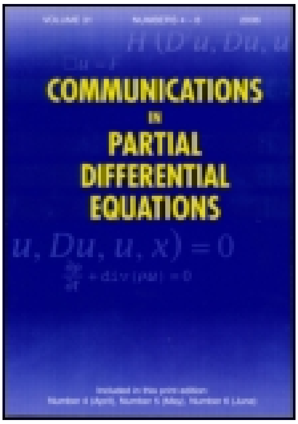

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Spectral Asymptotics for Nonsmooth Singular Green Operators

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Spectral Asymptotics for Nonsmooth Singular Green Operators

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Singular Green operators G appear typically as boundary correction terms in resolvents for elliptic boundary value problems on a domain $\Omega \subset \mathbb{R}^n$, and more generally they appear in the calculus of pseudodifferential boundary problems. In particular, the boundary term in a Krein resolvent formula is a singular Green operator. It is well-known in smooth cases that when G is of negative order $-t$ on a bounded domain, its eigenvalues or s-numbers have the behavior $(*) s_j(G) \sim cj^{-t/(n-1)}$ for $j \to \infty$, governed by the boundary dimension $n-1$. In some nonsmooth cases, upper estimates $(**) s_j(G) \leq Cj^{-t/(n-1)}$ are known.

We show that $(*)$ holds when G is a general selfadjoint nonnegative singular Green operator with symbol merely Hölder continuous in x. We also show $(*)$ with $t = 2$ for the boundary term in the Krein resolvent formula comparing the Dirichlet and a Neumann-type problem for a strongly elliptic second-order differential operator (not necessarily selfadjoint) with coefficients in $W^1_q(\Omega)$ for some $q > n$.

Keywords Elliptic boundary value problems; Krein resolvent formula; Leading spectral asymptotics; Nonsmooth coefficients; Nonsmooth domains; Pseudodifferential boundary operators; Singular Green operator.

2000 MSC 35J25, 35P20, 35S15, 47A10, 47G30.

Introduction

Singular Green operators arise typically as boundary correction terms in solution formulas for elliptic boundary value problems. For example, if A is a strongly elliptic second-order differential operator with smooth coefficients on $\mathbb{R}^n$, with inverse Q, and $\Omega \subset \mathbb{R}^n$ is smooth bounded, then the solution operator for the Dirichlet problem $Au = f$ on Ω, $u|_{\partial\Omega} = 0$, has the form

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$$A_\gamma^{-1} = Q_+ + G_\gamma, \tag{i}$$

where Q_+ is the truncation of the pseudodifferential operator (ψdo) Q to Ω and G_γ is a singular Green operator. Another typical singular Green operator is the difference between the solution operators for two different boundary value problems for A,

$$G = \widetilde{A}^{-1} - A_\gamma^{-1}. \tag{ii}$$

In the study of the spectral behavior it is found that whereas the eigenvalues or singular values (s-numbers) of $\widetilde{A}^{-1}$ and A_γ^{-1} have the behavior $s_j(\widetilde{A}^{-1}) \sim c_A j^{-2/n}$, the s-numbers of G_γ and G behave like $s_j(G) \sim c_G j^{-2/(n-1)}$ (cf. e.g. Grubb [14, 15], Birman-Solomyak [8]). Here the boundary dimension $n-1$ enters since these operators have their essential effect near the boundary.

This spectral behavior of G is well-known for operators A with smooth coefficients, whereas in cases with nonsmooth coefficients, generally only upper estimates $s_j(G) \leq Cj^{-2/(n-1)}$ are known (from Birman [6], when coefficients are continuous with bounded first derivatives and the boundary is C^2).

We shall here address the question of showing asymptotic estimates for singular Green operators in cases with nonsmooth coefficients. The main tool will be the calculus of nonsmooth pseudodifferential boundary operators (ψdbo's) developed by Abels [1, 2] (as a generalization of the smooth ψdbo's, Boutet de Monvel [10], Grubb [15, 16]). Our results will deal both with general selfadjoint nonnegative singular Green operators with C^τ-smoothness in the x-variable, and with the special, not necessarily selfadjoint operators in (i), (ii), with W^1_q-smoothness of the coefficients of A ($q > n$), building on the resolvent construction in Abels et al. [4].

Contents. In Section 1 we recall the Krein resolvent formula in the smooth setting and show a precise spectral asymptotic estimate in the selfadjoint case. Section 2 deals with spectral asymptotic results for nonsmooth ψdo's, recalling an early result of Birman and Solomyak, and proving a result for C^τ-smooth ψdo's of negative order (any $\tau > 0$), based on the calculus for such ψdo's by Marschall. In Section 3 we recall the theory of nonsmooth ψdbo's by Abels, with some supplements. Section 4 gives the proof of spectral asymptotic estimates for nonsmooth selfadjoint singular Green operators of negative order and class 0, defined on $\mathbb{R}^n_+$ or on a smooth bounded set $\Omega \subset \mathbb{R}^n$. In Section 5, similar results are obtained for the singular Green term in the Dirichlet resolvent (i), and for the singular Green term in the Krein formula (ii), by a different method based on work of Abels, Grubb and Wood; here nonselfadjointness is allowed. In Section 6, the results of Section 5 are extended to the case of domains Ω with $B^{\frac{3}{2}}_{p,2}$-boundary; this includes $C^{\frac{3}{2}+\varepsilon}$-domains. Finally, the Appendix written by Abels gives the proof of one of the theorems used in Section 3.

1. Results in the Smooth Case

1.1. *Some Notation*

For $x \in \mathbb{R}^n$ we denote $x' = (x_1, \ldots, x_{n-1})$, so that $x = (x', x_n)$, and we denote $\mathbb{R}^n_\pm = \{x \mid x_n \gtrless 0\}$. Moreover, $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$.

When Ω is a smooth open subset of $\mathbb{R}^n$ with boundary $\partial\Omega = \Sigma$, we use the standard L_2-Sobolev spaces, with the following notation: $H^s(\mathbb{R}^n)$ ($s \in \mathbb{R}$) has the norm $\|v\|_s = \|\mathscr{F}^{-1}(\langle\xi\rangle^s \mathscr{F} v)\|_{L_2(\mathbb{R}^n)}$; here $\mathscr{F}$ is the Fourier transform and $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. Next, $H^s(\Omega) = r_\Omega H^s(\mathbb{R}^n)$ where r_Ω restricts to Ω, provided with the norm $\|u\|_s = \inf\{\|v\|_s \mid v \in H^s(\mathbb{R}^n),\ u = r_\Omega v\}$. Moreover, we denote by $H_0^s(\overline{\Omega})$ the space $\{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}$; it is a closed subspace of $H^s(\mathbb{R}^n)$, and there is an identification of the antidual space of $H^s(\Omega)$ (the space of antilinear, i.e., conjugate linear, functionals), with $H_0^{-s}(\overline{\Omega})$ for any $s \in \mathbb{R}$, with a duality consistent with the $L_2(\Omega)$ scalar product. Spaces over the boundary, $H^s(\Sigma)$, are defined by local coordinates from $H^s(\mathbb{R}^{n-1})$, $s \in \mathbb{R}$. Here there is an identification of $H^{-s}(\Sigma)$ with the antidual space of $H^s(\Sigma)$.

Occasionally we shall also refer to some L_q-based Sobolev spaces, $1 < q < \infty$, namely the Bessel-potential spaces $H_q^s(\Omega)$; for $s = k \in \mathbb{N}_0$ they are also denoted $W_q^k(\Omega)$. They are defined in a similar way as above from $H_q^s(\mathbb{R}^n)$, provided with the norm $\|v\|_{s,p} = \|\mathscr{F}^{-1}(\langle\xi\rangle^s \mathscr{F} v)\|_{L_q(\mathbb{R}^n)}$. One has for $k \in \mathbb{N}_0$ that $H_q^k(\Omega) = W_q^k(\Omega) = \{u \in L_q(\Omega) \mid D^\alpha u \in L_q(\Omega) \text{ for } |\alpha| \le k\}$ (allowing also $q = \infty$).

For $\tau \ge 0$, we denote by $C^\tau(\Omega)$ the space of continuous functions $f(x)$ such that when $[\tau]$ is the largest integer $\le \tau$,

$$\|f\|_{C^\tau} \equiv \sum_{|\beta| \le [\tau]} \sup_x |D^\beta f(x)| + \sum_{|\beta| = [\tau]} \sup_{x \ne y} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x - y|^{\tau - [\tau]}} < \infty. \tag{1.1}$$

For $\tau = k \in \mathbb{N}_0$, they are also denoted $C_b^k(\Omega)$; when $\tau = k + \sigma$ with $k \in \mathbb{N}_0$ and $\sigma \in]0, 1[$, they are the Hölder spaces $C^{k,\sigma}(\Omega)$. We denote $\bigcap_{\tau \ge 0} C^\tau = C_b^\infty$.

1.2. *The Krein Resolvent Formula*

As a point of departure, consider the Krein resolvent formula comparing the resolvents of the Dirichlet realization A_γ and a Neumann-type realization $\widetilde{A}$. The operators are defined from a second-order strongly elliptic operator A with smooth coefficients on a smooth bounded or exterior domain Ω with boundary Σ (cf. Section 5) by the boundary conditions, assumed elliptic,

$$\gamma_0 u = 0, \text{ resp. } \chi u = C\gamma_0 u, \quad \text{on } \Sigma,$$

with a first-order tangential differential operator C; here $\gamma_0 u = u|_\Sigma$ and χu is the conormal derivative (5.12). Then one has when $0 \in \varrho(A_\gamma) \cap \varrho(\widetilde{A})$ (the resolvent sets) that

$$\widetilde{A}^{-1} - A_\gamma^{-1} = K_\gamma L^{-1} (K_\gamma')^*, \tag{1.2}$$

where K_γ and K_γ' are the Poisson operators for the Dirichlet problem for A resp. A', L is the realization of the first-order elliptic ψdo $C - P_{\gamma,\chi}$ with $D(L) = H^{\frac{3}{2}}(\Sigma)$, and $P_{\gamma,\chi} = \chi K_\gamma$, the Dirichlet-to-Neumann operator. Cf. Grubb [13, 14], and formulas (2.10), (2.15), (3.45) in Brown-Grubb-Wood [9] (based on [13, 14]); see also e.g. [17]. Abstract versions have been known for many years and various concrete applications to elliptic PDE given recently, cf. e.g. Malamud [27], Gesztesy-Mitrea [11], and their listings of other contributions.

Consider for precision in this introductory section the case where the operators A_γ and $\widetilde{A}$ are selfadjoint; then so is L, as an operator from $H^{-\frac{1}{2}}(\Sigma)$ to $H^{\frac{1}{2}}(\Sigma)$. Then also $K_\gamma = K_\gamma'$. (Nonselfadjoint cases, where principal estimates of s-numbers can be obtained, are treated in [20], Sect. 10.)

The operator $G = K_\gamma L^{-1} K_\gamma^*$ is a singular Green operator of order -2, and its spectral asymptotic behavior can be found in the following way: Denote the positive resp. negative eigenvalues by $\mu_j^\pm$, monotonely ordered and repeated according to multiplicity (one of the sequences may be finite and then needs no treatment, we leave this aspect out of the explanation). Then we have, since $\mu_j^\pm(B_1 B_2) = \mu_j^\pm(B_2 B_1)$:

$$\mu_j^\pm(G) = \mu_j^\pm(L^{-1} K_\gamma^* K_\gamma) = \mu_j^\pm(L^{-1} P_1), \tag{1.3}$$

where $P_1 = K_\gamma^* K_\gamma$ is a selfadjoint positive ψdo on Σ of order -1 (it is also used in [11] Th. 3.4, which gives more information). Let $P_2 = P_1^{\frac{1}{2}}$, positive selfadjoint of order $-\frac{1}{2}$, then we have furthermore:

$$\mu_j^\pm(G) = \mu_j^\pm(L^{-1} P_2^2) = \mu_j^\pm(P_2 L^{-1} P_2) = \mu_j^\pm(P_3), \quad P_3 = P_2 L^{-1} P_2. \tag{1.4}$$

Estimates of the eigenvalues $\mu_j^\pm$ are connected in a known way with estimates of the corresponding counting functions $N'^\pm(t; G) = N'^\pm(t; P_3)$; here $N'^\pm(t; S)$ indicates the number of positive, resp. negative eigenvalues of S outside the interval $]-1/t, 1/t[$.

For the counting functions we apply the results of Hörmander [23] and Ivrii [24] to the selfadjoint elliptic ψdo P_3^{-1} of order 2. This gives:

Theorem 1.1. *In the smooth selfadjoint case one has for the operator* $\widetilde{A}^{-1} - A_\gamma^{-1} = K_\gamma L^{-1} K_\gamma^* = G$ *that the eigenvalues satisfy*

$$N'^\pm(t; \widetilde{A}^{-1} - A_\gamma^{-1}) = C^\pm t^{(n-1)/2} + O(t^{(n-2)/2}) \text{ for } t \to \infty, \tag{1.5}$$

where $C^\pm$ *are determined from the principal symbol* $p_3^0(x', \xi')$ *of* P_3 (*defined through* (1.3)–(1.4)). *Moreover, if* $(p_3^0)^{-1}$ *satisfies Ivrii's condition from* [24] (*the bicharacteristics through points of* $T^*(\Sigma) \setminus 0$ *are nonperiodic except for a set of measure zero*), *there are constants* $C_1^\pm$ *such that*

$$N'^\pm(t; \widetilde{A}^{-1} - A_\gamma^{-1}) = C^\pm t^{(n-1)/2} + C_1^\pm t^{(n-2)/2} + o(t^{(n-2)/2}) \text{ for } t \to \infty. \tag{1.6}$$

The estimate (1.5) follows from Hörmander [23] in the scalar case, Ivrii allows arbitrary elliptic systems and has the precision in (1.6). (The result does not seem to have been formulated with this precision before.)

Note that in the scalar case, if Σ is connected, the selfadjointness and ellipticity prevents the principal symbol of L from changing between positive and negative values, since it must be real. So in that case only one of the sequences μ_j^+ or μ_j^- is infinite.

In rough cases, we do not expect to get the fine remainder estimates, but will aim for principal asymptotic (Weyl-type) estimates, as obtained in [15] for general singular Green operators in the smooth case.

2. Spectral Estimates for Nonsmooth Pseudodifferential Operators

2.1. *Weak Schatten Classes*

As in [15] we denote by $\mathfrak{C}_p(H, H_1)$ the p-th Schatten class consisting of the compact operators B from a Hilbert space H to another H_1 such that $(s_j(B))_{j\in\mathbb{N}} \in \ell_p(\mathbb{N})$.

Here $s_j(B) = \mu_j(B^*B)^{\frac{1}{2}}$, where $\mu_j(B^*B)$ denotes the j-th positive eigenvalue of B^*B, arranged nonincreasingly and repeated according to multiplicities. The so-called weak Schatten class consists of the compact operators B such that

$$s_j(B) \leq Cj^{-1/p} \text{ for all } j; \text{ we set } \mathbf{N}_p(B) = \sup_{j \in \mathbb{N}} s_j(B) j^{1/p}; \tag{2.1}$$

the notation $\mathfrak{S}_{(p)}(H, H_1)$ was used in [15] for this space. (The indication (H, H_1) is replaced by (H) if $H = H_1$; it can be omitted when it is clear from the context.) Different notation is used in some other works; $\mathfrak{C}_p$ is sometimes called $\mathfrak{S}_p$, and $\mathfrak{S}_{(p)}$ is also sometimes called Σ_p, $\mathfrak{S}_p$ or $\mathfrak{S}_{p,\infty}$. To avoid confusion, we shall in the present paper use the notation $\mathfrak{S}_{p,\infty}$ for the p-th weak Schatten class.

We recall (cf. e.g. [15] for details and references) that $\mathbf{N}_p(B)$ is a quasinorm on $\mathfrak{S}_{p,\infty}$, satisfying

$$\begin{gathered} \mathbf{N}_p\Big(\sum_{k=1}^{k_0} B_k\Big) \leq C_{p,\gamma} \sum_{k=1}^{k_0} \mathbf{N}_p(B_k) k^\gamma, \\ \text{with } \gamma = 0 \text{ if } p > 1, \quad \gamma > p^{-1} - 1 \text{ if } p \leq 1; \end{gathered} \tag{2.2}$$

here $C_{p,\gamma}$ is independent of k_0. Recall also that

$$\mathfrak{S}_{p,\infty} \cdot \mathfrak{S}_{q,\infty} \subset \mathfrak{S}_{r,\infty}, \text{ where } r^{-1} = p^{-1} + q^{-1}, \tag{2.3}$$

and

$$s_j(B^*) = s_j(B), \quad s_j(EBF) \leq \|E\| s_j(B) \|F\|, \tag{2.4}$$

when $E: H_2 \to H$ and $F: H_1 \to H_3$ are bounded linear maps between Hilbert spaces.

Moreover, we recall that when Ξ is a bounded open subset of $\mathbb{R}^m$ and reasonably regular, then the injection $H^t(\Xi) \hookrightarrow L_2(\Xi)$ is in $\mathfrak{S}_{m/t,\infty}$ when $t > 0$. It follows that when B is a linear operator in $L_2(\Xi)$ that is bounded from $L_2(\Xi)$ to $H^t(\Xi)$, then $B \in \mathfrak{S}_{m/t,\infty}$, with

$$\mathbf{N}_{m/t}(B) \leq C\|B\|_{\mathscr{L}(L_2(\Xi), H^t(\Xi))}. \tag{2.5}$$

2.2. *Results of Birman and Solomyak*

We shall study pseudodifferential operators with C^τ-smoothness in the x-variable further below, but let us first consider some results of Birman and Solomyak [7]. They show an asymptotic result under weak smoothness hypotheses both in the x-and the ξ-variable:

Theorem 2.1. [7] *On a closed manifold Ξ of dimension m, let P be defined in local coordinates from symbols $p(x, \xi)$ that are homogeneous in ξ of degree $-t < 0$. Denote $m/t = \mu$.*

1° *One has that $P \in \mathfrak{S}_{\mu,\infty}$ (i.e., $s_j(P)$ is $O(j^{-t/m})$) under the following hypotheses on the symbol in local coordinates*:

If $\mu \leq 1$ (i.e., $t \geq m$), assume that the symbols restricted to $\xi \in S^{m-1} = \{|\xi| = 1\}$ are in $L_\infty(S_\xi^{m-1}, C_x^\varepsilon)$ for some $\varepsilon > 0$.

If $\mu > 1$ *(i.e.,* $t < m$*) assume that the symbols at* $\xi \in S^{m-1}$ *are in* $L_\infty(S^{m-1}, W_p^l)$*, where*

$$p \geq 2, \quad pl > m, \quad 1/p > 1/2 - 1/q_1, \tag{2.6}$$

for some $2 < q_1, q_2 \leq \infty$ *with* $q_1^{-1} + q_2^{-1} = \mu^{-1}$, $l \in \mathbb{R}_+$.

2° *There is an asymptotic estimate*

$$s_j(P) j^{t/m} \to c, \tag{2.7}$$

when the properties under 1° *hold with* L_∞ *replaced by* C^0.

The constant c is the same as described further below in Theorem 2.5.

It is particularly interesting here that these estimates allow nonsmoothness in ξ, namely just boundedness or continuity, with no requirement on ξ-derivatives.

Also some nonsmoothness in x is allowed, best in low dimensions. Let us see what it means for $t = 2$.

Here 1° applies when when $\mu \leq 1$, i.e., $m \leq 2$, so only C^ε-smoothness in x is needed then.

When $2 < m \leq 4$, so that $1 < \mu \leq 2$, $1/2 \leq 1/\mu < 1$, we can in 2° take $1/q_1 = 1/2 - \delta$ with a small positive δ (hereby $q_1 > 2$) and $1/q_2 = 1/\mu - 1/q_1 = 1/\mu - 1/2 + \delta$ (which is $< 1/2$ for $\delta < 1 - 1/\mu$, hereby $q_2 > 2$). Then the requirement $1/p > 1/2 - 1/q_1$ allows taking $1/p$ arbitrarily close to 0. Now $pl > m$ can be fulfilled for arbitrarily small l by taking p sufficiently large, so W_p^l can be taken to contain C^ε, for a given small ε.

When $m > 4$, the inequalities will put a positive lower limit on the possible l that can enter in (2.6). For example, for $m = 5$, $\mu = 5/2$ and $1/\mu = 2/5$, then we can at best take $q_1 = 5/2$ and $q_2 = \infty$, which restricts p by

$$1/p > 1/2 - 1/q = 1/2 - 2/5 = 1/10,$$

so that $p < 10$. Then $pl > m$ is at best obtained with $l > 1/2$.

So already for $m = 5$, hence for the boundary of a set of dimension 6, the Birman-Solomyak result will not give asymptotic estimates for the most general situation in [4] where the symbols are only C^τ in x with a $\tau < 1/2$.

It should be noted that our symbol classes have a high degree of smoothness in ξ in contrast to those of Birman and Solomyak; we do not need their generality for the present purposes, and have to find another point of view.

2.3. *Approximation of ψdo's by Operators with Smooth Symbols*

We now turn to the symbol classes with Hölder smoothness in x and full smoothness in ξ, as defined in Kumano-go and Nagase [25], Marschall [28], Taylor [31] and other places:

Definition 2.2. *Let* $d \in \mathbb{R}$, $\tau > 0$, $m \in \mathbb{N}$, $N \in \mathbb{N}_0$. *The space* $C^\tau S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N)$ *consists of the functions* $p(x, \xi)$ *of* $x, \xi \in \mathbb{R}^m$ *such that*

$$\|D_\xi^\alpha p(x, \xi)\|_{C^\tau} \text{ is } O(\langle \xi \rangle^{d-|\alpha|}) \text{ for } |\alpha| \leq N. \tag{2.8}$$

We denote $\bigcap_{N\in\mathbb{N}_0} C^\tau S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N) = C^\tau S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m)$.

The symbol $p(x,\xi)$ *is said to be polyhomogeneous* (*with step* 1), *when there is an asymptotic expansion in symbols* $p_{d-j}(x,\xi)$ *homogeneous of order* $d-j$ *in* ξ *for* $|\xi| \geq 1$, *in the sense that each* $p_{d-j} \in C^\tau S^{d-j}_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N)$, *and for all* J, $p - \sum_{j<J} p_{d-j}$ *is in* $C^\tau S^{d-J}_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N)$. *For the subspaces of polyhomogeneous symbols we use the notation* $C^\tau S^d$ *instead of* $C^\tau S^d_{1,0}$.

Our convention for the Fourier transform is: $\mathcal{F}_{x\to\xi} u = \int_{\mathbb{R}^m} e^{-ix\cdot\xi} u(x)\,dx$; then $\mathcal{F}^{-1}_{\xi\to x} f = \int_{\mathbb{R}^m} e^{ix\cdot\xi} f(\xi)\,đ\xi$, with $đ\xi = (2\pi)^{-n} d\xi$.

A symbol in $C^\tau S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m)$ defines a ψdo P by

$$Pu = \mathrm{OP}(p(x,\xi))u = \int e^{i(x-y)\cdot\xi} p(x,\xi) u(y)\,dy đ\xi, \text{ also called } p(x, D_x)u, \tag{2.9}$$

with a suitable interpretation of the integral (as an oscillatory integral). We recall in passing that one can also define operators "in y-form" resp. "in (x,y)-form" by formulas

$$\begin{aligned} \mathrm{OP}(p(y,\xi))u &= \int e^{i(x-y)\cdot\xi} p(y,\xi) u(y)\,dy đ\xi, \text{ resp.} \\ \mathrm{OP}(p(x,y,\xi))u &= \int e^{i(x-y)\cdot\xi} p(x,y,\xi) u(y)\,dy đ\xi; \end{aligned} \tag{2.10}$$

the latter is in some texts said to be defined from a double symbol. We say that (2.9) is "in x-form".

It is well-known that P in (2.9) satisfies

$$P\colon H^d(\mathbb{R}^m) \to L_2(\mathbb{R}^m) \text{ and hence } P^*\colon L_2(\mathbb{R}^m) \to H^{-d}(\mathbb{R}^m). \tag{2.11}$$

(More information is given below in Theorem 2.3.) Hence when d is a negative number $-t$, and P is defined on a compact m-dimensional manifold from such symbols in local coordinates, $P \in \mathfrak{S}_{m/t,\infty}$ in view of (2.5).

This gives upper spectral estimates, and we shall obtain the asymptotic spectral estimates by approximation of the ψdo symbols by smooth polyhomogeneous symbols for which the estimates are known. The so-called *symbol smoothing*, where p is written as $p^\sharp + p^\flat$, $p^\sharp \in S^m_{1,\delta}$ and $p^\flat$ of lower order, is not useful here, since the polyhomogeneity is lost in this decomposition. What we do is in fact simpler, namely approximation by convolution in the x-variable with an approximate unit.

There is one small obstacle here, namely that C^∞-functions are not dense in C^τ when τ is not integer. For example, if one compares a C^∞-function $u(t)$ on $[-1,1]$ with the function $|t|^\tau \in C^\tau([-1,1])$ for some $\tau \in]0,1[$, one finds that

$$\sup_{t\neq s} \frac{|(u(t)-|t|^\tau) - (u(s)-|s|^\tau)|}{|t-s|^\tau} \geq \sup_{t\neq 0} \frac{|u(t)-u(0)-|t|^\tau|}{|t|^\tau} \geq 1.$$

However, it is well-known (cf. e.g. Lunardi [26] Ch. 1) that the so-called little-Hölder space $h^\tau(\mathbb{R}^m)$, consisting of functions $u(x) \in C^\tau(\mathbb{R}^m)$ such that

$$\lim_{h\to 0} \sup_{0<|x-y|\leq h} \frac{|u(x)-u(y)|}{|x-y|^\tau} = 0, \tag{2.12}$$

is a closed subspace of $C^\tau(\mathbb{R}^m)$ for $0 < \tau < 1$ that equals the closure of $C_b^\infty(\mathbb{R}^m)$ in the C^τ-norm. Here, when ϱ_k is an approximate unit, i.e., $\varrho_k(x) = k^m \varrho(kx)$ for $k \in \mathbb{N}$, for some $\varrho \in C_0^\infty(\mathbb{R}^m)$ with $\|\varrho\|_{L_1(\mathbb{R}^m)} = 1$, one can check that if $u \in h^\tau(\mathbb{R}^m)$, then $\varrho_k * u \to u$ in $h^\tau(\mathbb{R}^m)$ for $k \to \infty$.

Now for $0 < \tau < \tau_1 < 1$,

$$C^{\tau_1}(\mathbb{R}^m) \hookrightarrow h^\tau(\mathbb{R}^m) \hookrightarrow C^\tau(\mathbb{R}^m), \tag{2.13}$$

so a function $u \in C^{\tau_1}(\mathbb{R}^m)$ is approximated in C^τ-norm by $\varrho_k * u$ for $k \to \infty$.

Similarly, one can check that when $p(x, \xi)$ is a symbol in $C^\tau S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N)$, then

$$\varrho_k(x) * p(x, \xi) \to p(x, \xi) \text{ in } C^{\tau'} S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N), \text{ when } 0 < \tau' < \tau < 1; \tag{2.14}$$

here $\varrho_k * p \in C_b^\infty S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N)$.

There are analogous definitions of symbol spaces where C^τ is replaced by H^r_q-spaces (Bessel-potential spaces), that we shall refer to in Section 5 below; ψdo's defined from such symbols were studied by Marschall in [29]. When $p(x, \xi) \in H^r_q S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N)$, then $\varrho_k * p \in C_b^\infty S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N)$, and

$$\varrho_k(x) * p(x, \xi) \to p(x, \xi) \text{ in } H^r_q S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N). \tag{2.15}$$

One can also define symbols valued in Banach spaces. Let X be a Banach space, then $C^\tau S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N; X)$ consists of the functions $p(x, \xi)$ from $(x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m$ to X such that

$$\|D_\xi^\alpha p(x, \xi)\|_{C^\tau(\mathbb{R}^m_x, X)} \text{ is } O(\langle\xi\rangle^{d-|\alpha|}) \text{ for } |\alpha| \le N, \tag{2.16}$$

where $C^\tau(\mathbb{R}^m_x, X)$ is provided with the norm in (1.1) with absolute values replaced by X-norms. X can in particular be a space of bounded linear operators $X = \mathcal{L}(X_0, X_1)$ between Banach spaces X_0, X_1. The use of special Fréchet spaces such as $\mathcal{S}_+$ in the place of X is discussed below in Section 3.

The following result was shown (in a greater generality) in Marschall [28] Th. 2.1:

Theorem 2.3. *For* $d \in \mathbb{R}$, $|s| < \tau$, *one has that when* $p(x, \xi) \in C^\tau S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N)$ *with* $N > m/2 + 1$, *then* $\mathrm{OP}(p)$ *is continuous*:

$$\|\mathrm{OP}(p)\|_{\mathcal{L}(H^{s+d}(\mathbb{R}^m), H^s(\mathbb{R}^m))} < \infty. \tag{2.17}$$

In other words, the linear map OP from the Banach space $C^\tau S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N)$ to the Banach space $\mathcal{L}(H^{s+d}(\mathbb{R}^m), H^s(\mathbb{R}^m))$ is bounded for each $|s| < \tau$, when $N > m/2 + 1$. Then also

$$\|\mathrm{OP}(p) - \mathrm{OP}(p_k)\|_{\mathcal{L}(H^{s+d}(\mathbb{R}^m), H^s(\mathbb{R}^m))} \to 0 \text{ for } k \to \infty, \text{ when } p_k = \varrho_k * p. \tag{2.18}$$

This holds for each $|s| < \tau$, since there is room for a $\tau' \in \,]0, \tau[$ such that $|s| < \tau'$, and the symbol convergence holds in $C^{\tau'} S^d_{1,0}$.

To show spectral asymptotic estimates, we shall use what is known in smooth cases and extend it to nonsmooth cases by use of suitable perturbation results for s-numbers:

Lemma 2.4.
1° *If* $s_j(B)j^{1/p} \to C_0$ *and* $s_j(B')j^{1/p} \to 0$ *for* $j \to \infty$, *then* $s_j(B+B')j^{1/p} \to C_0$ *for* $j \to \infty$.
2° *If* $B = B_M + B'_M$ *for each* $M \in \mathbb{N}$, *where* $s_j(B_M)j^{1/p} \to C_M$ *for* $j \to \infty$ *and* $s_j(B'_M)j^{1/p} \le c_M$ *for* $j \in \mathbb{N}$, *with* $C_M \to C_0$ *and* $c_M \to 0$ *for* $M \to \infty$, *then* $s_j(B)j^{1/p} \to C_0$ *for* $j \to \infty$.

The statement in 1° is the Weyl-Ky Fan theorem (cf. e.g. [12] Th. II 2.3), and 2° is a refinement shown in [15] Lemma 4.2.2°.

We have the following result for nonsmooth ψdo's on closed smooth manifolds.

Theorem 2.5. *Let E be an M-dimensional smooth vector bundle E over a smooth compact boundaryless m-dimensional manifold* Ξ. *Let* $t > 0$ *and* $0 < \tau < 1$, *and let P be a* C^τ-*smooth* ψ*do acting in E, with symbol defined in local trivializations from symbols* $p(x, \xi)$ *in* $C^\tau S^{-t}(\mathbb{R}^m \times \mathbb{R}^m) \otimes \mathcal{L}(\mathbb{C}^M)$. *Then the s-numbers of P satisfy the asymptotic estimate*

$$s_j(P)j^{t/m} \to c(p^0)^{t/m} \text{ for } j \to \infty, \tag{2.19}$$

where

$$c(p^0) = \frac{1}{m(2\pi)^m} \int_\Xi \int_{|\xi|=1} \operatorname{tr}\big((p^0(x, \xi)^* p^0(x, \xi))^{m/2t}\big)\, d\omega dx. \tag{2.20}$$

Proof. Using the approximation (2.14) in localizations, we can approximate P by a sequence of operators P_k with polyhomogeneous C^∞-symbols (locally in $C^\infty S^{-t}$), converging in the topology of symbols in $C^{\tau'} S^{-t}$ for $0 < \tau' < \tau$. Then the norm of $P - P_k$ in $\mathcal{L}(H^{-t}(\Xi), L_2(\Xi))$ goes to 0 for $k \to \infty$. The statement of the theorem holds for the P_k, as a corollary to Seeley [30], cf. [15], Lemma 4.5ff. Moreover, since $(P - P_k)^* \to 0$ in $\mathcal{L}(L_2(\Xi), H^t(\Xi))$,

$$\sup_j s_j(P - P_k)j^{t/m} \to 0 \text{ for } k \to \infty,$$

cf. (2.5). We also have that $c(p_k^0) \to c(p^0)$ for $k \to \infty$, since the symbol sequence converges in $C^{\tau'}$. Then the conclusion follows for P by Lemma 2.4 2°. □

3. Nonsmooth Pseudodifferential Boundary Operators

3.1. *Boundary Symbols with Hölder Smoothness*

We want to generalize the results of [15] on singular Green operators to nonsmooth symbols. This takes a larger effort, since the continuity from $H^{-t}(\Omega)$ to $L_2(\Omega)$, when G is of order $-t$ and class 0 on the manifold $\overline{\Omega}$ with boundary, only implies $G \in \mathfrak{S}_{n/t,\infty}$, not $G \in \mathfrak{S}_{(n-1)/t,\infty}$ as is known in the smooth case.

We denote by $\mathcal{S}_\pm = \mathcal{S}(\overline{\mathbb{R}}_\pm)$ the space of restrictions $r^\pm$ to $\overline{\mathbb{R}}_\pm$ of functions in $\mathcal{S}(\mathbb{R})$, and by $\mathcal{S}_{++} = \mathcal{S}(\overline{\mathbb{R}}^2_{++})$ the space of restrictions to $\overline{\mathbb{R}}^2_{++} = \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$ of functions in $\mathcal{S}(\mathbb{R}^2)$; here $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing C^∞-functions.

Singular Green operators of class 0 are defined in the smooth case on $\overline{\mathbb{R}}^n_+$ from functions $\tilde{g}(x', x_n, y_n, \xi') \in S^d_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathscr{S}_{++})$ by

$$Gu = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{g}(x', x_n, y_n, \xi') \mathscr{F}_{y' \to \xi'} u(y', y_n)\, dy_n đ\xi'.$$

The tool that we shall use to invoke the effect of the boundary dimension is a decomposition of the spaces $\mathscr{S}_+$ and $\mathscr{S}_{++}$ in terms of (a variant of) Laguerre functions, using expansions in such functions not only for the symbols but also on the operator level (Section 3.3).

Let us briefly recall the Laguerre expansions (introduced in [10], their role being further explained in [15, 16, 18]). Define

$$\text{when } k \geq 0,\ \varphi_k(x_n, \sigma) = \begin{cases} (2\sigma)^{\frac{1}{2}}(\sigma - \partial_{x_n})^k (x_n^k e^{-x_n\sigma})/k! & \text{for } x_n \geq 0, \\ 0 & \text{for } x_n < 0; \end{cases}.$$

$$\text{when } k < 0,\ \varphi_k(x_n, \sigma) = \varphi_{-k-1}(-x_n, \sigma).$$

Here σ can be any positive number. In the considerations of symbols, we shall take it as

$$\sigma(\xi') = [\xi'],\ \text{a smooth positive function of } \xi' \text{ that equals } |\xi'| \text{ for } |\xi'| \geq 1. \quad (3.1)$$

The functions form an orthonormal basis of $L_2(\mathbb{R})$; those with $k \geq 0$ span $L_2(\mathbb{R}_+)$, and those with $k < 0$ span $L_2(\mathbb{R}_-)$. Their Fourier transforms are

$$\hat{\varphi}_k(\xi_n, \sigma) = (2\sigma)^{\frac{1}{2}} \frac{(\sigma - i\xi_n)^k}{(\sigma + i\xi_n)^{k+1}},\ \text{where we denote } \hat{\varphi}'_k(\xi_n, \sigma) = \frac{(\sigma - i\xi_n)^k}{(\sigma + i\xi_n)^{k+1}}, \quad (3.2)$$

forming orthogonal bases of $L_2(\mathbb{R})$.

The point is that the $\varphi_k(x_n, \sigma)$, $k \geq 0$, belong to $\mathscr{S}_+$ and span its elements in rapidly decreasing series, in the sense that the following two systems of seminorms on the Fréchet space $\mathscr{S}_+$ are equivalent:

$$\begin{aligned} &\|x_n^l D_{x_n}^{l'} u(x_n)\|_{L_2(\mathbb{R}_+)},\ l, l' \in \mathbb{N}_0,\ \text{resp.} \\ &\|\{\langle j\rangle^M b_j\}\|_{\ell_2(\mathbb{N}_0)},\ M \in \mathbb{N}_0,\ \text{where } u(x_n) = \sum_{j\in\mathbb{N}_0} b_j \varphi_j(x_n, \sigma); \end{aligned} \quad (3.3)$$

cf. e.g. [16], Lemma 2.2.1, or [18], Lemma 10.14. This holds in such a way that an estimate in one of the systems is dominated by a fixed finite set of estimates in the other system. (A sequence $\{b_j\}$ is called rapidly decreasing, when $\sup_j \langle j\rangle^M |b_j| < \infty$ for all M.)

Similarly, the products $\varphi_j(x_n, \sigma)\varphi_k(y_n, \sigma)$ belong to $\mathscr{S}_{++}$, and here there is equivalence of the systems of seminorms

$$\begin{aligned} &\|x_n^l D_{x_n}^{l'} y_n^m D_{y_n}^{m'} u(x_n, y_n)\|_{L_2(\mathbb{R}^2_{++})},\ l, l', m, m' \in \mathbb{N}_0,\ \text{resp.} \\ &\|\{\langle j\rangle^M \langle k\rangle^{M'} c_{jk}\}\|_{\ell_2(\mathbb{N}_0\times\mathbb{N}_0)},\ M, M' \in \mathbb{N}_0,\ \text{where } u(x_n, y_n) \\ &\quad = \sum_{j,k\in\mathbb{N}_0} c_{jk}\varphi_j(x_n, \sigma)\varphi_k(y_n, \sigma), \end{aligned} \quad (3.4)$$

again with finite interdependence.

There are the following rules for differentiations in ξ' and ξ_n, when σ is as in (3.1) (then $\partial_{\xi_j}\sigma = \xi_j\sigma^{-1}$ for $|\xi'| \geq 1$):

$$\partial_{\xi_j}\hat{\varphi}_k(\xi_n, \sigma) = \big(k\hat{\varphi}_{k-1} - \hat{\varphi}_k - (k+1)\hat{\varphi}_{k+1}\big)(2\sigma)^{-1}\partial_{\xi_j}\sigma, \quad j < n,$$
$$\partial_{\xi_n}\hat{\varphi}_k(\xi_n, \sigma) = -i\big(k\hat{\varphi}_{k-1} + (2k+1)\hat{\varphi}_k + (k+1)\hat{\varphi}_{k+1}\big)(2\sigma)^{-1}. \tag{3.5}$$

Let us also recall that, with $e^{\pm}$ denoting extension by zero for $x_n \lessgtr 0$, Fourier transformation gives the space $\mathscr{H}^+ = \mathscr{F}_{x_n\to\xi_n}(e^+\mathscr{S}_+)$; its conjugate space is $\mathscr{H}^-_{-1} = \mathscr{F}_{x_n\to\xi_n}(e^-\mathscr{S}_-)$; and both are contained in $\mathscr{H} = \mathscr{H}^+\dot{+}\mathscr{H}^-_{-1}\dot{+}\mathbb{C}[t]$, where $\mathbb{C}[t]$ consists of the polynomials on $\mathbb{R}$. The projections of $\mathscr{H}$ to the components $\mathscr{H}^+$ resp. $\mathscr{H}^-_{-1}$ are denoted h^+ resp. h^-_{-1}. We refer to the indicated books for more information; this complex notation will not be important in the present paper.

We are now ready to give the technical definitions of the nonsmooth symbol spaces (that may be taken lightly in a first reading). Boundary operator symbols with C^τ-smoothness are defined as in Abels [1], and we in addition formulate the estimates in terms of Laguerre expansions.

Definition 3.1. *Let $d \in \mathbb{R}$ and $\tau > 0$. The space $C^\tau S^d_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{S}_+)$ consists of the functions $\tilde{f}(x', x_n, \xi')$ of $x', \xi' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}_+$, such that*

$$\|x_n^l D_{x_n}^{l'} D_{\xi'}^\alpha \tilde{f}(x', x_n, \xi')\|_{C^\tau(\mathbb{R}^{n-1}, L_2(\mathbb{R}_+))} \text{ is } O(\langle\xi'\rangle^{d+\frac12 - l + l' - |\alpha|})$$
$$\text{for } l, l' \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^{n-1}. \tag{3.6}$$

Equivalently, with $f = \mathscr{F}_{x_n\to\xi_n}(e^+\tilde{f})$,

$$\|h^+\xi_n^{l'} D_{\xi_n}^{l'} D_{\xi'}^\alpha f(x', \xi', \xi_n)\|_{C^\tau(\mathbb{R}^{n-1}, L_2(\mathbb{R}))} \text{ is } O(\langle\xi'\rangle^{d+\frac12 - l + l' - |\alpha|}) \text{ for } l, l' \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^{n-1}.$$

Likewise equivalently, with $\tilde{f}(x', x_n, \xi') = \sum_{k\in\mathbb{N}_0} b_k(x', \xi')\varphi_k(x_n, \sigma(\xi'))$,

$$\|\{\langle k\rangle^M D_{\xi'}^\alpha b_k(x', \xi')\}\|_{C^\tau(\mathbb{R}^{n-1}, \ell_2(\mathbb{N}_0))} \text{ is } O(\langle\xi'\rangle^{d+\frac12-|\alpha|}) \text{ for } M \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^{n-1}, \tag{3.7}$$

i.e., the coefficient sequence $\{b_k(x', \xi')\}_{k\in\mathbb{N}_0}$ is rapidly decreasing in $C^\tau S^{d+\frac12}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$.

Introduce also for $N \in \mathbb{N}_0$ the notation $C^\tau S^d_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, N; \mathscr{S}_+)$ for the space of functions $\tilde{f}(x', x_n, \xi')$ satisfying the estimates (3.6) for $l, l' \in \mathbb{N}_0$, $|\alpha| \leq N$; then $C^\tau S^d_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{S}_+) = \bigcap_{N\in\mathbb{N}_0} C^\tau S^d_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, N; \mathscr{S}_+)$.

The functions $\tilde{f}$ and f are said to be *polyhomogeneous*, when there moreover is an asymptotic expansion of f in functions $f_{d-j}(x', \xi)$ homogeneous of degree $d - j$ in (ξ', ξ_n) for $|\xi'| \geq 1$, in a similar way as in Definition 2.2. For the subspaces of polyhomogeneous symbols we use the notation $C^\tau S^d$ instead of $C^\tau S^d_{1,0}$.

The functions $\tilde{f}$ and f serve as symbol-kernels resp. symbols of *Poisson operators of order* $d + 1$, here they are usually denoted $\tilde{k}$ resp. k. First there is the definition of an operator with respect to the x_n-variable:

$$\mathrm{OPK}_n(\tilde{k})v = \tilde{k}(x', x_n, \xi') \cdot v, \text{ also denoted } k(x', \xi', D_n)v, \tag{3.8}$$

going from $\mathbb{C}$ to $\mathscr{S}_+$ for each (x', ξ'); this is the *boundary symbol operator*. Then the full operator is defined for $v \in \mathscr{S}(\mathbb{R}^{n-1})$ by using the pseudodifferential definition with respect to (x', ξ') (denoted OP$'$):

$$\mathrm{OPK}(\tilde{k})v = \mathrm{OP}'\mathrm{OPK}_n(\tilde{k})v = \int_{\mathbb{R}^{2n-2}} e^{i(x'-y')\cdot\xi'}\tilde{k}(x', x_n, \xi')v(y')\, dy'\, đ\xi'. \tag{3.9}$$

(One also writes $\mathrm{OPK}(\tilde{k})$ as $\mathrm{OPK}(k)$, and $\mathrm{OPK}_n(\tilde{k})$ as $\mathrm{OPK}_n(k)$.)

The same classes of functions $\tilde{f}$ serve as symbol-kernels for *trace operators of order d and class zero*; here they are usually denoted $\tilde{t}$. The associated symbol t is the conjugate Fourier transform of $e^+\tilde{t}$ in x_n, $t(x', \xi', \xi_n) = \int_0^\infty e^{ix_n\xi_n}\tilde{t}(x', x_n, \xi')\, dx_n = \overline{\mathscr{F}}_{x_n\to\xi_n} e^+\tilde{t}$. The definition of an operator with respect to the x_n-variable is, for $u \in \mathscr{S}_+$:

$$\mathrm{OPT}_n(\tilde{t})u = \int_0^\infty \tilde{t}(x', x_n, \xi')u(x_n)\, dx_n, \text{ also denoted } t(x', \xi', D_n)u, \tag{3.10}$$

going from $\mathscr{S}_+$ to $\mathbb{C}$ for each (x', ξ'); the boundary symbol operator. Then the full operator is defined for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+) = \mathscr{S}(\mathbb{R}^n)|_{\overline{\mathbb{R}}^n_+}$ by:

$$\mathrm{OPT}(\tilde{t})u = \mathrm{OP}'\mathrm{OPT}_n(\tilde{t})u = \int_{\mathbb{R}^{2n-2}} \int_0^\infty e^{i(x'-y')\cdot\xi'}\tilde{t}(x', y_n, \xi')u(y', y_n)\, dy_n dy'\, đ\xi'. \tag{3.11}$$

Definition 3.2. *Let $d \in \mathbb{R}$ and $\tau > 0$. The space $C^\tau S^d_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{S}_{++})$ consists of the functions $\tilde{g}(x', x_n, y_n, \xi')$ of $x', \xi' \in \mathbb{R}^{n-1}$, $x_n, y_n \in \mathbb{R}_+$, such that*

$$\|x_n^l D_{x_n}^{l'} y_n^m D_{y_n}^{m'} D_{\xi'}^\alpha \tilde{g}(x', x_n, y_n, \xi')\|_{C^\tau(\mathbb{R}^{n-1}, L_2(\mathbb{R}^2_{++}))} \text{ is } O(\langle\xi'\rangle^{d+1-l+l'-m+m'-|\alpha|}) \tag{3.12}$$

for $l, l', m, m' \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^{n-1}$.

Equivalently, with $g = \mathscr{F}_{x_n\to\xi_n}\overline{\mathscr{F}}_{y_n\to\eta_n}(e^+_{x_n}e^+_{y_n}\tilde{g})$,

$$\|h^+_{\xi_n} h^-_{-1,\eta_n} \xi_n^{l'} D^l_{\xi_n} \eta_n^{m'} D^m_{\eta_n} D^\alpha_{\xi'} g(x', \xi', \xi_n, \eta_n)\|_{C^\tau(\mathbb{R}^{n-1}, L_2(\mathbb{R}^2))} \text{ is } O(\langle\xi'\rangle^{d+1-l+l'-m+m'-|\alpha|}),$$

for $l, l', m, m' \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^{n-1}$.

Likewise equivalently, with $\tilde{g}(x', x_n, y_n, \xi') = \sum_{j,k\in\mathbb{N}_0} c_{jk}(x', \xi')\varphi_j(x_n, \sigma)\varphi_k(y_n, \sigma)$,

$$\|\{\langle j\rangle^M \langle k\rangle^{M'} D^\alpha_{\xi'} c_{jk}(x', \xi')\}\|_{C^\tau(\mathbb{R}^{n-1}, \ell_2(\mathbb{N}_0\times\mathbb{N}_0))} \text{ is } O(\langle\xi'\rangle^{d+1-|\alpha|}), \tag{3.13}$$

for $M, M' \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^{n-1}$, i.e., the coefficient sequence $\{c_{jk}(x', \xi')\}_{j,k\in\mathbb{N}_0}$ is rapidly decreasing in $C^\tau S^{d+1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$.

Introduce also the notation $C^\tau S^d_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, N; \mathscr{S}_{++})$ for the space of functions $\tilde{g}(x', x_n, y_n, \xi')$ satisfying (3.12) *for $l, l', m, m' \in \mathbb{N}_0$, $|\alpha| \le N$.*

The functions $\tilde{g}$ and g are said to be *polyhomogeneous*, when there moreover is an asymptotic expansion of g in functions $g_{d-j}(x', \xi', \xi_n, \eta_n)$ homogeneous of degree $d - j$ in (ξ', ξ_n, η_n) for $|\xi'| \ge 1$, in a similar way as in Definition 2.2.

The functions $\tilde{g}$ and g serve as symbol-kernels resp. symbols of *singular Green operators of order* $d+1$ *and class zero*. The definition of an operator with respect to the x_n-variable is

$$\mathrm{OPG}_n(\tilde{g})u = \int_0^\infty \tilde{g}(x', x_n, y_n, \xi')u(y_n)\,dy_n, \text{ also denoted } g(x', \xi', D_n)u, \quad (3.14)$$

acting in $\mathscr{S}_+$ for each (x', ξ'); the boundary symbol operator. Then the full operator is defined for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ by:

$$\mathrm{OPG}(\tilde{g})u = \mathrm{OP}'\mathrm{OPG}_n(\tilde{g})u = \int_{\mathbb{R}^{2n-2}} e^{i(x'-y')\cdot\xi'} \int_0^\infty \tilde{g}(x', x_n, y_n, \xi') \\ u(y', y_n)\,dy_n dy' đ\xi'. \quad (3.15)$$

The operators defined as in (3.9), (3.15) can for precision be said to be *in x'-form*, to distinguish this from the case where the functions $\tilde{k}, \tilde{t}, \tilde{g}$ in the integrals depend on y' in the place of x'. Such cases also define Poisson, trace and singular Green operators; the operators are said to be *in y'-form*, denoted $\mathrm{OPK}(\tilde{k}(y', x_n, \xi'))$, etc. Also (x', y')-forms can occur. The adjoint of the Poisson operator $\mathrm{OPK}(\tilde{f}(x', x_n, \xi'))$ is the trace operator $\mathrm{OPT}(\overline{\tilde{f}}(y', x_n, \xi'))$.

In the following we often leave out $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ from the notation for the symbol spaces.

For the consideration of the full operators as operator-valued ψdo's, the following properties of the boundary symbol operators will be very useful:

Lemma 3.3. *Let* $\tilde{k} \in C^\tau S^{d-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{S}_+)$, $\tilde{t} \in C^\tau S^{d}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{S}_+)$, *and* $\tilde{g} \in C^\tau S^{d-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{S}_{++})$. *Then the boundary symbol operators* $k(x', \xi', D_n)$, $t(x', \xi', D_n)$ *and* $g(x', \xi', D_n)$ (*cf.* (3.8), (3.10), (3.14)) *satisfy, for* $s', s'' \geq 0$:

$$\begin{aligned} k(x', \xi', D_n) &\in C^\tau S^{d-\frac{1}{2}+s'}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{L}(\mathbb{C}, H^{s'}(\mathbb{R}_+)), \\ t(x', \xi', D_n) &\in C^\tau S^{d+\frac{1}{2}+s'}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{L}(H_0^{-s'}(\overline{\mathbb{R}}_+), \mathbb{C})), \\ g(x', \xi', D_n) &\in C^\tau S^{d+s'+s''}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{L}(H_0^{-s'}(\overline{\mathbb{R}}_+), H^{s''}(\mathbb{R}_+)). \end{aligned} \quad (3.16)$$

Here, when $M \in \mathbb{N}_0$, *the Poisson boundary symbol operator seminorms*

$$\sup_{\xi'} \langle\xi'\rangle^{-d+\frac{1}{2}+|\alpha|-s'} \|D^\alpha_{\xi'} k(x', \xi', D_n)\|_{C^\tau(\mathbb{R}^{n-1}, \mathscr{L}(\mathbb{C}, H^{s'}(\mathbb{R}_+)))}, \quad |\alpha| \leq N, s' \leq M, \quad (3.17)$$

are dominated by the system of symbol seminorms

$$\sup_{\xi'} \langle\xi'\rangle^{-d+\frac{1}{2}+|\alpha|-l} \|D^l_{x_n} D^\alpha_{\xi'} \tilde{k}(x', x_n, \xi')\|_{C^\tau(\mathbb{R}^{n-1}, L_2(\mathbb{R}_+))}, \quad |\alpha| \leq N, l \leq M. \quad (3.18)$$

They are also dominated by the system of seminorms in terms of Laguerre expansions $\tilde{k}(x', x_n, \xi') = \sum_{j \in \mathbb{N}_0} b_j(x', \xi')\varphi_j(x_n, \sigma(\xi'))$:

$$\sup_{\xi'} \langle\xi'\rangle^{-d+\frac{1}{2}+|\alpha|} \|\{\langle j\rangle^{M+1} D^\alpha_{\xi'} b_j(x', \xi')\}_{j \in \mathbb{N}_0}\|_{C^\tau(\mathbb{R}^{n-1}, \ell_2(\mathbb{N}_0))}, \quad |\alpha| \leq N. \quad (3.19)$$

Likewise, the trace and singular Green boundary symbol operator seminorms

$$\begin{aligned}&\sup_{\xi'}\langle\xi'\rangle^{-d-\frac12+|\alpha|-s'}\|D^\alpha_{\xi'}t(x',\xi',D_n)\|_{C^\tau(\mathbb{R}^{n-1},\mathcal{L}(H_0^{-s'}(\overline{\mathbb{R}}_+),\mathbb{C}))},\quad |\alpha|\le N,\ s'\le M,\ \textit{resp.}\\&\sup_{\xi'}\langle\xi'\rangle^{-d+|\alpha|-s'-s''}\|D^\alpha_{\xi'}g(x',\xi',D_n)\|_{C^\tau(\mathbb{R}^{n-1},\mathcal{L}(H_0^{-s'}\overline{\mathbb{R}}_+,H^{s''}(\mathbb{R}_+)))},\quad |\alpha|\le N,\ s'\le M,\end{aligned}\tag{3.20}$$

are dominated by the systems of symbol seminorms

$$\begin{aligned}&\sup_{\xi'}\langle\xi'\rangle^{-d-\frac12+|\alpha|-l}\|D^l_{x_n}D^\alpha_{\xi'}\tilde t(x',x_n,\xi')\|_{C^\tau(\mathbb{R}^{n-1},L_2(\mathbb{R}_+))},\ |\alpha|\le N,\ l\le M,\ \textit{or}\\&\sup_{\xi'}\langle\xi'\rangle^{-d-\frac12+|\alpha|}\|\{\langle j^{M+1}D^\alpha_{\xi'}b_j(x',\xi')\}_{j\in\mathbb{N}_0}\|_{C^\tau(\mathbb{R}^{n-1},\ell_2(\mathbb{N}_0))},\ |\alpha|\le N,\ \textit{resp.}\\&\sup_{\xi'}\langle\xi'\rangle^{-d+|\alpha|-l}\|D^l_{x_n}D^m_{y_n}D^\alpha_{\xi'}\tilde g(x',x_n,y_n,\xi')\|_{C^\tau(\mathbb{R}^{n-1},L_2(\mathbb{R}^2_{++}))},\ |\alpha|\le N,\ l,m\le M,\ \textit{or}\\&\sup_{\xi'}\langle\xi'\rangle^{-d+|\alpha|}\|\{\langle(j,k)\rangle^{2M+1}D^\alpha_{\xi'}c_{jk}(x',\xi')\}_{j,k\in\mathbb{N}_0}\|_{C^\tau(\mathbb{R}^{n-1},\ell_2(\mathbb{N}_0^2))},\ |\alpha|\le N.\end{aligned}\tag{3.21}$$

Here $\tilde t=\sum_{j\in\mathbb{N}_0}b_j\varphi_j(x_n,\sigma)$ *and* $\tilde g=\sum_{j,k\in\mathbb{N}_0}c_{jk}\varphi_j(x_n,\sigma)\varphi_k(y_n,\sigma)$.

Proof. This is a variant of a result shown (in more generality including L_p-spaces and weighted norms) in Abels [1], Lemma 4.6, as a generalization of [21], Th. 3.7.

The present L_2-version is straightforward to show. Consider $\tilde k$. The estimates

$$\langle\xi'\rangle^{-d+\frac12-l}\|D^l_{x_n}\tilde k(x',x_n,\xi')\|_{C^\tau(\mathbb{R}^{n-1},L_2(\mathbb{R}_+))}=O(1),\quad l\le M,$$

imply

$$\langle\xi'\rangle^{-d+\frac12-M}\|\tilde k(x',x_n,\xi')\|_{C^\tau(\mathbb{R}^{n-1},H^M(\mathbb{R}_+))}=O(1),$$

so since $k(x',\xi',D_n)$ is multiplication by $\tilde k$, the finiteness of (3.17) follows for $\alpha=0$, $s'=M$. A similar treatment of $D^\alpha_{\xi'}\tilde k$ includes general α. Noninteger values $s'\in[0,M]$ are included by interpolation, since $H^{s'}$ interpolates between H^0 and H^M; here the symbol seminorms for $l\le M$ suffice for the estimates. This shows the statement on domination of the seminorms (3.17) by those in (3.18).

The second statement on domination follows from the fact that by [16], (2.2.20),

$$\langle\xi'\rangle^{-d+\frac12-l}\|D^l_{x_n}\tilde k(x',x_n,\xi')\|_{L_2(\mathbb{R}_+)}\le c\|\{\langle j\rangle^{l+\varepsilon}b_j(x',\xi')\}_{j\in\mathbb{N}_0}\|_{\ell_2(\mathbb{N}_0)}.$$

The proofs for the other types of operators are similar (for $\tilde t$ one can simply note that $t(x',\xi',D_n)$ is the adjoint of the Poisson operator $v\mapsto\bar{\tilde t}(x',x_n,\xi')v$, and $H_0^{-s'}(\overline{\mathbb{R}}_+)$ is the dual space of $H^{s'}(\mathbb{R}_+)$). □

One of the consequences derived in [1] is that the operators of order d have the Sobolev space continuities:

$$\begin{aligned}\mathrm{OPK}(\tilde k):&\ H^{s+d-\frac12}(\mathbb{R}^{n-1})\to H^s(\mathbb{R}^n_+)\text{ for }|s|<\tau,\\\mathrm{OPT}(\tilde t):&\ H^{s+d}(\mathbb{R}^n_+)\to H^{s-\frac12}(\mathbb{R}^{n-1})\text{ for }|s-\tfrac12|<\tau,\ s+d>-\tfrac12,\\\mathrm{OPG}(\tilde g):&\ H^{s+d}(\mathbb{R}^n_+)\to H^s(\mathbb{R}^n_+)\text{ for }|s|<\tau,\ s+d>-\tfrac12;\end{aligned}\tag{3.22}$$

when $\tilde{k}$, $\tilde{t}$ and $\tilde{g}$ are as in Lemma 3.3. We return to the proof below in Theorem 3.8.

Remark 3.4. The special case where the operators map from H^s-space to H^s-space for $|s| < \tau$, is particularly convenient in composition rules:

$$\begin{aligned} \mathrm{OPK}(\tilde{k}) &: \ H^s(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^n_+) \text{ for } |s| < \tau, \text{ when } d = \tfrac{1}{2}, \\ \mathrm{OPT}(\tilde{t}) &: \ H^s(\mathbb{R}^n_+) \to H^s(\mathbb{R}^{n-1}) \text{ for } |s| < \tau,\ s > -\tfrac{1}{2}, \text{ when } d = -\tfrac{1}{2}, \\ \mathrm{OPG}(\tilde{g}) &: \ H^s(\mathbb{R}^n_+) \to H^s(\mathbb{R}^n_+) \text{ for } |s| < \tau,\ s > -\tfrac{1}{2}, \text{ when } d = 0; \end{aligned} \tag{3.23}$$

here $\tilde{k}$ and $\tilde{t} \in C^\tau S^{-\frac{1}{2}}_{1,0}(\mathscr{S}_+)$, $\tilde{g} \in C^\tau S^{-1}_{1,0}(\mathscr{S}_{++})$. (For the statement on $\mathrm{OPT}(\tilde{t})$ we replaced $s - \frac{1}{2}$ in (3.22) by s.) We say that these operators are *of neutral order*.

The order conventions, introduced originally by Boutet de Monvel in [10], may seem a little confusing. They respect the principle that a composition of two operators of orders d_1 resp. d_2 is of order $d_1 + d_2$, but the order is not preserved when one passes from a Poisson operator of order d to its adjoint, which is a trace operator of order $d - 1$ and class 0. To compensate for this phenomenon, one might think of redefining the orders by adding or subtracting $\frac{1}{2}$, but this is not really helpful, since in considerations of mappings between L_p-based Sobolev spaces, the role of $\frac{1}{2}$ is taken over by $\frac{1}{p}$ and $\frac{1}{p'}$.

3.2. *Rules of Calculus*

Let us now recall some composition rules. We here let $a(x', \xi', D_n)$ play the role of any of the boundary symbol operators introduced above, such that the resulting full operator can be written as $A = \mathrm{OP}'a(x', \xi', D_n)$. We also include functions $s(x', \xi')$ that are symbols of ψdo's on $\mathbb{R}^{n-1}$, and on the boundary symbol level simply act as multiplications. The composition of two boundary symbol operators a_1 and a_2 is denoted $a_1 \circ_n a_2$. The compositions can of course only be applied when A_1 is defined on the range of A_2 (which can be a Sobolev space over $\mathbb{R}^{n-1}$ or $\mathbb{R}^n_+$). The notation $\tilde{a}_1 \circ_n \tilde{a}_2 = \tilde{a}_3$ is sometimes also used with the $\tilde{a}_i$ denoting the corresponding symbol-kernels. Full details are found in [16, 18] or [1]; some examples are:

$$\begin{aligned} \tilde{k} \circ_n \tilde{t} &= \tilde{k}(x', x_n, \xi')\tilde{t}(x', y_n, \xi'), \\ \tilde{t} \circ_n \tilde{k} &= \int_0^\infty \tilde{t}(x', x_n, \xi')\tilde{k}(x', x_n, \xi')\, dx_n, \\ \tilde{g} \circ_n \tilde{k} &= \int_0^\infty \tilde{g}(x', x_n, y_n, \xi')\tilde{k}(x', y_n, \xi')\, dy_n, \\ \tilde{t} \circ_n \tilde{g} &= \int_0^\infty \tilde{t}(x', x_n, \xi')\tilde{g}(x', x_n, y_n, \xi')\, dx_n, \\ \tilde{g} \circ_n \tilde{g}' &= \int_0^\infty \tilde{g}(x', x_n, z_n, \xi')\tilde{g}'(x', z_n, y_n, \xi')\, dz_n, \\ \tilde{k} \circ_n s &= \tilde{k}(x', x_n, \xi')s(x', \xi'), \\ s \circ_n \tilde{t} &= s(x', \xi')\tilde{t}(x', x_n, \xi'). \end{aligned} \tag{3.24}$$

As usual, $\mathrm{OP}'(a_1 \circ a_1)$ is a good approximation to A_1A_2; the following theorem gives more information on this for the Hölder-smooth symbol classes.

Theorem 3.5. *For $i = 1, 2$, let $\tilde{k}_i(x', x_n, \xi')$ be Poisson symbol-kernels of order $d_i + \frac{1}{2}$, let $\tilde{t}_i(x', x_n, \xi')$ be trace symbol-kernels of order $d_i - \frac{1}{2}$ and class 0, let $\tilde{g}_i(x', x_n, y_n, \xi')$ be singular Green symbol-kernels of order d_i and class 0, and let $s_i(x', \xi')$ be ψdo symbols on $\mathbb{R}^{n-1}$ of orders d_i, with τ_i-smoothness, respectively, defining operators K_i, T_i, G_i, S_i. Let $d = d_1 + d_2$ and $\tau = \min\{\tau_1, \tau_2\}$, and let $\theta \in]0, 1[$ with $\theta < \tau$. Then the composed operators satisfy*

$$\begin{aligned} &A_1A_2 - \mathrm{OP}'(a_1 \circ_n a_2)\colon H^{s+d-\theta} \to H^s, \\ &\textit{when } |s| < \tau,\ s > \theta - \tau_2,\ \theta - \tau_2 < s + d_1 < \tau_2; \end{aligned} \tag{3.25}$$

in addition the class condition $s + d - \theta > -\frac{1}{2}$ must be satisfied if A_2 is a trace or singular Green operator. (The H^t stand for Sobolev spaces over $\mathbb{R}^{n-1}$ or $\mathbb{R}^n_+$ depending on the context where they are used.)

The rule in the case where a_1 and a_2 are ψdo's on $\mathbb{R}^{n-1}$ is known from Taylor [31], Prop. I 2.1D. The other rules (on ψdbo's) are proved in Abels [1], Th. 4.13, by use of an extension of Taylor's result to vector-valued operators, cf. also [2]. The above statement differs from the formulation in [1] by referring to orders $d_i \pm \frac{1}{2}$ for Poisson and trace operators; this allows a unified formulation. One can also describe rules where a trace operator or singular Green operator contains standard traces γ_j (i.e., has positive class); they can be deduced from the above by combination with mapping properties of the γ_j.

We moreover need rules for compositions with ψdo's P on $\mathbb{R}^n$ satisfying the transmission condition at $x_n = 0$, and truncated to $\mathbb{R}^n_+$ as $P_+ = r^+Pe^+$ (where $r^\pm$ denotes restriction to $\mathbb{R}^n_\pm$, $e^\pm$ stands for extension by zero on $\mathbb{R}^n \setminus \mathbb{R}^n_\pm$). Here the composition rules are more complicated, already on the symbol level. Moreover, the symbol $p(x, \xi)$ of P is in general x_n-dependent, whereas the boundary symbol operator $p(x', \xi', D_n)_+$ is defined from the symbol at $x_n = 0$. When one of the factors A_i in a composition is of the form P_+, it is the true operator that enters as A_i and the x_n-independent boundary symbol operator that enters as a_i. It is proved in Section 5 of [1] that the statement in Theorem 3.5 is valid also for these cases:

Theorem 3.6. *The conclusion of Theorem 3.5 holds also when G_1 or G_2 is replaced by a C^{τ_i}-smooth ψdo $P_{i,+}$ of order d_i satisfying the transmission condition at $x_n = 0$.*

We here view the truncated ψdo's as being of class 0. There is a refinement allowing operators of negative class, but we shall not need it here and refer to the quoted works for details on this.

For a composition of two truncated ψdo's $P_{1,+}$ and $P_{2,+}$ one uses the formula

$$P_{1,+}P_{2,+} = (P_1P_2)_+ - L(P_1, P_2), \tag{3.26}$$

where the composition P_1P_2 follows a rule as for S_1S_2 in Theorem 3.5, and the singular Green term $L(P_1, P_2)$ is treated by use of other rules in Theorem 3.5, supplied with considerations of standard trace operators γ_j. The outcome is essentially as in (3.25) (more details are given in [1], Sect. 5.3).

For the purpose of spectral estimates we need a sharpening of the information in (3.22), (3.23), taking into account how many symbol seminorm estimates are needed for the operator norm estimates. This uses the following generalization of Marschall's result Theorem 2.3:

Theorem 3.7. *Let H_0 and H_1 be Hilbert spaces, and consider operator-valued symbols $p \in C^\tau S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N; \mathscr{L}(H_0, H_1))$, where the symbol seminorms* (2.8) *are taken in $C^\tau(\mathbb{R}^m, \mathscr{L}(H_0, H_1))$. For $N \geq m+1$, $\mathrm{OP}(p)$ is continuous*

$$\mathrm{OP}(p)\colon H^{s+d}(\mathbb{R}^m, H_0) \to H^s(\mathbb{R}^m, H_1) \text{ for } |s| < \tau.$$

The proof is given in the Appendix (written by Helmut Abels). We use the theorem to show that a fixed finite set of symbol seminorms suffices for estimates of operator norms.

Theorem 3.8. *Let $\tau \in]0, 1[$. Let $\tilde{k}$, $\tilde{t}$ and $\tilde{g}$ be as in Lemma* 3.3.

1° *For each $|s| < \tau$, the norm of $\mathrm{OPK}(\tilde{k})$ in* (3.22) *is bounded by a finite system of seminorms*:

$$\begin{aligned}&\|\mathrm{OPK}(\tilde{k})\|_{\mathscr{L}(H^{s+d-\frac{1}{2}}(\mathbb{R}^{n-1}), H^s(\mathbb{R}^n_+))}\\ &\leq C_s \sup_{\xi'} \langle \xi' \rangle^{|\alpha|-d+\frac{1}{2}} \|D^l_{x_n} D^\alpha_{\xi'} \tilde{k}\|_{C^\tau(\mathbb{R}^{n-1}, L_2(\mathbb{R}_+))}, \ l = 0, 1, \ |\alpha| \leq n. \end{aligned} \tag{3.27}$$

The symbol seminorms in (3.27) *are estimated by the following Laguerre symbol seminorms, where $\tilde{k}(x', x_n, \xi') = \sum_{k \in \mathbb{N}_0} b_k(x', \xi') \varphi_k(x_n, \sigma(\xi'))$,*

$$\sup_{\xi'} \langle \xi' \rangle^{|\alpha|-d+\frac{1}{2}} \|\{\langle k \rangle^2 D^\alpha_{\xi'} b_k\}\|_{C^r(\mathbb{R}^{n-1}, \ell_2(\mathbb{N}_0))}, \ |\alpha| \leq n. \tag{3.28}$$

2° *For each $|s' - \frac{1}{2}| < \tau$, resp. $|s| < \tau$, the norms of $\mathrm{OPT}(\tilde{t})$ and $\mathrm{OPG}(\tilde{g})$ in* (3.22) *are bounded by finite systems of seminorms*:

$$\begin{aligned}&\|\mathrm{OPT}(\tilde{t})\|_{\mathscr{L}(H^{s'+d}(\mathbb{R}^n_+), H^{s'-\frac{1}{2}}(\mathbb{R}^n_+))}\\ &\leq C_{s'} \sup_{\xi'} \langle \xi' \rangle^{|\alpha|-d-\frac{1}{2}} \|D^l_{x_n} D^\alpha_{\xi'} \tilde{t}\|_{C^\tau(\mathbb{R}^{n-1}, L_2(\mathbb{R}_+))}, \ l = 0, 1, \ |\alpha| \leq n, \\ &\|\mathrm{OPG}(\tilde{g})\|_{\mathscr{L}(H^{s+d}(\mathbb{R}^n_+), H^s(\mathbb{R}^n_+))}\\ &\leq C_s \sup_{\xi'} \langle \xi' \rangle^{|\alpha|-d} \|D^l_{x_n} D^m_{y_n} D^\alpha_{\xi'} \tilde{g}\|_{C^\tau(\mathbb{R}^{n-1}, L_2(\mathbb{R}^2_{++}))}, \ l, m = 0, 1, \ |\alpha| \leq n. \end{aligned} \tag{3.29}$$

The operator norms can also be estimated in terms of finite systems of Laguerre seminorms, as in Lemma 3.3.

Proof. The proof follows that of [1], Th. 4.8 in a simplified version, but taking the dependence on specific finite seminorm systems into account.

Consider $K = \mathrm{OPK}(\tilde{k}) = \mathrm{OP}'(k(x', \xi', D_n))$. By Lemma 3.3, $k(x', \xi', D_n)$ is in $C^\tau S^{d-\frac{1}{2}+s'}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{L}(\mathbb{C}, H^{s'}(\mathbb{R}_+))$ for all $s' \geq 0$. Applying Theorem 3.7 with

$N = (n-1) + 1 = n$, we see that K is continuous

$$K\colon H^{s+d-\frac{1}{2}+s'}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^{n-1}, H^{s'}(\mathbb{R}_+)) \tag{3.30}$$

for $|s| < \tau$, $s' \geq 0$. Here we let $s' \in [0, 1]$, then the operator norm is estimated by (3.18) or (3.19) with $N = n$ and $M = 1$. If $s \in [0, \tau[$, observe that

$$H^s(\mathbb{R}^n_+) = H^s(\mathbb{R}^{n-1}, H^0(\mathbb{R}_+)) \cap H^0(\mathbb{R}^{n-1}, H^s(\mathbb{R}_+)) \text{ for } s \geq 0, \tag{3.31}$$

and apply this to (3.30) with (s, s') of the form $(s, 0)$ and of the form $(0, s)$; then we find that

$$K\colon H^{s+d-\frac{1}{2}}(\mathbb{R}^{n-1}) \to H^s(\overline{\mathbb{R}}_+) \tag{3.32}$$

holds for $s \geq 0$. If $s \in]-\tau, 0]$, we obtain (3.32) by taking $s' = 0$ in (3.30) and using that

$$H^s(\mathbb{R}^{n-1}, H^0(\mathbb{R}_+)) \hookrightarrow H^s(\mathbb{R}^n_+) \text{ if } s \leq 0. \tag{3.33}$$

Since $s' \in [0, 1]$, the operator norm is estimated as asserted in the theorem.
The proofs for T and G are adapted from [1] Th. 4.8 in a similar way. □

Also rules in Theorems 3.4 and 3.5 may be sharpened to a dependence on only finitely many symbol seminorms, by an extension of results of Marschall [28] to vector-valued operators, but since we do not need them in the present paper, we shall not pursue this here.

Remark 3.9. Let us mention one more rule, namely that the difference between operators defined in x'-form resp. y'-form from the same symbol is of lower order; it is not included in [1, 2]. Since the result is not essential for the present work, we only indicate some ingredients in a proof: A special case of Corollary 4.6 of [28] is: If $a(x, \xi) \in C^\tau S^d_{1,0}(\mathbb{R}^m \times \mathbb{R}^m, N; \mathscr{L}(H_0, H_1))$, with $H_0 = H_1 = \mathbb{C}$, then

$$\mathrm{OP}(a(x, \xi)) - \mathrm{OP}(a(y, \xi))\colon H^{s+d-\theta}(\mathbb{R}^m, H_0) \to H^s(\mathbb{R}^m, H_1) \tag{3.34}$$

is bounded, provided that $N > \frac{3}{2}m + 2$, $\theta \in [0, 1]$, $\tau > \theta$, $|s| < \tau$ and $|s + d - \theta| < \tau$. (This allows d and $s + d$ to run in a small interval around 0.)

One needs a generalization of the statement to general Hilbert spaces H_0, H_1, and then one can prove by use of Lemma 3.3 that when $K = \mathrm{OPK}(\tilde{k}(x', x_n, \xi'))$ and $K' = \mathrm{OPK}(\tilde{k}(y', x_n, \xi'))$ are Poisson operators defined in x'-form resp. y'-form from a symbol $\tilde{k}(x', x_n, \xi') \in C^\tau S^{-\frac{1}{2}}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{S}_+)$, $\tau < 1$, then

$$K - K'\colon H^{s-\theta}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^n_+) \text{ for } -\tau + \theta < s < \tau. \tag{3.35}$$

There are similar statements for trace and singular Green operators.

3.3. *Laguerre Boundary Operators*

Let us introduce a notation for the special Poisson and trace operators on $\mathbb{R}^n_+$—Laguerre boundary operators—defined from Laguerre functions with $k \geq 0$ (as in Grubb and Schrohe [22]):

$$\Phi_k = \mathrm{OPK}(\varphi_k(x_n, \sigma(\xi'))), \quad \Phi_k^* = \mathrm{OPT}(\varphi_k(x_n, \sigma(\xi'))); \tag{3.36}$$

here Φ_k is a Poisson operator of order $\frac{1}{2}$, and the adjoint Φ_k^* is a trace operator of order $-\frac{1}{2}$ and class 0 (note that they are of neutral order as defined in Remark 3.4). The operator Φ_k maps $L_2(\mathbb{R}^{n-1})$ continuously, in fact isometrically, into $L_2(\mathbb{R}^n_+)$. Moreover, because of the orthonormality and completeness of the φ_k,

$$\Phi_j^*\Phi_k = \delta_{jk} I_{\mathbb{R}^{n-1}}, \quad \sum_{k\in\mathbb{N}_0} \Phi_k\Phi_k^* = I_{\mathbb{R}^n_+}, \tag{3.37}$$

where $I_{\mathbb{R}^{n-1}}$ resp. $I_{\mathbb{R}^n_+}$ stands for the identity operator on functions on $\mathbb{R}^{n-1}$ resp. $\mathbb{R}^n_+$. Denoting by H_j the range of Φ_j in $L_2(\mathbb{R}^n_+)$, we have that the H_j are mutually orthogonal closed subspaces, and that

$$L_2(\mathbb{R}^n_+) = \bigoplus_{j\in\mathbb{N}_0} H_j; \text{ with isometries } \Phi_j\colon L_2(\mathbb{R}^{n-1}) \xrightarrow{\sim} H_j,$$
$$\Phi_j^*\colon H_j \xrightarrow{\sim} L_2(\mathbb{R}^{n-1}), \tag{3.38}$$

the latter acting as an inverse of Φ_j.

Lemma 3.10. *When G is a singular Green operator on $\mathbb{R}^n_+$ of order d and class 0, defined from a symbol-kernel $\tilde{g}(x', \xi', x_n, y_n) \in C^\tau S^{d-1}_{1,0}(\mathbb{R}^n_+ \times \mathbb{R}^{n-1}; \mathscr{S}_{++})$, it can be written in the form*

$$G = \sum_{k\in\mathbb{N}_0} K_k\Phi_k^*, \quad K_k = \mathrm{OPK}(\tilde{k}_k(x', x_n, \xi')), \tag{3.39}$$

with Poisson operators K_k of order $d + \frac{1}{2}$, their symbol-kernels $(\tilde{k}_k)_{k\in\mathbb{N}_0}$ forming a rapidly decreasing sequence in $C^\tau S^{d-\frac{1}{2}}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{S}_+)$. Here

$$\tilde{k}_k(x', x_n, \xi') = \int_0^\infty \tilde{g}(x', x_n, y_n, \xi')\varphi_k(y_n, \sigma(\xi'))\, dy_n, \quad K_k = G\Phi_k. \tag{3.40}$$

Proof. The symbol-kernel $\tilde{g}$ of G has a Laguerre expansion:

$$g(x', x_n, y_n, \xi') = \sum_{j,k\in\mathbb{N}_0} c_{jk}(x', \xi')\varphi_j(x_n, \sigma(\xi'))\varphi_k(y_n, \sigma(\xi')), \tag{3.41}$$

with $(c_{jk})_{j,k\in\mathbb{N}_0}$ rapidly decreasing in $C^\tau S^d_{1,0}$, i.e., the relevant symbol seminorms on $\langle k\rangle^M \langle j\rangle^{M'} c_{jk}$ are bounded in j, k for any $M, M' \in \mathbb{N}_0$. Set

$$\tilde{k}_k(x', x_n, \xi') = \sum_{j\in\mathbb{N}_0} c_{jk}(x', \xi')\varphi_j(x_n, \sigma).$$

It is a rapidly decreasing sequence of symbols in $C^\tau S_{1,0}^{d-\frac{1}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathscr{S}_+)$, in view of the estimates of the c_{jk} and the (x'-independent) φ_j. Then

$$G = \sum_{k\in\mathbb{N}_0} \mathrm{OPK}\big(\tilde{k}_k(x', x_n, \xi')\big) \circ \mathrm{OPT}\big(\varphi_k(x_n, \sigma)\big) = \sum_{k\in\mathbb{N}_0} K_k \Phi_k^*,$$

where $K_k = \mathrm{OPK}(\tilde{k}_k)$. The Poisson operators K_k and their symbols satisfy (3.40) in view of the orthonormality of the φ_k and (3.37).

This shows the assertion. □

4. Spectral Estimates for Nonsmooth Singular Green Operators

4.1. *Upper Estimates*

We now turn to eigenvalue estimates for s.g.o.s. First we establish upper spectral estimates. (Both here and in the proof of Theorem 4.3 below, the ideas from Ch. 4 of [15] are used wherever convenient.) Let ψ and ψ_1 be functions in $C_0^\infty(\mathbb{R}^n)$, supported in a ball B_R. Denote $B_R \cap \overline{\mathbb{R}}^n_+ = B_{R,+}$, $B_R \cap (\mathbb{R}^{n-1} \times \{0\}) = B'_R$.

Theorem 4.1. *Let G be as in Lemma* 3.10, *of negative order $d = -t$. Then $\psi G \psi_1 \in \mathfrak{S}_{(n-1)/t,\infty}$. Moreover, when G is written as a series $\sum_{k\in\mathbb{N}_0} K_k \Phi_k^*$,*

$$\mathbf{N}_{(n-1)/t}(\psi G \psi_1) \le C \sum_{k\in\mathbb{N}_0} k^\gamma \mathbf{N}_{(n-1)/t}(\psi K_k \Phi_k^* \psi_1) < \infty, \tag{4.1}$$

where $\gamma = 0$ if $t < n-1$, $\gamma > t/(n-1) - 1$ if $t \ge n-1$.

Proof. The crucial step is to show that each $\psi K_k \Phi_k^* \psi_1$ is compact as an operator in $L_2(B_{R,+})$, with the desired estimate of s-numbers. To see this, note that K_k is bounded from $H^{-t}(\mathbb{R}^{n-1})$ to $L_2(\mathbb{R}^n_+)$, where ψK_k maps into functions supported in $B_{R,+}$, with a bound depending on the symbol estimates and ψ. Then the adjoint $K_k^*\psi$ is bounded from $L_2(B_{R,+})$ to $H^t(\mathbb{R}^{n-1})$. But we cannot conclude from this that ψK_k and $K_k^*\psi$ would be in $\mathfrak{S}_{(n-1)/t,\infty}$, since there is no support restriction in $\mathbb{R}^{n-1}$.

Instead we take ψ_1 into the picture, and moreover insert a cutoff function $\zeta \in C_0^\infty(\mathbb{R}^{n-1})$ that is 1 on B'_R, to the right of K_k. Write

$$\psi K_k \Phi_k^* \psi_1 = \psi K_k \zeta \Phi_k^* \psi_1 + \psi K_k (1-\zeta) \Phi_k^* \psi_1. \tag{4.2}$$

The first term will have the $\mathfrak{S}_{(n-1)/t,\infty}$-property, since $\psi K_k \zeta$ has it and the Φ_k^* are bounded from $L_2(\mathbb{R}^n_+)$ to $L_2(\mathbb{R}^{n-1})$ with norm 1. For the second term, we focus instead on $(1-\zeta)\Phi_k^*\psi_1$. As shown in [15], proof of Prop. 4.7, it is of order $-\infty$ in such a way that for its estimates as a mapping of a fixed low order, the symbol seminorms that enter have a polynomial growth in k (in view of the formulas (3.5)). Then for a fixed large a, the norm of $(1-\zeta)\Phi_k^*\psi_1$ from $H_0^{-a}(\overline{\mathbb{R}}^n_+)$ to $H^{-t}(\mathbb{R}^{n-1})$ grows at most polynomially with k. Let $\widetilde{B}$ be a smooth bounded open subset of $\mathbb{R}^n_+$ containing $B_{R,+}$. The operator $\psi K_k(1-\zeta)\Phi_k^*\psi_1$ is bounded from $H_0^{-a}(\overline{\widetilde{B}})$ to $L_2(\widetilde{B})$, and its adjoint is bounded from $L_2(\widetilde{B})$ to $H^a(\widetilde{B})$, hence belongs to $\mathfrak{S}_{n/a,\infty}$. We can take a so large that $n/a \le (n-1)/t$, then we can conclude that $\psi K_k \Phi_k^* \psi_1 \in \mathfrak{S}_{(n-1)/t,\infty}$.

We have shown in Theorem 3.8 that the norm on operators K_k from a fixed Sobolev space to another can be estimated in terms of a specific finite subset of the symbol seminorms. Then since the symbol seminorms of the K_k go to zero rapidly for $k \to \infty$, and the estimates of the operators $(1-\zeta)\Phi_k^*\psi_1$ grow at most polynomially in k, the quasinorms $\mathbf{N}_{(n-1)/t}(\psi K_k \Phi_k^* \psi_1)$ go to zero rapidly for $k \to \infty$, so the series in (4.1) is convergent. This completes the proof. □

Corollary 4.2. *When G is a C^τ-smooth singular Green operator of negative order $-t$ and class* 0 *on a bounded smooth subset Ω of $\mathbb{R}^n$ with boundary Σ, then $G \in \mathfrak{S}_{(n-1)/t,\infty}$.*

Proof. Let $\{\varrho_1, \dots, \varrho_{J_0}\}$ be a partition of unity subordinate to a relatively open cover $\{U_1, \dots, U_{I_1}\}$ of $\overline{\Omega}$, with coordinate maps $\kappa_i \colon U_i \to V_i$ to relatively open subsets V_i of $\overline{\mathbb{R}}^n_+$, such that any two $\varrho_{j_1}, \varrho_{j_2}$ have support in some U_i, $i = i(j_1, j_2)$. (More on partitions of unity subordinate to covers e.g. in [18], Lemma 8.4.) We can then write

$$G = \sum_{j_1, j_2 \le J_0} \varrho_{j_1} G \varrho_{j_2}, \tag{4.3}$$

where each piece $\varrho_{j_1} G \varrho_{j_2}$ carries over to an s.g.o. $\underline{\varrho}_{j_1} \underline{G} \underline{\varrho}_{j_2}$ on $\mathbb{R}^n_+$ of the type considered in Theorem 4.1, by the coordinate change for U_i. We apply Theorem 4.1 to the localized pieces. Since the coordinate change gives rise to L_2-bounded mappings (for compact subsets of U_i resp. V_i containing $\operatorname{supp}\varrho_{j_1} \cup \operatorname{supp}\varrho_{j_2}$ resp. $\operatorname{supp}\underline{\varrho}_{j_1} \cup \operatorname{supp}\underline{\varrho}_{j_2}$), we see in view of (2.4) that $\varrho_{j_1} G \varrho_{j_2} \in \mathfrak{S}_{p,\infty}$ if and only if $\underline{\varrho}_{j_1} \underline{G} \underline{\varrho}_{j_2} \in \mathfrak{S}_{p,\infty}$, with equivalent quasinorms $\mathbf{N}_p$. It follows that the operators $\varrho_{j_1} G \varrho_{j_2}$ are in $\mathfrak{S}_{(n-1)/t,\infty}$, then so is their sum by (2.2), and the quasinorm can be estimated in terms of the quasinorms of the localized pieces. □

4.2. *Asymptotic Estimates in Selfadjoint Cases*

Next, we shall show Weyl-type spectral estimates in polyhomogeneous selfadjoint cases.

Theorem 4.3. *Let G be a C^τ-smooth polyhomogeneous singular Green operator on $\mathbb{R}^n_+$ of order $-t < 0$ and class* 0, *selfadjoint and ≥ 0, and let $\psi(x) = \psi_0(x')\psi_n(x_n)$, where $\psi_0 \in C_0^\infty(\mathbb{R}^{n-1}, \mathbb{R})$, and $\psi_n \in C_0^\infty(\mathbb{R}, \mathbb{R})$ equals* 1 *on a neighborhood of* 0. *Then the eigenvalues of $\psi G \psi$ satisfy the asymptotic estimate*

$$\mu_j(\psi G \psi) j^{t/(n-1)} \to c(\psi_0^2 g^0)^{t/(n-1)} \text{ for } j \to \infty, \tag{4.4}$$

where

$$c(\psi_0^2 g^0) = \frac{1}{(n-1)(2\pi)^{(n-1)}} \int_{\mathbb{R}^{n-1}} \int_{|\xi'|=1} \operatorname{tr}((\psi_0(x')^2 g^0(x', \xi', D_n))^{(n-1)/t})\, d\omega dx'. \tag{4.5}$$

Proof. In addition to the decomposition of G as in Lemma 3.10 using Laguerre operators to the right, we can decompose by Laguerre operators to the left, writing G in the form

$$G = \sum_{l,m \in \mathbb{N}_0} \Phi_l C_{lm} \Phi_m^*, \text{ by defining } C_{lm} = \Phi_l^* G \Phi_m = \Phi_l K_m, \tag{4.6}$$

for each l, m. For with this definition,

$$\sum_{l,m\in\mathbb{N}_0} \Phi_l C_{lm} \Phi_m^* = \sum_{l,m\in\mathbb{N}_0} \Phi_l \Phi_l^* G \Phi_m \Phi_m^* = G,$$

in view of (3.37). Note that

$$C_{lm}^* = \Phi_m G^* \Phi_l^* = C_{ml},$$

since $G = G^*$.

Each C_{lm} is an operator of pseudodifferential type on $\mathbb{R}^{n-1}$, of the form $\Phi_l K_m$ with K_m as defined in Lemma 3.10; here since we compose with Φ_l to the left, we get from the composition rules that

$$\begin{aligned} &C_{lm} = C_{lm}^0 + C_{lm}', \quad C_{lm}^0 = \mathrm{OP}'(c_{lm}(x', \xi')) \text{ with} \\ &c_{lm} = \varphi_l(x_n, \sigma) \circ_n k_m(x', \xi', D_n) \in C^\tau S^{-t}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}), \\ &C_{lm}' : \ H^{s-t-\theta}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^{n-1}) \text{ for } |s| < \tau, -\tau + \theta < s < \tau. \end{aligned}$$

The symbol $c_{lm}(x', \xi')$ is a polyhomogeneous C^τ-smooth ψdo symbol on $\mathbb{R}^{n-1}$ of order $-t$.

We shall also consider finite sums

$$G_M = \sum_{l,m<M} \Phi_l C_{lm} \Phi_m^*.$$

Defining

$$\mathscr{K}_M = (\Phi_0 \quad \Phi_1 \quad \cdots \quad \Phi_{M-1}), \quad \mathscr{C}_M = \left(C_{lm}\right)_{l,m=0,\dots,M-1}, \tag{4.7}$$

a row matrix of Poisson operators, resp. an $M \times M$-matrix of ψdo-type operators, we can write

$$G_M = \mathscr{K}_M \mathscr{C}_M \mathscr{K}_M^*,$$

it ranges in $\bigoplus_{j<M} H_j$ and vanishes on $\left(\bigoplus_{j<M} H_j\right)^\perp$, cf. (3.37)–(3.38). Since

$$G_M = \mathscr{K}_M \mathscr{K}_M^* G \mathscr{K}_M \mathscr{K}_M^*,$$

the nonnegativity of G implies that of G_M; $\mathscr{C}_M$ is likewise ≥ 0.

Denote

$$G - G_M = G_M^\dagger, \quad G_M^\dagger = \sum_{l \text{ or } m\geq M} \Phi_l C_{lm} \Phi_m^*.$$

The first thing we show is that the contribution from $\psi G_M^\dagger \psi$ to the spectral asymptotics of $\psi G \psi$ comes from a contributing term where the constant goes to 0 for $M \to \infty$, plus a term in a better weak Schatten class. Write $\sum_{l \text{ or } m\geq M} = \sum_{l\in\mathbb{N}_0, m\geq M} + \sum_{l\geq M, m<M}$, and note that

$$\sum_{l\in\mathbb{N}_0, m\geq M} \Phi_l C_{lm} \Phi_m^* = \sum_{m\geq M} K_m \Phi_m^*,$$

and, since $C_{lm}^* = C_{ml}$,

$$(\sum_{l\geq M, m<M} \Phi_l C_{lm} \Phi_m^*)^* = \sum_{l\geq M, m<M} \Phi_m C_{lm}^* \Phi_l^* = \sum_{m<M, l\geq M} \Phi_m \Phi_m^* K_l \Phi_l^* = \Psi_M \sum_{l\geq M} K_l \Phi_l^*,$$

with $\Psi_M = \sum_{m<M} \Phi_m \Phi_m^* = \mathscr{K}_M \mathscr{K}_M^*$ a selfadjoint projection singular Green operator in $L_2(\mathbb{R}_+^n)$. Thus

$$G_M^\dagger = \sum_{l \text{ or } m\geq M} \Phi_l C_{lm} \Phi_m^* = \sum_{m\geq M} K_m \Phi_m^* + (\Psi_M \sum_{m\geq M} K_m \Phi_m^*)^*.$$

By the calculations in the proof of Theorem 4.1,

$$\mathbf{N}_{(n-1)/t}(\psi \sum_{m\geq M} K_m \Phi_m^* \psi) \leq C \sum_{m\geq M} m^\gamma \mathbf{N}_{(n-1)/t}(\psi K_m \Phi_m^* \psi) \to 0 \text{ for } M\to\infty.$$

This takes care of the first term in $\psi G_M^\dagger \psi$, and for the last term we insert a cutoff function η, equal to 1 on a ball containing $\operatorname{supp}\psi$, leading to:

$$\psi \Psi_M K_m \Phi_m^* \psi = \psi \Psi_M \eta K_m \Phi_m^* \varphi + \eta \psi \Psi_M (1-\eta) K_m \Phi_m^* \psi.$$

Here the first term defines a sequence that is rapidly decreasing in m with respect to the $\mathbf{N}_{(n-1)/t}$-quasinorm, since $\psi\Psi_M$ has bounded L_2-norm independent of M. So the $\mathbf{N}_{(n-1)/t}$-quasinorm of the sum over $m \geq M$ goes to 0 for $M \to \infty$. In the second term, $\psi\Psi_M(1-\eta)$ is of order $-\infty$ and the norm estimates from H^{-a} to L_2 are at most polynomially growing in M for each fixed a, so the summation in m is in $\mathfrak{S}_{p,\infty}$ for all $p > 0$. Thus

$$\psi G_M^\dagger \psi = G_{M,1,\psi}^\dagger + G_{M,2,\psi}^\dagger, \tag{4.8}$$

where $\mathbf{N}_{(n-1)/t}(G_{M,1,\psi}^\dagger) \to 0$ for $M \to \infty$, and $G_{M,2,\psi}^\dagger \in \bigcap_{p>0} \mathfrak{S}_{p,\infty}$ for all M. We view the operators as perturbations of $\psi G_M \psi$, where Lemma 2.4 1° can be used for $G_{M,2,\psi}$, and Lemma 2.4 2° will later be used for $G_{M,1,\psi}$.

Now consider $\psi G_M \psi$. Let B_R' be a ball containing $\operatorname{supp}\psi_0$, and let $\zeta \in C_0^\infty(\mathbb{R}^{n-1})$ be such that $\zeta(x') = 1$ on $\overline{B_R'}$. Then $\psi_0 = \zeta\psi_0$, and we write

$$\psi G_M \psi = \sum_{l,m<M} \psi \Phi_l \zeta C_{lm} \Phi_m^* \psi + \sum_{l,m<M} \psi \Phi_l (1-\zeta) C_{lm} \Phi_m^* \psi.$$

Here each $\psi\Phi_l(1-\zeta)$ is of order $-\infty$, so $\psi\Phi_l(1-\zeta)C_{lm}\Phi_m^*\psi$ is continuous from $H^{-t}(\mathbb{R}_+^n)$ to $H^a(\mathbb{R}_+^n)$ for any $a \in \mathbb{R}$, hence is in $\mathfrak{S}_{p,\infty}$ for all $p > 0$ (note the compactly supported factor ψ to the right). Similarly, we can insert ζ between C_{lm} and Φ_m^*, making an error in $\bigcap_{p>0} \mathfrak{S}_{p,\infty}$, and this leaves us with having to show the asymptotic estimate for

$$\sum_{l,m<M} \psi \Phi_l \zeta C_{lm} \zeta \Phi_m^* \psi.$$

As a next step, we observe that

$$(1-\psi_n(x_n))\psi'(x')\Phi_l\zeta C_{lm}\zeta\Phi_m^*\psi$$

has the factor $(1-\psi_n)\Phi_l$ which is also of order $-\infty$, so that the operator it enters in, contributes with a term in $\bigcap_{p>0}\mathfrak{S}_{p,\infty}$. Then ψ_n can be omitted from the operator we have to study. The same holds to the right. In both cases we use that a factor ζ inside the expression assures that the composed map passes via a compact set.

Now the problem is reduced to studying a sum of terms

$$\psi_0\Phi_l\zeta C_{lm}\zeta\Phi_m^*\psi_0.$$

Here we note that

$$\psi_0\Phi_l\zeta C_{lm}\zeta\Phi_m^*\psi_0 = [\psi_0, \Phi_l]\zeta C_{lm}\zeta\Phi_m^*\psi_0 + \Phi_l\psi_0 C_{lm}\zeta\Phi_m^*\psi_0$$

since $\psi_0\zeta = \psi_0$; here the commutator $[\psi_0, \Phi_l]$ is of 1 step lower order than Φ_l, so the operator is in $\mathfrak{S}_{(n-1)/(t+1),\infty}$ and will not contribute to the asymptotic formula. Doing a similar commutation to the right, we end up with having to study

$$G_{M,\psi_0} = \sum_{l,m<M}\Phi_l\psi_0 C_{lm}\psi_0\Phi_m^*.$$

With the notation (4.7),

$$G_{M,\psi_0} = \mathcal{K}_M\mathcal{C}_{M,\psi_0}\mathcal{K}_M^*, \text{ where } \mathcal{C}_{M,\psi_0} = \big(\psi_0 C_{lm}\psi_0\big)_{l,m=0,\dots,M-1}.$$

The operator is selfadjoint ≥ 0, so we can use that eigenvalues are preserved under commutation, to calculate:

$$\begin{aligned} s_j(G_{M,\psi_0}) &= \mu_j(G_{M,\psi_0}) = \mu_j(\mathcal{K}_M\mathcal{C}_{M,\psi'}\mathcal{K}_M^*) = \\ &\mu_j(\mathcal{C}_{M,\psi_0}\mathcal{K}_M^*\mathcal{K}_M) = \mu_j(\mathcal{C}_{M,\psi'}) = s_j(\mathcal{C}_{M,\psi_0}), \end{aligned} \tag{4.9}$$

since $\mathcal{K}_M^*\mathcal{K}_M$ equals the identity matrix in view of (3.37).

We have now arrived at the consideration of the operator of ψdo-type $\mathcal{C}_{M,\psi_0}$. It is of the form

$$\mathcal{C}_{M,\psi_0} = \mathrm{OP}'(c_{M,\psi_0}) + R, \quad c_{M,\psi_0} = (\psi_0(x')^2 c_{lm}(x',\xi'))_{l,m<M},$$

where R: $H^{-t-\theta}(\mathbb{R}^{n-1}) \to H^0(B'_R)$, hence is in $\mathfrak{S}_{(n-1)/(t+\theta),\infty}$ (for some $\theta > 0$). A spectral asymptotics formula is obtained for $\mathrm{OP}'(c_{M,\psi_0})$ by application of Theorem 2.5, and it extends to $\mathcal{C}_{M,\psi_0}$ by Lemma 2.4 1°, giving:

$$s_j(\mathcal{C}_{M,\psi_0})j^{t/(n-1)} \to c(c^0_{M,\psi_0})^{t/(n-1)} \text{ for } j\to\infty, \tag{4.10}$$

where the constant is defined as in (2.20) and c^0_{M,ψ_0} is the principal part of c_{M,ψ_0}. Hence $s_j(G_{M,\psi_0})$ likewise has this behavior (cf. (4.9)), and so has $s_j(\psi G_M\psi)$, since $G_{M,\psi_0} - \psi G_M\psi$ was shown above to be in $\mathfrak{S}_{(n-1)/(t+\varepsilon),\infty}$ for some $\varepsilon > 0$.

Since $\operatorname{tr} c^0_{M,\psi_0}(x',\xi') = \sum_{m<M} \psi_0^2 c^0_{mm}(x',\xi') = \operatorname{tr}\psi_0^2 g^0(x',\xi',D_n)$ (since these symbols are related in a similar way as the operators), and a similar rule holds for traces of powers of these nonnegative selfadjoint operators, we have that

$$c(c^0_{M,\psi_0}) = c(\psi_0^2 g^0_M)$$

defined as in (4.5) from the symbol of $\psi_0 G_M \psi_0$. So we can now write

$$s_j(\psi G_M \psi) j^{t/(n-1)} \to c(\psi_0^2 g^0_M)^{t/(n-1)} \text{ for } j \to \infty. \tag{4.11}$$

Finally we combine this with the information we have on $G^\dagger_M$. It gives that

$$\psi G \psi = \psi G_M \psi + G^\dagger_{M,1,\psi_0} + G^\dagger_{M,2,\psi_0},$$

where $\mathbf{N}_{(n-1)/t}(G^\dagger_{M,1,\psi}) \to 0$ for $M \to \infty$, and $G^\dagger_{M,2,\psi} \in \bigcap_{p>0} \mathfrak{S}_{p,\infty}$ for all M. By Lemma 2.4 1°, $\psi G_M \psi + G^\dagger_{M,2,\psi'}$ likewise has the spectral asymptotic behavior (4.11). Moreover, $c(\psi_0^2 g^0_M) \to c(\psi_0^2 g^0)$ for $M \to \infty$ (using that $g^0_M \to g^0$ in the appropriate symbol norms). Then since $\sup_j s_j(G^\dagger_{M,1,\psi_0}) j^{t/(n-1)} \to 0$ for $M \to \infty$, an application of Lemma 2.4 2° gives (4.4) with (4.5). □

We have a similar result in the curved domain situation:

Theorem 4.4. *Let G be a selfadjoint nonnegative C^τ-smooth polyhomogeneous singular Green operator of negative order $-t$ and class 0 on a bounded smooth subset Ω of $\mathbb{R}^n$ with boundary Σ. Then the positive eigenvalues of G satisfy an asymptotic estimate*

$$\mu_j(G) j^{t/(n-1)} \to c(g^0)^{t/(n-1)} \textit{ for } j \to \infty, \tag{4.12}$$

where

$$c(g^0) = \tfrac{1}{(n-1)(2\pi)^{(n-1)}} \int_\Sigma \int_{|\xi'|=1} \operatorname{tr}(g^0(x',\xi',D_n)^{(n-1)/t})\, d\omega dx'. \tag{4.13}$$

Proof. On a tubular neighborhood of Σ we can construct a family of C^∞-smooth Poisson operators $\widetilde{\Phi}_l$, $l \in \mathbb{N}_0$, such that their symbol-kernels at Σ in local coordinates have principal part equal to $\varphi_l(x_n, \sigma(\xi'))$ at each $x' \in \Sigma$.

Define

$$\begin{aligned} \widetilde{C}_{lm} &= \widetilde{\Phi}_l^* G \widetilde{\Phi}_m \text{ for } l, m \in \mathbb{N}_0, \\ \widetilde{G}_M &= \sum_{l,m<M} \widetilde{\Phi}_l \widetilde{C}_{lm} \widetilde{\Phi}_m^*, \\ \widetilde{G}^\dagger_M &= G - \widetilde{G}_M. \end{aligned} \tag{4.14}$$

Note that $\widetilde{C}^*_{lm} = \widetilde{C}_{ml}$, since G is selfadjoint. We shall show that these operators act in a similar way as in the case studied in Theorem 4.3.

First consider $\widetilde{G}_M$. Defining

$$\widetilde{\mathcal{K}}_M = (\widetilde{\Phi}_0 \quad \widetilde{\Phi}_1 \quad \cdots \widetilde{\Phi}_{M-1}), \quad \widetilde{\mathcal{C}}_M = \big(\widetilde{C}_{lm}\big)_{l,m=0,\dots,M-1}, \tag{4.15}$$

a row matrix of Poisson operators, resp. an $M \times M$-matrix of ψdo-type operators, we can write

$$\widetilde{G}_M = \widetilde{\mathcal{K}}_M \widetilde{\mathcal{C}}_M \widetilde{\mathcal{K}}^*_M,$$

Then since it is selfadjoint ≥ 0:

$$s_j(\widetilde{G}_M) = \mu_j(G_M) = \mu_j(\widetilde{\mathcal{K}}_M \widetilde{\mathcal{C}}_M \widetilde{\mathcal{K}}^*_M) = \mu_j(\widetilde{\mathcal{C}}_M \widetilde{\mathcal{K}}^*_M \widetilde{\mathcal{K}}_M).$$

Here $\widetilde{\mathcal{C}}_M$ is selfadjoint nonnegative, since for $\underline{u} = (u_0, \dots, u_{M-1}) \in L_2(\Sigma)^M$,

$$(\widetilde{\mathcal{C}}_M \underline{u}, \underline{u})_{L_2(\Sigma)^M} = \sum_{l,m<M} (\widetilde{C}_{lm} u_m, u_l)_{L_2(\Sigma)} = \sum_{l,m<M} (G \widetilde{\Phi}^*_m u_m, \widetilde{\Phi}^*_l u_l)_{L_2(\Omega)} \geq 0.$$

Moreover, $\widetilde{\mathcal{K}}^*_M \widetilde{\mathcal{K}}_M$ a selfadjoint nonnegative $M \times M$-matrix of C^∞-smooth ψdo's with principal part I_M (the identity matrix). Since it is elliptic nonnegative, its square root P_M is well-defined (Seeley [30]). Then we can continue the spectral calculation as follows:

$$\mu_j(\widetilde{\mathcal{C}}_M \widetilde{\mathcal{K}}^*_M \widetilde{\mathcal{K}}_M) = \mu_j(\widetilde{\mathcal{C}}_M P_M^2) = \mu_j(P_M \widetilde{\mathcal{C}}_M P_M) = \mu_j(\widetilde{\mathcal{C}}_M + R) = s_j(\widetilde{\mathcal{C}}_M + R),$$

where R is selfadjoint of the form $R = \widetilde{\mathcal{C}}_M(P_M - I_M) + (P_M - I_M)\widetilde{\mathcal{C}}_M P_M$. Since $\widetilde{\mathcal{C}}_M \in \mathfrak{S}_{(n-1)/t,\infty}$, $P_M - I_M \in \mathfrak{S}_{(n-1),\infty}$ and P_M is bounded, $R \in \mathfrak{S}_{(n-1)/(t+1),\infty}$ by (2.3), (2.4). To $\widetilde{\mathcal{C}}_M$ we can apply Theorem 2.4 getting a spectral asymptotics formula, and to $\widetilde{\mathcal{C}}_M + R$ we can apply the perturbation rule Lemma 2.4 1° for s-numbers. This gives, in view of the above identifications of eigenvalues:

$$s_j(\widetilde{G}_M) j^{t/(n-1)} \to c(c^0_M)^{t/(n-1)} \text{ for } j \to \infty, \tag{4.16}$$

where the constant is defined as in (2.20) and c^0_M denotes the principal symbol of $\widetilde{\mathcal{C}}_M$.

Now we turn to the study of $\widetilde{G}^\dagger_M$. Here we use a localization as in the proof of Corollary 4.2, leading to the study of finitely many localized pieces

$$\widetilde{G}^\dagger_{M,i,j} = \underline{\varrho}_i(\underline{G} - \sum_{l,m<M} \underline{\widetilde{\Phi}}_l \underline{\widetilde{C}}_{lm} \underline{\widetilde{\Phi}}^*_m)\underline{\varrho}_j,$$

acting in $\mathbb{R}^n_+$. Consider one of these pieces. We can assume that $\underline{G}$ is defined on all of $\mathbb{R}^n_+$ although it only enters with cut-off functions around it. Write

$$\underline{G} = \sum_{l,m\in\mathbb{N}_0} \Phi_l \underline{C}_{lm} \Phi^*_m, \quad \underline{G}^\dagger_M = \sum_{l \text{ or } m \geq M} \Phi_l \underline{C}_{lm} \Phi^*_m,$$

as in the proof of Theorem 4.3, then

$$\begin{aligned}\widetilde{G}^\dagger_{M,i,j} &= \underline{\varrho}_i(\sum_{l,m\in\mathbb{N}_0} \Phi_l \underline{C}_{lm} \Phi^*_m - \sum_{l,m<M} \underline{\widetilde{\Phi}}_l \underline{\widetilde{C}}_{lm} \underline{\widetilde{\Phi}}^*_m)\underline{\varrho}_j \\ &= \underline{\varrho}_i(\underline{G}^\dagger_M + \sum_{l,m<M} (\Phi_l \underline{C}_{lm} \Phi^*_m - \underline{\widetilde{\Phi}}_l \underline{\widetilde{C}}_{lm} \underline{\widetilde{\Phi}}^*_m))\underline{\varrho}_j.\end{aligned}$$

Here

$$\underline{\varrho}_i(\Phi_l \underline{C}_{lm} \Phi_m^* - \underline{\widetilde{\Phi}}_l \underline{\widetilde{C}}_{lm} \underline{\widetilde{\Phi}}_m^*)\underline{\varrho}_j = \underline{\varrho}_i((\Phi_l - \underline{\widetilde{\Phi}}_l)\underline{C}_{lm}\Phi_m^* + \underline{\widetilde{\Phi}}_l(\underline{C}_{lm} - \underline{\widetilde{C}}_{lm})\Phi_m^*$$
$$\underline{\widetilde{\Phi}}_l \underline{\widetilde{C}}_{lm}(\Phi_m^* - \underline{\widetilde{\Phi}}_m^*))\underline{\varrho}_j,$$

which is in $\mathfrak{S}_{(n-1)/(t+\theta),\infty}$ with $\theta > 0$. This is seen by inspecting each term, e.g.,

$$\underline{\varrho}_i(\Phi_l - \underline{\widetilde{\Phi}}_l)\underline{C}_{lm}\Phi_m^* \underline{\varrho}_j = \underline{\varrho}_i(\Phi_l - \underline{\widetilde{\Phi}}_l)\zeta \underline{C}_{lm}\Phi_m^* \underline{\varrho}_j + \underline{\varrho}_i(\Phi_l - \underline{\widetilde{\Phi}}_l)(1-\zeta)\underline{C}_{lm}\Phi_m^* \underline{\varrho}_j,$$

with $\zeta \in C_0^\infty(\mathbb{R}^{n-1})$ equal to 1 on B'_R, $B_R \supset \operatorname{supp} \underline{\varrho}_i$, where the first term is composed of operators in $\mathfrak{S}_{n-1,\infty}$ and $\mathfrak{S}_{(n-1)/t,\infty}$, and the second term goes from $H^{-t}(\mathbb{R}^n_+)$ to $H^a(B_{R,+})$, any $a > 0$. We also use that $\underline{C}_{lm} - \underline{\widetilde{C}}_{lm}$ is of order $-t-1$, since the two operators have the same principal symbol.

Finally, $\underline{\varrho}_i \underline{G}_M^\dagger \underline{\varrho}_j$ is of the type already analyzed in the proof of Theorem 4.3;

$$\underline{\varrho}_i \underline{G}_M^\dagger \underline{\varrho}_j = \underline{\varrho}_i \underline{G}_{M,1}^\dagger \underline{\varrho}_j + \underline{\varrho}_i \underline{G}_{M,2}^\dagger \underline{\varrho}_j,$$

where $\underline{\varrho}_i \underline{G}_{M,2}^\dagger \underline{\varrho}_j \in \mathfrak{S}_{(n-1)/(t+\theta),\infty}$ and $\mathbf{N}_{(n-1)/t}(\underline{\varrho}_i \underline{G}_{M,1}^\dagger \underline{\varrho}_j)$ goes to 0 for $M \to \infty$. Carrying the operators back to Ω and summing over i, j, we find that

$$\widetilde{G}_M^\dagger = \widetilde{G}_{M,1}^\dagger + \widetilde{G}_{M,2}^\dagger$$

with similar properties. The proof can now be completed in the same way as the proof of Theorem 4.3. □

For a nonselfadjoint singular Green operator G one gets the result of Theorem 4.3 when G^*G can be brought on the form $G_1 + R$ with G_1 as in Theorem 4.3 and R of order $< -tn/(n-1)$. This holds for τ large, in particular when G is C^∞-smooth, and hereby provides another proof of Theorem 4.10 in [15].

5. Results for Boundary Terms Arising from Resolvents

5.1. *The Dirichlet Resolvent*

As we saw in Section 4, a severe difficulty in the discussion of spectral asymptotics formulas for C^τ-smooth singular Green operators of order $-t$ is that although the remainders in compositions are of lower order, say $-t-\theta$, they only gain an $\mathfrak{S}_{n/(t+\theta),\infty}$-estimate, which is generally not better than the estimate in $\mathfrak{S}_{(n-1)/t,\infty}$ one is aiming for.

In the treatment of singular Green terms coming from resolvents of differential boundary problems, we shall use another strategy that is based on the special product form of the s.g.o.-terms. This will even allow nonselfadjointness.

Consider a second-order strongly elliptic differential operator A in divergence form

$$A = -\sum_{j,k=1}^n \partial_j a_{jk}(x)\partial_k + \sum_{j=1}^n a_j(x)\partial_j + a_0(x), \tag{5.1}$$

where the a_{jk} and a_j are functions in $H_q^1(\Omega) = \{u \in L_q(\Omega) \mid Du \in L_q(\Omega)^n\}$ and $a_0 \in L_q(\Omega)$, for some $q > n$. Let $\tau \in]0, 1 - n/q]$; recall that $H_q^1(\Omega) \hookrightarrow C^\tau(\overline{\Omega})$.

In the present section we consider a smooth Ω; this will be relaxed in Section 6 where there are additional bounds on τ adapted to the boundary regularity as in [4].

It follows from [4] that the solution operator for the nonhomogeneous Dirichlet problem for A (which exists uniquely, by a variational construction, when a sufficiently large constant has been added to A) maps as follows:

$$\mathcal{A} = \begin{pmatrix} A \\ \gamma_0 \end{pmatrix} \text{ has the inverse } \mathcal{B} = \begin{pmatrix} R_\gamma & K_\gamma \end{pmatrix} : \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix} \to H^{s+2}(\Omega),$$
$$\text{for } -\tau < s \leq 0, \tag{5.2}$$

belonging to the nonsmooth ψdbo calculus in the sense that R_γ and K_γ have the form of C^τ-smooth ψdbo's composed with order-reducing operators, plus lower-order remainders. Moreover, K_γ extends to a bounded mapping from $H^{s'-\frac{1}{2}}(\Sigma)$ to $H^{s'}(\Omega)$ for $s' \in [0, 2]$, belonging to the calculus in a suitable sense.

Furthermore, when A is extended to a uniformly strongly elliptic operator on $\mathbb{R}^n$ with positive lower bound, the inverse Q there is the sum of a C^τ-smooth ψdo composed with an order-reducing operator, plus a remainder of lower order. Let Q_+ be the truncation to Ω, i.e., $Q_+ = r_\Omega Q e_\Omega$, where r_Ω restricts to Ω and e_Ω extends by zero. Then the resolvent can be written

$$R_\gamma = Q_+ + G_\gamma, \quad G_\gamma = -K_\gamma \gamma_0 Q_+. \tag{5.3}$$

Here the product structure of G_γ, passing via the $(n-1)$-dimensional manifold Σ, will be useful in Schatten class considerations.

The choice of the space H_q^1 for the coefficients in (5.1) is governed by the fact that a result of Marschall in [29] shows that this allows a comparison with the x-form operator $A^\times = -\sum_{j,k=1}^n a_{jk}(x)\partial_j\partial_k$, assuring that $A - A^\times$ is continuous from $H^{s+2-\theta}(\Omega)$ to $H^s(\Omega)$ for $-\tau + \theta < s \leq 0$.

Our strategy will be to approximate G_γ by a composition of C^∞-coefficient operators, obtained by approximation the original boundary value problem by C^∞-coefficient problems, where the desired estimates are well-known.

Proposition 5.1. *There is a sequence of C^∞-smooth invertible ψdbo systems $\mathcal{A}_k = \begin{pmatrix} A_k \\ \gamma_0 \end{pmatrix}$, with inverses $A_k^{-1} = Q_k$ on $\mathbb{R}^n$, $\mathcal{A}_k^{-1} = \mathcal{B}_k = \begin{pmatrix} R_{\gamma,k} & K_{\gamma,k} \end{pmatrix}$ on Ω, such that the symbol seminorms of $A - A_k$ converge to 0 in $H_q^1 S^2(\mathbb{R}^n \times \mathbb{R}^n)$, and in $C^{\tau'} S^2(\mathbb{R}^n \times \mathbb{R}^n)$ for all $\tau' \in]0, \tau[$, and when $k \to \infty$,*

$$\begin{aligned}
\|A - A_k\|_{\mathcal{L}(H^{s+2}(\mathbb{R}^n), H^s(\mathbb{R}^n))} &\to 0 \textit{ for each } |s| < \tau, \\
\|\mathcal{A} - \mathcal{A}_k\|_{\mathcal{L}(H^{s+2}(\Omega), H^s(\Omega) \times H^{s+\frac{3}{2}}(\Sigma))} &\to 0 \textit{ for each } |s| < \tau, \\
\|Q - Q_k\|_{\mathcal{L}(H^s(\mathbb{R}^n), H^{s+2}(\mathbb{R}^n))} &\to 0 \textit{ for each } s \in]-\tau, 0], \\
\|\mathcal{B} - \mathcal{B}_k\|_{\mathcal{L}(H^s(\Omega) \times H^{s+\frac{3}{2}}(\Sigma), H^{s+2}(\Omega))} &\to 0 \textit{ for each } s \in]-\tau, 0], \\
\|K_\gamma - K_{\gamma,k}\|_{\mathcal{L}(H^{s'-\frac{1}{2}}(\Sigma), H^{s'}(\Omega))} &\to 0 \textit{ for each } s' \in [0, 2].
\end{aligned} \tag{5.4}$$

Proof. As accounted for in Section 2, $A_k = \varrho_k * A$ converges to A in $H^1_q S^2$ and in $C^{\tau'} S^2$ for any $\tau' \in]0, \tau[$. Starting with a sufficiently large k and relabeling, we can assure that the operators A_k are uniformly strongly elliptic on $\mathbb{R}^n$, uniformly in k. (Note that the A_k are differential operators and only their coefficients are modified by convolution with ϱ_k.) We can assume that a sufficiently large constant has been added to A such that all the operators have a lower bound greater than a positive constant.

By Theorem 2.3 ff., the first two lines in (5.4) are valid. For the second line we note that for differential operators, the restriction to functions on Ω is tacitly understood, and that γ_0 cancels out.

The other statements will be obtained by following each step of the construction in [4]. First there is the construction in Section 4.2 there, where parametrices are constructed from the principal symbols in a localized situation. The principal interior symbol and boundary symbol are determined pointwise in x or x' by the standard theory; when $C^{\tau'}$-dependence is taken into account, one similarly gets the $C^{\tau'}$-dependence for the inverted symbols. Then the convergence of the principal symbol of A_k to that of A in $C^{\tau'}$ implies a convergence in $C^{\tau'}$ of the inverse principal symbol and boundary symbol. By Theorem 2.3 ff. and Theorem 3.8, the resulting operator families converge in the asserted Sobolev norms; note also that the Sobolev operator norms are bounded uniformly in k.

Next, we go to the λ-dependent construction in Sect. 4.3 of [4], applied to the operators on Ω. Here the remainder in the calculation of $\mathscr{A}_k(\lambda)\mathscr{B}^0_k(\lambda) - I$ will have not only lower order, but also a small operator-norm in terms of λ, uniformly in k.

It is found that for sufficiently large λ in a sector around $\mathbb{R}_-$ the exact inverses are determined by Neumann series arguments and have the form of the sum of a parametrix coming from the calculus and a lower-order term. The convergence in the third and fourth lines of (5.4) follows from the estimates, for the values of s allowed in the inverse construction for the nonsmooth problem, namely for $s \in]-\tau, 0]$.

For the last statement in (5.4) we use the additional information worked out in Section 5.2 of [4], again examined with a view to the approximation in $C^{\tau'}$ for $k \to \infty$. □

Theorem 5.2.

1° *The ψdo term Q_+ in* (5.3) *has the spectral behavior*

$$s_j(Q_+)j^{2/n} \to c(q^0_+)^{2/n} \text{ for } j \to \infty, \tag{5.5}$$

where the constant is defined from the principal symbol q^0 (or $a^0 = (q^0)^{-1}$):

$$\begin{aligned} c(q^0_+) &= \tfrac{1}{n(2\pi)^n} \int_\Omega \int_{|\xi|=1} (q^{0*}(x,\xi) q^0(x,\xi))^{n/4}\, d\omega dx \\ &= \tfrac{1}{n(2\pi)^n} \int_\Omega \int_{|\xi|=1} (a^{0*} a^0)^{-n/4}\, d\omega dx. \end{aligned} \tag{5.6}$$

2° *The singular Green term $G_\gamma = -K_\gamma \gamma_0 Q_+$ (cf.* (5.3)) *in the solution operator to the Dirichlet problem for A has the spectral behavior*

$$s_j(G_\gamma)j^{2/(n-1)} \to c(g^0)^{2/(n-1)} \text{ for } j \to \infty, \tag{5.7}$$

where the constant is defined from the principal symbol g^0 of G_γ:

$$c(g^0) = \tfrac{1}{(n-1)(2\pi)^{(n-1)}} \int_\Sigma \int_{|\xi'|=1} tr((g^{0*}(x',\xi',D_n)g^0(x',\xi',D_n))^{(n-1)/4})\, d\omega dx', \quad (5.8)$$

cf. also (5.11) *below.*

Proof. We approximate Q and K_γ by the sequences Q_k and $K_{\gamma,k}$ as in Proposition 5.1. It is known in the C^∞-case that the exact solution operators Q_k, $\mathcal{B}_k = \big(R_{\gamma,k}\ K_{\gamma,k}\big)$ belong to the ψdbo calculus (cf. e.g. [16] or the recent account in [18], Sect. 11.3).

1°. The statements (5.5)–(5.6) are well-known for $Q_{k,+}$, see e.g. [16] Proposition 4.5.3. It follows from the third line in (5.4) that $Q_+ - Q_{k,+} \to 0$ in the norm of $\mathcal{L}(H^0(\Omega), H^2(\Omega))$, hence in the quasinorm of $\mathfrak{S}_{n/2,\infty}(L_2(\Omega))$. Then the result follows by an application of Lemms 2.4 2°.

2°. The statements (5.7)–(5.8) for $G_{\gamma,k} = -K_{\gamma,k}\gamma_0 Q_{k,+}$ are known for these operators from [15].

Now we have

$$G_\gamma - G_{\gamma,k} = (-K_\gamma + K_{\gamma,k})\gamma_0 Q_+ - K_{\gamma,k}(\gamma_0 Q_+ - \gamma_0 Q_{k,+}).$$

Here $\gamma_0 Q_+$ is bounded from $H^0(\Omega)$ to $H^{\frac{3}{2}}(\Sigma)$. Writing it as $\Lambda_0^{-\frac{3}{2}}\Lambda_0^{\frac{3}{2}}\gamma_0 Q_+$, where the Λ_0^r are homeomorphisms from $H^s(\Sigma)$ to $H^{s-r}(\Sigma)$ for all s, with $(\Lambda_0^r)^{-1} = \Lambda_0^{-r}$, we see that $\gamma_0 Q_+$ belongs to $\mathfrak{S}_{(n-1)/(3/2),\infty}(L_2(\Omega), L_2(\Sigma))$, as the composition of the operator $\Lambda_0^{-\frac{3}{2}} \in \mathfrak{S}_{(n-1)/(3/2),\infty}(L_2(\Sigma))$ and the bounded operator $\Lambda_0^{\frac{3}{2}}\gamma_0 Q_+$ from $L_2(\Omega)$ to $L_2(\Sigma)$. Similarly, since $K_\gamma - K_{\gamma,k} = (K_\gamma - K_{\gamma,k})\Lambda_0^{\frac{1}{2}}\Lambda_0^{-\frac{1}{2}}$, where $\Lambda_0^{-\frac{1}{2}} \in \mathfrak{S}_{(n-1)/(1/2),\infty}(L_2(\Sigma))$, and the norm of $(K_\gamma - K_{\gamma,k})\Lambda_0^{\frac{1}{2}}$ from $L_2(\Sigma)$ to $L_2(\Omega)$ goes to 0 in view of (5.4),

$$\mathbf{N}_{(n-1)/(1/2)}(K_\gamma - K_{\gamma,k}) \to 0 \text{ for } k \to \infty. \quad (5.9)$$

It follows by (2.3) that

$$\mathbf{N}_{(n-1)/2}((K_\gamma - K_{\gamma,k})\gamma_0 Q_+) \to 0 \text{ for } k \to \infty.$$

For the other term, note that $\mathbf{N}_{(n-1)/(1/2)}(K_{\gamma,k})$ is bounded in k, and $\mathbf{N}_{(n-1)/(3/2)}(\gamma_0 Q_+ - \gamma_0 Q_{k,+}) \to 0$ for $k \to \infty$ by (5.4), so also

$$\mathbf{N}_{(n-1)/2}(K_{\gamma,k}(\gamma_0 Q_+ - \gamma_0 Q_{k,+})) \to 0 \text{ for } k \to \infty.$$

Together these statements show that

$$\mathbf{N}_{(n-1)/2}(G_\gamma - G_{\gamma,k}) \to 0 \text{ for } k \to \infty.$$

We can then apply Lemma 2.4 2° to the decomposition $G_\gamma = G_{\gamma,k} + (G_\gamma - G_{\gamma,k})$, concluding the assertion in the theorem, since $g_k^0 \to g^0$ for $k \to \infty$. □

Remark 5.3. Because of the special structure of G_γ it is easy to determine the constant, in the following way: when all operators are smooth we have, denoting $\gamma_0 Q_+ = T$:

$$s_j(G_\gamma)^2 = \mu_j(G_\gamma^* G_\gamma) = \mu_j(T^* K_\gamma^* K_\gamma T) = \mu_j(K_\gamma^* K_\gamma TT^*) = \mu_j(P_1 P_1') = \mu_j(P_2 (P_2')^2 P_2),$$

where $P_1 = K_\gamma^* K_\gamma$, $P_1' = TT^*$ are selfadjoint nonnegative elliptic operators with square roots P_2 resp. P_2'. Then

$$s_j(G_\gamma) = s_j(P_2' P_2).$$

To calculate the principal symbols at a point $x' \in \Sigma$ we consider the principal symbol of A in local coordinates for $|\xi'| \geq 1$:

$$\begin{aligned} a^0(x', \xi', \xi_n) &= s_0(x')(\xi_n - \lambda^+(x', \xi'))(\xi_n - \lambda^-(x', \xi')) \\ &= s_0(x')(i\xi_n + \kappa^+(x', \xi'))(-i\xi_n + \kappa^-(x', \xi')), \end{aligned}$$

where $\operatorname{Im} \lambda^\pm \gtrless 0$, and $\kappa^+ = -i\lambda^+$, $\kappa^- = i\lambda^-$ have positive real part. Then Q has principal symbol

$$q^0 = \frac{1}{a^0} = \frac{1}{s_0(\kappa^+ + \kappa^-)} \Big(\frac{1}{\kappa^+ + i\xi_n} + \frac{1}{\kappa^- - i\xi_n} \Big),$$

and it follows from the rules of calculus for ψdbo's that T and K_γ have principal symbols

$$t^0 = \frac{1}{s_0(\kappa^+ + \kappa^-)(\kappa^- - i\xi_n)}, \quad k^0 = \frac{1}{\kappa^+ + i\xi_n}. \tag{5.10}$$

Then $P_1' = TT^*$ and $P_1 = K_\gamma^* K_\gamma$ have principal symbols

$$p_1'^0 = \frac{1}{|s_0(\kappa^+ + \kappa^-)|^2 2 \operatorname{Re} \kappa^-}, \quad p_1^0 = \frac{1}{2 \operatorname{Re} \kappa^+},$$

and (cf. Theorem 2.5)

$$\begin{aligned} c(g^0) = c(p_2'^0 p_2^0) = \tfrac{1}{(n-1)(2\pi)^{(n-1)}} \int_\Sigma \int_{|\xi'|=1} &(4|s_0(\kappa^+ + \kappa^-)|^2 \\ &\operatorname{Re} \kappa^- \operatorname{Re} \kappa^+)^{-(n-1)/4} \, d\omega dx'. \end{aligned} \tag{5.11}$$

The formula carries over to the nonsmooth case since the symbols converge in $C^{\tau'}$ ($\tau' < \tau$) for $k \to \infty$.

5.2. *Resolvent Differences*

Next, we consider the difference between the Dirichlet resolvent and the resolvent of a Neumann problem for A defined by a boundary condition $\chi u - C\gamma_0 u = 0$ as studied in [4], Section 7. Here χu is the conormal derivative,

$$\chi u = \sum_{j,k=1}^{n} n_j \gamma_0 a_{jk} \partial_k u, \tag{5.12}$$

where $\vec{n} = (n_1, \dots, n_n)$ is the interior unit normal on Σ. Moreover, C is a first-order tangential differential operator $C = c \cdot D_\tau + c_0$, where $c = (c_1, \dots, c_n)$, all $c_j \in H^1_q(\Sigma)$ ($D_\tau = -i\partial_\tau$ as defined in [4], Section 2.4). To assure invertibility, let us assume that the sesquilinear form

$$a_{\chi,C}(u,v) = \sum_{j,k=1}^{n} (a_{jk}\partial_k u, \partial_j v)_{L_2(\Omega)} + (\sum_{j=1}^{n} a_j \partial_j u + a_0 u, v)_{L_2(\Omega)} + (C\gamma_0 u, \gamma_0 v)_{L_2(\Sigma)} \tag{5.13}$$

satisfies $\operatorname{Re} a_{\chi,C}(u,u) \geq c_0 \|u\|^2_{H^1(\Omega)} - k\|u\|^2_{L_2(\Omega)}$ for $u \in H^1(\Omega)$; then we can add a constant (absorbed in a_0) such that the realization $A_{\chi,C}$ in $L_2(\Omega)$ defined from $a_{\chi,C}$ by variational theory has its spectrum in a sector in $\{\operatorname{Re} z > 0\}$. As shown in [4] there holds a Krein-type resolvent formula

$$A^{-1}_{\chi,C} - A^{-1}_\gamma = K_\gamma L^{-1}(K'_\gamma)^*; \tag{5.14}$$

here L acts like $C - P_{\gamma,\chi}$, where $P_{\gamma,\chi}$ is the Dirichlet-to-Neumann operator. The boundary value problem is elliptic, hence so is $L = C - P_{\gamma,\chi}$, so $D(L) = H^{\frac{3}{2}}(\Sigma)$ (cf. [4] Th. 7.2), mapping this space homeomorphically onto $H^{\frac{1}{2}}(\Sigma)$.

It is shown in Sect. 5 of [4] that

$$P_{\gamma,\chi} \colon H^{s-\frac{1}{2}}(\Sigma) \to H^{s-\frac{3}{2}}(\Sigma)$$

is continuous for $s \in [0,2]$, and since multiplication by the coefficients c_j in C preserves $H^s(\Sigma)$ for $|s| \leq 1$ (cf. [4] (2.29)), C has this mapping property for $s \in [\frac{1}{2}, 2]$. So L extends to a continuous map of this kind for $s \in [\frac{1}{2}, 2]$.

When we now also approximate C by smoothed out operators C_k, and $P_{\gamma,\chi}$ is approximated by smooth ψdo's $P_{\gamma,\chi,k}$ as a result of the earlier mentioned smoothing of A, we get a sequence $L_k = C_k - P_{\gamma,\chi,k}$ such that

$$\|L - L_k\|_{\mathscr{L}(H^{s-\frac{1}{2}}(\Sigma), H^{s-\frac{3}{2}}(\Sigma))} \to 0 \text{ for } k \to \infty, \text{ all } s \in [\tfrac{1}{2}, 2].$$

Since L is an invertible operator from $H^{\frac{3}{2}}(\Sigma)$ to $H^{\frac{1}{2}}(\Sigma)$, L_k is likewise so for large k, say $k \geq k_0$, by a Neumann series argument. Moreover, the inverses L_k^{-1} are estimated uniformly in $k \geq k_0$, and

$$\|L^{-1} - L_k^{-1}\|_{\mathscr{L}(H^{\frac{1}{2}}, H^{\frac{3}{2}})} = \|L^{-1}(L_k - L)L_k^{-1}\|_{\mathscr{L}(H^{\frac{1}{2}}, H^{\frac{3}{2}})} \to 0 \text{ for } k \to \infty.$$

We can then show:

Theorem 5.4. *The singular Green term* $G_C = K_\gamma L^{-1}(K'_\gamma)^*$ *in the Krein resolvent formula* (5.14) *has the spectral behavior*

$$s_j(G_C) j^{2/(n-1)} \to c(g^0_C)^{2/(n-1)} \text{ for } j \to \infty, \tag{5.15}$$

where the constant is defined from the principal symbol g_C^0 of G_C by formula (5.8), *cf. also* (5.17) *below.*

Proof. Define $G_{C,k} = K_{\gamma,k} L_k^{-1} (K'_{\gamma,k})^*$, and write

$$G_C - G_{C,k} = (K_\gamma - K_{\gamma,k}) L^{-1} (K'_\gamma)^* + K_{\gamma,k} (L^{-1} - L_k^{-1})(K'_\gamma)^* + K_{\gamma,k} L_k^{-1} ((K'_\gamma)^* - (K'_{\gamma,k})^*). \tag{5.16}$$

We shall show that all three terms have $\mathbf{N}_{(n-1)/2}$-quasinorms going to 0 for $k \to \infty$. Then since the statements in the theorem are well-known for $G_{C,k}$, the result follows for G_C by application of Lemma 2.4 2°.

Since K'_γ is of the same kind as K_γ, its adjoint satisfies for all $s' \in [0, 2]$ that $(K'_\gamma)^* \in \mathcal{L}(H_0^{-s'}(\overline{\Omega}), H^{-s'+\frac{1}{2}}(\Sigma))$, and $(K'_\gamma)^* - (K'_{\gamma,k})^*$ goes to zero in these operator norms for $k \to \infty$.

In the first term in (5.16), the boundedness of $(K'_\gamma)^*$ from $H^0(\Omega)$ to $H^{\frac{1}{2}}(\Sigma)$ and that of L^{-1} from $H^{\frac{1}{2}}(\Sigma)$ to $H^{\frac{3}{2}}(\Sigma)$ imply that $L^{-1}(K'_\gamma)^* \in \mathcal{L}(H^0(\Omega), H^{\frac{3}{2}}(\Sigma))$; hence it lies in $\mathfrak{S}_{(n-1)/(3/2),\infty}$ (by an argumentation as in the proof of Theorem 5.2). Together with (5.9) this implies that the composed operator goes to 0 in $\mathfrak{S}_{(n-1)/2,\infty}$ for $k \to \infty$.

For the second term, we use that the norm of $(L^{-1} - L_k^{-1})(K'_\gamma)^*$ in $\mathcal{L}(H^0(\Omega), H^{\frac{3}{2}}(\Sigma))$ goes to zero, and for the last term we use that the norm of $L_k^{-1}((K'_\gamma)^* - (K'_{\gamma,k})^*)$ in $\mathcal{L}(H^0(\Omega), H^{\frac{3}{2}}(\Sigma))$ goes to zero; both are combined with the fact that $K_\gamma \in \mathfrak{S}_{(n-1)/(1/2),\infty}$. This completes the proof. □

Remark 5.5. Also here, the constant can be determined from a formula with ψdo's on Σ. In local coordinates, K_γ has the principal symbol (5.11), and since A' has the principal symbol $\overline{a}^0$, K'_γ has the principal symbol $(\overline{\kappa^-} + i\xi_n)^{-1}$, and that of $(K'_\gamma)^*$ is $(\kappa^- - i\xi_n)^{-1}$. Let $P''_1 = (K'_\gamma)^* K'_\gamma$, with square root P''_2; here $p''^0_2 = (2\operatorname{Re}\kappa^-)^{-1/2}$. Then, with notation from Remark 5.3 (recall $p_2^0 = (2\operatorname{Re}\kappa^+)^{-1/2}$), one has in case of smooth operators:

$$s_j(G_C)^2 = \mu_j(K'_\gamma (L^*)^{-1} K_\gamma^* K_\gamma L^{-1} (K'_\gamma)^*) = \mu_j((L^*)^{-1} P_2^2 L^{-1} (P''_2)^2) = s_j(P_2 L^{-1} P''_2)^2,$$

and hence, with l^0 denoting the principal symbol of L,

$$c(g_C^0) = \tfrac{1}{(n-1)(2\pi)^{(n-1)}} \int_\Sigma \int_{|\xi'|=1} (4(l^0)^2 \operatorname{Re}\kappa^+ \operatorname{Re}\kappa^-)^{-(n-1)/4} \, d\omega dx'. \tag{5.17}$$

The formula extends to nonsmooth cases by the approximation used in Theorem 5.4.

Corollary 5.6. *The operators A_γ^{-1} and $A_{\chi,C}^{-1}$ have the same spectral asymptotic behavior as Q_+ in Theorem* 5.2.

Proof. The statement for A_γ^{-1} is found by application of Lemma 2.4 1° and Theorem 5.2 to (5.3). Next, the statement for $A_{\chi,C}$ follows similarly from (5.14) and Theorem 5.4. □

Remark 5.7. The estimates extend to differences between resolvents at arbitrary points of $\varrho(A_\gamma) \cap \varrho(A_{\chi,C})$ in the same way as in [19], using (3.7) there.

6. Nonsmooth Boundaries

In [4] also open sets with nonsmooth boundaries were allowed. While the coefficients a_{jk}, a_j were taken in $H^1_q(\Omega)$ ($a_0 \in L_q(\Omega)$) as above, the domain Ω was taken to be $B^{\frac{3}{2}}_{p,2}$ (meaning that the boundary is locally the graph of a function in $B^{\frac{3}{2}}_{p,2}(\mathbb{R}^{n-1})$), possibly with different choices of p and q. More precisely it was required that $p, q > 2$, $p < \infty$, with

$$1 - \tfrac{n}{q} \geq \tfrac{1}{2} - \tfrac{n-1}{p} \equiv \tau_0 > 0; \tag{6.1}$$

then τ was taken as τ_0. This domain regularity includes $C^{\frac{3}{2}+\varepsilon}$ for small ε, and is included in $C^{1+\tau_0}$.

One interest of the Besov space $B^{\frac{3}{2}}_{p,2}$ is that it hits the differentiability index $\frac{3}{2}$ in an exact way, better that $C^{\frac{3}{2}+\varepsilon}$. It comes in as the boundary value space (trace space) for the anisotropic Sobolev space

$$W^2_{(2,p)}(\mathbb{R}^n_+) = \{f \mid D^\alpha f \in L_2(\mathbb{R}_+; L_p(\mathbb{R}^{n-1})), |\alpha| \leq 2\}. \tag{6.2}$$

Proposition 6.1. 1° *Let $0 < \delta < \delta_1 < 1$. When Ω is a bounded $C^{1+\delta_1}$-domain, there exists a sequence of bounded C^∞-domains Ω_l with $C^{1+\delta}$-diffeomorphisms $\lambda_l\colon \overline{\Omega} \xrightarrow{\sim} \overline{\Omega}_l$ (sending $\partial\Omega = \Sigma$ onto $\partial\Omega_l = \Sigma_l$), such that $\lambda_l \to \mathrm{Id}$ in $C^{1+\delta}$ on a neighborhood Ω' of $\overline{\Omega}$, and in this sense $\overline{\Omega}_l \to \overline{\Omega}$, for $l \to \infty$.*

2° *When Ω is a bounded domain with $B^{\frac{3}{2}}_{p,2}$-boundary, it is $C^{1+\tau_0}$-smooth, and there exists a sequence of bounded C^∞-domains Ω_l with $C^{1+\tau_0}$-diffeomorphisms $\lambda_l\colon \overline{\Omega} \xrightarrow{\sim} \overline{\Omega}_l$ (associated to the $B^{\frac{3}{2}}_{p,2}$-boundary structure as in* [4]*) such that $\lambda_l \to \mathrm{Id}$ in $C^{1+\tau_0}$ on a neighborhood Ω' of $\overline{\Omega}$, and in this sense $\overline{\Omega}_l \to \overline{\Omega}$, for $l \to \infty$. Moreover, the estimates for the associated operators studied in* [4] *for second-order problems hold boundedly in l.*

Proof. 1°. The $C^{1+\delta_1}$-structure is defined by a family of $C^{1+\delta_1}$-charts $\kappa_j\colon V_j \to U_j$, $j = 1, \dots, N$, where V_j and U_j are open subsets of $\mathbb{R}^n$ and κ_j is a $C^{1+\delta_1}$-diffeomorphism from V_j to U_j, sending $V_{j,+} = V_j \cap \overline{\mathbb{R}}^n_+$ resp. $V_{j,0} = V_j \cap (\mathbb{R}^{n-1} \times \{0\})$ onto $U_{j,+} = U_j \cap \overline{\Omega}$ resp. $U_{j,0} = U_j \cap \partial\Omega$, and where $U \equiv \bigcup_{j\leq N} U_j \supset \overline{\Omega}$. We can choose bounded open subsets V_j', V_j'', mapped onto U_j', U_j'', such that $\overline{V}_j'' \subset V_j' \subset \overline{V}_j' \subset V_j$, $\overline{U}_j'' \subset U_j' \subset \overline{U}_j' \subset U_j$, with $U'' \equiv \bigcup_{j\leq N} U_j'' \supset \overline{\Omega}$, and there is a smooth partition of unity $\{\psi_j\}_{j\leq N}$ for $\overline{U}''$ with $\operatorname{supp}\psi_j \subset U_j'$, $\psi_j \geq c_j > 0$ on U_j'' for each j. With ϱ_l denoting an approximative unit with $\operatorname{supp}\varrho_l \subset B_{1/l}$, the convolution $\varrho_l * \kappa_j$ is well-defined on $V_{j,l} = \{x \in V_j \mid \operatorname{dist}(x, \partial V_j) > 1/l\}$, and

$$\varrho_l * \kappa_j \to \kappa_j \text{ in } C^{1+\delta}(V_j') \text{ for } l \to \infty,\ l \geq l_0,$$

with l_0 so large that $\overline{V}_j' \subset V_{j,l_0}$ for all $j \leq N$. Then define the mapping λ_l on the neighborhood $U = \bigcup U_j$ of $\overline{\Omega}$ by

$$\lambda_l = \sum_{j\leq N} (\varrho_l * \kappa_j) \circ (\psi_j \kappa_j^{-1}); \tag{6.3}$$

it converges to the identity on $\overline{U}''$, in the $C^{1+\delta}$-norm.

Let $\Omega_l = \lambda_l(\Omega)$; for large l it is contained in a small neighborhood of $\overline{\Omega}$. The mappings $\varrho_l * \kappa_j$ from V'_j to neighborhoods of $\overline{U}'_j$ are C^∞ and are bijective from the sets V''_j to their images $\widetilde{U}_{j,l}$ in the neighborhoods of $\overline{U}''_j$, for l sufficiently large. Note that they are C^∞ from the subset $V''_{j,+}$ (with the flat boundary piece $V''_{j,0} \subset \mathbb{R}^{n-1} \times \{0\}$) to $\widetilde{U}_{j,l} \cap \overline{\Omega}_l$, so they provide a system of C^∞-charts for $\overline{\Omega}_l$. This is then indeed a C^∞-smooth subset of $\mathbb{R}^n$.

Since $\lambda_l \to \mathrm{Id}$ on $\overline{U}''$, it is a $C^{1+\delta}$-diffeomorphism from U'' to $\lambda_l(U'')$ for l sufficiently large. In particular, it then defines a $C^{1+\delta}$-diffeomorphism from $\overline{\Omega}$ to $\overline{\Omega}_l$, converging to the identity for $l \to \infty$; in this sense $\overline{\Omega}_l \to \overline{\Omega}$ for $l \to \infty$.

2°. In the case of a domain with $B^{\frac{3}{2}}_{p,2}$-boundary, one can likewise approximate with C^∞-domains. It is shown in [4] that such domains are $C^{1+\tau_0}$, so one can use 1° to do a $C^{1+\tau}$-approximation for any $\tau < \tau_0$. One can also do a more precise construction adapted to the structure explained in [4]:

We refer to the notation there, see in particular Lemma 2.2, Prop. 2.7 and Remark 2.9 there. Recall that $\overline{\Omega}$ is covered by patches $U_0, \dots, U_J$, where $\overline{U}_0 \subset \Omega$, and the other U_j, together covering Σ, are such that $U_j \cap \Omega$ is of the form $U_j \cap \{x \mid x_n > \gamma_j(x')\}$ for a function $\gamma_j \in B^{\frac{3}{2}}_{p,2}(\mathbb{R}^{n-1})$ (after a rotation of coordinates). From γ_j one constructs (in Prop. 2.7 of [4]) a $C^{1+\tau_0}$-diffeomorphism F_j on $\mathbb{R}^n$ mapping $\mathbb{R}^n_+$ to $\{x \mid x_n > \gamma_j(x')\}$. For each of these U_j, we can smoothen out the boundary defining function $\gamma_j(x')$ by convolution with an approximate identity $\varrho_l(x')$; then $\varrho_l * \gamma_j$ converges to γ_j in $B^{\frac{3}{2}}_{p,2}$-norm for $l \to \infty$. The associated diffeomorphisms $F_{j,l}$ constructed as in Prop. 2.7 are C^∞ (since the lifting used in Lemma 2.2 of [4] lifts C^∞ to C^∞), and converge to F_j in $C^{1+\tau_0}$. For U_0 we simply use the identity diffeomorphism for all l. As under 1°, the approximations can be pieced together by a suitably chosen partition of unity, leading to an approximation of Ω by C^∞-domains Ω_l, such that the boundary converges in $B^{\frac{3}{2}}_{p,2}$ and the diffeomorphisms converge in $C^{1+\tau_0}$. Moreover, the various other constructions in [4] can be followed through these coordinate changes. □

For C^1-domains it is not hard to prove that 1° holds with approximation in C^1.

Let $\Omega_l = \underline{\Omega}$ be one of the approximating domains, with a $C^{1+\tau}$-diffeomorphism $\lambda\colon \overline{\Omega} \to \underline{\Omega}$. When u is a function on $\overline{\Omega}$, denote by $\underline{u}$ the function on $\overline{\underline{\Omega}}$ defined by

$$\underline{u}(y) = u(\lambda^{-1}(y)) \text{ for } y \in \overline{\underline{\Omega}}, \text{ also denoted } u \circ \lambda^{-1} \text{ or } (\lambda^{-1})^* u. \tag{6.4}$$

An operator B in $L_2(\Omega)$ carries over to the operator $\underline{B}$ in $L_2(\underline{\Omega})$ by

$$\underline{B}\,\underline{u} = \underline{(Bu)}, \text{ also written as } (\lambda^{-1})^* B \lambda^* \underline{u}.$$

We have by the rules for coordinate changes:

$$(Bu, v)_{L_2(\Omega)} = \int_\Omega Bu(x)\bar{v}(x)\,dx = \int_{\underline{\Omega}} \underline{B}\,\underline{u}(y)\underline{\bar{v}}(y)J(y)\,dy = (\underline{B}\,\underline{u}, J\underline{\bar{v}})_{L_2(\underline{\Omega})}, \tag{6.5}$$

where J denotes the Jacobian $|\det(\partial(\lambda^{-1})_j/\partial y_k)_{j,k=1,\dots,n}|$; it is a positive C^τ-function on $\overline{\underline{\Omega}}$. Denote $J(y)^{\frac{1}{2}} = \eta(y)$, then moreover,

$$(Bu, v)_{L_2(\Omega)} = (\underline{B}\,\underline{u}, \eta^2\underline{\bar{v}})_{L_2(\underline{\omega})} = (\eta\underline{B}\eta^{-1}\eta\underline{u}, \eta\underline{v})_{L_2(\underline{\omega})}; \tag{6.6}$$

in particular one gets for $B = I$ that $\|u\|_{L_2(\Omega)} = \|\eta\underline{u}\|_{L_2(\underline{\omega})}$.

Thus the mapping $\Phi : u \mapsto \eta\underline{u} = \eta(\lambda^{-1})^* u$ is an *isometry of* $L_2(\Omega)$ *onto* $L_2(\underline{\omega})$, i.e., a unitary operator $\Phi = \eta(\lambda^{-1})^*$, and by use of this mapping, B is carried over to the operator on $L_2(\underline{\omega})$

$$\widetilde{B} = \eta\underline{B}\,\eta^{-1} = \Phi B \Phi^{-1}. \tag{6.7}$$

The analysis of the spectrum of B can then be performed as an analysis of the spectrum of $\widetilde{B}$. In particular, when B is compact, $s_j(B) = s_j(\widetilde{B})$ for all j.

We shall now show that the results of Section 5 extend to the general case of [4], where also the boundary is nonsmooth.

Theorem 6.2. *Theorems* 5.2 *and* 5.4 (*with corollaries and remarks*) *extend to the situation where* Ω *is a bounded open set in* $\mathbb{R}^n$ *with* $B^{\frac{3}{2}}_{p,2}$*-boundary, satisfying* (6.1).

Proof. Consider the Dirichlet and Neumann-type problems for A on Ω, as defined in [4]. For each l, apply the diffeomorphism λ_l of Proposition 6.1 2° to carry the operators over to $\underline{\omega} = \Omega_l$; as shown in [4] this gives operators A_l, C_l of the kind treated in Section 5. For the resulting ψdo $Q_{l,+}$ and singular Green terms $G_{\gamma,l} = -K_{\gamma,l}\gamma_0 Q_{l,+}$ and $G_{C_l} = K_{\gamma,l}L_l^{-1}(K'_{\gamma,l})^*$ from the solution operators, the spectral asymptotics results of Section 5 are valid.

Consider, for example, $B = G_C$ on Ω, carried over to $\underline{B}_l = G_{C_l}$ on Ω_l (the operators Q_+ and G_γ are treated similarly). Theorem 5.4 and Remark 5.5 applied to $\underline{B}_l$ give that

$$s_j(\underline{B}_l)j^{2/(n-1)} \to c(g^0_{C_l})^{2/(n-1)} \text{ for } j \to \infty, \text{ where}$$
$$c(g^0_{C_l}) = \tfrac{1}{(n-1)(2\pi)^{(n-1)}} \int_{\Sigma_l}\int_{|\xi'|=1} (4(l^0_l)^2 \operatorname{Re}\kappa_l^+ \operatorname{Re}\kappa_l^-)^{-(n-1)/4}\, d\omega dx'. \tag{6.8}$$

To obtain similar results for B we define as in (6.7):

$$\widetilde{B}_l = \eta_l \underline{B}_l \eta_l^{-1}; \tag{6.9}$$

here the s-numbers satisfy for all j, l,

$$s_j(B) = s_j(\widetilde{B}_l). \tag{6.10}$$

The results of Section 5 will not be applied directly to the operator $\eta_l\underline{B}_l\eta_l^{-1}$, since the multiplication by the C^{τ_0}-functions η_l, η_l^{-1} does not fit easily into the calculus. Instead we shall use their properties for $l \to \infty$. Since λ_l converges to the identity in $C^{1+\tau_0}$ for $l \to \infty$ on a neighborhood Ω' of $\overline{\Omega}$, the Jacobian J_l converges to 1 in $C^{\tau_0}(\Omega')$, and so does its square root η_l as well as the inverse function η_l^{-1}.

Now by (2.4), noting also that $\underline{B}_l = \eta_l^{-1}\widetilde{B}_l\eta_l$,

$$\|\eta_l\|_{L_\infty} s_j(\underline{B}_l) \|\eta_l^{-1}\|_{L_\infty} \geq s_j(\widetilde{B}_l) \geq \frac{1}{\|\eta_l^{-1}\|_{L_\infty}} s_j(\underline{B}_l) \frac{1}{\|\eta_l\|_{L_\infty}},$$

for all j, l (norms in $L_\infty(\Omega')$). Hence, in view of (6.10),

$$\limsup_{j\to\infty} s_j(B) j^{2/(n-1)} \leq \|\eta_l\|_{L_\infty}\|\eta_l^{-1}\|_{L_\infty} c(g^0_{C_l})^{2/(n-1)},$$

$$\liminf_{j\to\infty} s_j(B) j^{2/(n-1)} \geq \frac{1}{\|\eta_l\|_{L_\infty}\|\eta_l^{-1}\|_{L_\infty}} c(g^0_{C_l})^{2/(n-1)}, \tag{6.11}$$

for all l. Since $\|\eta_l\|_{L_\infty}$ and $\|\eta_l^{-1}\|_{L_\infty}$ converge to 1 and $c(g^0_{C_l})^{2/(n-1)} \to c(g^0_C)^{2/(n-1)}$ for $l \to \infty$, we conclude that

$$\lim_{j\to\infty} s_j(B) j^{2/(n-1)} = c(g^0_C)^{2/(n-1)}, \tag{6.12}$$

as was to be shown. □

Appendix. Proof of Theorem 3.7

For the following let $(\varphi_j(\xi))_{j\in\mathbb{N}_0}$ be a smooth partition of unity on $\mathbb{R}^n$ such that

$$\operatorname{supp}\{\varphi_0\} \subseteq \overline{B_2(0)}, \qquad \operatorname{supp}\{\varphi_0\} \subseteq \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\} \quad \text{for } j \in \mathbb{N}$$

satisfying $\varphi_j(-\xi) = \varphi_j(\xi)$ for all $\xi \in \mathbb{R}^n$, $j \in \mathbb{N}_0$ and

$$|\partial_\xi^\alpha \varphi_j(\xi)| \leq C_\alpha 2^{-j|\alpha|} \qquad \text{for all } j \in \mathbb{N}_0$$

for arbitrary $\alpha \in \mathbb{N}_0^n$. This implies

$$|\partial_\xi^\alpha \varphi_j(\xi)| \leq C_\alpha \langle\xi\rangle^{-|\alpha|} \quad \text{uniformly in } j \in \mathbb{N}_0 \tag{A.1}$$

for all $\alpha \in \mathbb{N}_0^n$, since $2^{j-1} \leq |\xi| \leq 2^{j+1}$ on $\operatorname{supp}\varphi_j$ if $j \geq 1$ and the estimate for $j = 0$ is trivial.

Lemma A.1. *Let Z be a Banach space, $N \in \mathbb{N}_0$, and let g: $\mathbb{R}^n \to Z$ be an N-times continuously differentiable function, $N \in \mathbb{N}_0$, with compact support. Then*

$$\|\mathscr{F}^{-1}[g](x)\|_Z \leq C_N |\operatorname{supp} g| |x|^{-N} \sup_{|\beta|=N} \|\partial_\xi^\beta g\|_{L_\infty(\mathbb{R}^n;Z)}, \tag{A.2}$$

uniformly in $x \neq 0$ and g.

Proof. See e.g. Lemma 5.10 in Abels [3]. □

Lemma A.2. *Let X_0, X_1 be Banach spaces, let $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n, N; \mathscr{L}(X_0, X_1))$, $m \in \mathbb{R}, N \in \mathbb{N}, \tau > 0$, and let $p_j(x,\xi) = p(x,\xi)\varphi_j(\xi)$, $j \in \mathbb{N}_0$. Then for all $\alpha \in \mathbb{N}_0^n$, $M \in \{0, \dots, N\}$ there is a constant $C_{\alpha,M}$ such that*

$$\|\partial_z^\alpha k_j(x,z)\|_{\mathscr{L}(X_0,X_1)} \leq C_{\alpha,M} |z|^{-M} 2^{j(n+m+|\alpha|-M)} \tag{A.3}$$

uniformly in $j \in \mathbb{N}_0$, $z \neq 0$, where

$$k_j(x,z) := \mathscr{F}^{-1}_{\xi\mapsto z}[p_j(x,\xi)] \quad \textit{for all } x, z \in \mathbb{R}^n.$$

Proof. The lemma follows from Lemma A.1 applied to $g_x(\xi) = (i\xi)^\alpha p_j(x,\xi)$, where $x \in \mathbb{R}^n$ is considered as a fixed parameter, cf. Lemma 5.14 in Abels [3]. Since some of the arguments will be modified below, we include the details. Because of (A.1) and the Leibniz formula, it is easy to show that for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le N$,

$$\|\partial_\xi^\alpha p_j(x,\xi)\|_{\mathscr{L}(X_0,X_1)} \le C_\alpha \langle \xi \rangle^{m-|\alpha|}$$

uniformly in $x, \xi \in \mathbb{R}^n$ and $j \in \mathbb{N}_0$. Moreover, this implies that for all $\alpha, \gamma \in \mathbb{N}_0^n$ with $|\gamma| \le N$,

$$\left\| \partial_\xi^\gamma \big((i\xi)^\alpha p_j(x,\xi) \big) \right\|_{\mathscr{L}(X_0,X_1)} \le C_\alpha \langle \xi \rangle \xi^{m+|\alpha|-|\gamma|}$$

uniformly in $x, \xi \in \mathbb{R}^n$ and $j \in \mathbb{N}_0$, by another use of the Leibniz formula. Therefore Lemma A.1 yields

$$\begin{aligned} \|\partial_z^\alpha k_j(x,z)\|_{\mathscr{L}(X_0,X_1)} &\le C_N |\mathrm{supp} p_j(x,\cdot)| |z|^{-M} \sup_{|\gamma|=M} \left\| \partial_\xi^\gamma \big((i\xi)^\alpha p_j(x,\xi) \big) \right\|_{\mathscr{L}(X_0,X_1)} \\ &\le C_{\alpha,\beta,M} |z|^{-M} 2^{j(n+m+|\alpha|-M)}, \end{aligned}$$

since $|\mathrm{supp} p_j(x,.)| \le C2^{jn}$ independently of $x \in \mathbb{R}^n$. □

Note that for the last lemma no smoothness in x is needed; in fact boundedness in x would suffice.

Recall the notation $p(x, D_x) = \mathrm{OP}(p(x,\xi))$, cf. (2.9).

Lemma A.3.

Let X_0, X_1 be Banach spaces and let $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n, n+1; \mathscr{L}(X_0, X_1))$, $m \in \mathbb{R}$, $\tau > 0$, and let $p_j(x, D_x) = p(x, D_x)\varphi_j(D_x)$. Then for any $1 \le q < \infty$,

$$\|p_j(x,D_x)\|_{\mathscr{L}(L_q(\mathbb{R}^n;X_0),L_q(\mathbb{R}^n;X_1))} \le C2^{jm} \quad \text{for all } j \in \mathbb{N}_0, \tag{A.4}$$

where C does not depend on j.

Proof. The proof is variant of the proof of Lemma 6.20 in Abels [3]. We include the details for the convenience of the reader and since some of its arguments will be modified below.

First of all

$$p_j(x,D_x)f = \int_{\mathbb{R}^n} k_j(x, x-y) f(y) dy \quad \text{for all } f \in \mathscr{S}(\mathbb{R}^n; X_0),$$

where k_j satisfies (A.3). According to these estimates

$$\int_{\mathbb{R}^n} \|k_j(x,z)\|_{\mathscr{L}(X_0,X_1)} dz \le C\Big(\int_{|z|\le 2^{-j}} 2^{j(n+m)} dz + \int_{|z|>2^{-j}} |z|^{-n-1} 2^{j(m-1)} dz \Big).$$

The first comes from (A.3) for $M = 0$ and the second comes from the choice $M = n+1$. Hence we get by a simple calculation

$$\int_{\mathbb{R}^n} \|k_j(x,z)\|_{\mathscr{L}(X_0,X_1)} dz \le C2^{jm},$$

which proves (A.4), since

$$\left\| \int_{\mathbb{R}^n} k_j(x, x-y) f(y) dy \right\|_{L_q(\mathbb{R}^n; X_1)} \le \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \|k_j(x, z)\|_{\mathscr{L}(X_0, X_1)} dz \|f\|_{L_q(\mathbb{R}^n; X_0)}$$

for all $f \in \mathscr{S}(\mathbb{R}^n; X_0)$ because of Young's inequality for convolution integrals. □

Lemma A.4. *Let X_0, X_1 be Banach spaces and let $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n, n+1; \mathscr{L}(X_0, X_1))$, $m \in \mathbb{R}$, $\tau > 0$, and let $p_{j,k}(x, \xi) = \varphi_k(D_x) p(x, \xi) \varphi_j(\xi)$ for all $j, k \in \mathbb{N}_0$. Then for any $1 \le q < \infty$,*

$$\|p_{j,k}(x, D_x)\|_{\mathscr{L}(L_q(\mathbb{R}^n; X_0), L_q(\mathbb{R}^n; X_1))} \le C 2^{-\tau k} 2^{jm} \quad \textit{for all } j, k \in \mathbb{N}_0, \tag{A.5}$$

where C does not depend on j, k.

Proof. First of all we have that

$$\begin{aligned} &\sup_{x \in \mathbb{R}^n} \|\partial_\xi^\alpha \varphi_k(D_x) p(x, \xi)\|_{\mathscr{L}(X_0, X_1)} \\ &\le C 2^{-\tau k} \|\partial_\xi^\alpha p(\cdot, \xi)\|_{C^\tau(\mathbb{R}^n; \mathscr{L}(X_0, X_1))} \le C' \langle \xi \rangle^{m - |\alpha|} 2^{-\tau k}, \end{aligned}$$

since $C^\tau(\mathbb{R}^n; Z) \subseteq B^\tau_{\infty,\infty}(\mathbb{R}^n; Z)$ with continuous embedding, for a general Banach space Z. If $Z = \mathbb{C}$, then the statement follows e.g. from Theorem 6.2.5 in [5]. For a general Banach space Z the proof directly carries over. Hence by the Leibniz formula

$$\begin{aligned} &\|\partial_\xi^\beta \big(\varphi_k(D_x) p(x, \xi) \varphi_j(\xi)\big)\|_{\mathscr{L}(X_0, X_1)} \\ &\le C 2^{-\tau k} \|\partial_\xi^\beta \big(p(\cdot, \xi) \varphi_j(\xi)\big)\|_{C^\tau(\mathbb{R}^n; \mathscr{L}(X_0, X_1))} \le C' \langle \xi \rangle^{m - |\alpha|} 2^{-\tau k} \end{aligned}$$

uniformly in $j, k \in \mathbb{N}_0$, $\xi \in \mathbb{R}^n$. Now, if $k_{j,k}(x, z) = \mathscr{F}^{-1}_{\xi \mapsto z}[p_{j,k}(x, \xi)]$, then

$$p_{j,k}(x, D_x) f = \int_{\mathbb{R}^n} k_{j,k}(x, x-y) f(y) dy \quad \text{for all } f \in \mathscr{S}(\mathbb{R}^n; X_0).$$

Hence as in the proof of Lemma A.2 one shows that

$$\|\partial_z^\alpha k_{j,k}(x, z)\|_{\mathscr{L}(X_0, X_1)} \le C_{\alpha, M} |z|^{-M} 2^{j(n+m+|\alpha|-M)} 2^{-\tau k}$$

uniformly in $j, k \in \mathbb{N}_0$, $z \ne 0$, for all $|\alpha| = M \in \{0, \dots, n+1\}$. Therefore the same estimates as in the proof of Lemma A.3 (with an additional factor $2^{-\tau k}$) prove (A.5). □

Lemma A.5. *Let X_0, X_1 be Banach spaces, let $p \in C^\tau S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n, n+1; \mathscr{L}(X_0, X_1))$, and let $p_j(x, \xi) = p(x, \xi) \varphi_j(\xi)$. Then for any $1 \le q < \infty$, there is a constant C such that*

$$\|\varphi_i(D_x) p_j(x, D_x)\|_{\mathscr{L}(L_q(\mathbb{R}^n; X_0), L_q(\mathbb{R}^n; X_1))} \le \begin{cases} C 2^{-i\tau} & \textit{if } i \ge j+4, \\ C & \textit{if } j-3 \le i \le j+3, \\ C 2^{-j\tau} & \textit{if } j \ge i+4, \end{cases} \tag{A.6}$$

uniformly in $i, j \in \mathbb{N}_0$.

Proof. First of all, if $j-3 \leq i \leq j+3$, then the estimate follows from (A.4). To prove the other cases, we use that

$$\varphi_i(D_x)p_j(x, D_x) = \sum_{k=0}^{\infty} \varphi_i(D_x)p_{j,k}(x, D_x),$$

where $p_{j,k}(x, \xi) = \varphi_k(D_x)p_j(x, \xi)$. Moreover,

$$\begin{aligned}
\int_{\mathbb{R}^n} p_{j,k}(x, D_x)f(x)\overline{g(x)}\,dx &= \int_{\mathbb{R}^{2n}} e^{ix\cdot\xi}\varphi_k(D_x)p_j(x, \xi)\hat{f}(\xi)\overline{g(x)}\,đ\xi dx \\
&= \int_{\mathbb{R}^{2n}} p_j(x, \xi)\hat{f}(\xi)\overline{\varphi_k(D_x)e^{-ix\cdot\xi}g(x)}\,đ\xi dx \\
&= \int_{\mathbb{R}^{3n}} e^{-i\eta\cdot x}p_j(x, \xi)\hat{f}(\xi)\varphi_k(\eta)\overline{\hat{g}(\xi+\eta)}\,đ\xi đ\eta dx \\
&= \int_{\mathbb{R}^{3n}} e^{i(\xi-\xi')\cdot x}p(x, \xi)\varphi_j(\xi)\hat{f}(\xi)\varphi_k(\xi'-\xi)\overline{\hat{g}(\xi')}\,đ\xi đ\xi' dx
\end{aligned}$$

for all $f \in \mathscr{S}(\mathbb{R}^n; X_1)$, $g \in \mathscr{S}(\mathbb{R}^n)$. Here for every $x \in \mathbb{R}^n$ the support of

$$\int_{\mathbb{R}^n} e^{i(\xi-\xi')\cdot x}p(x, \xi)\varphi_j(\xi)\hat{f}(\xi)\varphi_k(\xi'-\xi)\overline{\hat{g}(\xi')}đ\xi$$

with respect to ξ' is contained in

$$\begin{aligned}
&\{2^{k-1} \leq |\xi| \leq 2^{k+1}\} + \{2^{j-1} \leq |\xi| \leq 2^{j+1}\} \\
&\subseteq \begin{cases} \{2^{k-2} \leq |\xi| \leq 2^{k+2}\} & \text{if } k \geq j+3, \\ \{2^{j-2} \leq |\xi| \leq 2^{j+2}\} & \text{if } j \geq k+3, \\ \{|\xi| \leq 2^{j+2}\} & \text{if } j \geq k, \\ \{|\xi| \leq 2^{k+2}\} & \text{if } k \geq j. \end{cases}
\end{aligned}$$

Now let us consider first the case $i \geq j+4$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varphi_i(D_x)p_{j,k}(x, D_x)f(x)\overline{g(x)}\,dx \\
&\quad = \int_{\mathbb{R}^{3n}} e^{i(\xi-\xi')\cdot x}p(x, \xi)\varphi_j(\xi)\hat{f}(\xi)\varphi_k(\xi'-\xi)\overline{\varphi_i(\xi')\hat{g}(\xi')}đ\xi đ\xi' dx = 0
\end{aligned}$$

for all $f \in \mathscr{S}(\mathbb{R}^n; X_1)$, $g \in \mathscr{S}(\mathbb{R}^n)$, if $|k-i| \geq 4$, since

$$\begin{aligned}
\{2^{k-2} \leq |\xi| \leq 2^{k+2}\} \cap \{2^{i-1} \leq |\xi| \leq 2^{i+1}\} &= \emptyset \qquad \text{if } k \geq i+4, \\
\{|\xi| \leq 2^{k+2}\} \cap \{2^{i-1} \leq |\xi| \leq 2^{i+1}\} &= \emptyset \qquad \text{if } j \leq k \leq i-4, \\
\{|\xi| \leq 2^{j+2}\} \cap \{2^{i-1} \leq |\xi| \leq 2^{i+1}\} &= \emptyset \qquad \text{if } k < j.
\end{aligned}$$

Therefore

$$\varphi_i(D_x)p_j(x, D_x) = \sum_{k=i-3}^{i+3} \varphi_i(D_x)p_{j,k}(x, D_x),$$

which implies

$$\|\varphi_i(D_x)p_j(x,D_x)\|_{\mathscr{L}(L_q(\mathbb{R}^n;X_0),L_q(\mathbb{R}^n;X_1))} \le \sum_{k=i-3}^{i+3} \|\varphi_i(D_x)p_{j,k}(x,D_x)\|_{\mathscr{L}(L_q(\mathbb{R}^n;X_0),L_q(\mathbb{R}^n;X_1))} \le C2^{-\tau i}$$

due to (A.5).

Finally, we consider the case $j \ge i+4$.

$$\int_{\mathbb{R}^n} \varphi_i(D_x)p_{j,k}(x,D_x)f(x)\overline{g(x)}\,dx = \int_{\mathbb{R}^{3n}} e^{i(\xi-\xi')\cdot x} p(x,\xi)\varphi_j(\xi)\hat{f}(\xi)\varphi_k(\xi'-\xi)\overline{\varphi_i(\xi')\hat{g}(\xi')}\,đ\xi đ\xi' dx = 0$$

for all $f \in \mathscr{S}(\mathbb{R}^n; X_1)$, $g \in \mathscr{S}(\mathbb{R}^n)$ if $|k-j| \ge 4$, since

$$\{2^{k-2} \le |\xi| \le 2^{k+2}\} \cap \{2^{i-1} \le |\xi| \le 2^{i+1}\} = \emptyset \qquad \text{if } k \ge j+4,$$
$$\{2^{j-2} \le |\xi| \le 2^{j+2}\} \cap \{2^{i-1} \le |\xi| \le 2^{i+1}\} = \emptyset \qquad \text{if } k \le j-4.$$

Therefore

$$\varphi_i(D_x)p_j(x,D_x) = \sum_{k=i-3}^{i+3} \varphi_i(D_x)p_{j,k}(x,D_x),$$

which implies

$$\|\varphi_i(D_x)p_j(x,D_x)\|_{\mathscr{L}(L_q(\mathbb{R}^n;X_0),L_q(\mathbb{R}^n;X_1))} \le \sum_{k=j-3}^{j+3} \|\varphi_i(D_x)p_{j,k}(x,D_x)\|_{\mathscr{L}(L_q(\mathbb{R}^n;X_0),L_q(\mathbb{R}^n;X_1))} \le C2^{-\tau j}$$

due to (A.5) again. □

Proof of Theorem 3.7. First of all, by a composition from the right with $\langle D_x\rangle^{-m}$ we can always reduce to the case $m=0$, which we consider in the following.

We use that $f = \sum_{j=0}^{\infty} f_j$, where $f_j = \varphi_j(D_x)f$, $f \in \mathscr{S}(\mathbb{R}^n; H_0)$. Since $\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_k = \emptyset$ if $|j-k| > 1$,

$$p(x,D_x)f = \sum_{j=0}^{\infty} p_j(x,D_x)f = \sum_{k=0}^{\infty} p_j(x,D_x)(f_{j-1}+f_j+f_{j+1}) = \sum_{j=0}^{\infty} p_j(x,D_x)\tilde{f}_j,$$

where $\tilde{f}_j = f_{j-1}+f_j+f_{j+1}$ and we have set $f_{-1}=0$. Moreover, we use that $\|f\|_{H^s(\mathbb{R}^n;H)}$ is equivalent to

$$\|f\|_{B^s_{2,2}(\mathbb{R}^n;H)} := \|(2^{sj}f_j)_{j\in\mathbb{N}_0}\|_{\ell_2(\mathbb{N}_0,L_2(\mathbb{R}^n;H))}$$

if H is a Hilbert space. This follows easily from Plancherel's theorem and the fact that $\langle\xi\rangle^s$ is equivalent to 2^{js} on $\operatorname{supp}\varphi_j$.

If $f \in \mathscr{S}(\mathbb{R}^n; H_0)$, then

$$\|(2^{sj}\tilde{f}_j)_{j\in\mathbb{N}_0}\|_{\ell_2(\mathbb{N}_0,L_2(\mathbb{R}^n;H_0))} = \|(2^{sj}(f_{j-1}+f_j+f_{j+1}))_{k\in\mathbb{N}_0}\|_{\ell_2(\mathbb{N}_0,L_2(\mathbb{R}^n;H_0))}$$
$$\leq C\left\|\left(2^{sj}\varphi_j(D_x)f\right)_{j\in\mathbb{N}_0}\right\|_{\ell_2(\mathbb{N}_0,L_2(\mathbb{R}^n;H_0))} = C\|f\|_{B^s_{2,2}(\mathbb{R}^n;H_0)}.$$

Next, we use

$$2^{is}\|\varphi_i(D_x)p_j(x,D_x)\tilde{f}_j\|_{L_2(\mathbb{R}^n;H_1)}$$
$$\leq \begin{cases} C2^{-(\tau-s)i-js}2^{js}\|\tilde{f}_j\|_{L_2(\mathbb{R}^n;H_0)} & \text{if } i\geq j+4,\\ C2^{js}\|\tilde{f}_j\|_{L_2(\mathbb{R}^n;H_0)} & \text{if } j-3\leq i\leq j+3,\\ C2^{(i-j)s-\tau j}2^{js}\|\tilde{f}_j\|_{L_2(\mathbb{R}^n;H_0)} & \text{if } j\geq i+4,\end{cases}$$
$$\leq C2^{-(\tau-|s|)|i-j|}2^{js}\|\tilde{f}_j\|_{L_2(\mathbb{R}^n;H_0)}$$

Now let $a_j = 2^{-(\tau-|s|)|j|}$, $b_j = 2^{sj}\|\tilde{f}_j\|_{L_2(\mathbb{R}^n;H_0)}$ for $j\in\mathbb{N}_0$ and $a_j=b_j=0$ for $j\in\mathbb{Z}\setminus\mathbb{N}_0$. Then

$$\|p(x,D_x)f\|_{B^s_{2,2}(\mathbb{R}^n;H_1)}$$
$$\leq C\left\|(a_j)_{j\in\mathbb{Z}} * (b_j)_{j\in\mathbb{Z}}\right\|_{\ell_2(\mathbb{Z})} \leq C\|(a_j)_{j\in\mathbb{Z}}\|_{\ell_1(\mathbb{Z})}\|(b_j)_{j\in\mathbb{Z}}\|_{\ell_2(\mathbb{Z})}$$
$$= C\|(2^{sj}\tilde{f}_j)_{j\in\mathbb{N}_0}\|_{\ell_2(\mathbb{N}_0,L_2(\mathbb{R}^n;H_0)))} \leq C\|f\|_{B^s_{2,2}(\mathbb{R}^n;H_0)},$$

where the sequence $(c_j)_{j\in\mathbb{Z}} = (a_j)_{j\in\mathbb{Z}} * (b_j)_{j\in\mathbb{Z}}$ is the convolution of $(a_j)_{j\in\mathbb{Z}}$ and $(b_j)_{j\in\mathbb{Z}}$. (Recall that $B^s_{2,2} = H^s$.) Since $\mathscr{S}(\mathbb{R}^n; H_0)$ is dense in $H^s(\mathbb{R}^n; H_0)$, the theorem follows. □

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