\mathcal{U} \in \mathrm{cov}(Y)$ and $\mathcal{V} \in \mathrm{cov}(X)$ there exist an open cover $\mathcal{W} \in \mathrm{cov}(X)$ and a homeomorphism $h\colon Y \to Y$ such that $f \circ h \in B(f, \mathcal{V})$ and $hf^{-1}(\mathcal{W}) \prec \mathcal{U}$.

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- A closed surjection $f\colon Y \to X$ between Polish spaces is a near- homeomorphism if and only if the following condition is satisfied:
  - ($\star\star$) For each $\mathcal{U} \in \mathrm{cov}(Y)$ and each $\mathcal{V} \in \mathrm{cov}(X)$ there exists a homeomorphism $h\colon Y \to Y$ such that $f \circ h \in B(f, \mathcal{V})$ and the collection $\{hf^{-1}(x)\colon x \in X\}$ refines $\mathcal{U}$.

Z-sets

- A closed subset A of a space X is said to be a *Z-set* in X if the set $\{f \in C(X,X) \colon f(X) \cap A = \emptyset\}$ is dense in the space $C(X,X)$. If the set $\{f \in C(X,X) \colon \mathrm{cl}_X f(X) \cap A = \emptyset\}$ is dense in $C(X,X)$, then we say that A is a strong Z-set in X.

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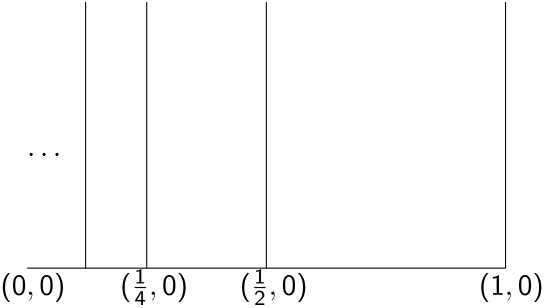
- The concepts of the Z-set and the strong Z-set differ even for very simple spaces.

Z-sets

- Consider the subset

$$X = ([0,1] \times \{0\}) \bigcup \left(\bigcup_{n=1}^{\infty} \left\{ \left\{ \frac{1}{n} \right\} \times [0,1] \right\} \right)$$

of the plane:



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- Compact subsets are Z-sets in the Hilbert space.

Z-set Unknotting Theorems

- Z-set Unknotting Theorem: Let $f \colon Z \to F$ be a homeomorphism between Z-sets of $\mathbb{I}^\omega$ (or $\mathbb{R}^\omega$). Then there is an autohomeomorphism $F \colon \mathbb{I}^\omega \to \mathbb{I}^\omega$ (respectively, $F \colon \mathbb{R}^\omega \to \mathbb{R}^\omega$) which extends f.

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Universality Properties; Disjoint (Discrete) Disks Properties

- Strong universality: A locally compact (Polish) space X is strongly universal if for any compact (Polish) space B, its closed subset A, any $\mathcal{U} \in \mathrm{cov}(X)$ and any map $f\colon B \to X$ such that the restriction $f|A$ is a Z-embedding, there is a Z-embedding $g\colon B \to X$ which is $\mathcal{U}$-close to f and such that $g|A = f|A$.

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- A space X has DD^{n}P if the set $\{f \in C(D_1^n \oplus D_2^n, X)\colon f(D_1^n) \cap f(D_2^n = \emptyset\}$ is dense in $C(D_1^n \oplus D_2^n, X)$.

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- A space X has a Strong Discrete Approximation Property (SDAP) if the set

$$f \in C(\oplus\{D^n\colon n \in \omega\}, X)\colon \{f(D^n)\colon n \in \omega\}\text{is discrete}\}$$

is dense in $C(\oplus\{D^n\colon n \in \omega\}, X)$.

- A (locally) compact A(N)E is homeomorphic to $\mathbb{I}^\omega$ (to a $\mathbb{I}^\omega$-manifold) if and only if it is strongly universal for compact spaces (or equivalently, has the $\mathrm{DD}^n\mathrm{P}$ for each n).

Topological Characterization of $\mathbb{I}^\omega$- and $\mathbb{R}^\omega$-manifolds

- A (locally) compact A(N)E is homeomorphic to $\mathbb{I}^\omega$ (to a $\mathbb{I}^\omega$-manifold) if and only if it is strongly universal for compact spaces (or equivalently, has the DD^{n}P for each n).
- A Polish A(N)E is homeomorphic to $\mathbb{R}^\omega$ (to a $\mathbb{R}^\omega$-manifold) if and only if it is strongly universal for Polish spaces (equivalently, has the SDAP).

Properies of $\mathbb{I}^\omega$- and $\mathbb{R}^\omega$-manifolds

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- If X and Y are homotopy equivalent $\mathbb{I}^\omega$-manifolds, then $X \times [0,1) \approx Y \times [0,1)$.