Chapman's Complement Theorem

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Categorical Definition of Shape

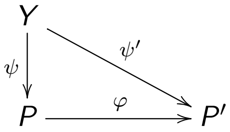
- Let us define the category SHAPE.
  (i) Objects are compact spaces.
  (ii) Morphism $\alpha\colon X \to Y$ is defined as a map

$$\alpha\colon \bigcup\{[Y,P]\colon P \in ANE\} \to \bigcup\{[X,P]\colon P \in ANE\}$$

  satisfying the following conditions:
  (a) If P is an ANE compactum, then $\alpha([Y,P]) \subset [X,P]$.
  (b) If P, P' are compact ANE's and $\psi \in [Y,P]$, $\psi' \in [Y,P']$ and $\varphi \in [P,P']$ are such that $\varphi \circ \psi \simeq \psi'$, then $\varphi \circ \alpha(\psi) \simeq \alpha(\psi')$.

Categorical Definition of Shape

- Here is condition (b) in terms of diagrams:
  Homotopy commutativity of the following diagram



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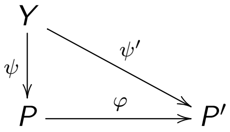

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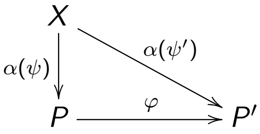

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- Chapman's Complement Theorem: Let A and B are Z-sets in the Hilbert cube $\mathbb{I}^\omega$. Then the following conditions are equivalent:
  (i) $Sh(A) = Sh(B)$.
  (b) $\mathbb{I}^\omega \setminus A$ is homeomorphic to $\mathbb{I}^\omega \setminus B$.

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  (i) $Sh(A) = Sh(B)$.
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- Proof is complicated and involves full machinery of the infinite-dimensional topology as presented in previous talks.
- Let us at least discuss some key ideas of the proof.

Chapman's Complement Theorem

- Let us see how existence of a homeomorphism $h\colon \mathbb{I}^\omega \setminus A \to \mathbb{I}^\omega \setminus B$ defines a shape morphism from A to B. Recall that A is a Z-set in $\mathbb{I}^\omega$ which guarantees that there is a homotopy $H\colon \mathbb{I}^\omega \times [0,1] \to \mathbb{I}^\omega$ such that:

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- $(\star\star)$ $H(\mathbb{I}^\omega, (0,1]) \subset \mathbb{I}^\omega \setminus A$.

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- $(\star)$ $H(x,0) = x$ for any $x \in \mathbb{I}^\omega$ and
- $(\star\star)$ $H(\mathbb{I}^\omega, (0,1]) \subset \mathbb{I}^\omega \setminus A$.
- This homotopy defines a sequence $\alpha = \{f_n\}$ of maps of $\mathbb{I}^\omega$ into itself as follows:

$$f_n(x) = h \circ F(x, \frac{1}{n}), \quad x \in \mathbb{I}^\omega,$$

which turns out is a shape morphism $\alpha\colon A \to B$.

- Claim. For each neighborhood G of A in $\mathbb{I}^\omega$ there exist a neighborhood U of A in $\mathbb{I}^\omega$ and a number $t_0 \in (0, 1]$ such that $H(U, [0, t_0]) \subset G$.

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- Consider the set $H^{-1}(G)$. This is a neighborhood of $A \times \{0\}$ in $\mathbb{I}^\omega \times [0,1]$ (by $(*)$).

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- Clearly

$$H(U \times [0,t_0]) \subset H(H^{-1}(G)) = G.$$

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- Then $H^{-1}(V \setminus B) \cup A$ is a neighborhood of A in $\mathbb{I}^\omega$.

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- Then $H^{-1}(V \setminus B) \cup A$ is a neighborhood of A in $\mathbb{I}^\omega$.
- By Claim, there exist a neighborhood U of A in $\mathbb{I}^\omega$ and a number $t_0 \in (0,1]$ such that $H(U,[0,t_0]) \subset H^{-1}(V \setminus B) \cup A$.
- Take N such that $\frac{1}{N} < t_0$, then $f_n|U \simeq f_{n+1}|U$ in V for any $n \geq N$.

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- This shows that the sequence $\alpha = \{f_n\}$, where

$$f_n(x) = h \circ F(x, \frac{1}{n}), \quad x \in \mathbb{I}^\omega,$$

indeed defines a shape morphism $\alpha \colon A \to B$.

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$$f_n(x) = h \circ F(x, \frac{1}{n}), \quad x \in \mathbb{I}^\omega,$$

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- Similarly,

$$g_n(x) = h^{-1} \circ G(x, \frac{1}{n}), \quad x \in \mathbb{I}^\omega,$$

where $G \colon \mathbb{I}^\omega \times [0,1] \to \mathbb{I}^\omega$ is a homotopy associated with the Z-set B, defines a shape morphism $\beta \colon B \to A$.

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- For a neighborhood U of A we need to find a neighborhood V of A and an integer N such that $g_n \circ f_n \simeq \mathrm{id}_V$ in U for each $n \geq N$.

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- For a neighborhood U of A we need to find a neighborhood V of A and an integer N such that $g_n \circ f_n \simeq \mathrm{id}_V$ in U for each $n \geq N$.
- Choose a neighborhood W of B and $\epsilon_1 > 0$ such that $h^{-1}(W \setminus B) \subset U$ and $h^{-1}G_t(W) \subset U$ for each t with $0 < t \leq \epsilon_1$.

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- For a neighborhood U of A we need to find a neighborhood V of A and an integer N such that $g_n \circ f_n \simeq \mathrm{id}_V$ in U for each $n \geq N$.
- Choose a neighborhood W of B and $\epsilon_1 > 0$ such that $h^{-1}(W \setminus B) \subset U$ and $h^{-1}G_t(W) \subset U$ for each t with $0 < t \leq \epsilon_1$.
- Also choose a neighborhood $V \subset U$ of A and ϵ_2, $0 < \epsilon_2 \leq \epsilon_1$ such that $hF_t(V) \subset W$ for each t with $0 < t \leq \epsilon_2$.

- Then for $n \geq \frac{1}{\epsilon_2}$ we have

$$g_n f_n|V = h^{-1} G_{\frac{1}{n}} h F_{\frac{1}{n}}|V \simeq h^{-1} h F_{\frac{1}{n}}|V = F_{\frac{1}{n}}|V \simeq \mathrm{id}_V \quad \text{in} \quad U$$

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- The first homotopy comes from homotopy $G_{\frac{1}{n}} | W$ to id_W via $G_t | W$, $0 \leq t \leq \frac{1}{n}$.

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- The first homotopy comes from homotopy $G_{\frac{1}{n}}|W$ to id_W via $G_t|W$, $0 \leq t \leq \frac{1}{n}$.
- The second homotopy comes from homotopy $F_{\frac{1}{n}}|V$ to id_V via $F_t|V$, $0 \leq t \leq \frac{1}{n}$.

- How to construct a homeomorphism from a shape isomorphism?

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- This is a difficult part of Chapman's Theorem. Proof is complicated and uses facts we discussed yesterday - Z-set unkniotting theorem for $\mathbb{I}^\omega$-manifolds, etc.