A unified approach to classification of nuclear C^*-algebras

Marius Dădărlat and Søren Eilers

October 17, 1998

Slides and paper available via
http://www.math.unm.edu/~eilers/papers/unified.html

Classification

$$K_0(A) \cong K_0(B) \Longrightarrow A \cong B$$
$$K_*(A) \cong K_*(B) \Longrightarrow A \cong B$$
$$\underline{\mathbf{K}}(A) \cong \underline{\mathbf{K}}(B) \Longrightarrow A \cong B$$
$$(K_*(A), TA) \simeq (K_*(B), TB) \Longrightarrow A \cong B$$

We concentrate on the *real rank zero* case.

Key concepts:

$\boxed{\exists}$: Existence theorem

$\boxed{!}$: Uniqueness theorem

$\boxed{\exists}$ *and* $\boxed{\textbf{!}}$ *blueprints*

A $*$-homomorphism $\varphi : A \longrightarrow B$ induces elements in the groups

$$\mathrm{Hom}(K_0(A), K_0(B))$$
$$\mathrm{Hom}(K_*(A), K_*(B))$$
$$\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B))$$
$$KK(A, B)$$

Let us denote all these elements by φ_*.

$\boxed{\exists}$: For any group element f there exists $\varphi : A \longrightarrow B$ with

$$f = \varphi_*$$

(on a finite set of generators).

$\boxed{\textbf{!}}$: If $\varphi, \psi : A \longrightarrow B$ satisfy

$$\varphi_* = \psi_*$$

then φ is (approximately) (stably) unitarily equivalent to ψ.

$\boxed{\exists}$ *and* $\boxed{!}$ *examples, I*

Let A and B be AF algebras.

Proposition [Elliott $\boxed{\exists}$, 1976] If

$$f \in \mathrm{Hom}(K_0(A), K_0(B))$$

is positive, there exists a $*$-homomorphism $\varphi : A \longrightarrow B$ with $f = \varphi_*$.

Proposition [Elliott $\boxed{!}$, 1976] If $\varphi_* = \psi_*$ in

$$\mathrm{Hom}(K_0(A), K_0(B))$$

then φ and ψ are approximately unitarily equivalent, i.e.

$$\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$$

for finitely many a.

$\boxed{\exists}$ *and* $\boxed{!}$ *examples, II*

Let $A = C(X)$ and $B = \mathcal{L}(H)/\mathcal{K}(H)$.

Proposition [BDF $\boxed{\exists}$, 1977] If

$$f \in KK(A, B),$$

there exists a $*$-homomorphism $\varphi : A \longrightarrow B$ with $f = \varphi_*$.

Proposition [BDF $\boxed{!}$, 1977] If $\varphi_* = \psi_*$ in

$$KK(A, B)$$

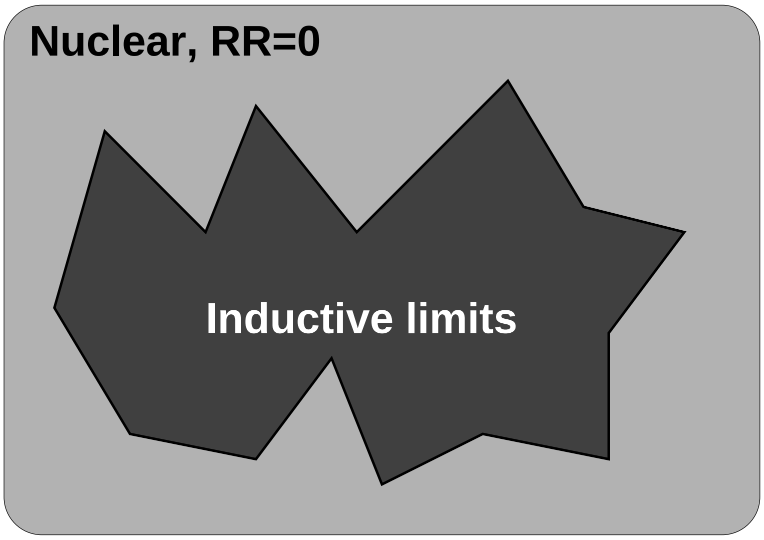
then φ and ψ are unitarily equivalent.

Historical remark

Most $\boxed{\exists}$ and $\boxed{\textbf{!}}$ results have been restricted to narrow classes of C^*-algebras, and inductive limits of them.

AF, AT, AD, AH, ASH, $\ldots$

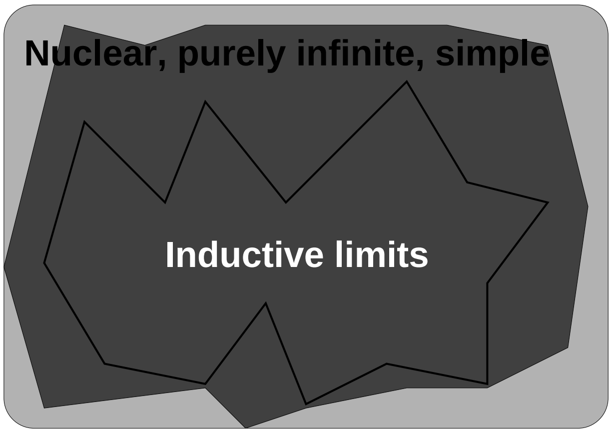
Pressing question: What is the size of the inductive limit classes?



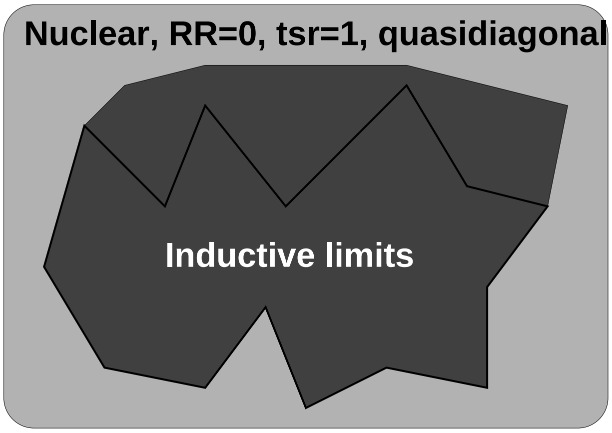
Possible method: Classify classes not given by inductive limit descriptions.

Progress

Infinite C^*-algebras [Kirchberg-Phillips]



Finite C^*-algebras [H. Lin, D&E]:



$\boxed{\exists}$ *difficulties*

$$\chi_t : C(\mathbb{T}) \longrightarrow C(\mathbb{T}) \qquad \chi_t(f) = f(e^{\pi i t})$$

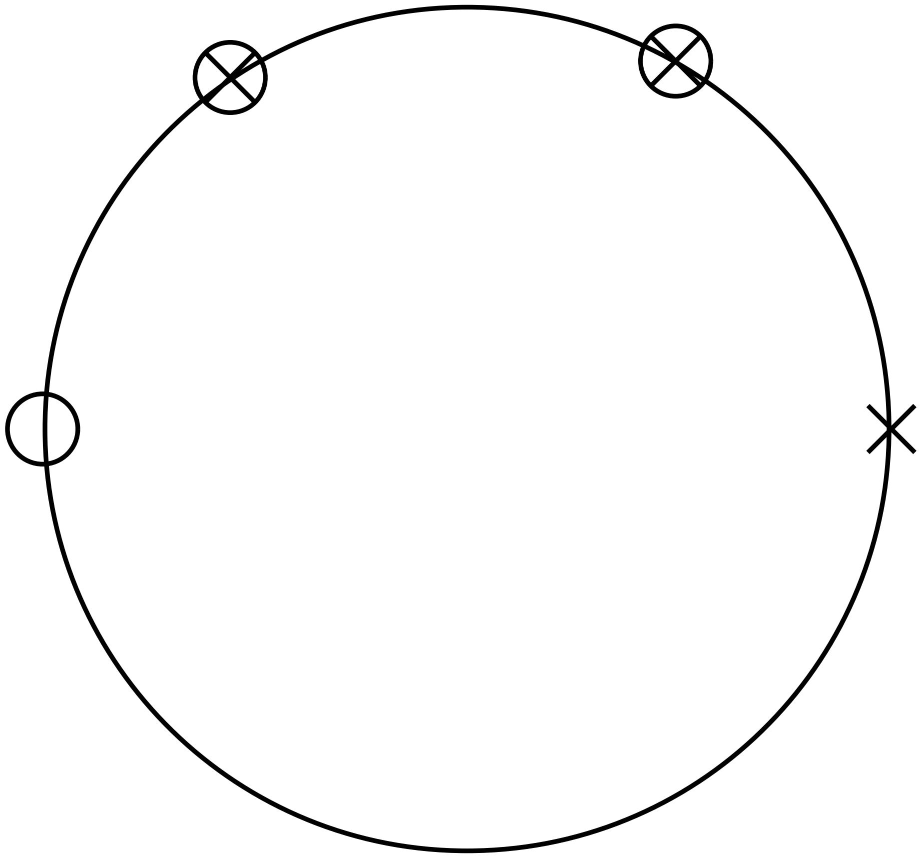
- Consider $f = 0 \oplus 1 : K_*(C(\mathbb{T})) \longrightarrow K_*(C(\mathbb{T}))$
  - For all $*$-homomorphisms $\varphi : C(\mathbb{T}) \longrightarrow \mathbf{M}_n(C(\mathbb{T}))$, $f \neq \varphi_*$.
  - <u>However</u>, $f = (\mathrm{id})_* - (\chi_0)_*$.

- Consider $f = [1 \quad 1] : K_0(C(\mathbb{T}^2)) \longrightarrow K_0(\mathbb{C})$
  - For all $*$-homomorphisms $\varphi, \psi : C(\mathbb{T}^2) \longrightarrow \mathbf{M}_n(\mathbb{C})$, $f \neq \varphi_* - \psi_*$.
  - <u>However</u>, there is an "almost morphism" $\sigma : C(\mathbb{T}^2) \longrightarrow \mathbf{M}_n(\mathbb{C})$ with $f = \sigma_\sharp$ on a finite set of generators.

$\boxed{!}$ *difficulties*

$$\chi_t : C(\mathbb{T}) \longrightarrow C(\mathbb{T}) \qquad \chi_t(f) = f(e^{\pi i t})$$

- Consider $\varphi, \psi : C(\mathbb{T}) \longrightarrow C(\mathbb{T})$ given by $\varphi = \chi_0, \psi = \chi_1$.
  - Then $\varphi_* = \psi_*$, yet
  $$\|u\varphi(\text{id})u^* - \psi(\text{id})\| = 2$$
  for all u.
  - <u>However</u>,
  $$\left\| u \begin{bmatrix} \varphi(\text{id}) & & \\ & \chi_{1/3}(\text{id}) & \\ & & \chi_{2/3}(\text{id}) \end{bmatrix} u^* - \begin{bmatrix} \psi(\text{id}) & & \\ & \chi_{1/3}(\text{id}) & \\ & & \chi_{2/3}(\text{id}) \end{bmatrix} \right\|$$
  is smaller.



Challenge

Note *padding* in

$$f = (\mathrm{id})_* - (\chi_0)_*$$

$$\left\| u \begin{bmatrix} \varphi(\mathrm{id}) & & \\ & \chi_{1/3}(\mathrm{id}) & \\ & & \chi_{2/3}(\mathrm{id}) \end{bmatrix} u^* - \ldots \right\|$$

The challenge is to achieve general pairs of $\boxed{\exists}$ and $\boxed{!}$ theorems, utilizing only padding which can be absorbed, and mutually adjusted so they can interact to yield classification theorems.

Let

$$\Gamma : A \longrightarrow \mathcal{L}_B(E)$$
$$\Delta : A \longrightarrow \mathcal{L}_B(F)$$

be unital representations. We write $\Gamma \sim \Delta$ if there exists a sequence $U_m \in \mathcal{L}_B(E, F)$, consisting of unitaries, such that

(i) $\|\Gamma(a) - U_m \Delta(a) U_m^*\| \longrightarrow 0$, $m \longrightarrow \infty$

(ii) $\Gamma(a) - U_m \Delta(a) U_m^* \in \mathcal{K}_B(E)$

for any $a \in A$.

Quasidiagonal & absorbing representations

Consider a unital representation

$$\Gamma : A \longrightarrow M(\mathcal{K} \otimes B) = \mathcal{L}_B(\mathbf{H}_B)$$

Definition We say Γ is *absorbing* if for any other unital representation $\Gamma' : A \longrightarrow \mathcal{L}_B(E')$, $\Gamma \oplus \Gamma' \sim \Gamma$.

Definition We say Γ is *quasidiagonal* if there exists an approximate unit of projections (e_n) for $\mathcal{K} \otimes B$ with the property that

$$[e_n, \Gamma(a)] \longrightarrow 0$$

for all $a \in A$.

Convention Define "almost morphisms"

$$\gamma_n = e_n \Gamma e_n : A \longrightarrow \mathbf{M}_{r_n}(B).$$

and write

$$\Gamma \ggg (\gamma_n)$$

when Γ is quasidiagonal and absorbing.

Remainder of talk

(i) State general $\boxed{\exists}$ and $\boxed{!}$ theorems.

(ii) Discuss existence of quasidiagonal and absorbing representations.

(iii) Classification: The infinite case (old!)

(iv) Classification: The finite case (new!)

(v) Sketch proof of $\boxed{!}$.

$\boxed{\boxed{\textbf{Theorem}}}$ [D&E $\boxed{\exists}$, 1998] Let A, B be unital C^*-algebras and assume that A is separable and nuclear. Assume furthermore that a quasidiagonal absorbing representation $\Gamma : A \longrightarrow M(\mathcal{K} \otimes B)$ exists, and write $\Gamma \ggg (\gamma_n)$.

For any $\alpha \in KK(A, B)$ there exists N and an "almost morphism" $\sigma : A \longrightarrow \mathbf{M}_{2r_N}(B)$ with

$$\alpha_* = \sigma_\sharp - (\gamma_N)_\sharp$$

on a finite set of elements in $\underline{\mathbf{K}}(A)$.

[The case "A quasidiagonal" obtained independently by H. Lin]

$\boxed{\textbf{Theorem}}$ [D&E $\boxed{!}$, 1998] Let A, B be unital C^*-algebras and assume that A is separable and nuclear. Assume furthermore that a quasidiagonal absorbing representation $\Gamma : A \longrightarrow M(\mathcal{K} \otimes B)$ exists, and write $\Gamma \ggg (\gamma_n)$.

Suppose that

$$\varphi, \psi : A \longrightarrow B$$

are two unital $*$-homomorphisms with $\varphi_* = \psi_*$ in $KK(A, B)$. Then there exists N and u with

$$\left\| u \begin{bmatrix} \varphi(a) & \\ & \gamma_N(a) \end{bmatrix} u^* - \begin{bmatrix} \psi(a) & \\ & \gamma_N(a) \end{bmatrix} \right\| < \varepsilon$$

for finitely many a.

[Preceded by theorem of Lin in “UCT” case]

Quasidiagonal & absorbing representations

Assume that there is a representation $\pi : A \longrightarrow \mathcal{L}(\mathbf{H})$ for which an increasing sequence of projections (e_n) exists with

$$\dim(e_n) < \infty$$
$$e_n \nearrow 1_{\mathbf{H}}$$
$$[e_n, \pi(a)] \longrightarrow 0$$

[classical notion of quasidiagonality]

If furthermore A is nuclear and π is of infinite multiplicity, then

$$\Gamma : A \longrightarrow M(\mathcal{K} \otimes B) \qquad \Gamma(a) = \pi(a) \otimes 1_B$$

is quasidiagonal and absorbing, for any B.

Note that $\Gamma \ggg (\gamma_n)$ with

$$\dim(\gamma_n(A)) < \infty.$$

Quasidiagonal & absorbing representations

Assume that there is a *unital simple inclusion* $\iota : A \longrightarrow B$, i.e. a unital $*$-monomorphism which can be factored

$$A \hookrightarrow C \hookrightarrow B$$

where C is simple.

If furthermore A is nuclear, then

$$\Gamma : A \longrightarrow M(\mathcal{K} \otimes B) \qquad \Gamma(a) = 1_{\mathcal{L}(\mathbf{H})} \otimes \iota(a)$$

is quasidiagonal and absorbing [Lin].

Note that $\Gamma \ggg (\gamma_n)$ with

$$\gamma_n(a) = \begin{bmatrix} \iota(a) & & & \\ & \iota(a) & & \\ & & \ddots & \\ & & & \iota(a) \end{bmatrix}$$

$\boxed{\exists}$, *the infinite case*

Let A, B be purely infinite simple C^*-algebras in standard position.

$\boxed{\exists_+}$: For any $\alpha \in KK(A,B)$ there exists an "almost morphism" $\sigma : A \longrightarrow B$ with

$$\alpha_* = \sigma_\sharp$$

on a finite set of elements in $\underline{\mathbf{K}}(A)$.

$\boxed{\textbf{Proof}}$ A unital simple inclusion

$$\iota : A \hookrightarrow \mathcal{O}_2 \hookrightarrow B$$

We get from D&E $\boxed{\exists}$:

$$\begin{aligned} \alpha_* &= \sigma_\sharp - (\Gamma_N)_\sharp \\ &= \sigma_\sharp - (\iota \oplus \cdots \oplus \iota)_* \\ &= \sigma_\sharp \end{aligned}$$

$\boxed{\textbf{!}}$, *the infinite case*

Let A, B be purely infinite simple C^*-algebras.

$\boxed{\textbf{!}_+}$: Suppose that

$$\varphi, \psi : A \longrightarrow B$$

are two unital $*$-homomorphisms with $\varphi_* = \psi_*$ in $KK(A, B)$. Then φ and ψ are approximately unitarily equivalent.

$\boxed{\textbf{Proof}}$ Ingredients:

(i) D&E $\boxed{\textbf{!}}$: $\varphi \oplus \eta \oplus \cdots \oplus \eta$ and $\psi \oplus \eta \oplus \cdots \oplus \eta$ are approximately unitarily equivalent for $\eta : A \longrightarrow B$.

(ii)
$$\begin{tikzcd} A \arrow[rr, equal] \arrow[dr] & & A \\ & pAp \oplus \mathbf{M}_n(\mathcal{O}_2) \arrow[ur, hook] & \end{tikzcd}$$

(iii) Rørdam $\boxed{\textbf{!}}$ for maps $\mathcal{O}_2 \longrightarrow B$

(iv) $K_*(\prod B_i)$

Classification, the finite case

Let A, B be unital, separable, nuclear and simple C^*-algebras. Suppose further that

(i) A and B have Lin's TAF property

(ii) A and B satisfy the UCT

(iii) $K_0(A) = \mathbb{Q}$

Then

$$K_*(A) \cong K_*(B) \Longrightarrow A \cong B$$

$\boxed{\exists}$, *the finite case*

$\boxed{\exists_+}$: For any $\alpha \in KK(A, B)$ there exists "almost morphisms"

$$\sigma : A \longrightarrow \mathbf{M}_r(B) \qquad \mu : A \longrightarrow \mathbf{M}_r(\mathbb{C})$$

with

$$\alpha_* = \sigma_\sharp - \mu_\sharp$$

on a finite set of elements in $\underline{\mathbf{K}}(A)$.

$\boxed{\textbf{!}}$, *the finite case*

Let A, B be unital nuclear simple C^*-algebras with TAF and UCT.

$\boxed{\textbf{!}_+}$: Suppose that

$$\varphi, \psi : A \longrightarrow B$$

are two unital $*$-homomorphisms with $\varphi_* = \psi_*$ in $KK(A, B)$. Then φ and ψ are approximately unitarily equivalent.

$\boxed{\textbf{Proof}}$ Ingredients:

(i) D&E $\boxed{\textbf{!}}$: $\varphi \oplus \eta \oplus \cdots \oplus \eta$ and $\psi \oplus \eta \oplus \cdots \oplus \eta$ are approximately unitarily equivalent for $\eta : A \longrightarrow B$.

(ii)
$$\begin{tikzcd} A \arrow[rr, equal] \arrow[dr] & & A \\ & pAp \oplus \mathbf{M}_n(\mathbb{C}) \arrow[ur, hook] & \end{tikzcd}$$

(iii) Well-known $\boxed{\textbf{!}}$ for maps $\mathbb{C} \longrightarrow B$

(iv) $K_*(\prod B_i)$

Sketch of proof of $\boxed{\textbf{!}}$

α°	$\varphi_* = \psi_*$ in $KK(A,B)$
β°	$0 \longrightarrow \mathcal{K}\otimes B \longrightarrow C_\varphi \longrightarrow A \longrightarrow 0$ $\cong$ $0 \longrightarrow \mathcal{K}\otimes B \longrightarrow C_\psi \longrightarrow A \longrightarrow 0$
γ°	$u_t \begin{bmatrix} t\varphi(a) & \\ & \Gamma(a) \end{bmatrix} {u_t}^* - \begin{bmatrix} t\psi(a) & \\ & \Gamma(a) \end{bmatrix}$ $\in C_0((0,1), \mathcal{K}\otimes B)$
δ°	With $v = u_t$ for large t, 1° $\left\Vert v \begin{bmatrix} \varphi(a) & \\ & \Gamma(a) \end{bmatrix} v^* - \begin{bmatrix} \psi(a) & \\ & \Gamma(a) \end{bmatrix} \right\Vert < \varepsilon$ 2° $[v] = 0$ in $K_0(D_\Gamma)$

Here

$$D_\Gamma = \{x \in M(\mathcal{K}\otimes B) \mid [x, \Gamma(A)] \in \mathcal{K}\otimes B\}$$