Overview of Shape Theory

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ANE-sequences

- We now consider only metrizable compact spaces. For each such X there exists an ANE-sequence associated with it. In other words, there exists an inverse sequence $\mathcal{S}_X = \{X_n, p_n^{n+1}, \omega\}$ such that $X = \lim \mathcal{S}_X$.

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- To see this embed X into the Hilbert cube $\mathbb{I}^\omega$, fix a metric d on it and for each n choose a closed ANE-neighborhood X_n of X such that $d(x, X) < \frac{1}{n}$ for every point $x \in X_n$. We may assume that the sequence of X_n's in decreeasing. Then this sequence together with inclusion maps forms an ANE-sequence.

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- More sophisticated observation (Freudenthal): Any (n-dimensional) metrizable compactum is the limit of an inverse sequence consisting of (n-dimensional) ANE-compacta (even polyhedra) with surjective projections.

ANE-sequences

- Freudenthal's theorem is not valid for non-metrizable compact spaces. There exists a 1-dimensional compact space that cannot be represented as the limit space of an inverse spectrum consisting of metrizable ANE-compacta and surjective limit projections. Any compactum with $1 = \dim X < \operatorname{ind} X$ will serve as an example.

Shape in terms of ANE-sequences

- By a morphism

$$\alpha\colon \mathcal{S}_X = \{X_n, p_n^{n+1}, \omega\} \to \mathcal{S}_Y = \{Y_n, q_n^{n+1}, \omega\}$$

between two ANE-sequences we understand a system $(\varphi, \{f_n\})$, where $\varphi\colon \omega \to \omega$ is an increasing function and maps $f_n\colon X_{\varphi(n)} \to Y_n$ are such that $f_m \circ p_{\varphi(m)}^{\varphi(n)} \simeq q_m^n \circ f_n$ for every $m \leq n$

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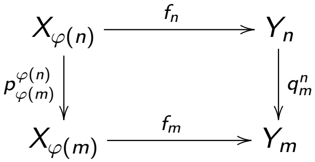
- Here is the diagram (commutes homotopically!)



Shape in terms of ANE-sequences

- Suppose that we have two morphisms

$$\alpha = (\varphi, \{f_n\})\colon \mathcal{S}_X = \{X_n, p_n^{n+1}, \omega\} \to \mathcal{S}_Y = \{Y_n, q_n^{n+1}, \omega\}$$

and

$$\beta = (\psi, \{g_n\})\colon \mathcal{S}_X = \{X_n, p_n^{n+1}, \omega\} \to \mathcal{S}_Y = \{Y_n, q_n^{n+1}, \omega\}$$

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  between two ANE-sequences.
- Let us introduce a homotopy relation between α and β. We say that $\alpha \simeq \beta$ if for each n there exists m such that $m \geq \varphi(n), \psi(n)$ and $f_n \circ p^m_{\varphi(n)} \simeq g_n \circ p^m_{\psi(n)}$.

Shape in terms of ANE-sequences

- Here is the diagram (commutes homotopically!)

$$
\begin{array}{ccc}
X_m & \xrightarrow{\;p^m_{\varphi(n)}\;} & X_{\varphi(n)} \\
\Big\downarrow{\scriptstyle p^m_{\psi(n)}} & & \Big\downarrow{\scriptstyle f_n} \\
X_{\psi(n)} & \xrightarrow{\;g_n\;} & Y_n
\end{array}
$$

Spectral Definition of Shape

- Suppose we have two morphisms $\alpha\colon \mathcal{S}_X \to \mathcal{S}_Y$ and $\beta\colon \mathcal{S}_Y \to \mathcal{S}_X$. We say that $\mathcal{S}_X \simeq \mathcal{S}_Y$ if $\alpha \circ \beta \simeq \mathrm{id}_{\mathcal{S}_Y}$ and $\beta \circ \alpha \simeq \mathrm{id}_{\mathcal{S}_X}$

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- We say that two compact spaces X and Y have the same shape (notation: $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$) if X and Y have homotopy equivalent ANE- sequences associated with them.

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- Turns out that the relation $\simeq$ is an equivalence relation for ANE-sequences.
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- We say that two compact spaces X and Y have the same shape (notation: $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$) if X and Y have homotopy equivalent ANE- sequences associated with them.
- Note that choice of ANE-sequences is irrelevant.

Borsuk's (original) Definition of Shape

- Suppose A and B are compact subsets of the Hilbert cube $\mathbb{I}^\omega$. Translating above definitions to this situation by a shape morphism from A to B we mean a sequence of maps $\mathbf{f} = \{f_n \colon \mathbb{I}^\omega \to \mathbb{I}^\omega\}$ with the following properties:
  - ($\star$) for each neighborhood V of B there exists a neighborhood U of A and an integer N such that for each $n \geq N$, $f_n(U) \subset V$ and $f_n|U \simeq f_{n+1}|U$ in V.

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- If $\mathbf{f} = \{f_n\}$ and $\mathbf{g} = \{g_n\}$ are two shape morphisms from A to B then we say that $\mathbf{f}$ and $\mathbf{g}$ are homotopic if for any neighborhood V of B there exist a neighborhood U of A and an integer N such that for each $n \geq N$ we have $f_n|U \simeq g_n|U$ in V

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- Let us note that if we have two morphisms $\mathbf{f}\colon A \to B$ and $\mathbf{g}\colon B \to A$ such that $\mathbf{gf} \simeq \mathrm{id}_A$ and $\mathbf{fg} \simeq \mathrm{id}_B$, then $Sh(A) = Sh(B)$ as defined above (routine verification - pen and paper).

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- This definition does not depend on the given embedding of A (or B) into $\mathbb{I}^\omega$.
- A good and simple exercise: if A and B are ANE's then $Sh(A) = Sh(B)$ if and only if $A \simeq B$.