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Solution of Parabolic Pseudo-differential Initial–Boundary Value Problems

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1. INTRODUCTION

The motivation for the present work comes from a study of initial–boundary value problems for the Navier–Stokes equations [G–S1, 2, 3]. These problems (or rather, their linearized versions) are *degenerate parabolic*, and we had found a method to get rid of the degeneracy by carrying the problems over to pseudo-differential *non-degenerate parabolic* problems, where the results of the book [Gr2] could be used. However, the solvability statements for parabolic problems formulated in [Gr2] could be made more interesting and useful by some extra developments, in particular a deeper analysis of the necessary compatibility conditions and an inclusion of *all* values of the Sobolev space exponents, also those that have an exceptional role in boundary value problems.

Since this study has an interest not only for the Navier–Stokes equations, but also more generally (e.g., whenever a reduction of a differential operator problem leads to a parabolic pseudo-differential boundary value problem), we decided to present it in a separate article.

We consider a problem

$$\begin{aligned} &\text{(i)} \quad \partial_t u + P_\Omega u + Gu = f && \text{for} \quad (x, t) \in Q = \Omega \times I, \\ &\text{(ii)} \quad Tu = \varphi && \text{for} \quad (x, t) \in S = \Gamma \times I, \qquad (1.1) \\ &\text{(iii)} \quad u|_{t=0} = u_0 && \text{for} \quad x \in \Omega, \end{aligned}$$

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where Ω is a smooth bounded open set in $\mathbf{R}^n$ with boundary Γ, and I is an interval $I =]0, b[$, $b \leqslant +\infty$. With notation as in Boutet de Monvel [BM] (cf. also [Gr2]), P is a pseudo-differential operator on $\mathbf{R}^n$ of order $d > 0$ having the transmission property at Γ, P_Ω is its restriction to Ω, and G is a singular Green operator of order and class $\leqslant d$:

$$G = \sum_{0 \leqslant j < d} K_j \gamma_j + G'. \tag{1.2}$$

Here $\gamma_j u = (-i\partial/\partial n)^j u|_\Gamma$, the K_j are Poisson operators, and G' is a term that is well-defined on L^2. The operators act in a C^∞ vector bundle E over $\bar{\Omega}$. T is a normal system of trace operators T_j of orders $j < d$. We denote $P_\Omega + G = M$. ($\bar{\Omega}$ could also be taken as an n-dimensional manifold with boundary, as described in [BM], [Gr2].) The pseudo-differential formulation contains the usual parabolic differential operator problems as a special case.

The associated L^2-realization A, defined by

$$Au = (P_\Omega + G)\, u, \qquad D(A) = \{u \in H^d(\Omega, E) \mid Tu = 0\}, \tag{1.3}$$

is also studied.

Our results contain a complete description of the necessary and sufficient compatibility conditions on the data $\{f, \varphi, u_0\}$ for solutions to exist in specific spaces.

Here is an overview of the contents:

In Section 2, we recall the properties of Sobolev–Slobodetskiĭ spaces that we shall use, in particular the anisotropic spaces $H^{r,s}$ of order $r \geqslant 0$ in the space variable x and of order $s \geqslant 0$ in the time variable t. The spaces $H^{r,r/d}$ over Q and S are particularly relevant for the problem (1.1).

In Section 3, we briefly recall the definition of the various pseudo-differential boundary operators P_Ω, G, K, T, and S entering in the theory, and we establish their mapping properties in relation to the anisotropic spaces $H^{r,s}$.

Section 4 contains the definition of parabolicity (where, in particular, d is an even integer), and recalls the fundamental result of [Gr2] on the existence and estimates for the resolvent $R_\lambda = (A - \lambda)^{-1}$. We supply this with a theorem on how to modify A by a finite rank ps.d.o. of order $-\infty$ to get an operator with the spectrum lying in an arbitrary right half plane $\{\operatorname{Re} \lambda \geqslant a\}$.

Section 5 gives a complete discussion of the compatibility conditions needed to find a function $u \in H^{r,s}$ with given initial data $\partial_t^l u|_{t=0}$ on $\Omega \times \{0\}$ and boundary data $T_j u$ on S; they have to match at the "corner" $\Gamma \times \{0\}$. We here allow non-local (pseudo-differential, normal) trace operators T_j, extending results of Grisvard [Gri]; and for special values of

r and s we give a new integral compatibility condition, equivalent in a certain sense to that of [Gri], but invariantly formulated. Finally, we extend the interpolation theorem of Grisvard to non-local trace operators, showing that

$$[H_T^d(\Omega, E), L^2(\Omega, E)]_\theta = H_T^{(1-\theta)d}(\Omega, E) \qquad \text{for} \quad \theta \in]0, 1[, \qquad (1.4)$$

where H_T^s is the space of functions $u \in H^s$ such that those of the conditions $T_j u = 0$ that make sense are valid (with a special interpretation for $s - \frac{1}{2}$ integer).

In Section 6, we derive the main result on the solvability of (1.1), namely an existence and uniqueness theorem, *for all* $r \geq 0$, of a solution $u \in H^{r+d, r/d+1}(Q, \underline{E})$ (with estimates), for all sets of data

$$\begin{aligned} &f \in H^{r, r/d}(Q, \underline{E}), \\ &\varphi = \{\varphi_j\} \in \prod_{0 \leq j < d} H^{r+d-j-1/2, (r+d-j-1/2)/d}(S, \underline{F}_j), \qquad (1.5) \\ &u_0 \in H^{r+d/2}(\Omega, E), \end{aligned}$$

satisfying the appropriate compatibility conditions. (More precisely, the estimates in the case $I = \mathbf{R}_+$ depend on the lower bound of the spectrum of A, cf. also Purmonen [Pu].)

Section 7 finally gives some complements to this, namely an extension of the solvability to initial data u_0 in $H^s(\Omega, E)$, all $s \geq 0$, a study of the connection with holomorphic semigroup theory, and a proof of estimates for the solutions of (1.1) with $f = 0$, $\varphi = 0$,

$$\|u(t)\|_{H^r_{T,M}} \leq C_{r,s,a_1} t^{-(r-s)/d} e^{-a_1 t} \|u_0\|_{H^s_{T,M}} \qquad \text{for all} \quad t > 0, r \geq s \geq 0. \qquad (1.6)$$

This holds for all $u_0 \in H^s_{T,M}$ when a_1 is less than the real lower bound of the spectrum of A; but we furthermore show that (1.6) can be obtained for *any* a_1, globally in t, for initial values u_0 in a suitable closed subspace V of $L^2(\Omega, E)$ of finite codimension (depending on a_1). This is seen by use of the perturbation result from Section 4. We also get improved estimates for data $\{f, 0, 0\}$ with f valued in V.

2. Anisotropic Sobolev–Slobodetskiĭ Spaces

We denote by Ω a bounded open subset of $\mathbf{R}^n$ with smooth boundary $\partial\Omega = \Gamma$; or in some cases $\Omega = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_n > 0\}$, with $\partial\Omega = \mathbf{R}^{n-1}$, whose points are denoted $x' = (x_1, ..., x_{n-1})$. We write ∂_j or ∂_{x_j} for $\partial/\partial x_j$, and D_j or D_{x_j} for $-i\partial_j$, where $i = (-1)^{1/2}$, and we use the multiindex notation $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ with $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{N}^n$, etc., here $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $t \in \mathbf{R}$, $[t]$ denotes the largest integer $\leqslant t$. The scalar product of x and y in $\mathbf{R}^n$ is written $x \cdot y$, and the norm is $|x| = (x \cdot x)^{1/2}$. We shall use the notation, also for $x \in \mathbf{C}$,

$$\langle x \rangle = (1 + |x|^2)^{1/2}.$$

The restriction to the boundary is denoted $\gamma_0 : u \to u|_\Gamma$. We denote by $\vec{n} = (n_1, ..., n_n)$ the unit normal vector field defined near Γ, and we write $\gamma_0 D_\nu^j f = \gamma_j f$, where $D_\nu = \sum_{j=1}^n n_j D_j$. (In second order problems it is customary to use the real definition, where D_ν is replaced by $\partial_\nu = \sum_{j=1}^n n_j \partial_j$, but the above definition, also used in [Gr2], gives more convenient Green's formulas in higher order cases.)

Let us first recall the classical Sobolev–Slobodetskiĭ spaces, cf. [So, Sl, L–M]. Among several possibilities we have chosen to use the notation of Lions and Magenes [L–M] in the present paper (with some small modifications), since we also refer to their presentation of the interpolation theory. $H^r(\mathbf{R}^n)$ (also called $W_2^r(\mathbf{R}^n)$) is the space of distributions $v \in \mathscr{S}'(\mathbf{R}^n)$ for which $\|v\|_r \equiv \|\langle \xi \rangle^r \hat{v}(\xi)\|_{L^2(\mathbf{R}^n)}$ is finite; and $H^r(\Omega)$ is the space of restrictions $u = r_\Omega v$ to Ω, with norm

$$\|u\|_{H^r(\Omega)} \equiv \|u\|_r = \inf\{ \|v\|_r \mid v \in H^r(\mathbf{R}^n), u = r_\Omega v \}. \tag{2.1}$$

We denote by $H_0^r(\bar{\Omega})$ the closed subspace of $H^r(\mathbf{R}^n)$ consisting of the elements supported in $\bar{\Omega}$. Here $H_0^r(\bar{\Omega})$ coincides with $H_0^r(\Omega)$ as defined in [L–M], *except when* $r = m + 1/2$, $m \in \mathbf{N}$, where $H_0^{m+1/2}(\bar{\Omega})$ equals the space called $H_{00}^{m+1/2}(\Omega)$ in [L–M]; also $H^{-m-1/2}(\Omega)$ deviates from that of [L–M], when $m \in \mathbf{N}$. For every $r \in \mathbf{R}$, $H_0^r(\bar{\Omega})$ and $H^{-r}(\Omega)$ can be identified with one another's duals; and $C_0^\infty(\Omega)$ is dense in $H_0^r(\bar{\Omega})$ and $C^\infty(\bar{\Omega}) \cap H^r(\Omega)$ is dense in $H^r(\Omega)$ (cf., e.g., [H1, Section 2.5]). The present spaces have the interpolation property for *all* s, t (real interpolation):

$$\begin{aligned} &[H^s(\Omega), H^t(\Omega)]_\theta = H^{(1-\theta)s+\theta t}(\Omega), \\ &[H_0^s(\bar{\Omega}), H_0^t(\bar{\Omega})]_\theta = H_0^{(1-\theta)s+\theta t}(\bar{\Omega}), \qquad \text{for} \quad s \geqslant t,\ s \text{ and } t \in \mathbf{R},\ \theta \in]0, 1[. \end{aligned} \tag{2.2}$$

(This is shown in [L–M] for $s \geqslant t \geqslant 0$ in the first line and for $0 \geqslant s \geqslant t$ in the second line (and for general indices avoiding the half-integers); and it then follows in full generality, e.g., by use of the homeomorphism $\Lambda_r = OP((\langle \xi' \rangle + i\xi_n)^r)$ from $H^s(\mathbf{R}^n)$ to $H^{s-r}(\mathbf{R}^n)$ that sends $H_0^s(\bar{\mathbf{R}}_+^n)$ onto $H_0^{s-r}(\bar{\mathbf{R}}_+^n)$.) We recall the general interpolation inequality

$$\|u\|_{[X,Y]_\theta} \leqslant C \|u\|_X^{1-\theta} \|u\|_Y^\theta \qquad \text{for} \quad u \in X, \tag{2.3}$$

where X and Y are Hilbert spaces with $X \subset Y$ (continuous injection), X dense in Y.

For $r \geqslant 0$, r not integer, the norm in $H^r(\mathbf{R}^n)$ can also be described by

$$\|u\|^2_{H^r(\mathbf{R}^n)} \simeq \|u\|^2_{H^{[r]}(\mathbf{R}^n)} + \sum_{\alpha = [r]} \|D^\alpha u\|^2_{H^{r-[r]}_*(\mathbf{R}^n)}$$
$$\simeq \|u\|^2_{L^2(\mathbf{R}^n)} + \sum_{0 \leqslant |\alpha| \leqslant [r]} \|u\|^2_{H^{r-|\alpha|}_*(\mathbf{R}^n)}, \tag{2.4}$$

where the H^s_* seminorm is defined for $s \in \mathbf{R}_+ \backslash \mathbf{N}$ by

$$\|u\|^2_{H^s_*(\mathbf{R}^n)} = \sum_{|\beta| = [s]} \int_{\mathbf{R}^{2n}} |D^\beta u(x) - D^\beta u(y)|^2 \frac{dx\, dy}{|x-y|^{n+2(s-[s])}}, \tag{2.5}$$

following Aronszajn [Ar] and Slobodetskiĭ [Sl]. (For integer r, $\|u\|^2_r \simeq \sum_{|\alpha| \leqslant r} \|D^\alpha u\|^2_{L^2}$.) For $r \geqslant 0$, the elements of $H^r_0(\bar{\Omega})$ are identified with the functions they define on Ω (extended by zero on $\mathbf{R}^n \backslash \Omega$). Then when $r - \frac{1}{2}$ is not integer, $H^r_0(\bar{\Omega})$ identifies with the closed subspace of $H^r(\Omega)$ of functions u such that $\gamma_j u = 0$ for $0 \leqslant j < r - \frac{1}{2}$. For $r = m + \frac{1}{2}$, $m \in \mathbf{N}$, $H^r_0(\bar{\Omega})$ is a non-closed subspace of $H^r(\Omega)$ with a strictly stronger norm, satisfying

$$\|u\|^2_{H^r_0(\Omega)} \simeq \|u\|^2_{H^r(\Omega)} + \sum_{|\alpha| = [r]} \int_\Omega |D^\alpha u(x)|^2 \frac{dx}{\operatorname{dist}(x, \Gamma)^{2(r-[r])}}. \tag{2.6}$$

This formula is valid generally for $r \in \mathbb{R}_+$.

The definitions are extended to Γ by use of local coordinates, and they generalize easily to vector valued functions and distributions.

We now consider the spaces in the case with an extra variable t. Let $b \in \,]0, \infty]$, and denote $]0, b[\, = I$, or I_b for precision, $\Omega \times I = Q$ or Q_b, and $\Gamma \times I = S$ or S_b. When E is a vector bundle over Ω, we denote by $\underline{E}$ its lifting to Q (or to $\Omega \times I'$ for other intervals I'), and use a similar notation for liftings from Γ to S, etc. The basic type of space to be used is the anisotropic Sobolev space of order r in the x-variable and of order s in the t-variable, where r and $s \geqslant 0$:

$$H^{r,s}(Q, \underline{E}) = L^2(I; H^r(\Omega, E)) \cap H^s(I; L^2(\Omega, E)), \text{ with norm}$$
$$\|u\|_{r,s} = (\|u\|^2_{L^2(I; H^r)} + \|u\|^2_{H^s(I; L^2)})^{1/2};$$
$$\text{e.g., for } r, s \in \,]0, 1[, \ Q = \mathbf{R}^n \times \mathbf{R}, \tag{2.7}$$
$$\|u\|^2_{r,s} \simeq \int_{\mathbf{R}^{2n+1}} \frac{|u(x,t) - u(y,t)|^2}{|x-y|^{n+2r}} dx\, dy\, dt$$
$$+ \int_{\mathbf{R}^{n+2}} \frac{|u(x,t) - u(x,\tau)|^2}{|t-\tau|^{1+2s}} dx\, dt\, d\tau + \|u\|^2_{L^2(Q)}.$$

(Similar notation for spaces over Γ). When E is a trivial bundle, eg., $E=\Omega\times\mathbf{C}^N$, we usually just write $H^{r,s}(Q,\underline{E})$ as $H^{r,s}(Q)^N$, etc. (with N omitted in case $N=1$). By use of local coordinates, of the inequality, valid for $0\leqslant r'<r$, $s'=s(r-r')/r$,

$$\langle\xi\rangle^{r'}\langle\tau\rangle^{s'}=\langle\xi\rangle^{r'}(\langle\tau\rangle^{s/r})^{r-r'}\leqslant C_{r,s,r'}(\langle\xi\rangle^r+\langle\tau\rangle^s), \tag{2.8}$$

and of restriction and extension operators between spaces over $\mathbf{R}^n$ and Ω, one can show that

$$\|u\|_{H^{s'}(I;\,H^{r'}(\Omega))}\leqslant C'_{r,s,r'}\|u\|_{H^{r,s}(Q)},\qquad\text{for}\quad 0\leqslant r'<r,\ s'=s(r-r')/r. \tag{2.9}$$

As shown in [L–M, Proposition 4.2.1], the spaces have the interpolation property

$$[H^{r,s}(Q),H^{\rho,\sigma}(Q)]_\theta=H^{(1-\theta)r+\theta\rho,(1-\theta)s+\theta\sigma}(Q)$$
$$\text{for}\quad r\geqslant\rho\geqslant 0,\ s\geqslant\sigma\geqslant 0,\ \theta\in\,]0,1[. \tag{2.10}$$

The cases $H^{r,r/d}(Q,\underline{E})$, where d is an even integer, are of special interest for parabolic problems. In each of the spaces, the set of C^∞ sections is dense.

For precision, we sometimes use the notation for the restriction of a function $u(x,t)$ to $t=t_0$ (most often for $t_0=0$)

$$r_{t_0}u=u|_{t=t_0}, \tag{2.11}$$

and we sometimes denote $\Omega\times\{t_0\}=\Omega_{(t_0)}$ and $\Gamma\times\{t_0\}=\Gamma_{(t_0)}$ instead of just Ω or Γ.

We denote by $H^{r,s}_{(0)}(Q)$ the closed subspace of $H^{r,s}(\Omega\times\,]-\infty,b[)$ consisting of the functions supported in $\{t\geqslant 0\}$. It can be identified with a subspace of $H^{r,s}(Q)$, closed if $s-\frac12\notin\mathbf{N}$ (then it is the subspace of functions u with $r_0\partial_t^j u=0$ for $j<s-\frac12$), but not closed if $s-\frac12\in\mathbf{N}$; and the norm is equivalent to

$$\|u\|^2_{H^{r,s}_{(0)}(Q)}\simeq\|u\|^2_{H^{r,s}(Q)}+\int_0^b\|\partial_t^{[s]}u\|^2_{L^2(\Omega)}\frac{dt}{t^{2(s-[s])}}. \tag{2.12}$$

These spaces coincide with the spaces called $H^{r,s}_{,0}(Q)$ in [L–M], when $s-\frac12\notin\mathbf{N}$. With the present definition, the spaces have unrestrictedly the interpolation property

$$[H^{r,s}_{(0)}(Q),H^{\rho,\sigma}_{(0)}(Q)]_\theta=H^{(1-\theta)r+\theta\rho,(1-\theta)s+\theta\sigma}_{(0)}(Q)$$
$$\text{for}\quad r\geqslant\rho\geqslant 0,\ s\geqslant\sigma\geqslant 0,\ \theta\in\,]0,1[, \tag{2.13}$$

proved by a variant of the proof of [L–M, Proposition 4.2.1].

The space $H^{r,s}_{(0)}(S)$ is defined analogously, and the concepts extend of course to spaces of sections in vector bundles.

One can also define spaces with negative exponents r and s, but since this is quite complicated, we refrain from a systematic treatment (referring to [L–M]).

The following statements are well-known (see, e.g., [L–M, Theorem 4.2.1 and Proposition 4.2.3]):

LEMMA 2.1. *Let r and $s \in \bar{\mathbf{R}}_+$, and let $a(x,t) \in C^\infty(\bar{Q})$. The mappings $a(x,t)$, D_x^α, D_t^j, γ_0, and r_0, going from $C^\infty(\bar{Q})$ to $C^\infty(\bar{Q})$, $C^\infty(\bar{S})$, and $C^\infty(\bar{\Omega}_{(0)})$, respectively, extend to continuous mappings:*

$$\begin{aligned}
&a(x,t)\colon H^{r,s}(Q) \to H^{r,s}(Q);\\
&D_x^\alpha\colon H^{r,s}(Q) \to H^{r-|\alpha|,(r-|\alpha|)s/r}(Q) && \text{for } r>0,\ |\alpha| \leqslant r;\\
&D_t^j\colon H^{r,s}(Q) \to H^{(s-j)r/s,\,s-j}(Q) && \text{for } s>0, j \leqslant s; \qquad (2.14)\\
&\gamma_0\colon H^{r,s}(Q) \to H^{r-1/2,(r-1/2)s/r}(S) && \text{for } r>1/2;\\
&r_0\colon H^{r,s}(Q) \to H^{(s-1/2)r/s}(\Omega_{(0)}) && \text{for } s>1/2.
\end{aligned}$$

The mappings γ_0 and r_0 described here are surjective.

In particular, when $s = r/q$ for some $q>0$, one has the continuity properties

$$\begin{aligned}
&a(x,t)\colon H^{r,r/q}(Q) \to H^{r,r/q}(Q);\\
&D_x^\alpha\colon H^{r,r/q}(Q) \to H^{r-|\alpha|,(r-|\alpha|)/q}(Q) && \text{for } r>0,\ |\alpha| \leqslant r;\\
&D_t^j\colon H^{r,r/q}(Q) \to H^{r-jq,\,r/q-j}(Q) && \text{for } r>0, j \leqslant r/q; \qquad (2.15)\\
&\gamma_0\colon H^{r,r/q}(Q) \to H^{r-1/2,(r-1/2)/q}(S) && \text{for } r>1/2;\\
&r_0\colon H^{r,r/q}(Q) \to H^{r-q/2}(\Omega_{(0)}) && \text{for } r>q/2,
\end{aligned}$$

with γ_0 and r_0 surjective.

On the spaces $H^{r,s}(S)$, the differential operators and r_0 act in a similar way.

When $b = +\infty$, the mappings are of course just extended from mappings defined on $C^\infty(\bar{Q}_\infty) \cap H^{r,s}(Q_\infty)$. A similar convention is used in the following.

One has moreover the continuity and surjectiveness of the system of trace operators (see, e.g., [L–M, Theorem 4.2.3 and its proof]):

$$\text{(i)} \qquad \{\gamma_0, \ldots, \gamma_m\}\colon H^{r,r/q}(Q) \to \prod_{j=0}^{m} H^{r-j-1/2,(r-j-1/2)/q}(S) \qquad \text{for } r > m + 1/2, \qquad (2.16)$$

$$\text{(ii)}\qquad \{r_0, ..., r_0\partial_t^m\}: H^{r,r/q}(Q) \to \prod_{j=0}^{m} H^{r-jq-q/2}(\Omega_{\{0\}}) \qquad \text{for} \quad r/q > m + 1/2.$$

These operators have continuous right inverses. Also, $r_t u$ is defined for $t > 0$ when $u \in H^{r,r/q}(Q)$ with $r/q > \frac{1}{2}$; in fact, $r_t u$ (also called $u(t)$) depends continuously on t,

$$H^{r,r/q}(Q) \subset C^0(\bar{I}, H^{r-q/2}(\Omega)), \tag{2.17}$$

as a special case of the general trace theorem (valid for $X \subset Y$ densely and continuously)

$$L^2(I; X) \cap H^1(I; Y) \subset C^0(\bar{I}; [X, Y]_{1/2}). \tag{2.18}$$

3. Pseudo-differential Boundary Operators

Let us very briefly recall the structure of the operators entering in the Boutet de Monvel calculus. (Cf. Boutet de Monvel [BM], Grubb [Gr1, 2], or, e.g., Rempel and Schulze [R-S] for more details.)

First there are the pseudo-differential operators (ps.d.o.'s). Here one looks at operators P of order d defined on $\mathbf{R}^n$ (or on a neighborhood Ω' of $\bar{\Omega}$) by the usual formula

$$Pu(x) = OP(p)\, u(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi}\, p(x, \xi)\, \hat{u}(\xi)\, d\xi, \tag{3.1}$$

where $\hat{u}$ is the Fourier transform of u, and where the symbol $p(x, \xi)$ is a C^∞ function, developed in a series of terms $p^j(x, \xi)$ ($j \in \mathbf{N}$) homogeneous of degree $d - j$ in ξ for $|\xi| \geq 1$, of which p is an asymptotic sum, in the sense that

$$\left| D_x^\beta D_\xi^\alpha \left(p(x, \xi) - \sum_{0 \leq j < N} p^j(x, \xi) \right) \right| \leq c(x)\, \langle \xi \rangle^{d - |\alpha| - N} \qquad \text{for all} \quad \alpha, \beta, N,$$

with $c(x)$ continuous in x and dependent on the indices. The truncated operator P_Ω on Ω is defined by

$$P_\Omega = r_\Omega P e_\Omega, \tag{3.2}$$

where e_Ω denotes extension by zero on $\mathbf{R}^n \setminus \Omega$ and r_Ω denotes restriction from $\mathbf{R}^n$ to Ω. To assure that P_Ω has good continuity properties in the Sobolev spaces over $\bar{\Omega}$ (and in particular maps $C^\infty(\bar{\Omega})$ into $C^\infty(\bar{\Omega})$), P is assumed to have the so-called transmission property at Γ; this is in particular satisfied by operators composed of differential operators and inverses of elliptic differential operators. We take the order d of P to be integer.

The operators P_Ω of order $-\infty$ are precisely the integral operators with C^∞ kernel on $\bar{\Omega}$, sometimes called smoothing operators or negligible operators.

Sometimes P is given, with the help of local trivializations, as a mapping between vector bundles, $P: C^\infty(\Omega_1, \tilde{E}) \to C^\infty(\Omega_1, \tilde{E}')$, where $\tilde{E}$ and $\tilde{E}'$ are C^∞ vector bundles over $\Omega_1 \supset \bar{\Omega}$, with $\tilde{E}|_\Omega = E$, $\tilde{E}'|_\Omega = E'$. Then we likewise define $P_\Omega: C^\infty(\bar{\Omega}, E) \to C^\infty(\bar{\Omega}, E')$ by (3.2), where e_Ω now denotes extension by 0 on $\Omega_1 \setminus \Omega$.

Pseudo-differential operators S over the $(n-1)$-dimensional manifold Γ are defined from ps.d.o.'s on $\mathbf{R}^{n-1}$ with the help of local coordinates.

Poisson operators of order $d \in \mathbf{R}$ (going from $\mathbf{R}^{n-1}$ to $\mathbf{R}^n_+$) look basically as follows:

$$Kv(x) = OPK(\tilde{k})\, v(x) = (2\pi)^{1-n} \int_{\mathbf{R}^{n-1}} e^{ix'\cdot\xi'}\, \tilde{k}(x', x_n, \xi')\, \hat{v}(\xi')\, d\xi'. \tag{3.3}$$

Here $\tilde{k}$ (called the *symbol-kernel* in [Gr2], it is the inverse Fourier transform in ξ_n of the *symbol* $k(x', \xi)$ introduced in [BM]) satisfies estimates

$$\|D^\beta_{x'} x_n^m D^{m'}_{x_n} D^\alpha_{\xi'} \tilde{k}(x', x_n, \xi')\|_{L^2_{x_n}(\mathbf{R}_+)} \leqslant c(x') \langle \xi' \rangle^{d - 1/2 - |\alpha| - m + m'} \tag{3.4}$$

for all indices α, β, m, and m', and is an asymptotic series of quasi-homogeneous terms

$$\tilde{k}^j(x', \mu^{-1} x_n, \mu\xi') = \mu^{d-j} \tilde{k}^j(x', x_n, \xi') \qquad \text{for} \quad x' \geqslant 1, \mu \geqslant 1,$$

with $\tilde{k} - \sum_{0 \leqslant j < N} \tilde{k}^j$ satisfying (3.4) with d replaced by $d - N$. The corresponding terms in k are homogeneous in (ξ', ξ_n) of degree $d - 1 - j$ for $|\xi'| \geqslant 1$. Poisson operators going from Γ to Ω are defined from the ones above with the help of local coordinates.

A trace operator of order $d \in \mathbf{R}$, going from Ω to Γ, is an operator of the form

$$Tu = \sum_{0 \leqslant k \leqslant l-1} S_k \gamma_k u + T'u, \tag{3.5}$$

where the γ_k are the usual trace operators and the S_k are pseudo-differential operators on Γ of order $d - k$, respectively, and T' is a special kind of trace operator that in local coordinates (where Ω and Γ are replaced by $\mathbf{R}^n_+$ and $\mathbf{R}^{n-1}$) has the form

$$T'u(x') = OPT(\tilde{t}')\,u(x')$$
$$= (2\pi)^{1-n} \int_{\mathbf{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{t}'(x', x_n, \xi')\,\acute{u}(\xi', x_n)\,dx_n\,d\xi'; \quad (3.6)$$

here $\acute{u}(\xi', x_n)$ stands for $\mathscr{F}_{x'\to\xi'} u(x', x_n)$, and $\tilde{t}'$ is a symbol-kernel with the same properties as those of $\tilde{k}$ described above, only with $d-\frac{1}{2}$ replaced by $d+\frac{1}{2}$.

The number l in (3.5) is called the *class* of T, and T' is then said to be *of class* 0; T' is well-defined on $L^2(\Omega)$, whereas the sum $\sum_{0\leqslant k\leqslant l-1} S_k\gamma_k u$ is only well-defined on $H^r(\Omega)$ for $r>l-\frac{1}{2}$. One assigns the symbol $\sum_{0\leqslant k\leqslant l-1} s_k(x', \xi')\,\xi_n^k$ to this sum. Trace operators going from Ω to Γ are defined from the ones above with the help of local coordinates.

A scalar trace operator T (3.5) is said to be *normal* when $l=d+1$ and the coefficient S_d is an invertible function. More generally, when T maps sections in a vector bundle E over $\bar{\Omega}$ to sections in a vector bundle F over Γ (in particular, if T is matrix formed), normality means that $l=d+1$ and the coefficient S_d in the representation (3.5) is a surjective morphism (resp. matrix) from $E|_\Gamma$ onto F. We shall also consider systems of trace operators $\{T_{d_1}, ..., T_{d_m}\}$ going from E to $F_1\oplus\cdots\oplus F_m$ of orders $\{d_1, ..., d_m\}$; such a system is said to be normal when all the orders are different, and each T_{d_j} is normal (in particular, the class of T_{d_j} is d_j+1).

Finally, a singular Green operator G (s.g.o.) of order $d\in\mathbf{R}$ and class $l\in\mathbf{N}$ on Ω is an operator

$$Gu = \sum_{0\leqslant k\leqslant l-1} K_k\gamma_k u + G'u, \quad (3.7)$$

where the K_k are Poisson operators of order $d-k$ and G' has the form, in local coordinates where Ω is replaced by $\mathbf{R}^n_+$,

$$G'u(x) = OPG(\tilde{g}')\,u(x)$$
$$= (2\pi)^{1-n} \int_{\mathbf{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{g}'(x', x_n, y_n, \xi')\,\acute{u}(\xi', y_n)\,dy_n\,d\xi'. \quad (3.8)$$

Here the *symbol-kernel* $\tilde{g}'(x', x_n, y_n, \xi')$ (the inverse Fourier transform in ξ_n and inverse co-Fourier transform in η_n of the *symbol* $g'(x', \xi', \xi_n, \eta_n)$) satisfies estimates

$$\|D^\beta_{x'} x_n^m D^{m'}_{x_n} y_n^k D^{k'}_{y_n} D^\alpha_{\xi'}\,\tilde{g}'(x', x_n, y_n, \xi')\|_{L^2_{x_n, y_n}(\mathbf{R}_+\times\mathbf{R}_+)}$$
$$\leqslant c(x')\langle\xi'\rangle^{d-|\alpha|-m+m'-k+k'} \quad (3.9)$$

for all indices, and has an asymptotic expansion in a series of quasi-homogeneous terms

$$\tilde{g}'^j(x', \mu^{-1}x_n, \mu^{-1}y_n, \mu\xi') \\ = \mu^{d-j+1}\tilde{g}'^j(x', x_n, y_n, \xi') \qquad \text{for} \quad |\xi'| \geqslant 1, \mu \geqslant 1,$$

such that $\tilde{g}' - \sum_{0 \leqslant j < N} \tilde{g}'^j$ satisfies (3.9) with d replaced by $d-N$. The corresponding terms in g' are homogeneous in (ξ', ξ_n, η_n) of degree $d-1-j$ for $|\xi'| \geqslant 1$.

Again, the *class* l of G indicates how many of the classical traces it contains. The term G' is of class 0 and is defined on all of $L^2(\Omega)$. One assigns the symbol $\sum_{0 \leqslant k \leqslant l-1} k_k(x', \xi) \eta_n^k$ to the sum $\sum_{0 \leqslant k \leqslant l-1} K_k \gamma_k$.

Singular Green operators arise typically when Poisson and trace operators are composed (as in $G = KT$); another source is the composition of truncated ps.d.o.'s (cf. (3.2)), where

$$L(P, Q) \equiv (PQ)_\Omega - P_\Omega Q_\Omega \tag{3.10}$$

is a singular Green operator.

The five types of operators P_Ω, S, K, T, and G introduced above have the property that compositions among them again lead to operators belonging to these types. They are usually considered together in systems, generally of the form

$$\mathscr{A} = \begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(\bar{\Omega}, E) \\ \times \\ C^\infty(\Gamma, F) \end{matrix} \rightarrow \begin{matrix} C^\infty(\bar{\Omega}, E') \\ \times \\ C^\infty(\Gamma, F') \end{matrix}, \tag{3.11}$$

where E and E' are C^∞ vector bundles over $\bar{\Omega}$, F and F' are C^∞ vector bundles over Γ. When such a system satisfies a certain ellipticity condition, then it has a parametrix (or inverse) belonging to the calculus.

The operators have the continuity properties with respect to Sobolev spaces expressed briefly in the formula

$$\mathscr{A} = \begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix} : \begin{matrix} H^s(\Omega, E) \\ \times \\ H^{s-1/2}(\Gamma, F) \end{matrix} \rightarrow \begin{matrix} H^{s-d}(\Omega, E') \\ \times \\ H^{s-d-1/2}(\Gamma, F') \end{matrix}, \tag{3.12}$$

where each entry is of order d; $s \geqslant 0$ for P, G and T; $s - d \geqslant 0$ for G, T and K; and G and T are of class $\leqslant d$. (More refined information is given in [BM], [Gr2], and later in this paper.) G and T do not have a meaning on the full space $L^2(\Omega, E)$ unless their class is zero.

In the treatment of parabolic problems, we use the above pseudo-differential operators considered not only on $\bar{\Omega}$ but also on $\bar{\Omega} \times I$. For C^∞ functions, the operators are just defined as above with respect to x and considered as constant in t, and this definition extends by continuity to more general spaces. The operators can then be seen as special cases of t-dependent pseudo-differential boundary operators (or more general operators "acting in t" as well). We shall not study such a calculus systematically here. Let us just mention one point, which is amply discussed in [Gr2] (cf., e.g., Definition 1.5.14 and Proposition 2.3.14 there), namely that when the symbols of the *above* operators are considered as τ-dependent (where τ is the variable in the Fourier transform with respect to t) then one has to take the so-called *regularity number* ν into account. This is essentially *the Hölder continuity* (generalized to include negative values) *of the strictly homogeneous principal symbols at* $\xi' = 0$ (in suitable symbol norms, as in (3.4) and (3.9)); and really good estimates are only obtained when the regularity is $\geqslant 0$.

For example, when G is of the form $G = K\gamma_0$, where K is a Poisson (operator of order d with principal symbol-kernel $\tilde{k}^0$, then since the $L^2(\mathbf{R}_+)$ norm of $\tilde{k}^0$ is $\mathcal{O}(|\xi'|^{d-1/2})$, G is of regularity $d - \frac{1}{2}$. In this example, G *is of class* 1 (as defined above). Generally, when G if of order d and class l, the regularity ν is $\geqslant 0$ if and only if $d \geqslant \max\{l - \frac{1}{2}, 0\}$; more presicely,

$$G \text{ has regularity } \nu = \min\{d, d - l + \tfrac{1}{2}\}. \tag{3.13}$$

We do not explicitly use the regularity concept in the present paper, only refer to it when it comes up naturally. The systems $\{P_\Omega + G, T\}$ studied in the sequel have regularity $\nu \geqslant \frac{1}{2}$.

We shall now see how the various operators act in anisotropic Sobolev spaces. To begin with something that is really different from differential operators, consider Poisson operators.

PROPOSITION 3.1. *Let K be a Poisson operator of order $m \in \mathbf{R}$ (cf. (3.3), etc.), originally defined as an operator from $C^\infty(\Gamma)$ to $C^\infty(\bar{\Omega})$ and extended, as an operator that is constant in t, to map $C^\infty(\bar{S})$ into $C^\infty(\bar{Q})$. Then for any $r \geqslant \max\{m, \frac{1}{2}\}$ and $s \geqslant 0$, K extends to an operator with the continuity property*

$$K\colon H^{r-1/2,s}(S) \to H^{r-m,s}(Q), \qquad \text{if } m \leqslant \tfrac{1}{2}, \tag{3.14}$$

$$K\colon H^{r-1/2,(r-1/2)s/r}(S) \to H^{r-m,(r-m)s/r}(Q), \qquad \text{if } m \geqslant \tfrac{1}{2}. \tag{3.15}$$

Proof. It is known from [BM] that K is continuous,

$$K\colon H^{r-1/2}(\Gamma) \to H^{r-m}(\Omega), \tag{3.16}$$

for $r \geqslant m$. When the t-variable is included, this readily gives the continuity properties

$$\begin{aligned} K&: L^2(I; H^{r-1/2}(\Gamma)) \to L^2(I; H^{r-m}(\Omega)), \\ K&: H^s(I; H^{m-1/2}(\Gamma)) \to H^s(I; L^2(\Omega)). \end{aligned} \tag{3.17}$$

When $m \leqslant \frac{1}{2}$, this implies (3.14), since $L^2(\Omega)$ is continuously imbedded in $H^{m-1/2}(\Omega)$. When $m \geqslant \frac{1}{2}$, we get (3.15) from (3.17), when we furthermore apply a version of (2.9) with Ω replaced by Γ, r replaced by $r - \frac{1}{2}$, and r' taken equal to $m - \frac{1}{2}$. ∎

Remark 3.2. The Poisson operators have a more refined structure than what is apparent from (3.16). In fact, the operator K, considered in local coordinates (where Ω and Γ are replaced by $\mathbf{R}^n_+$ and $\mathbf{R}^{n-1}$), is continuous,

$$K: H^{r+r'-1/2}_{x'}(\mathbf{R}^{n-1}) \to H^{\{r-m,r'\}}_{x}(\mathbf{R}^n_+), \tag{3.18}$$

for all r and $r' \in \mathbf{R}$, where $H^{\{a,b\}}_x(\mathbf{R}^n_+)$ stands for the space of restrictions to $\mathbf{R}^n_+$ of distributions U on $\mathbf{R}^n$ satisfying

$$\langle \xi \rangle^a \langle \xi' \rangle^b \hat{U}(\xi) \in L^2(\mathbf{R}^n) \tag{3.19}$$

(cf., e.g., [Gr2, Theorem 2.5.1] for fixed μ). For example, (3.18) implies easily the following rule, allowing negative norms over the boundary:

$$\begin{aligned} &\|Kv\|_{L^2(I; H^{\{m',r'-m'-m\}}(\mathbf{R}^n_+))} \\ &\qquad \leqslant C \|v\|_{L^2(I; H^{r'-1/2}(\mathbf{R}^{n-1}))} \qquad \text{for all} \quad m' \geqslant 0, r' \in \mathbf{R}. \end{aligned} \tag{3.20}$$

One point of interest in this is that γ_j of any order can then be applied to Kv, giving $\gamma_j Kv \in L^2(I; H^{r'-1/2-m-j}(\mathbf{R}^{n-1}))$; in fact, $\gamma_j K$ is a pseudo-differential operator on Γ of order $m + j$.

There are related (x', x_n)-anisotropic estimates for the other types of pseudo-differential boundary operators, which we invoke whenever they are needed. However, we refrain from a systematic analysis here of estimates that are anisotropic in all three variables x', x_n, and t at the same time, in order to keep notations relatively simple. Actually, one can base such an analysis on the considerations already made in [Gr2] concerning μ-dependent norms, for μ^d (or $\mu^{r/s}$) there plays the same role as $|\tau|$ here (where τ is the variable corresponding to t by Fourier transformation).

For the other types of operators in the Boutet de Monvel calculus, one finds

PROPOSITION 3.3. (1) *Let P be a pseudo-differential operator on $\mathbf{R}^n$ of order $m \in \mathbf{Z}$, having the transmission property at Γ; define P_Ω on $C^\infty(\bar{\Omega})$ by*

(3.2) *and extend it to* $C^\infty(\bar{Q})$ *as an operator that is constant in* t. *Then for any* $r \geq \max\{m, 0\}$ *and any* $s \geq 0$, P_Ω *extends to an operator with the continuity property*

$$P_\Omega: H^{r,s}(Q) \to H^{r-m,s'}(Q), \qquad \textit{where} \quad s' = \min\{(r-m)\,s/r, s\}. \tag{3.21}$$

Here and below, $(r-m)\,s/r$ *is read as* s *when* $r=0$. *Similarly, when* S *is a ps.d.o. of order* $m \in \mathbf{R}$ *on* Γ, *extended to an operator in* $C^\infty(\bar{S})$ *that is constant in* t, *then* S *extends by continuity to a continuous operator*

$$S: H^{r,s}(S) \to H^{r-m,s'}(S) \tag{3.22}$$

when r, s, *and* s' *are as above.*

(2) *Let* T *be a trace operator of order* $m \in \mathbf{R}$ *and class* $l \in \mathbf{N}$, $T = \sum_{0 \leq k \leq l-1} S_k \gamma_k + T'$ (*cf.* (3.5)), *defined as an operator from* $C^\infty(\bar{Q})$ *to* $C^\infty(\bar{S})$ *that is constant in* t. *Let* G *be a singular Green operator of order* $m \in \mathbf{R}$ *and class* $l \in \mathbf{N}$, $G = \sum_{0 \leq k \leq l-1} K_k \gamma_k + G'$ (*cf.* (3.7)), *defined as an operator in* $C^\infty(\bar{Q})$ *that is constant in* t. *Let* $r \geq \max\{m + \frac{1}{2}, 0\}$ *for* T *and* $r \geq \max\{m, 0\}$ *for* G; *then for* $s \geq 0$ *and* $r > l - \frac{1}{2}$, T *and* G *extend to continuous operators*

$$\begin{aligned}
&T: H^{r,s}(Q) \to H^{r-m-1/2,s'}(S), \qquad \textit{where}\\
&\qquad s' = \min\{(r-m-\tfrac{1}{2})\,s/r, s\}, \qquad \textit{if} \quad l=0;\\
&\qquad s' = \min\{(r-m-\tfrac{1}{2})\,s/r, (r-l+\tfrac{1}{2})\,s/r\}, \qquad \textit{if} \quad l \geq 1;\\
&G: H^{r,s}(Q) \to H^{r-m,s'}(Q), \qquad \textit{where}\\
&\qquad s' = \min\{(r-m)\,s/r, s\}, \qquad \textit{if} \quad l=0;\\
&\qquad s' = \min\{(r-m)\,s/r, (r-l+\tfrac{1}{2})\,s/r\}, \qquad \textit{if} \quad l \geq 1.
\end{aligned} \tag{3.23}$$

Proof. (1) S has the continuity property

$$S: H^r(\Gamma) \to H^{r-m}(\Gamma), \tag{3.24}$$

for all r. When the t-variable is included, this gives the continuity properties

$$\begin{aligned}
&S: L^2(I; H^r(\Gamma)) \to L^2(I; H^{r-m}(\Gamma)),\\
&S: H^s(I; H^m(\Gamma)) \to H^s(I; L^2(\Gamma)).
\end{aligned} \tag{3.25}$$

For $m \leq 0 \leq r$, we then get (3.22) by use of the fact that $L^2(\Gamma) \subset H^m(\Gamma)$; and for $r \geq m \geq 0$, we use instead a version of (2.9) with Ω replaced by Γ and $r' = m$.

For P, [BM] shows that

$$P_\Omega : H^r(\Omega) \to H^{r-m}(\Omega). \tag{3.26}$$

As above, this gives the continuity properties, since $r \geqslant m$,

$$\begin{aligned} P_\Omega &: L^2(I; H^r(\Omega)) \to L^2(I; H^{r-m}(\Omega)), \\ P_\Omega &: H^s(I; H^m(\Omega)) \to H^s(I; L^2(\Omega)). \end{aligned} \tag{3.27}$$

For $m \leqslant 0 \leqslant r$, we then get (3.21) by use of the fact that $L^2(\Omega) \subset H^m(\Omega)$, and for $r \geqslant m \geqslant 0$, we use (2.9) with $r' = m$.

(2) Now consider T. For the terms $S_k \gamma_k$ we use that by Lemma 2.1, γ_k maps $H^{r,s}(Q)$ continuously into $H^{r-k-1/2,(r-k-1/2)s/r}(S)$, since $r > l - \frac{1}{2} \geqslant k + \frac{1}{2}$. Then (3.22) gives that the $S_k \gamma_k$ are continuous:

$$\begin{aligned} S_k \gamma_k &: H^{r,s}(Q) \to H^{r-m-1/2,(r-m-1/2)s/r}(S) \quad \text{if} \quad k \leqslant m, \\ S_k \gamma_k &: H^{r,s}(Q) \to H^{r-m-1/2,(r-k-1/2)s/r}(S) \quad \text{if} \quad k \geqslant m. \end{aligned} \tag{3.28}$$

For the term T' of class 0 it is known (cf., e.g., [Gr2, Proposition 2.5.2]) that in local coordinates

$$\|T'u\|_{H^{\sigma-m-1/2}_{x'}(\mathbf{R}^{n-1})} \leqslant C\, \|u\|_{L^2(\mathbf{R}_{+,x_n}, H^\sigma(\mathbf{R}^{n-1}_{x'}))} \quad \text{for all} \quad \sigma \in \mathbf{R}. \tag{3.29}$$

It follows, by application of $\mathscr{F}_{t \to \tau}$ to $T'u$ and u and integration in τ, that

$$\begin{aligned} &\|T'u\|_{L^2(\mathbf{R}_t; H^{\sigma-m-1/2}(\mathbf{R}^{n-1}_{x'}))} \\ &\qquad \leqslant C\, \|u\|_{L^2(\mathbf{R}_t, L^2(\mathbf{R}_{+,x_n}, H^\sigma(\mathbf{R}^{n-1}_{x'})))} \quad \text{for all} \quad \sigma \in \mathbf{R}. \end{aligned} \tag{3.30}$$

When $m \leqslant -\frac{1}{2}$, we use that (3.29) gives in particular, for $\sigma = m + \frac{1}{2} \leqslant 0$,

$$\|T'u\|_{L^2_{x'}(\mathbf{R}^{n-1})} \leqslant C\, \|u\|_{L^2_x(\mathbf{R}^n_+)},$$

from which

$$\|T'u\|_{H^{0,s}(\mathbf{R} \times \mathbf{R}^{n-1})} \leqslant C\, \|u\|_{H^{0,s}(\mathbf{R} \times \mathbf{R}^n_+)}$$

follows as usual. This, together with (3.30) for $\sigma = r$, shows the continuity of

$$T' : H^{r,s}(\mathbf{R} \times \mathbf{R}^n_+) \to H^{r-m-1/2,s}(\mathbf{R} \times \mathbf{R}^{n-1}) \quad \text{when} \quad m \leqslant -\tfrac{1}{2}. \tag{3.31}$$

When $m \geqslant -\frac{1}{2}$, we have from (3.29) that

$$\|T'u\|_{L^2_{x'}(\mathbf{R}^{n-1})} \leqslant C\, \|u\|_{H^{m+1/2}_x(\mathbf{R}^n_+)}.$$

Taking $H^{(r-m-1/2)s/r}$ norms in the t-variable we find, by application of (2.9) with $r' = r - m - \frac{1}{2}$,

$$\|T'u\|_{H^{0,(r-m-1/2)s/r}(\mathbf{R}\times\mathbf{R}^{n-1})} \leqslant C\,\|u\|_{H^{r,s}(\mathbf{R}\times\mathbf{R}^n_+)}.$$

Together with (3.30) for $\sigma = r$, this shows the continuity of

$$T'\colon H^{r,s}(\mathbf{R}\times\mathbf{R}^n_+) \to H^{r-m-1/2,(r-m-1/2)s/r}(\mathbf{R}\times\mathbf{R}^{n-1}), \qquad \text{when} \quad m \geqslant -\tfrac{1}{2}. \tag{3.32}$$

(3.28), (3.31), and (3.32) together imply the statements on T in (3.23).

(3) The singular Green operator G is treated as follows. The terms $K_k\gamma_k$ satisfy, by Lemma 2.1 and Proposition 3.1,

$$\begin{aligned} K_k\gamma_k\colon H^{r,s}(Q) &\to H^{r-m,(r-m-1/2)s/r}(Q) && \text{if} \quad k+\tfrac{1}{2}\leqslant m,\\ K_k\gamma_k\colon H^{r,s}(Q) &\to H^{r-m,(r-k-1/2)s/r}(Q) && \text{if} \quad k+\tfrac{1}{2}\geqslant m. \end{aligned} \tag{3.33}$$

For G' one has, by [BM], that G' is continuous,

$$G'\colon H^r(\Omega) \to H^{r-m}(\Omega). \tag{3.34}$$

Then one proceeds exactly as for P, obtaining

$$\begin{aligned} G'\colon H^{r,s}(Q) &\to H^{r-m,(r-m)s/r}(Q) && \text{for} \quad r\geqslant m\geqslant 0,\\ G'\colon H^{r,s}(Q) &\to H^{r-m,s}(Q) && \text{for} \quad r\geqslant 0\geqslant m. \end{aligned} \tag{3.35}$$

Taken together, the statements show (3.23). ∎

Remark 3.4. The various rules are a little complicated in the way one distinguishes between the cases where s' takes the most natural, proportional value ($s' = (r-m)\,s/r$ for P, G, and K; $s' = (r-m-\frac{1}{2})\,s/r$ for T) and the other cases. For P and K, this is linked directly with the *order* m of the operator. But for G and T it is linked also with the *class* l; if $l > \max\{m+\frac{1}{2}, 0\}$, the class takes over in the definition of s'. These cases are precisely the ones with negative regularity ν, as defined in (3.13) and in [Gr2, Sects. 1.5, 2.3]; and the whole discussion could be formulated in terms of the regularity concept analyzed in [Gr2].

In the direct calculations connected with parabolic pseudo-differential problems, the operators are of positive regularity, so the proportional values enter. However, the general statements above are needed in some finer calculations, as in [G–S3].

COROLLARY 3.5. *Let $q>0$. For the operators considered in Propositions* 3.1 *and* 3.3, *one has in particular the continuity properties*

$$K\colon H^{r-1/2,(r-1/2)/q}(S)\to H^{r-m,r'}(Q), \qquad \text{where} \tag{3.36}$$

$$r'=\min\{(r-\tfrac{1}{2})/q,\ (r-m)/q\};$$

$$P_\Omega\colon H^{r,r/q}(Q)\to H^{r-m,\min\{(r-m)/q,r/q\}}(Q); \tag{3.37}$$

$$S\colon H^{r,r/q}(S)\to H^{r-m,\min\{(r-m)/q,r/q\}}(S); \tag{3.38}$$

$$T\colon H^{r,r/q}(Q)\to H^{r-m-1/2,r'}(S), \qquad \text{where} \tag{3.39}$$

$$r'=\min\{(r-m-\tfrac{1}{2})/q,\ r/q\}, \qquad \text{if} \quad l=0;$$

$$r'=\min\{(r-m-\tfrac{1}{2})/q,\ (r-l+\tfrac{1}{2})/q\}, \qquad \text{if} \quad l\geqslant 1;$$

$$G\colon H^{r,r/q}(Q)\to H^{r-m,r'}(Q), \qquad \text{where} \tag{3.40}$$

$$r'=\min\{(r-m)/q,\ r/q\}, \qquad \text{if} \quad l=0;$$

$$r'=\min\{(r-m)/q,\ (r-l+\tfrac{1}{2})/q\}, \qquad \text{if} \quad l\geqslant 1.$$

In all these statements, $r\geqslant\max\{m,0\}$; *moreover*, $r\geqslant\frac{1}{2}$ *in* (3.36) *and* $r>l-\frac{1}{2}$ *in* (3.39) *and* (3.40).

All the rules extend without difficulty to operators between vector bundles.

4. PARAMETER-ELLIPTIC AND PARABOLIC PROBLEMS

The systems we consider are of the following kind: E is an N-dimensional C^∞ vector bundle over $\bar{\Omega}$, extended to a bundle $\tilde{E}$ over a neighborhood Ω_1 of $\bar{\Omega}$. There is given a pseudo-differential operator P of order d (integer >0) in $\tilde{E}$ having the transmission property at Γ, and a singular Green operator G in E of order d and class $\leqslant d$. The boundary condition is defined in terms of a column vector of trace operators $T=\{T_0,\dots,T_{d-1}\}$, where the T_l go from E to bundles F_l over Γ of fiber dimension N'_l ($N'_l\geqslant 0$), with T_l of order l and class $\leqslant l+1$, i.e., of the form

$$T_l=\sum_{0\leqslant j\leqslant l} S_{lj}\gamma_j+T'_l,$$

with T' of class 0. We denote $\sum_{l=0}^{d-1} N'_l=N'$ and $F_0\oplus\cdots\oplus F_{d-1}=F$. (The zero-dimensional bundles over Γ have been included for notational convenience—to avoid having to specify the subset of $\{0,\dots,d-1\}$ consisting of the orders of the non-trivial trace operators.) Then we define:

DEFINITION 4.1. (1) Let $\theta \in \mathbf{R}$. The system

$$\mathscr{A}_\mu = \begin{pmatrix} P_\Omega + G + \mu^d e^{i\theta} \\ T \end{pmatrix}, \tag{4.1}$$

depending on the parameter μ, is said to be parameter-elliptic when the following properties hold (expressed in local coordinates):

(I) The principal interior symbol $p^0(x, \xi) + \mu^d e^{i\theta}$ is bijective for all x, all $(\xi, \mu) \in \bar{\mathbf{R}}^{n+1}_+$ with $|\xi|^2 + |\mu|^2 \geq 1$.

(II) The principal boundary symbol operator

$$a^0(x', \xi', \mu, D_n) = \begin{pmatrix} p^0(x', 0, \xi', D_n)_\Omega + g^0(x', \xi', D_n) + \mu^d e^{i\theta} \\ t^0(x', \xi', D_n) \end{pmatrix}:$$

$$H^d(\mathbf{R}_+)^N \to L^2(\mathbf{R}_+)^N \times \mathbf{C}^{N'} \tag{4.2}$$

is a bijection for all x', all $(\xi', \mu) \in \bar{\mathbf{R}}^n_+$, with $|\xi'| \geq 1$, $\mu \geq 0$.

(III) For each $\mu > 0$, each x', the strictly homogeneous principal boundary symbol operator $a^h(x', \xi', \mu, D_n)$ (coinciding with (4.2) for $|\xi'| \geq 1$) converges in the operator norm for $\xi' \to 0$ to a limit operator

$$a^h(x', 0, \mu, D_n) \colon H^d(\mathbf{R}_+)^N \to L^2(\mathbf{R}_+)^N \times \mathbf{C}^{N'}, \tag{4.3}$$

which is bijective.

(2) The time-dependent system $\{\partial_t + P_\Omega + G, T\}$ is said to be parabolic when $\{P_\Omega + \mu^d e^{i\theta} + G, T\}$ is parameter-elliptic for all $\theta \in [-\pi/2, \pi/2]$.

The above formulation is a version of Definitions 1.5.5 and 3.1.3 in [Gr2]. One small difference is that we here consider the boundary symbol operators as going from $H^d(\mathbf{R}_+)^N$ to $L^2(\mathbf{R}_+)^N \times \mathbf{C}^{N'}$ instead of from $\mathscr{S}(\mathbf{R}_+)^N$ to $\mathscr{S}(\mathbf{R}_+)^N \times \mathbf{C}^{N'}$, which does not change the content of the ellipticity property, as explained after Definition 3.1.3 there. Also, the convergence in symbol norm, defined generally in [Gr2], coincides with the convergence in operator norm in the present case.

We recall from [Gr2, Remark 1.5.10] that when (2) holds, d is even and $N' = Nd/2$.

We also recall from [Gr2, Lemma 1.5.7] that when (1) holds on a ray, T is necessarily *normal* (the contribution from T to the limit boundary symbol operator $a^h(x', 0, \mu, D_n)$ is the standard trace operator $\{S_{00}\gamma_0, \ldots, S_{d-1,d-1}\gamma_{d-1}\}$, where the S_{ll} are surjective vector bundle morphisms).

One of the main results in the book [Gr2] is that the parameter-ellipticity implies resolvent estimates in the following way (cf. [Gr2, Theorem 3.3.1, Corollary 3.3.2]):

THEOREM 4.2. *Let P, G, and T be as defined in Definition 4.1, and let θ be a value for which (1) holds. Let A be the associated L^2-realization*

$$Au = (P_\Omega + G)\, u, \qquad D(A) = \{u \in H^d(\Omega, E) \mid Tu = 0\}, \tag{4.4}$$

and let $R_\lambda = (A - \lambda)^{-1}$ be the resolvent, i.e., the solution operator for the problem

$$\begin{aligned} (P_\Omega + G - \lambda)\, u &= f \quad \text{in} \quad \Omega, \\ Tu &= 0 \quad \text{on} \quad \Gamma, \end{aligned} \tag{4.5}$$

defined for the $\lambda \in \mathbf{C}$ for which it exists as a bounded operator in $L^2(\Omega, E)$.

There are an ε and a constant r_0 such that the resolvent exists on the rays $\lambda = re^{i\theta'}$, $r \geqslant r_0$, $\theta' \in [\theta - \varepsilon, \theta + \varepsilon]$, satisfying for each $s \in \bar{\mathbf{R}}_+$ the estimate

$$\begin{aligned} &\|R_\lambda f\|_{s+d} + \langle \lambda \rangle^{s/d+1} \|R_\lambda f\|_0 \\ &\quad \leqslant C_s(\|f\|_s + \langle \lambda \rangle^{s/d} \|f\|_0) \qquad \text{for} \quad f \in H^s(\Omega, E). \end{aligned} \tag{4.6}$$

Moreover, the spectrum $\sigma(A)$ is discrete, and R_λ is holomorphic on $\mathbf{C} \setminus \sigma(A)$.

In particular, if $\{\partial_t + P_\Omega + G, T\}$ is parabolic, the spectrum of A has a real lower bound $a_0 = \min\{Re\, \lambda \mid \lambda \in \sigma(A)\} > -\infty$ and lies in a sectorial region

$$\mathscr{V} = \{\lambda \mid \operatorname{Re} \lambda \geqslant a_0\} \cap \{\lambda \mid |\lambda| \leqslant r_1 \ or\ |arg\, \lambda| \leqslant \pi/2 - \varepsilon_1\}, \tag{4.7}$$

for some $\varepsilon_1 > 0$, $r_1 \geqslant 0$; and for any $\delta > 0$, (4.6) holds (with C_s depending on δ) in

$$\begin{aligned} \mathscr{W}_\delta = \{\lambda \mid \operatorname{Re} \lambda \leqslant a_0 - \delta\} \cup \{\lambda \mid |\lambda| \geqslant r_1 + \delta \\ and\ arg\, \lambda \in [\pi/2 - \varepsilon_1 + \delta, 3\pi/2 + \varepsilon_1 - \delta\}. \end{aligned} \tag{4.8}$$

The discreteness of the spectrum follows from the compactness of R_c for a constant c in the resolvent set, and this compactness follows from the continuity of R_c as an operator from $L^2(\Omega, E)$ to $H^d(\Omega, E)$.

The points in the spectrum are eigenvalues of A, but the eigenfunctions need not form a basis of $L^2(\Omega, E)$ (they do of course if A is selfadjoint). It is of interest to know in the parabolic case whether one can remove the eigenvalues in $\{\operatorname{Re} \lambda \leqslant 0\}$ by a finite dimensional modification of A. This can in fact be done in a quite concrete way.

THEOREM 4.3. *Assume that parabolicity holds. Let* $\{\lambda_k\}$ *be the sequence of mutually distinct eigenvalues, ordered in such a way that*

$$\operatorname{Re}\lambda_0 \leqslant \operatorname{Re}\lambda_1 \leqslant \cdots \leqslant \operatorname{Re}\lambda_k \leqslant \cdots; \tag{4.9}$$

then $\operatorname{Re}\lambda_k \to \infty$. *Let* $k' \in \mathbf{N}$. *There is a decomposition of* $X = L^2(\Omega)$ *into closed subspaces* V *and* W *such that*

$$\begin{gathered} X = V \dot{+} W, \qquad W \text{ has finite dimension,} \\ D(A) = (D(A) \cap V) \dot{+} W, \\ A(D(A) \cap V) \subset V \qquad \text{and} \qquad A(W) \subset W, \end{gathered} \tag{4.10}$$

and the operator $A|_V$ *(with domain* $D(A) \cap V$*) has the spectrum*

$$\sigma(A|_V) = \{\lambda_k\}_{k \geqslant k'}. \tag{4.11}$$

For any linear operator B *in* W, *the operator* A_B *defined by*

$$A_B u = Av + Bw \qquad \text{for} \quad u = v + w \text{ with } v \in D(A) \cap V,\, w \in W, \tag{4.12}$$

has the spectrum

$$\sigma(A_B) = \{\lambda_k\}_{k \geqslant k'} \cup \sigma(B); \tag{4.13}$$

and $\mathscr{S} = A_B - A$ *is a ps.d.o. of order* $-\infty$ *and finite rank (written explicitly in* (4.20-21) *below). One can in particular choose* B *such that, e.g.,*

$$\sigma(A_B) \subset \{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda \geqslant \operatorname{Re}\lambda_{k'}\}, \qquad \text{or} \qquad \sigma(A_B) = \{\lambda_k\}_{k \geqslant k'}. \tag{4.14}$$

Proof. Let $c < \operatorname{Re}\lambda_0$; then $R_c = (A - c)^{-1}$ is a compact operator in X with spectrum equal to the set $\{\mu_k = (\lambda_k - c)^{-1}\}_{k \in \mathbf{N}} \subset \{\operatorname{Re}\lambda > 0\}$. By a theorem of F. Riesz (see, e.g., Riesz and Nagy [R–N, Sects. 77–80]), there is an integer $\nu_0 \geqslant 1$ such that

$$Z_0 = \{u \in X \mid (\mu_0 - R_c)^n u = 0 \text{ for some } n\} = \{u \in X \mid (\mu_0 - R_c)^{\nu_0} u = 0\},$$

Z_0 has finite dimension $r_0 \geqslant \nu_0$, and Z_0 has the closed complement V_0, where

$$V_0 = \bigcap_{n=1}^{\infty} (\mu_0 - R_c)^n X = (\mu_0 - R_c)^{\nu_0} X.$$

Clearly, Z_0 and V_0 are invariant under R_c, and the spectrum of $R_c|_{V_0}$ is precisely the set $\{\mu_k\}_{k \geqslant 1}$. The procedure is repeated, with

$$Z_j = \{u \in V_{j-1} \mid (\mu_j - R_c)^n u = 0 \text{ for some } n\}$$
$$= \{u \in V_{j-1} \mid (\mu_j - R_c)^{\nu_j} u = 0\},$$
$$V_j = \bigcap_{n=1}^{\infty} (\mu_j - R_c)^n V_{j-1} = (\mu_j - R_c)^{\nu_j} V_{j-1},$$

until we arrive at the closed subspace

$$V = V_{k'-1} = (\mu_{k'-1} - R_c)^{\nu_{k'-1}} \cdots (\mu_1 - R_c)^{\nu_1} (\mu_0 - R_c)^{\nu_0} X, \qquad (4.15)$$

which is invariant under R_c and has the complement W in X,

$$W = Z_0 \dot{+} Z_1 \dot{+} \cdots \dot{+} Z_{k'-1} \qquad (4.16)$$

of dimension $r = r_0 + r_1 + \cdots r_{k'-1}$. W is likewise invariant under R_c. The spectrum of $R_c|_V$ is the set $\{\mu_k\}_{k > k'}$. Since $A = R_c^{-1} + c$, V and W are also invariant under A (in the sense expressed in (4.10)); and the spectrum of $A|_V$ is the set $\{\lambda_k\}_{k > k'}$.

Now define the perturbation A_B of A by (4.12). Clearly, $A_B - \lambda$ is bijective from $D(A)$ to X if and only if both $A|_V - \lambda$: $D(A) \cap V \to V$ and $B - \lambda$: $W \to W$ are bijective, so (4.13) follows. We here have

$$A_B = A + (B - A)\,\pi = A + C\pi, \qquad (4.17)$$

where π is the projection of X onto W along V, and C is the linear operator $B - A|_W$ in W. By the ellipticity of A, W lies in $C^\infty(\bar{\Omega}, E)$, cf. (4.16). Since R_c^* is a resolvent associated with a normal elliptic pseudo-differential boundary problem adjoint to the given one (cf. [Gr2, Sect. 1.6]), the space $V^\perp$, equal to

$$V^\perp = \{z \in X \mid (\bar{\mu}_0 - R_c^*)^{\nu_0} (\bar{\mu}_1 - R_c^*)^{\nu_1} \cdots (\bar{\mu}_{k'-1} - R_c^*)^{\nu_{k'-1}} z = 0\}, \quad (4.18)$$

lies in $C^\infty(\bar{\Omega}, E)$; it has dimension r like W. Then the projection π can be written as

$$\pi u = \sum_{j,k=1}^{r} a_{jk}(u, z_k)\, w_j, \qquad (4.19)$$

where the z_k form a basis of $V^\perp$ and the w_j a basis of W, all C^∞ functions. It follows that $A_B - A$ can be written as

$$\mathscr{S}u = (A_B - A)\, u = C\pi u = \sum_{j,k=1}^{r} c_{jk}(u, z_k)\, w_j, \qquad (4.20)$$

so it is an integral operator with C^∞ kernel

$$K_{\mathscr{S}}(x, y) = \sum_{j,k=1}^{r} c_{jk} w_j(x) \bar{z}_k(y), \tag{4.21}$$

hence a ps.d.o. of order $-\infty$ (a negligible, smoothing operator) in the calculus.

For the last statement in the theorem, one takes B with spectrum in $\{\lambda \in \mathbf{C} \mid \operatorname{Re} \lambda \geq \operatorname{Re} \lambda_{k'}\}$, resp. with $B = \lambda_{k_1} I$ for some $k_1 \geq k'$. ▮

When A is selfadjoint, $v_0 = \cdots = v_{k'-1} = 1$, and $Z_0, ..., Z_{k'-1}$ are the eigenspaces associated with the eigenvalues $\lambda_0, ..., \lambda_{k'-1}$ for A; so V is then simply the closed span of the eigenspaces associated with the eigenvalues $\{\lambda_k\}_{k \geq k'}$.

Remark 4.4. In the above construction, it is of interest to know whether $C = B - A|_W$ can be chosen with special properties, e.g., with much lower rank than r, or with range in a special subspace of W. (Actually, one could also let C map out of W; then it is $A|_W + \pi C$ that determines the new spectrum.) In particular, when the operators in W are expressed as matrices in terms of a basis of W, one can ask whether C can be obtained on the form $C = EF$, where E is a given $(r \times q)$-matrix with $q < r$. Questions of this kind are well known in control theory; see, e.g., Wonham [W, Chap. 2] for an analysis of the possibility of choosing F such that

$$B = A|_W + EF$$

has prescribed eigenvalues (pole assignment) or eigenvalues in a particular set (pole shifting). (For example, if $\dim Z_0 = \cdots = \dim Z_{k'-1} = 1$, then $\sigma(B) \subset \mathbf{R}_+$ can be obtained by a C of rank 1!) In this connection, it is also interesting to see what it takes to interpret C as a "boundary feedback" term (cf. the articles of Lasiecka and Triggiani, e.g., [L–T], and a recent analysis by Pedersen [Pe]; these works treat cases where the eigenvectors form a basis).

5. Compatibility Conditions

According to Lemma 2.1, both of the mappings γ_0 and r_0 are defined for $u \in H^{r,s}(Q)$ when $r > \frac{1}{2}$ and $s > \frac{1}{2}$, with

$$\gamma_0 u \in H^{r-1/2,(r-1/2)s/r}(S), \qquad r_0 u \in H^{(s-1/2)r/s}(\Omega_{(0)}). \tag{5.1}$$

Here the restrictions to the "corner" $\Gamma_{(0)} = \{x \in \Gamma, t = 0\}$ are defined according to Lemma 2.1, when $(r-\frac{1}{2})\,s/r>\frac{1}{2}$, and $(s-\frac{1}{2})\,r/s>\frac{1}{2}$, i.e., when $1/r+1/s<2$, and then

$$r_0\gamma_0 u=\gamma_0 r_0 u\in H^{r-1/2-r/2s}(\Gamma_{(0)}),$$

by extension by continuity of the identity valid for smooth functions. So if an initial–boundary value problem is considered, where both $\gamma_0 u$ and $r_0 u$ are prescribed,

$$\gamma_0 u=\varphi, \qquad r_0 u=v,$$

then in order to have solutions in $H^{r,s}(Q)$ with $1/r+1/s<2$, we must necessarily prescribe the *compatibility condition* (or corner condition)

$$r_0\varphi=\gamma_0 v. \tag{5.2}$$

For the study of differential equations in higher order $H^{r,s}$ spaces, one also needs links between initial and boundary values of x and t-derivatives; more, the larger r is.

The limiting cases $1/r+1/s=2$ are not covered by Lemma 2.1. Also when r and s are larger, there occur exceptional cases, now for the derivatives of u. So in order to have solvability results for a full scale of values of r, we must include these cases in the study.

In the following, we set

$$r/s=q, \qquad \text{i.e., } s=r/q;$$

then the limiting cases for the consideration of (5.2) are the cases

$$r=(q+1)/2, \qquad s=(q+1)/2q=(1/q+1)/2, \qquad \text{where} \quad q\in\mathbf{R}_+.$$

Let us take a fixed $q\in\mathbf{R}_+$ and define the integral

$$I[\varphi,v]=\int_{t\in I}\int_{x'\in\Gamma_{(t)}}\int_{y\in\Omega_{(0)}}\frac{|\varphi(x',t)-v(y)|^2}{(|x'-y|^q+t)^{1+n/q}}\,dy\,d\sigma_{x'}\,dt \tag{5.3}$$

for functions $\varphi(x',t)$ on S and $v(y)$ on $\Omega_{(0)}$ ($\sigma_{x'}$ is the surface measure on Γ). When this integral is finite, we say that φ and v *coincide at* $\Gamma_{(0)}$. Related integrals (in $W_p^{r,r/2}$ spaces) appear in [Sa–S] and in [L–S–U, p. 317].

THEOREM 5.1. (1) *There is a constant C_1 such that when $u\in H^{(q+1)/2,(q+1)/2q}(Q)$, then $\gamma_0 u\in H^{q/2,1/2}(S)$ and $r_0 u\in H^{1/2}(\Omega_{(0)})$ coincide at $\Gamma_{(0)}$, with*

$$I[\gamma_0 u, r_0 u]\le C_1\,\|u\|^2_{H^{(q+1)/2,(q+1)/2q}(Q)}. \tag{5.4}$$

(2) *There is a constant C_2 such that if $\varphi \in H^{q/2,1/2}(S)$ and $v \in H^{1/2}(\Omega_{(0)})$ coincide at $\Gamma_{(0)}$, i.e., $I[\varphi, v] < \infty$, then there exists $u \in H^{(q+1)/2,(q+1)/2q}(Q)$ such that $\gamma_0 u = \varphi$ and $r_0 u = v$ and*

$$\|u\|^2_{H^{(q+1)/2,(q+1)/2q}(Q)} \leqslant C_2(\|\varphi\|^2_{H^{q/2,1/2}(S)} + \|v\|^2_{H^{1/2}(\Omega_{(0)})} + I[\varphi, v]). \tag{5.5}$$

(3) *In particular, when $\varphi \in H^{q/2,1/2}(S)$, it coincides with 0 at $\Gamma_{(0)}$ if and only if $\varphi \in H^{q/2,1/2}_{(0)}(S)$. In that case, there exists $u \in H^{(q+1)/2,(q+1)/2q}(Q)$ with $\gamma_0 u = \varphi$ and $r_0 u = 0$, i.e., $u \in H^{(q+1)/2,(q+1)/2q}_{(0)}(Q)$. If v is given in $H^{1/2}_0(\bar{\Omega}_{(0)})$, then there exists $u \in H^{(q+1)/2,(q+1)/2q}(Q)$ with $\gamma_0 u = 0$ and $r_0 u = v$.*

Proof. In local coordinates, (5.3) takes the form

$$I[\varphi, v] = \int_{t \in \mathbf{R}_+} \int_{x' \in \mathbf{R}^{n-1}} \int_{y \in \mathbf{R}^n_+} \frac{|\varphi(x', t') - v(y', y_n)|^2}{(|x' - y'|^q + y_n^q + t)^{1+n/q}} \, dy \, dx' \, dt, \tag{5.6}$$

and we shall show that when $\varphi \in H^{q/2,1/2}(\mathbf{R}^{n-1} \times \mathbf{R}_+)$ and $v \in H^{1/2}(\mathbf{R}^n_+)$, $I[\varphi, v]$ is finite if and only if the following integral introduced in Grisvard [Gri] is finite:

$$\begin{aligned} I'[\varphi, v] &= \int_{\mathbf{R}^n_+} |\varphi(x', t) - v(x', t^{1/q})|^2 \frac{dx' \, dt}{t} \\ &= q \int_{\mathbf{R}^n_+} |\varphi(x', \tau^q) - v(x', \tau)|^2 \frac{dx' \, d\tau}{\tau}. \end{aligned} \tag{5.7}$$

Observe that for $t > 0$,

$$\begin{aligned} \frac{1}{t} &= c_1 \int_{\mathbf{R}^n_+} \frac{dy' \, dy_n}{(|x' - y'|^q + y_n^q + t)^{1+n/q}} \\ &= c_2 \int_{\mathbf{R}^n_+} \frac{dy' \, ds}{(|x' - y'|^q + t^q + s)^{1+n/q}}. \end{aligned} \tag{5.8}$$

When $q \geqslant 1$, we insert the following decomposition in (5.6):

$$\varphi(x', t) - v(y', y_n) = [\varphi(x', t) - v(x', t^{1/q})] + [v(x', t^{1/q}) - v(y', y_n)].$$

The first bracket gives $I'[\varphi, v]$ in view of the first formulas in (5.7) and (5.8), and the second bracket gives an integral satisfying

$$\iiint \frac{|v(x', t^{1/q}) - v(y', y_n)|^2}{(|x'-y'|^q + y_n^q + t)^{1+n/q}}\, dy\, dx'\, dt$$
$$= \iiint \frac{|v(x', x_n) - v(y', y_n)|^2\, x_n^{q-1}}{(|x'-y'|^q + y_n^q + x_n^q)^{1+n/q}}\, dy\, dx'\, dx_n$$
$$\leqslant C \int_{x\in \mathbf{R}^n_+} \int_{y \in \mathbf{R}^n_+} \frac{|v(x', x_n) - v(y', y_n)|^2}{(|x'-y'|^q + |x_n - y_n|^q)^{1/q+n/q}}\, dy\, dx$$
$$\leqslant C \int_{x\in \mathbf{R}^n} \int_{y \in \mathbf{R}^n} \frac{|\tilde{v}(x) - \tilde{v}(y)|^2}{|x-y|^{1+n}}\, dy\, dx$$
$$\leqslant C' \, \|v\|^2_{H^{1/2}(\mathbf{R}^n_+)};$$

here $\tilde{v}$ denotes an extension of v to $H^{1/2}(\mathbf{R}^n)$, depending linearly and continuously on v, and we have used (2.4) and (2.5). This shows that

$$\begin{aligned} I[\varphi, v] &\leqslant C_1(I'[\varphi, v] + \|v\|^2_{1/2}), \\ I'[\varphi, v] &\leqslant C_2(I[\varphi, v] + \|v\|^2_{1/2}) \qquad \text{for} \quad q \geqslant 1; \end{aligned} \tag{5.9}$$

in this sense the two integrals are equivalent.

When $q \leqslant 1$, we insert instead in (5.6)

$$\varphi(x', t) - v(y', y_n) = [\varphi(x', t) - \varphi(y', t)] + [\varphi(y', t) - \varphi(y', y_n^q)] + [\varphi(y', y_n^q) - v(y', y_n)].$$

The third bracket gives $I'[\varphi, v]$ by the second formulas in (5.7) and (5.8), with $\tau = y_n$ and x' interchanged with y'. The first two brackets give integrals estimated by

$$\iiint \frac{|\varphi(x', t) - \varphi(y', t)|^2}{(|x'-y'|^q + y_n^q + t)^{1+n/q}}\, dx'\, dy dt$$
$$\leqslant C \iiint \frac{|\varphi(x', t) - \varphi(y', t)|^2}{|x'-y'|^{n-1+q}}\, dx'\, dy'\, dt$$
$$\leqslant C_1 \, \|\varphi\|^2_{H^{q/2,0}},$$
$$\iiint \frac{|\varphi(y', t) - \varphi(y', y_n^q)|^2}{(|x'-y'|^q + y_n^q + t)^{1+n/q}}\, dx'\, dy dt$$
$$\leqslant C' \iiint \frac{|\varphi(y', t) - \varphi(y', s)|^2}{|t-s|^2}\, dy'\, dt\, ds$$
$$\leqslant C_1' \, \|\varphi\|^2_{H^{0,1/2}};$$

here we have replaced y_n by $s^{1/q}$ and used that $s^{1/q-1} \leqslant (|x'-y'|^q + s+t)^{1/q-1}$ and $s+t \geqslant |t-s|$ for $s, t \in \mathbf{R}_+$. Then we can conclude (cf. (2.7))

$$I[\varphi, v] \leqslant C_1'(I'[\varphi, v] + \|\varphi\|^2_{q/2,1/2}),$$
$$I'[\varphi, v] \leqslant C_2'(I[\varphi, v] + \|\varphi\|^2_{q/2,1/2}) \qquad \text{for} \quad q \leqslant 1. \tag{5.10}$$

In view of (5.9) and (5.10), the theorem can be carried over to [Gri, Theorem 6.1] (with $r=(q+1)/2, s=(q+1)/2q$). More precisely, [Gri] shows the equivalent version of (1) and the surjectiveness in (2); then the estimate (5.5) follows. Now consider (3). Here we use that by (5.8), we have in local coordinates

$$I[\varphi, 0] = \iiint \frac{|\varphi(x', t)|^2}{(|x'-y'|^q + y_n^q + t)^{1+n/q}}\, dy\, dx'\, dt$$
$$= c_n \iint |\varphi(x', t)|^2 \frac{dx'\, dt}{t} \tag{5.11}$$

and hence (cf. (2.12))

$$\|\varphi\|^2_{H^{q/2,1/2}_{(0)}} \simeq \|\varphi\|^2_{H^{q/2,1/2}} + I[\varphi, 0], \tag{5.12}$$

which shows the first statement. The second statement is then a special case of (2), and the third statement is an analogous version with the rôles of S and $\Omega_{(0)}$ interchanged. ∎

The integral (5.3) has the advantage over (5.7) that it shows explicitly the invariant character of the coincidence condition.

Concerning higher order spaces, we have when $u \in H^{r,r/q}(Q)$,

$$\varphi_j = \gamma_j u \in H^{r-j-1/2,(r-j-1/2)/q}(S) \qquad \text{for} \quad 0 \leqslant j < r - \tfrac{1}{2},$$
$$v_l = \partial_t^l u \in H^{r-lq-q/2}(\Omega_{(0)}) \qquad \text{for} \quad 0 \leqslant l < r/q - \tfrac{1}{2},$$

so if $j + lq < r - (q+1)/2$, then necessarily

$$r_0 \partial_t^l \varphi_j = \gamma_j v_l \in H^{r-j-lq-(q+1)/2}(\Gamma_{(0)}); \tag{5.13}$$

and if $j + lq = r - (q+1)/2$, then, with χ denoting a function in $C_0^\infty(\Omega_1)$ that is 1 on a neighborhood of Γ,

$$I[\partial_t^l \varphi_j, D_\nu^j \chi v_l] < \infty.$$

A combination of the proof of Theorem 5.1 and the general trace theorem [Gri, Theorem 7.1] now gives immediately

THEOREM 5.2. *Let $r>0, q>0$. The mapping*

$$u \mapsto \mathscr{B}u = \{\{\gamma_j u\}_{0 \leqslant j < r-1/2}, \{r_0 \partial_t^l u\}_{0 \leqslant l < r/q-1/2}\} \tag{5.14}$$

is continuous and surjective from $H^{r,r/q}(Q)$ to the space of vectors $\Phi = \{\{\varphi_j\}_{0 \leqslant j < r-1/2}, \{v_l\}_{0 \leqslant l < r/q-1/2}\}$ satisfying

$$\begin{aligned} \varphi_j &\in H^{r-j-1/2,(r-j-1/2)/q}(S) && \text{for} \quad 0 \leqslant j < r-1/2, \\ v_l &\in H^{r-lq-q/2}(\Omega_{(0)}) && \text{for} \quad 0 \leqslant l < r/q-1/2, \\ r_0 \partial_t^l \varphi_j &= \gamma_j v_l && \text{for} \quad j+lq < r-(q+1)/2, \\ I[\partial_t^l \varphi_j, D_\nu^j \chi v_l] &< \infty && \text{for} \quad j+lq = r-(q+1)/2, \end{aligned} \tag{5.15}$$

provided with the norm $\|\Phi\|$ defined by

$$\begin{aligned} \|\Phi\|^2 = &\sum_{0 \leqslant j < r-1/2} \|\varphi_j\|^2_{H^{r-j-1/2,(r-j-1/2)/q}(S)} \\ &+ \sum_{0 \leqslant l < r/q-1/2} \|v_l\|^2_{H^{r-lq-q/2}(\Omega_{(0)})} \\ &+ \sum_{j+lq=r-(q+1)/2} I[\partial_t^l \varphi_j, D_\nu^j \chi v_l]; \end{aligned} \tag{5.16}$$

and $\mathscr{B}$ has a continuous right inverse in these spaces.

The theorem is easily generalized to vector valued functions and sections in vector bundles.

We shall also need an extension of the theorem to the more general trace operators occurring in the pseudo-differential calculus. Let $T = \{T_0, ..., T_{d-1}\}$ be a system of trace operators as defined in the beginning of Section 4, and assume that T is *normal* (this holds in particular if Definition 4.1 (1) is satisfied). It means that for each l, T_l is of the form

$$T_l = \sum_{0 \leqslant j \leqslant l} S_{lj} \gamma_j + T'_l, \tag{5.17}$$

where S_{ll} is a *surjective morphism* of $E|_\Gamma$ onto F_l (the S_{lj} with $j<l$ being ps.d.o.'s from $E|_\Gamma$ to F_l of order $l-j$) and T'_l is of class 0. Setting $\rho u = \{\gamma_0 u, ..., \gamma_{d-1} u\}$ (the Cauchy data of u on Γ with respect to the order d), we write

$$T = S\rho + T', \qquad \text{where } S \text{ is the matrix } S = (S_{lj})_{0 \leqslant l,j < d}; \tag{5.18}$$

here we have set $S_{lj} = 0$ for $j > l$, considered as ps.d.o.'s of order $l-j$.

We first remove the term T' by use of the following reduction introduced in [Gr2, Lemma 1.6.8]: One can find an operator Λ (of the form

$\Lambda = I + G'$, where G' is a s.g.o. of order and class 0), such that Λ is a homeomorphism of $H^r(\Omega, E)$ onto itself for all $r \geq 0$, and

$$T = S\rho + T' = S\rho\Lambda. \tag{5.19}$$

Λ then also defines a homeomorphism of $H^{r,s}(Q, \underline{E})$ onto itself for all $r \geq 0$, $s \geq 0$. When $T' = 0$, we just take $\Lambda = I$.

Now if S is the restriction to Γ of a (matrix formed) differential operator S on $\bar{\Omega}$, this first reduction suffices. Otherwise we need the following considerations:

Representing a neighborhood Σ of Γ as $\Gamma \times]-\delta, \delta[$, we can lift the vector bundles F_l to bundles F_l^e over Σ and the ps.d.o.'s S_{lj} on Γ to operators S_{lj}^e going from $\tilde{E}|_\Sigma$ to F_l^e (defined as "constant" in the variable $s \in]-\delta, \delta[$ and then carried over to the neighborhood of Γ in $\mathbf{R}^n$); and after truncations in the normal direction, we can view the sections $S_{lj}^e u$ as extended by zero to all of Ω. This is an extension to an extra variable s carried out in exactly the same way as the extension to the t-variable described in Proposition 3.3, so we get operators S_{lj}^e that are continuous from $H^r(\Omega, E)$ to $H^{r-l+j}(\Omega, E)$ for $r \geq l - j \geq 0$; for $l - j < 0$ they are 0. (Note that $H^r(\Gamma \times]-\delta, \delta[) = L^2(]-\delta, \delta[; H^r(\Gamma)) \cap H^r(]-\delta, \delta[; L^2(\Gamma))$.) We remark here that the extended operators S_{lj}^e are not ps.d.o.'s on Σ unless the S_{lj} are differential operators, in which case the extensions can be chosen as differential operators also.

Altogether, we can write

$$T_l u = \gamma_0 S_l^e \chi \Lambda u, \qquad \text{where } S_l^e = \sum_{0 \leq j \leq l} S_{lj}^e D_\nu^j,$$
$$\text{for each} \quad l \in \{0, ..., d-1\}. \tag{5.20}$$

THEOREM 5.3. *Let d be an integer ≥ 1 and let $r > 0$, $q > 0$. Let E be a vector bundle over $\bar{\Omega}$ of dimension N, and let $T = \{T_0, ..., T_{d-1}\}$ be a system of normal trace operators T_j of order j, going from E to bundles F_j over Γ of dimensions $N_j' \geq 0$. Introduce Λ, S_j^e, F_j^e, and χ as explained above, so that* (5.20) *holds. The mapping*

$$u \mapsto \mathcal{T} u = \{\{T_j u\}_{0 \leq j < r-1/2, j<d}, \{r_0 \partial_t^l u\}_{0 \leq l < r/q - 1/2}\} \tag{5.21}$$

is continuous and surjective from the space $H^{r,r/q}(Q, \underline{E})$ to the space of vectors of sections $\Phi = \{\{\varphi_j\}_{0 \leq j < r-1/2, j<d}, \{v_l\}_{0 \leq l < r/q - 1/2}\}$ satisfying

(i) $\varphi_j \in H^{r-j-1/2,(r-j-1/2)/q}(S, \underline{F}_j)$ *for* $0 \leq j < r - 1/2, j < d$,

(ii) $v_l \in H^{r-lq-q/2}(\Omega_{(0)}, E)$ *for* $0 \leq l < r/q - 1/2$,

$$\text{(iii)} \qquad r_0 \partial_t^l \varphi_j = T_j v_l \qquad \textit{for} \quad j + lq < r - (q+1)/2, j < d,$$

$$\text{(iv)} \qquad I[\partial_t^l \varphi_j, S_j^e \chi \Lambda v_l] < \infty \qquad \textit{for} \quad j + lq = r - (q+1)/2, j < d; \tag{5.22}$$

provided with the norm $\|\Phi\|$ *defined by*

$$\begin{aligned} \|\Phi\|^2 = & \sum_{\substack{0 \leq j < r - 1/2 \\ j < d}} \|\varphi_j\|^2_{H^{r-j-1/2,(r-j-1/2)/q}(S, \underline{F}_j)} \\ & + \sum_{0 \leq l < r/q - 1/2} \|v_l\|^2_{H^{r-lq-q/2}(\Omega_{(0)}, E)} \\ & + \sum_{\substack{j + lq = r - (q+1)/2 \\ j < d}} I[\partial_t^l \varphi_j, S_j^e \chi \Lambda v_l]; \end{aligned} \tag{5.23}$$

and $\mathscr{T}$ *has a continuous right inverse in these spaces.*

Proof. The continuity of $\mathscr{T}$ follows from the continuity properties established in Lemma 2.1, Proposition 3.3, and Theorem 5.2.

We show the surjectiveness as follows: As accounted for in [Gr2, Lemma 1.6.1], we can supply S with a *diagonal matrix* $S' = (S'_{lj})_{0 \leq l, j < d}$, where the S'_{ll} are morphisms of $E|_\Gamma$ onto certain bundles Z_l over Γ, such that

$$\tilde{S} = \begin{pmatrix} S \\ S' \end{pmatrix}, \qquad \text{understood as } \tilde{S} = (\tilde{S}_{lj})_{0 \leq l, j < d}, \ \tilde{S}_{lj} = \begin{pmatrix} S_{lj} \\ S'_{lj} \end{pmatrix},$$

is *bijective* from $\prod_{0 \leq j < d} H^{r-j}(E|_\Gamma)$ to $\prod_{0 \leq l < d} H^{r-l}(F_l \oplus Z_l)$; note that $\tilde{S}$ is triangular. Here, moreover, the "diagonal part", $\tilde{S}_{\text{diag}} = (\delta_{lj} \tilde{S}_{lj})_{0 \leq l, j < d}$, is a homeomorphism from $\prod_{0 \leq j < d} E|_\Gamma$ to $\prod_{0 \leq l < d} (F_l \oplus Z_l)$. We can then supply $S\rho$ with the normal trace operator $S'\rho$ and compose with $(\tilde{S})^{-1}$, obtaining that

$$\rho = (\tilde{S})^{-1} \begin{pmatrix} S\rho \\ S'\rho \end{pmatrix} : H^r(\Omega, E) \to \prod_{0 \leq l < d} H^{r-l-1/2}(\Gamma, E|_\Gamma). \tag{5.24}$$

The construction extends to the neighborhood Σ of Γ, giving operators S'^e_{lj}, $\tilde{S}^e_{lj}$ mapping into sections in Z^e_l, $F^e_l \oplus Z^e_l$, etc. We denote $S'^e_j = S'^e_{jj} D^j_\nu$. The operators and bundles extend as usual to the t-variable also. We define moreover $T_j = \gamma_j$ for $j = d, d+1, \ldots$

Let $w_l = \Lambda v_l$ for each l. We want to construct $w \in H^{r,r/q}(Q, \underline{E})$ such that $\sum S_{jk} \gamma_k w = \varphi_j$ for $j < r - \frac{1}{2}$, $j < d$, and $r_0 \partial_t^l w = w_l$ for $l < r/q - \frac{1}{2}$. In order to use the surjectiveness in Theorem 5.2, we shall supply the given section

$\varphi = \{\varphi_0, \ldots, \varphi_{d-1}\}$ in $\bigoplus_{j=0,\ldots,d-1} \underline{F}_j$ with a section $\psi = \{\psi_0, \ldots, \psi_{d-1}\}$ in $\bigoplus_{j=0,\ldots,d-1} \underline{Z}_j$, and sections $\psi_j \in \underline{E}|_\Gamma$ for $d \leqslant j < r - \frac{1}{2}$, such that the section $(\tilde{S})^{-1}\{\varphi, \psi\}$ in $\underline{E}|_\Gamma$, together with the ψ_j for $j \geqslant d$ and the w_l, satisfy the full compatibility conditions in Theorem 5.2. This means that we shall choose the ψ_j such that $r_0 \partial_t^l \psi_j = \gamma_0 S_j'^e w_l$ (possibly in the sense of coincidence) holds for $j + lq \leqslant r - (q+1)/2$, $j < d$, and $r_0 \partial_t^l \psi_j = \gamma_j w_l$ holds for $j + lq \leqslant r - (q+1)/2$, $j \geqslant d$. This is a purely differential operator problem. If $r/q - (q+1)/2q \notin \mathbf{N}$, the ψ_j can be chosen according to (2.16ii) with Q replaced by S. If $r/q - (q+1)/2q \in \mathbf{N}$, we apply (2.16ii) to the initial data w_l themselves to find a section W in $\underline{E}$ with $r_0 \partial_t^l W = w_l$, and then we take $\psi_j = \gamma_0 S_j'^e W$ for $j < d$ and $\psi_j = \gamma_j W$ for $j \geqslant d$, $j + lq \leqslant r - (q+1)/2$.

Altogether, we can supply the φ_j with sections ψ_j in $\underline{Z}_j$, resp. in $E|_\Gamma$, so that the sections $(\tilde{S})^{-1}\{\partial_t^l \varphi, \partial_t^l \psi\}$, $\{\psi_j\}_{j \geqslant d}$, match the w_l at the corner $\Gamma_{(0)}$ (cf. (5.24)). Then Theorem 5.2 shows the existence of w with initial values w_l and vertical Cauchy data $(\tilde{S})^{-1}\{\varphi, \psi\}$, $\{\psi_j\}_{j \geqslant d}$; and $u = \Lambda^{-1} w$ solves the original problem.

The continuity of the right inverse follows automatically. ∎

Remark 5.4. The formulation of the compatibility condition (5.22 iv) refers to the auxiliary operator Λ. Here one could use instead that the T_j' may be written as

$$T_j' = \gamma_0 P_j', \tag{5.25}$$

where the P_j' are ps.d.o.'s in $S_{1,0}^d$ defined on a neighborhood of $\bar{\Omega}$ (going from $\tilde{E}$ to F_j^e) and having the transmission property at Γ, as shown in [BM]. Then (5.22 iv) can be written in the form

$$I[\partial_t^l \varphi_j, (S_j^e \chi + P_j') v_l] < \infty \qquad \text{for} \quad j + lq = r - (q+1)/2. \tag{5.26}$$

As a corollary, we get a complete description of the interpolation spaces between $L^2(\Omega, E)$ and $\{u \in H^d(\Omega, E) \mid Tu = 0\}$, for general pseudo-differential trace operators, extending the result of Grisvard [Gri] for differential trace operators and the result of Grubb [Gr2, Theorem 4.4.2, Remark 4.4.3] for trace operators (5.18) with S differential:

COROLLARY 5.5. *Let d be an integer $\geqslant 1$, and let $T = \{T_0, \ldots, T_{d-1}\}$ be a normal system of trace operators as in Theorem* 5.3. *Let*

$$H_T^s(\Omega, E) = \{u \in H^s(\Omega, E) \mid T_j u = 0 \; for \; 0 \leqslant j < s - \tfrac{1}{2}\}$$
$$for \quad s \in [0, d], \; s - \tfrac{1}{2} \notin \mathbf{Z}, \tag{5.27}$$

it is a closed subspace of $H^s(\Omega, E)$; *and let* (*with auxiliary definitions as in Theorem* 5.3.)

$$H_T^{k+1/2}(\Omega, E) = \{u \in H^{k+1/2}(\Omega, E) \mid T_j u = 0 \text{ for } 0 \leq j < k,$$
$$\operatorname{dist}(x, \Gamma)^{-1/2} S_k^e \chi A u \in L^2(\Omega, F_k^e)\} \qquad \text{for} \quad k = 0, ..., d-1, \tag{5.28}$$

it is provided with the norm defined by

$$\|u\|^2_{H_T^{k+1/2}(\Omega, E)} = \|u\|^2_{k+1/2} + \|\operatorname{dist}(x, \Gamma)^{-1/2} S_k^e \chi A u\|^2_{L^2(\Omega, F_k^e)}. \tag{5.29}$$

Then for any $\theta \in]0, 1[$,

$$[H_T^d(\Omega, E), L^2(\Omega, E)]_\theta = H_T^{(1-\theta)d}(\Omega, E). \tag{5.30}$$

Proof. We use the following characterization of the interpolated space (cf. [Gri, Definition 2.2]):

$$v \in [X, Y]_\theta \qquad \text{if and only if}$$
$$v = r_0 u \qquad \text{for some } u \in L^2(\mathbf{R}_+; X) \cap H^{1/(2\theta)}(\mathbf{R}_+; Y). \tag{5.31}$$

Here $[X, Y]_\theta$ has the norm induced from $L^2(\mathbf{R}_+; X) \cap H^{1/(2\theta)}(\mathbf{R}_+; Y)$ by the mapping r_0:

$$\|v\|^2_{[X, Y]_\theta} = \inf\{\|u\|^2_{L^2(\mathbf{R}_+; X)} + \|u\|^2_{H^{1/(2\theta)}(\mathbf{R}_+; Y)} \mid r_0 u = v\}. \tag{5.32}$$

Thus in the present case the interpolated space consists of the initial values $v = r_0 u$ of functions $u \in H^{d, 1/(2\theta)}(Q, \underline{E})$ with $Tu = 0$. Theorem 5.3 applies with $r = d$, $q = 2\theta d$, to show that such v are described by the properties

$$v \in H^{d-q/2}(\Omega, E) = H^{(1-\theta)d}(\Omega, E),$$
$$T_j v = 0 \qquad \text{for} \quad j < d - (q+1)/2 = (1-\theta)\, d - 1/2, \tag{5.33}$$
$$I[0, S_j^e \chi A v] < \infty \qquad \text{for} \quad j = (1-\theta)\, d - 1/2.$$

The last statement, that enters only when $(1-\theta)\, d - 1/2 \in \mathbf{N}$, means that $S_j^e \chi A v \in H_0^{1/2}(\bar{\Omega}, F_j^e)$ (cf. Theorem 5.1), and can therefore be reformulated as the statement that $\operatorname{dist}(x, \Gamma)^{-1/2} S_j^e \chi A u \in L^2(\Omega, E)$; cf. (2.6). ∎

Remark 5.6. By the use of (5.25), one can also write

$$H_T^{k+1/2}(\Omega, E) = \{u \in H^{k+1/2}(\Omega, E) \mid T_j u = 0 \text{ for } 0 \leq j < k,$$
$$\operatorname{dist}(x, \Gamma)^{-1/2} (S_k^e \chi + P_k')\, u \in L^2(\Omega, F_k^e)\}$$
$$\text{for} \quad k = 0, ..., d-1. \tag{5.34}$$

6. Solution of Parabolic Problems

With the operators P, G, and $T = \{T_0, ..., T_{d-1}\}$ defined as in Section 4, we finally consider the time-dependent problem in the "cylinder" $Q = \Omega \times I$ with "vertical" boundary $S = \Gamma \times I$ and "bottom" Ω:

$$\begin{aligned} &\text{(i)} && \partial_t u + P_\Omega u + Gu = f && \text{for} \quad (x, t) \in Q, \\ &\text{(ii)} && Tu = \varphi && \text{for} \quad (x, t) \in S, \\ &\text{(iii)} && r_0 u \equiv u|_{t=0} = u_0 && \text{for} \quad x \in \Omega. \end{aligned} \tag{6.1}$$

Here $I =]0, b[$, $b \leqslant \infty$. Let us for brevity denote

$$P_\Omega + G = M. \tag{6.2}$$

Since ∂_t is of order 1 and P and G are of order d, it is natural to consider the problem for u in spaces of the type $H^{r, r/d}(Q, \underline{E})$; here the operators have the continuity properties:

$$\begin{aligned} &P_\Omega, G\colon H^{r+d, r/d+1}(Q, \underline{E}) \to H^{r, r/d}(Q, \underline{E}), \\ &\quad T_j\colon H^{r+d, r/d+1}(Q, \underline{E}) \to H^{r+d-j-1/2, (r+d-j-1/2)/d}(S, \underline{F}_j), \\ &\quad r_0\colon H^{r+d, r/d+1}(Q, \underline{E}) \to H^{r+d/2}(\Omega, E), \end{aligned} \tag{6.3}$$

for all $r \geqslant 0$, by Lemma 2.1 and Corollary 3.5 with $q = d$.

Recall that some of the bundles F_j may have dimension 0, and are then only included for notational convenience.

It is of course necessary for the solvability of (6.1) in $H^{r+d, r/d+1}(Q, \underline{E})$ that the boundary data φ_j match the initial value u_0 as described in Theorem 5.3:

$$\begin{aligned} r_0 \varphi_j = T_j u_0 \qquad & \text{for} \quad j < r + d - (d+1)/2 = r + d/2 - 1/2; \\ I[\varphi_j, S_j^e \chi \Lambda u_0] < \infty \qquad & \text{for} \quad j = r + d/2 - 1/2 \end{aligned}$$

(the latter condition only occurs when $r + (d-1)/2$ is one of the integers $j \in \{0, ..., d-1\}$ for which F_j is nonzero). But now we furthermore have to take the equation (6.1 i) into account. It implies, successively,

$$\partial_t^{l+1} u = -M \partial_t^l u + \partial_t^l f, \qquad l = 0, 1, ... \tag{6.4}$$

Applying r_0 and T_j here, we find that when u solves (6.1), one must necessarily have

$$r_0 \partial_t^{l+1} \varphi_j = r_0 \partial_t^{l+1} T_j u = T_j(-M r_0 \partial_t^l u + r_0 \partial_t^l f), \tag{6.5}$$

whenever these expressions have a sense. To formulate this as a condition purely on the data $\{f, \varphi, u_0\}$, we define successively the functions on Ω

$$u^{(0)} = u_0, \qquad u^{(l+1)} = -Mu^{(l)} + r_0 \partial_t^l f, \qquad \text{for} \quad l = 0, 1, \ldots; \tag{6.6}$$

then (6.5) gives rise to the *compatibility conditions*

$$T_j u^{(l)} = r_0 \partial_t^l \varphi_j, \tag{6.7}$$

applied when the expressions have a sense.

Altogether, we can formulate the complete compatibility requirements as follows:

DEFINITION 6.1. Let $r \in \bar{\mathbf{R}}_+$. A set of data

$$\begin{aligned} &f \in H^{r, r/d}(Q, \underline{E}), \\ &\varphi_j \in H^{r+d-j-1/2, (r+d-j-1/2)/d}(S, \underline{F}_j) \qquad \text{for} \quad j \in \{0, \ldots, d-1\}, \\ &u_0 \in H^{r+d/2}(\Omega, E), \end{aligned} \tag{6.8}$$

is said to satisfy the compatibility condition for (6.1) of order r, when the functions $u^{(l)}$ defined by (6.6) satisfy

$$\begin{aligned} r_0 \partial_t^l \varphi_j = T_j u^{(l)} \qquad &\text{for} \quad j + ld < r + d/2 - 1/2, \dim F_j > 0, \\ I[\partial_t^l \varphi_j, S_j^e \chi A u^{(l)}] < \infty \qquad &\text{for} \quad j + ld = r + d/2 - 1/2, \dim F_j > 0; \end{aligned} \tag{6.9}$$

in the latter expressions the auxiliary operators from Theorem 5.3 are used.

Here we have used that when the data satisfy (6.8), then $r_0 \partial_t^l f$ is defined for $l < r/d - \frac{1}{2}$ and lies in $H^{r-ld-d/2}(\Omega, E)$ by (2.16 ii), so

$$u^{(0)} \in H^{r+d/2}(\Omega, E), \ldots, u^{(l)} \in H^{r-ld+d/2}(\Omega, E), \ldots, \qquad \text{for} \quad l < r/d + 1/2; \tag{6.10}$$

and moreover, the t-derivatives of the vertical data satisfy

$$\partial_t^l \varphi_j \in H^{r-ld-j-1/2, (r-ld-j-1/2)/d}(S, \underline{F}_j). \tag{6.11}$$

Our main aim is to show that when the parabolicity condition in Definition 4.1 is satisfied, then the problem (6.1) has a unique solution in $H^{r+d, r/d+1}(Q, \underline{E})$ for any set of data (6.8) satisfying the compatibility condition of order r. First we establish an auxiliary result that is valid regardless of parabolicity:

PROPOSITION 6.2. *Let $M = P_\Omega + G$, where P is a ps.d.o. of order $d > 0$*

having the transmission property, and G is a s.g.o. of order and class d. Let $r \geq 0$. For any set of data

$$f \in H^{r, r/d}(Q, \underline{E}), \qquad u_0 \in H^{r+d/2}(\Omega, E), \tag{6.12}$$

there is a function $w \in H^{r+d, r/d+1}(Q, \underline{E})$ such that

$$r_0 \partial_t^l w = u^{(l)} \quad \text{for} \quad 0 \leq l < r/d + 1/2, \qquad (\partial_t + M) w - f \in H_{(0)}^{r, r/d}(Q, \underline{E}), \tag{6.13}$$

where the $u^{(l)}$ are defined from f and u_0 by (6.6). Here w can be chosen so that one has estimates

$$\|w\|^2_{H^{r+d,r/d+1}(Q,\underline{E})} + \|(\partial_t + M)w - f\|^2_{H^{r,r/d}_{(0)}(Q,\underline{E})} \leq C(\|f\|^2_{H^{r,r/d}(Q,\underline{E})} + \|u_0\|^2_{H^{r+d/2}(\Omega,E)}). \tag{6.14}$$

Proof. When $r/d - \frac{1}{2}$ is not integer, this is a simple application of the surjectiveness in (2.16 ii). Then $l_0 \equiv [r/d - \frac{1}{2}]$ is strictly less than $r/d - \frac{1}{2}$, and the $u^{(l)}$ satisfy

$$u^{(0)} \in H^{r+d/2}(\Omega, E), ..., u^{(l_0+1)} \in H^{r-l_0 d - d/2}(\Omega, E) \quad \text{for} \quad l_0 = [r/d - 1/2]. \tag{6.15}$$

According to (2.16 ii) there is a $w \in H^{r+d, r/d+1}(Q, \underline{E})$ with initial values

$$r_0 \, \partial_t^l w = u^{(l)}, \qquad \text{for} \quad l = 0, ..., l_0 + 1; \tag{6.16}$$

and since w can be chosen to depend linearly and continuously on the data, we obtain the estimate of the norm of w in (6.14) by inserting the formulas defining the $u^{(l)}$. These formulas also show that $f' = f - \partial_t w - Mw$ has initial values

$$r_0 \, \partial_t^l f' = 0 \qquad \text{for} \quad l = 0, ..., l_0, \tag{6.17}$$

and since $f' \in H^{r, r/d}(Q, \underline{E})$, this means that $f' \in H_{(0)}^{r, r/d}(Q, \underline{E})$. Here the $H_{(0)}^{r, r/d}$ norm is equivalent with the $H^{r, r/d}$ norm, so (6.14) follows.

When $r/d - \frac{1}{2} \in \mathbf{N}$ (i.e., $r/d - \frac{1}{2} = l_0$), this argument does not work, since $r_0 \, \partial_t^{l_0} f$ is not well-defined. But here we can take recourse to another device: Add an extra variable $x_{n+1} \in \mathbf{R}_+$ to the space coordinates, and regard Q as the boundary of $\hat{Q} = Q \times \mathbf{R}_+$, Ω as the boundary of $\hat{\Omega} = \Omega \times \mathbf{R}_+$. Since $r \geq d/2 > 0$, $f \in H^{r, r/d}(Q, \underline{E})$ is the boundary value according to (2.16 i) of a function $\hat{f} \in H^{r+1/2, (r+1/2)/d}(\hat{Q}, \hat{\underline{E}})$, and $u_0 \in H^{r+d/2}(\Omega, \underline{E})$ is the boundary value of a function $\hat{u}_0 \in H^{r+d/2+1/2}(\hat{\Omega}, \hat{E})$; and the new functions can be chosen to depend linearly and continuously on the given functions. Note that

$$H^r(\hat{\Omega}, \hat{E}) = L^2(\mathbf{R}_+; H^r(\Omega, E)) \cap H^r(\mathbf{R}_+; L^2(\Omega, E)),$$
$$H^{r,s}(\hat{Q}, \hat{\underline{E}}) = L^2(\mathbf{R}_+; H^{r,s}(Q, \underline{E})) \cap H^r(\mathbf{R}_+; L^2(Q, \underline{E})).$$

For the operator M extended to the new spaces we find the continuity properties, by use of (2.9), much as in Proposition 3.3:

$$M: L^2(\mathbf{R}_+; H^{r+d}(\Omega, E)) \to L^2(\mathbf{R}_+; H^r(\Omega, E)),$$
$$M: H^r(\mathbf{R}_+; H^d(\Omega, E)) \to H^r(\mathbf{R}_+; L^2(\Omega, E)), \qquad \text{and hence}$$
$$M: H^{r+d}(\hat{\Omega}, \hat{E}) \to H^r(\hat{\Omega}, \hat{E}); \qquad \text{and similarly}$$
$$M: H^{r+d, r/d+1}(\hat{Q}, \hat{\underline{E}}) \to H^{r, r/d}(\hat{Q}, \hat{\underline{E}}),$$

for $r \geq 0$. Since $(r+\frac{1}{2})/d - \frac{1}{2} = l_0 + 1/(2d)$ is not integer, we are now in a situation where the first part of the proof can be applied. This gives a function $\hat{w} \in H^{r+d+1/2, (r+d+1/2)/d}(\hat{Q}, \hat{\underline{E}})$ such that

$$r_0 \partial_t^l \hat{w} = \hat{u}^{(l)} \qquad \text{for} \quad 0 \leq l \leq l_0 = [(r+\tfrac{1}{2})/d - \tfrac{1}{2}],$$
$$\hat{f} - \partial_t \hat{w} - M\hat{w} \in H_{(0)}^{r+1/2, (r+1/2)/d}(\hat{Q}, \hat{\underline{E}});$$

and then $w = \hat{w}|_{x_{n+1}=0}$ solves the original problem. The estimates follow by restriction to $x_{n+1} = 0$, using that $\hat{f} - \partial_t \hat{w} - M\hat{w}$ can be regarded as an element of

$$H^{r+1/2, (r+1/2)/d}(\hat{\Omega} \times]-\infty, b[, \hat{\underline{E}})$$

supported in $\{t \geq 0\}$. ∎

We can finally show:

THEOREM 6.3. *Let P, G and $T = \{T_0, ..., T_{d-1}\}$ be as defined in Section 4, and assume moreover that the system $\{\partial_t + P_\Omega + G, T\}$ is parabolic, so in particular d is even and $N' = Nd/2$. Let $r \geq 0$. Consider systems of functions $\{f, \varphi, u_0\}$, given as in* (6.8) *and satisfying the compatibility condition of order r for* (6.1).

(1) *Let $I =]0, b[$, where $b \in]0, \infty[$. The evolution problem* (6.1) *has a unique solution u in $H^{d,1}(Q_b, \underline{E})$, and the following estimates hold*:

If $r-\frac{1}{2}$ is not integer, then there is a constant $C_{b,r}$ such that

$$\|u\|^2_{H^{r+d,r/d+1}(Q_b,\underline{E})} \leqslant C_{b,r}\Bigg(\sum_{0\leqslant j<d} \|\varphi_j\|^2_{H^{r+d-j-1/2,(r+d-j-1/2)/d}(S_b,\underline{F}_j)} + \|f\|^2_{H^{r,r/d}(Q_b,\underline{E})} + \|u_0\|^2_{H^{r+d/2}(\Omega,E)}\Bigg). \tag{6.18}$$

If $r-\frac{1}{2}$ is integer, then, writing $r=ld+k+\frac{1}{2}-d/2$ with $l\in\mathbf{N}$ and $k\in\{0,\ldots,d-1\}$,

$$\|u\|^2_{H^{r+d,r/d+1}(Q_b,\underline{E})} \leqslant C_{b,r}\Bigg(\sum_{0\leqslant j<d} \|\varphi_j\|^2_{H^{r+d-j-1/2,(r+d-j-1/2)/d}(S_b,\underline{F}_j)} + \|f\|^2_{H^{r,r/d}(Q_b,\underline{E})} + \|u_0\|^2_{H^{r+d/2}(\Omega,E)} + I[\partial_t^l\varphi_k, S_k^e\chi\Lambda u^{(l)}]\Bigg), \tag{6.19}$$

where the last term vanishes if $\dim F_k=0$.

(2) *Let $I=\,]0,\infty[$, i.e., $b=\infty$. If the real lower bound a_0 of the spectrum (cf. Theorem* 4.2) *is* >0, *then* (6.1) *has a unique solution in $H^{d,1}(Q_\infty,\underline{E})$, satisfying* (6.18) *when $r-\frac{1}{2}\notin\mathbf{N}$, resp.* (6.19) *when $r-\frac{1}{2}\in\mathbf{N}$.*

If $a_0\leqslant 0$, let $\rho>|a_0|$. When the given data $\{f,\varphi,u_0\}$ are merely such that (6.8) *holds for $\{e^{-\rho t}f, e^{-\rho t}\varphi, u_0\}$, and the compatibility condition of order r for* (6.1) *is satisfied, then* (6.1) *has a unique solution u with $e^{-\rho t}u\in H^{d,1}(Q_\infty,\underline{E})$; and $e^{-\rho t}u$ satisfies the estimates* (6.18) *resp.* (6.19) *in terms of $\{e^{-\rho t}f, e^{-\rho t}\varphi, u_0\}$.*

(3) *For each r, the operator norm of the solution operator*

$$\mathscr{R}_b\colon\{f,\varphi,u_0\}\mapsto u$$

is non-decreasing as a function of b. Thus the constants in (6.18), (6.19), *can be chosen such that $C_{b,r}\leqslant C_{b',r}$ for $b\leqslant b'<\infty$; here $b'=\infty$ is included when $a_0>0$.*

Proof. According to Proposition 6.2 there is a function w such that

$$r_0\,\partial_t^l w=u^{(l)} \qquad \text{for} \quad 0\leqslant l<r/d+1/2,$$
$$f'=f-\partial_t w-Mw\in H^{r,r/d}_{(0)}(Q,\underline{E}),$$

and (6.14) holds. Since the $u^{(l)}$ coincide with the $\partial_t^l\varphi_j$ at $\Gamma\times\{0\}$, the new boundary values

$$\varphi_j'=\varphi_j-T_j w$$

coincide with 0 at $\Gamma \times \{0\}$. Then if we introduce the new unknown function $u' = u - w$, the original problem is reduced to the problem

$$\begin{aligned} \partial_t u' + Mu' &= f' \quad && \text{in } Q, \\ Tu' &= \varphi' \quad && \text{on } S, \\ r_0 u' &= 0 \quad && \text{on } \Omega, \end{aligned} \tag{6.20}$$

where

$$f' \in H^{r,r/d}_{(0)}(Q, \underline{E}), \qquad \varphi' \in \prod_{0 \leq j < d} H^{r+d-j-1/2,(r+d-j-1/2)/d}_{(0)}(S, \underline{F}_j).$$

Since the spaces $H^{r,r/d}_{(0)}$ interpolate well, we only have to solve this directly in some particular cases. In fact, we let $r = 0, d, 2d, \ldots$. Consider $m \in \mathbf{N}$. For $r = md$,

$$\varphi'_j \in H^{(m+1)d-j-1/2,m+1-j/d-1/2d}_{(0)}(S, \underline{F}_j) \qquad \text{for each } j,$$

which means that $\varphi'_j \in H^{(m+1)d-j-1/2,m+1-j/d-1/2d}(S, \underline{F}_j)$ and the $\partial_t^l \varphi_j$ coincide with 0 at $t = 0$ for $l \leqslant m$ (note that $j/d + 1/2d \notin \mathbf{N}$). By Theorem 5.3, there exists $w' \in H^{(m+1)d,m+1}(Q, \underline{E})$ (depending continuously on φ') with $T_j w' = \varphi_j$ for $j = 0, \ldots, d-1$ and $r_0 \partial_t^l w' = 0$ for $l \leqslant m$; in particular, $w' \in H^{(m+1)d,m+1}_{(0)}(Q, \underline{E})$ and $\partial_t w' + Mw' \in H^{md,m}_{(0)}(Q, \underline{E})$. Denoting $u' - w' = v$, we can then replace the problem by that of finding $v \in H^{(m+1)d,m+1}(Q, \underline{E})$ such that

$$\begin{aligned} \partial_t v + Mv &= g \quad && \text{i } Q, \\ Tv &= 0 \quad && \text{on } S, \\ r_0 v &= 0 \quad && \text{on } \Omega, \end{aligned} \tag{6.21}$$

where $g = f' - \partial_t w' - Mw' \in H^{md,m}_{(0)}(Q, \underline{E})$. Here the formula (6.6) gives

$$v^{(0)} = \cdots = v^{(m)} = 0,$$

so we are in fact searching for a $v \in H^{(m+1)d,m+1}_{(0)}(Q, \underline{E})$. The problem (6.21) can then be formulated as a problem in $L^2(]-\infty, b[\,; L^2(\Omega, E))$.

$$\partial_t \tilde{v}(t) + A\tilde{v}(t) = \tilde{g}(t) \qquad \text{for} \quad t \in\,]-\infty, b[, \tag{6.22}$$

where $\tilde{v}$ and $\tilde{g}$ are the extensions by 0 on $\mathbf{R}_-$, and A is the realization of M defined by the boundary condition $Tv = 0$, cf. (4.4).

Now let us restrict the attention to the case where $b = \infty$ and $a_0 > 0$. We shall use the Laplace transform method as described in Lions and Magenes [L–M, Chap. 4], with some precisions of the kind made in Agranovič and

Vishik [A–V]. When $f(t)$ is a function on $\mathbf{R}_+$ and $\tilde{f}$ is its extension by zero on $\mathbf{R}_-$, we denote by $\hat{f}(\xi + i\eta)$ the Laplace transform

$$\hat{f}(\xi + i\eta) = \int_{\mathbf{R}} e^{-t\xi - it\eta} \tilde{f}(t)\, dt = \int_0^\infty e^{-t\xi - it\eta} f(t)\, dt$$

(the Fourier transform of $e^{-t\xi}\tilde{f}(t)$); this notation will also be used for functions valued in a Hilbert space X. We denote $\xi + i\eta = p$. In view of the Paley–Wiener theorem, the space of functions $f \in L^2(\mathbf{R}_+; X)$ carries over to the space of X-valued functions $\hat{f}(\xi + i\eta)$ holomorphic in $p = \xi + i\eta$ for $\xi > 0$ and with $\|\hat{f}(\xi + i\eta)\|_{L^2_\eta(\mathbf{R}; X)}$ uniformly bounded in ξ; in fact the following norms are equivalent:

$$\sup_{\xi > 0} \int_{\mathbf{R}} \|\hat{f}(\xi + i\eta)\|_X^2\, d\eta \simeq \|f\|^2_{L^2(\mathbf{R}_+; X)}. \tag{6.23}$$

Since $\partial_t^l \tilde{f}(t)$ carries over to $p^l \hat{f}(p)$ by the Laplace transformation, we have moreover

$$\sum_{0 \leq l \leq k} \sup_{\xi > 0} \int_{\mathbf{R}} |\xi + i\eta|^{2l}\, \|\hat{f}(\xi + i\eta)\|_X^2\, d\eta \simeq \|f\|^2_{H_0^k(\mathbf{R}_+; X)} \tag{6.24}$$

for $k \in \mathbf{N}$. Applying the Laplace transform to (6.22) (cf. [L–M, A–V]), we find the equation

$$(A + p)\, \hat{v}(p) = \hat{g}(p), \tag{6.25}$$

$p = \xi + i\eta$, where the given properties of g carry over to the informations, by (6.24),

$$\sup_{\xi > 0} \int_{\mathbf{R}} \langle p \rangle^{2m} \|\hat{g}(p)\|^2_{L^2(\Omega, E)}\, d\eta < \infty, \qquad \sup_{\xi > 0} \int_{\mathbf{R}} \|\hat{g}(p)\|^2_{H^{md}(\Omega, E)}\, d\eta < \infty.$$

Since $a_0 > 0$, $A + p$ has an inverse $(A + p)^{-1} = R_{-p}\colon L^2(\Omega, E) \to D(A)$ for all p with Re $p \geq 0$, satisfying (4.6) there (with $\lambda = -p$). Thus for Re $p \geq 0$, (6.25) is solved (uniquely in $D(A)$) by

$$\hat{v}(p) = (A + p)^{-1}\, \hat{g}(p),$$

which is holomorphic in $\{\text{Re } p > 0\}$, satisfying, by (4.6),

$$\begin{aligned} &\sup_{\xi > 0} \int_{\mathbf{R}} (\langle p \rangle^{2(m+1)} \|\hat{v}(p)\|^2_{L^2(\Omega, E)} + \|\hat{v}(p)\|^2_{H^{(m+1)d}(\Omega, E)})\, d\eta \\ &\quad \leq C \sup_{\xi > 0} \int_{\mathbf{R}} (\langle p \rangle^{2m} \|\hat{g}(p)\|^2_{L^2(\Omega, E)} + \|\hat{g}(p)\|^2_{H^{md}(\Omega, E)})\, d\eta \\ &\quad \simeq \|g\|^2_{H_{(0)}^{md, m}(Q, E)}. \end{aligned} \tag{6.26}$$

Let $\tilde{v}$ be the inverse Laplace transform of $\hat{v}$, then we conclude that it is supported in $\{t \geqslant 0\}$ and its restriction v to $\mathbf{R}_+$ lies in $H_{(0)}^{(m+1)d,m+1}(Q, \underline{E})$, with the norm estimated by the right hand side of (6.26).

This shows the solvability of (6.22), hence of (6.21), (6.20), and (6.1), for $r = md$, $m \in \mathbf{N}$ (in the case $b = \infty$, $a_0 > 0$). So there is a unique solution in $H^{d,1}(Q, \underline{E})$ (at least) for any r. We shall show that it satisfies the estimates (6.18), (6.19).

By interpolation between the values $m = 0, 1, 2, \ldots$, we get the continuity of the solution operator $\mathscr{P}$ for (6.20),

$$\mathscr{P}: \{f', \varphi'\} \in H_{(0)}^{r,r/d}(Q, \underline{E}) \times \prod_{0 \leqslant j < d} H_{(0)}^{r+d-j-1/2,(r+d-j-1/2)/d}(S, \underline{F}_j)$$
$$\mapsto u' \in H_{(0)}^{r+d,r/d+1}(Q, \underline{E}), \tag{6.27}$$

for all $r \in \bar{\mathbf{R}}_+$, cf. (2.13).

Then we have for $u = u' + w$, cf. (6.14) and (6.27),

$$\begin{aligned}
\|u\|^2_{H^{r+d,r/d+1}} &\leqslant 2\,\|u'\|^2_{H^{r+d,r/d+1}} + 2\,\|w\|^2_{H^{r+d,r/d+1}} \\
&\leqslant 2\,\|u'\|^2_{H_{(0)}^{r+d,r/d+1}} + 2\,\|w\|^2_{H^{r+d,r/d+1}} \\
&\leqslant C(\|f'\|^2_{H_{(0)}^{r,r/d}} + \|\varphi'\|^2_{\prod H_{(0)}^{r+d-j-1/2,(r+d-j-1/2)/d}} + \|f\|^2_{H^{r,r/d}} + \|u_0\|^2_{H^{r+d/2}}) \\
&\leqslant C_1(\|\varphi - Tw\|^2_{\prod H_{(0)}^{r+d-j-1/2,(r+d-j-1/2)/d}} + \|f\|^2_{H^{r,r/d}} + \|u_0\|^2_{H^{r+d/2}}).
\end{aligned}$$

When $r - \frac{1}{2} \notin \mathbf{N}$, the $H_{(0)}^{\sigma,\rho}$ norms in the above estimates are like $H^{\sigma,\rho}$ norms, and we get (6.18) by a simple use of continuity properties of T and (6.14). Otherwise, let $r = ld + k + \frac{1}{2} - d/2$ for some $l \in \mathbf{N}$ and $k \in \{0, \ldots, d-1\}$. The only nonstandard contribution is

$$\|\varphi_k - T_k w\|^2_{H_{(0)}^{ld+d/2,l+1/2}(S,\underline{F}_k)}$$
$$\simeq \|\varphi_k - T_k w\|^2_{H^{ld+d/2,l+1/2}(S,\underline{F}_k)} + \int_0^\infty \|\partial_t^l(\varphi_k - T_k w)\|^2_{L^2(\Gamma,F_k)} \frac{dt}{t};$$

cf. (2.12). We have in local coordinates, using (5.8), (5.20), (6.9), and (6.13),

$$\begin{aligned}
&\int_{\mathbf{R}_+} \int_{\mathbf{R}^{n-1}} |\partial_t^l(\varphi_k(x', t) - T_k w(x', t))|^2 \frac{dx'\,dt}{t} \\
&\quad = c_1 \int_{t \in \mathbf{R}_+} \int_{x' \in \mathbf{R}^{n-1}} \int_{y \in \mathbf{R}^n_+} \frac{|\partial_t^l(\varphi_k(x', t) - T_k w(x', t))|^2}{(|x' - y'|^d + y_n^d + t)^{1+n/d}}\, dy\, dx'\, dt \\
&\quad \leqslant 2c_1 \int\!\!\int\!\!\int \frac{|\partial_t^l \varphi_k(x', t) - S_k^e \chi A u^{(l)}(y)|^2}{(|x' - y'|^d + y_n^d + t)^{1+n/d}}\, dy\, dx'\, dt \\
&\qquad + 2c_1 \int\!\!\int\!\!\int \frac{|\partial_t^l T_k w(x', t) - S_k^e \chi A u^{(l)}(y)|^2}{(|x' - y'|^d + y_n^d + t)^{1+n/d}}\, dy\, dx'\, dt.
\end{aligned}$$

Thus

$$\int_0^T \int_{\Gamma} |\partial_t^l(\varphi_k(x', t) - T_k w(x', t))|^2 \frac{dx'\,dt}{t}$$
$$\leqslant C_2(I[\partial_t^l \varphi_k, S_k^e \chi A u^{(l)}] + I[\partial_t^l T_k w, S_k^e \chi A u^{(l)}]).$$

The last term here is estimated by $C_3(\|u_0\|_{H^{r-d/2}} + \|f\|_{H^{r,r/d}})$, since $u^{(l)} = r_0\,\partial_t^l w$ and w satisfies (6.14). Collecting the terms, we find (6.19).

If $a_0 \leqslant 0$, and $\rho > |a_0|$, then a replacement of u by $u_\rho = e^{-\rho t}u$ carries (6.1) over into the problem

$$\begin{aligned} \partial_t u_\rho + (M + \rho)\, u_\rho &= f_\rho, \\ T u_\rho &= \varphi_\rho, \\ r_0 u_\rho &= u_0, \end{aligned} \tag{6.28}$$

where $f_\rho = e^{-\rho t} f$ and $\varphi_\rho = e^{-\rho t}\varphi$. Then the statement in the theorem follows immediately from the preceding case.

Finally, let $b < \infty$. Here there are continuous extension operators from $H^{r,r/d}(Q_b, \underline{E})$ to $H^{r,r/d}(Q_\infty, \underline{E})$, mapping into functions supported in $\bar{Q}_{2b}$, say (so that also exponential estimates are valid); and there are similar extensions for the spaces over S_b. We can then apply the preceding results to the extended functions and restrict back to Q_b afterwards; this gives the estimates (6.18)–(6.19).

To see that the norm of $\mathscr{R}_b$ increases with b, we denote the norm of $\{f, \varphi, u_0\}$ indicated in the right hand side of (6.18) resp. (6.19) (taking the square root) by $N_{b,r}(f, \varphi, u_0)$. Then

$$\|\mathscr{R}_b\| = \sup\{ \|\mathscr{R}_b(f, \varphi, u_0)\|_{H^{r,r/d}(Q_b, \underline{E})} \mid N_{b,r}(f, \varphi, u_0) < 1 \}.$$

Let $b' > b$; then for any set $\{f, \varphi, u_0\}$ given over I_b with $N_{b,r}(f, \varphi, u_0) < 1$, there is a set $\{f', \varphi', u_0\}$ over $I_{b'}$ extending $\{f, \varphi, u_0\}$, such that $N(f', \varphi', u_0) < 1$ (cf. (2.1)). Then

$$\begin{aligned} \|\mathscr{R}_b\| &= \sup\{ \|\mathscr{R}_b(f, \varphi, u_0)\|_{H^{r,r/d}(Q_b, \underline{E})} \mid N_{b,r}(f, \varphi, u_0) < 1 \} \\ &= \sup\{ \|\mathscr{R}_b(f, \varphi, u_0)\|_{H^{r,r/d}(Q_b, \underline{E})} \mid \{f, \varphi, u_0\} \\ &\qquad \text{has ext. with } N_{b',r}(f', \varphi', u_0) < 1 \} \\ &\leqslant \sup\{ \|\mathscr{R}_b(f', \varphi', u_0)\|_{H^{r,r/d}(Q_{b'}, E)} \mid N_{b',r}(f', \varphi', u_0) < 1 \} \\ &= \|\mathscr{R}_{b'}\|. \end{aligned}$$

When $a_0 > 0$, $\mathscr{R}_\infty$ also has a finite norm, so the case $b' = \infty$ can be included then. ∎

In a recent work, Purmonen [Pu] has treated a more general type of parabolic pseudo-differential initial–boundary problems with ∂_t entering in higher powers, on the basis of the estimates in [Gr2], with a systematic formulation in terms of exponentially weighted spaces (and with some exceptional parameters).

7. Extensions and Complements

Of special interest is the solution operator $U: u_0(x) \mapsto u(x, t)$ for the problem where f and φ are 0,

$$\begin{aligned} \partial_t u + Mu &= 0 \quad && \text{in } Q, \\ Tu &= 0 \quad && \text{on } S, \\ r_0 u &= u_0 \quad && \text{on } \Omega. \end{aligned} \tag{7.1}$$

It is easy to show that the solutions of (7.1) are C^∞ up to the vertical boundary for $t > 0$. In fact one has:

THEOREM 7.1. *Let $r \geqslant 0$, and let u be the solution of* (6.1) *for a set of data $\{f, \varphi, u_0\}$ according to Theorem* 6.3. *If $f \in C^\infty(\Omega \times I, \underline{E})$ and $\varphi \in \prod_{0 \leqslant j < d} C^\infty(\Gamma \times I, \underline{F}_j)$, then $u \in C^\infty(\bar{\Omega} \times I, \underline{E})$.*

In particular, the solution operator U for (7.1) *maps $H^{d/2}(\Omega, E)$ into $C^\infty(\bar{\Omega} \times I, \underline{E})$.*

Proof. Let $I =]0, b[$, and let $\bar{I}_0 = [a_0, b_0]$, where $0 < a_0 < b_0 < b$; we shall show that $u \in C^\infty(\bar{\Omega} \times \bar{I}_0, \underline{E})$. For this purpose, let $\{\eta_k(t)\}_{k \in \mathbf{N}}$ be a sequence of functions in $C_0^\infty(I)$ such that $\eta_{k+1}\eta_k = \eta_{k+1}$ and $\eta_{k|I_0} = 1$ for all k. When u is the solution of (6.1) with data $\{f, \varphi, u_0\}$, then $u \in H^{r+d, r/d+1}(Q, \underline{E})$, and $\eta_k u$ solves

$$\begin{aligned} (\partial_t + M)(\eta_k u) &= \eta_k f + \eta_k' u \quad && \text{in } Q, \\ T(\eta_k u) &= \eta_k \varphi \quad && \text{on } S, \\ r_0(\eta_k u) &= 0 \quad && \text{on } \Omega \end{aligned}$$

for each k, where $\eta_k f \in \bigcap_{s \in \mathbf{R}_+} H_{(0)}^{s, s/d}(Q, \underline{E})$ and $\eta_k \varphi \in \prod_{0 \leqslant j < d} \bigcap_{s \in \mathbf{R}_+} H_{(0)}^{s, s/d}(S, \underline{F}_j)$. Since $\eta_0' u \in H^{r+d, r/d+1}(Q, \underline{E})$, we find by Theorem 6.3 that $\eta_0 u \in H_{(0)}^{r+2d, r/d+2}(Q, \underline{E})$. Now since η_k is 1 on supp η_{k+1},

$$(\partial_t + M)\, \eta_{k+1} u = \eta_{k+1} f + \eta_{k+1}' u = \eta_{k+1} f + \eta_{k+1}' \eta_k u \qquad \text{for each } k.$$

Then Theorem 6.3 gives successively

$$\eta_1 u \in H^{r+3d, r/d+3}(Q, \underline{E}), \ldots, \eta_k u \in H^{r+(k+2)d, r/d+k+2}(Q, \underline{E}), \ldots,$$

so we conclude that u is C^∞ on $\bar{\Omega} \times \bar{I}_0$. ∎

However, the smoothness at $t=0$ will of course depend on r.

In order to describe the compatibility condition of order r for $\{0, 0, u_0\}$, we recall the definition of $H^s_T(\Omega, E)$ in (5.27) and (5.28) for $0 \leqslant s \leqslant d$ (cf. also (5.34)), and set, for $m \in \mathbf{N}$, $0 \leqslant s \leqslant d$,

$$H^{md+s}_{T,M}(\Omega, E) = \{u \in H^{md+s}(\Omega, E) \mid M^k u \in H^d_T(\Omega, E) \text{ for } 0 \leqslant k < m,\ M^m u \in H^s_T(\Omega, E)\}. \tag{7.2}$$

We note here that the spaces $H^s_{T,M}(\Omega, E)$ form a full interpolating family:

THEOREM 7.2. *For all $r \geqslant s \geqslant 0$ and all $\theta \in]0, 1[$,*

$$[H^r_{T,M}(\Omega, E), H^s_{T,M}(\Omega, E)]_\theta = H^{(1-\theta)r+\theta s}_{T,M}(\Omega, E). \tag{7.3}$$

Proof. Let $r = md$ for an integer $m > 0$. Then since Γ is noncharacteristic for P (the principal symbol in local coordinates at the boundary is of the form

$$p^0(x', 0, \xi', \xi_n) = s_d(x')\, \xi_n^d + \mathcal{O}(\langle \xi' \rangle \langle \xi_n \rangle^{d-1})$$

with an invertible coefficient $s_d(x')$), the system $\{T, TM, \ldots, TM^{m-1}\}$ is, like T, a normal system of trace operators, now of orders up to $md-1$. Corollary 5.5 shows (7.3) in this case for any s, and the statement for general r follows from the reiteration property of the interpolation (cf. e.g., [L–M, Theorem 1.6.1]). ∎

Now it is clear from (6.6) and (6.7) that $\{0, 0, u_0\}$ *satisfies the compatibility condition of order r for* (6.1) *precisely when* $u_0 \in H^r_{T,M}(\Omega, E)$. We observe moreover that when u solves (7.1), $u(t)$ is a well-defined element of $C^\infty(\bar{\Omega}, E)$ for each $t > 0$ by Theorem 7.1, and hence, since $u(t_0 + t)$ is C^∞ for $(x, t) \in \bar{\Omega} \times [0, b - t_0[$ and solves the problem (7.1) there with initial value $u(t_0)$, one has that

$$u(t_0) \in \bigcap_{s \geqslant 0} H^s_{T,M}(\Omega, E) = C^\infty_{T,M}(\bar{\Omega}, E) \qquad \text{for} \quad t_0 > 0; \tag{7.4}$$

here we have introduced the notation $C^\infty_{T,M}(\bar{\Omega}, E)$ for the Fréchet space of C^∞ functions satisfying *all* compatibility conditions (6.7) with $f = 0$, $\varphi = 0$.

The problem (7.1) can be viewed more abstractly as a problem for $u(t): t \mapsto D(A)$,

$$\begin{aligned} \partial_t u(t) + Au(t) = 0 \qquad \text{for} \quad t > 0, \\ u(0) = u_0. \end{aligned} \tag{7.5}$$

So far, we have solved it for $u_0 \in H^s_{T,M}(\Omega, E)$ with $s \geq d/2$ (and shown that ∂_t, applied at first in a Sobolev space sense, is really taken in a C^∞ sense in I). But this can be extended down to $s \geq 0$. We assume in the following for simplicity that the real lower bound of the spectrum a_0 (as defined in Theorem 4.2) *is positive*, since this can always be obtained by a reduction as in (6.28). Then, in particular, A is bijective from $D(A)$ to $L^2(\Omega, E)$. Moreover, A maps $H^{(m+1)d}_{T,M}(\Omega, E)$ homeomorphically onto $H^{md}_{T,M}(\Omega, E)$ for all $m \in \mathbf{N}$, and therefore one has by interpolation

$$A: H^{d+s}_{T,M}(\Omega, E) \to H^s_{T,M}(\Omega, E) \qquad \text{homeomorphically, for all} \quad s \geq 0. \tag{7.6}$$

We also take $I = \mathbf{R}_+$, since the results for finite intervals can be deduced from this case.

First, consider $0 < s < d/2$. When u_0 is given in $H^s_T(\Omega, E)$, let $v_0 = A^{-1}u_0$; it lies in $H^{d+s}_{T,M}(\Omega, E)$, and hence (7.5) with initial value v_0 has a (unique) solution $v \in H^{3d/2+s, 3/2+s/d}(Q, \underline{E})$; here $r_t v = v(t)$ is as in (7.4) for $t > 0$. Setting $u = Av \in H^{d/2+s, 1/2+s/d}(Q, \underline{E})$, we see that it solves (7.5) with initial value u_0; the first line holds since the functions are in fact in $C^\infty(I; D(A^m))$ for any $m \in \mathbf{N}$, and the initial value is assumed in the sense of (2.16 ii).

Next, let $s = 0$. For $u \in L^2(\Omega, E)$ we can again take the solution v of the problem (7.5) with initial value $v_0 = A^{-1}u_0 \in H^d_T(\Omega, E)$, and then *define* $u = Av$ to be the solution with initial value u_0. But here, since $u \in H^{d/2, 1/2}(Q, \underline{E})$ only, the initial value is (at first sight) assumed only in an abstract sense. If M were a differential operator, we could use that M then maps $L^2(I; H^{d/2}(\Omega, E))$ continuously into $L^2(I; H^{-d/2}(\Omega, E))$ so that, since $\partial_t u = -Mu$,

$$\begin{aligned} u \in L^2(I; H^{d/2}(\Omega, E)) \qquad \text{and} \qquad \partial_t u \in L^2(I; H^{-d/2}(\Omega, E)) \\ \text{imply} \qquad u \in C^0(\bar{I}; L^2(\Omega, E)), \end{aligned}$$

cf. (2.18), with $u(0) = Av(0) = u_0$. However, the present operators $M = P_\Omega + G$ only map $H^r(\Omega, E)$ into $H^{r-d}(\Omega, E)$ for $r > l - \frac{1}{2}$, where l is the *class* of the singular Green operator G; it can be any integer between 0 and d. Other special arguments can be invoked when A is *variational* (i.e., is a realization of M defined from a coercive sesquilinear form, cf. [Gr2, Section 1.7]).

At any rate, thanks to the fact that the resolvent satisfies the estimates (4.6) not only in a halfplane $\{\operatorname{Re}\lambda \leqslant a_0 - \delta\}$, but also on rays $\{\lambda = re^{i\theta}\}$ with $\pm\theta = \pi/2 - \varepsilon$, cf. (4.8), we can involve another theory that is valid for all our realizations A, namely the theory of holomorphic semigroups (cf., e.g., Hille and Phillips [H–P], Friedman [F]; and [Gr2, Sect. 4.1]). Here we define the family of bounded operators $U(t)$ in $L^2(\Omega, E)$ by

$$U(t) = \frac{i}{2\pi} \int_{\mathscr{C}} e^{-\lambda t} R_\lambda \, d\lambda \qquad \text{for} \quad t > 0, \qquad U(0) = I, \tag{7.7}$$

where $\mathscr{C}$ is a curve in $\mathbf{C}$ going around the spectrum of A in the positive sense (e.g., $\mathscr{C}$ can be the boundary of $\mathscr{W}_\delta$ in (4.8)). $U(t)$ extends to an operator valued holomorphic function of $t \in \{z \in \mathbf{C} \setminus \{0\} \mid |\arg z| < \varepsilon_1\}$, where ε_1 is as in (4.7). It is well known that (7.5) has the solution

$$u(t) = U(t)\, u_0 \in C^0(\bar{I}; L^2(\Omega, E)) \cap C^0(I; D(A)) \cap C^1(I; L^2(\Omega, E)); \tag{7.8}$$

and the solution is unique in this space. As soon as $t > 0$, the solution is very smooth; it is in fact holomorphic in $\{t \in \mathbf{C} \setminus \{0\} \mid |\arg t| < \varepsilon_1\}$ with values in $D(A^m)$, for any $m > 0$.

Since $A^{-1} U(t) = U(t) A^{-1}$, the solution of the problem with initial value $v_0 = A^{-1} u_0$ is $A^{-1} U(t)\, u_0$; this must coincide with $v(t)$ constructed further above, since $v(t)$ is in $C^0(\bar{I}; L^2(\Omega, E)) \cap C^\infty(I; D(A^m))$, all m, cf. (2.17). Applying A, we see that the solution $u \in H^{d/2, 1/2}(Q, E)$ constructed above equals the semigroup solution. In particular, $u \in C^0(\bar{I}; L^2(\Omega, E))$.

Similarly, the solutions with more smooth initial data $u_0 \in H^s_{T,M}(\Omega, E)$ constructed above equal the semigroup solutions, so we can take advantage of all the general results known in connection with holomorphic semigroups, when we study our solutions.

We only do this when it is useful to us, not for its own sake—for example, the use of fractional powers A^σ, which is customary in some treatments, seems to be much too circumstantial, now that we already have a solvability theory for a full scale of interpolating spaces $H^s_{T,M}(\Omega, E)$, $s \geqslant 0$. (The precise characterization of the fractional powers A^σ is a complicated affair; see [Gr2, Sect. 4.4] for some partial results. Note that their *domains*, at least in the selfadjoint case, are completely characterized by Corollary 5.5 above, as the $H^\sigma_{T,M}$ spaces.)

To sum up, we have shown:

THEOREM 7.3. *Let* $a_0 > 0$, *and let* $I = \mathbf{R}_+$. *For any* $s \geqslant 0$, *any* $u_0 \in H^s_{T,M}(\Omega, E)$, *there is a solution of* (7.5) *in the space*

$$u \in H^{d/2 + s, 1/2 + s/d}(Q, E) \cap C^0(\bar{\mathbf{R}}_+; H^s_{T,M}(\Omega, E)) \cap C^\infty(\mathbf{R}_+; C^\infty_{T,M}(\bar{\Omega}, E)); \tag{7.9}$$

in fact the solution is holomorphic in $\{t \in \mathbf{C}\setminus\{0\} \mid |\arg t| < \varepsilon_1\}$ *with values in* $C^{\infty}_{T,M}(\bar{\Omega}, E)$. *The solution is unique in the space defined in* (7.8); *and for* $s \geqslant d/2$ *it is unique in* $H^{d,1}(Q, \underline{E})$.

For $a_0 \leqslant 0$, *there is a solution* u *such that similar statements hold for* $e^{-\rho t}u$, $\rho > |a_0|$.

We can also obtain estimates of norms of $u(t)$ for $t \to 0$.

THEOREM 7.4. *Let* $r \geqslant s \geqslant 0$. *Let* $a_0 > 0$, *and let* $I = \mathbf{R}_+$. *There is a constant* $C > 0$ *such that when* $u(t)$ *solves* (7.5) *with initial value* $u_0 \in H^s_{T,M}(\Omega, E)$, *then*

$$\|u(t)\|_{H^r_{T,M}(\Omega,E)} \leqslant C t^{-(r-s)/d} \|u_0\|_{H^s_{T,M}(\Omega,E)} \quad \text{for all} \quad t > 0. \tag{7.10}$$

When $a_0 \leqslant 0$, *the inequality holds with* C *replaced by* $Ce^{\rho t}$, *where* $\rho > |a_0|$.

Proof. Let $a_0 > 0$. It is well known from the semigroup theory (cf., e.g., [F, Sect. 2.2]) that the L^2 operator norms of $A^m U(t)$ satisfy

$$\|A^m U(t)\| \leqslant C_m t^{-m}, \qquad \text{for all} \quad t > 0, \qquad \text{all} \quad m \in \mathbf{N}. \tag{7.11}$$

When m and $k \in \mathbf{N}$, $m \geqslant k$, then by (7.6) and (7.11),

$$\begin{aligned}\|U(t)\,u_0\|_{H^{md}_{T,M}} &\leqslant C_1 \|A^m U(t)\,u_0\|_{L^2} = C_1 \|A^{m-k} U(t)\,A^k u_0\|_{L^2}\\ &\leqslant C_2 t^{-m+k} \|u_0\|_{H^{kd}_{T,M}}, \qquad \text{for} \quad u_0 \in H^{kd}_{T,M}(\Omega, E).\end{aligned}$$

Interpolating between m and $m+1$ and between k and $k+1$, we find

$$\begin{aligned}&\|U(t)\,u_0\|_{H^{md+\sigma}_{T,M}} \leqslant C_3 t^{-m+k} \|u_0\|_{H^{kd+\sigma}_{T,M}},\\ &\qquad\qquad \text{for} \quad 0 \leqslant \sigma \leqslant d, \quad u_0 \in H^{kd+\sigma}_{T,M}(\Omega, E),\end{aligned} \tag{7.12}$$

which shows (7.10) when $r - s$ is an integer multiple of d.

Now let $r > s \geqslant 0$. Choose $\theta \in\,]0, 1[$ and k integer such that $r - s = \theta k d$. Then in view of (2.3) and (7.12),

$$\begin{aligned}\|U(t)\,u_0\|_{H^r_{T,M}} &= \|U(t)\,u_0\|_{H^{s+\theta kd}_{T,M}} \leqslant C_4 \|U(t)\,u_0\|^{\theta}_{H^{s+kd}_{T,M}} \|U(t)\,u_0\|^{1-\theta}_{H^s_{T,M}}\\ &\leqslant C_5 (t^{-k} \|u_0\|_{H^s_{T,M}})^{\theta} \|u_0\|^{1-\theta}_{H^s_{T,M}} = C_5 t^{-(r-s)/d} \|u_0\|_{H^s_{T,M}},\end{aligned}$$

which completes the proof of (7.10).

If $a_0 \leqslant 0$, we replace $U(t)\,u_0$ by $e^{-\rho t} U(t)\,u_0$ with $\rho > |a_0|$, which solves the analogous problem for $A + \rho$, cf. (6.28). ∎

Besides having an interest in themselves, these estimates are useful in the treatment of nonlinear generalizations of the parabolic problems.

$U(t)$ can of course be studied further—for its kernel structure and asymptotic trace formulas see [Gr2].

We end by showing some improvements concerning exponentially decreasing solutions.

First, when $a_0 > 0$, we can shift M a little bit by replacing u by $e^{a_1 t}u$ for $a_1 \in]0, a_0[$, so that M is replaced by $M - a_1$, still an operator with positive lower bound of the spectrum; thus the solutions satisfy

$$\|U(t)\, u_0\|_{L^2} \leqslant C_{a_1} e^{-a_1 t} \|u_0\|_{L^2} \qquad \text{for all} \quad t \geqslant 0,$$

$$\|U(t)\, u_0\|_{H^r_{T,M}} \leqslant C_{r,s,a_1} t^{-(r-s)/d} e^{-a_1 t} \|u_0\|_{H^s_{T,M}} \qquad \text{for all} \quad t > 0, \quad r \geqslant s \geqslant 0. \tag{7.13}$$

It is perhaps more interesting that when $a_0 \leqslant 0$, we can *also* get exponentially decreasing solutions, now for u_0 in a subspace V of $L^2(\Omega, E)$ of *finite codimension*. In fact we can in both cases get solutions that decrease exponentially *as fast as we like*, for u_0 in a subspace of finite codimension.

THEOREM 7.5. *Let $I = \mathbf{R}_+$ and let a_1 be a given real number. There is a decomposition of $X = L^2(\Omega, E)$ into closed subspaces V and W satisfying* (4.10), *and an integral operator $\mathscr{S}$ with finite rank and C^∞ kernel* (*of the form* (4.20)–(4.21)), *such that the solution operator $U_1 : u_0 \mapsto u$ for the parabolic problem*

$$\begin{aligned} \partial_t u + Mu + \mathscr{S}u &= 0 \qquad \text{in } Q, \\ Tu &= 0 \qquad \text{on } S, \\ r_0 u &= u_0 \qquad \text{on } \Omega \end{aligned} \tag{7.14}$$

satisfies, for all $r \geqslant s \geqslant 0$,

$$\|U_1(t)\, u_0\|_{H^r_{T,M}} \leqslant C_{r,s,a_1} t^{-(r-s)/d} e^{-a_1 t} \|u_0\|_{H^s_{T,M}} \qquad \textit{for all} \quad t > 0. \tag{7.15}$$

Proof. Ordering the mutually distinct eigenvalues as in (4.9), we let k' be the first number such that $\operatorname{Re} \lambda_{k'} > a_1$; then Theorem 4.3 provides us with V, W, and $\mathscr{S}$ such that the lower bound of the spectrum of $A_B = A + \mathscr{S}$ is $> a_1$, when for B we take, e.g., $B = (a_1 + 1)I$ on W. Since $\mathscr{S}$ is a negligible ps.d.o. (a ps.d.o. of order $-\infty$), the principal part of the operators is unchanged, and the parabolic theory applies to the problem (7.14), defining a solution operator U_1 that can also be expressed as the semigroup $U_1(t)$ solving

$$\begin{aligned} \partial_t u(t) + A_B u(t) &= 0 \qquad \text{for} \quad t > 0, \\ u(0) &= u_0. \end{aligned} \tag{7.16}$$

In particular, (7.15) follows from the discussion preceding the theorem. ▮

COROLLARY 7.6. *For any $a_1 \in \mathbf{R}$ there is a closed subspace V of $L^2(\Omega, E)$ of finite codimension, such that the solutions $u(t)$ of* (7.1) (*or* (7.5)) *with initial values $u_0 \in V$ satisfy, for all $r \geqslant s \geqslant 0$,*

$$\|u(t)\|_{H^r_{T,M}} \leqslant C_{r,s,a_1} t^{-(r-s)/d} e^{-a_1 t} \|u_0\|_{H^s_{T,M}} \quad \textit{for all} \quad t > 0. \tag{7.17}$$

Proof. We take V and $A_B = A + \mathscr{S}$ as in Theorem 7.5. For λ outside $\sigma(A) \cup \sigma(A_B)$, the resolvents $(A_B - \lambda)^{-1}$ and $(A - \lambda)^{-1}$ coincide on V, in view of (4.10) and the construction of A_B. Thus for $u_0 \in V$,

$$\begin{aligned} U(t)\, u_0 &= \frac{i}{2\pi} \int_{\mathscr{C}} e^{-\lambda t} (A - \lambda)^{-1} u_0 \, d\lambda \\ &= \frac{i}{2\pi} \int_{\mathscr{C}} e^{-\lambda t} (A_B - \lambda)^{-1} u_0 \, d\lambda = U_1(t)\, u_0, \end{aligned} \tag{7.18}$$

where $\mathscr{C}$ goes around $\sigma(A) \cup \sigma(A_B)$. Then (7.17) follows from (7.15). ∎

We refer to Remark 4.4 for some observations concerning optimal choices of $\mathscr{S}$.

Also the results on the problem

$$\begin{aligned} \partial_t u + Mu &= f && \text{in } Q, \\ Tu &= 0 && \text{on } S, \\ r_0 u &= 0 && \text{on } \Omega, \end{aligned} \tag{7.19}$$

can be improved in this way. Take for simplicity $a_1 = 0$, and choose V and $\mathscr{S}$ according to Theorem 7.5; then the solution u of

$$\begin{aligned} \partial_t u + Mu + \mathscr{S}u &= f && \text{in } Q, \\ Tu &= 0 && \text{on } S, \\ r_0 u &= 0 && \text{on } \Omega, \end{aligned} \tag{7.20}$$

with f given in $H^{r, r/d}_{(0)}(Q, \underline{E})$, satisfies $u \in H^{r+d, r/d+1}_{(0)}(Q, \underline{E})$, with an estimate

$$\|u\|_{H^{r+d, r/d+1}_{(0)}(Q, \underline{E})} \leqslant C_r \|f\|_{H^{r, r/d}_{(0)}(Q, \underline{E})}. \tag{7.21}$$

Here u is determined from f by Laplace transformation of the resolvent, as described in the proof of Theorem 6.3. Now if f takes its values in V, more precisely, if

$$f \in L^2(\mathbf{R}_+; H^r(\Omega, E) \cap V) \cap H^{r/d}_0(\mathbf{R}_+; V), \tag{7.22}$$

then X can be replaced by V in (6.24), and $(A_B+p)^{-1}$ coincides with $(A+p)^{-1}$ on V, so we find that u solves (7.19), and

$$u \in L^2(\mathbf{R}_+; H^{r+d}(\Omega, E) \cap V) \cap H_0^{r/d+1}(\mathbf{R}_+; V). \tag{7.23}$$

When a_1 is general, V and $\mathscr{S}$ are chosen such that $A_B - a_1$ has its spectrum in $\{\operatorname{Re} \lambda > 0\}$; then for any $\sigma \leqslant a_1$,

$$\|e^{\sigma t} u\|_{H_{(0)}^{r+d, r/d+1}(Q, \underline{E})} \leqslant C_r \|e^{\sigma t} f\|_{H_{(0)}^{r, r/d}(Q, \underline{E})}. \tag{7.24}$$

We formulate the consequences for (7.19) in a theorem:

THEOREM 7.7. *Let $I = \mathbf{R}_+$ and let a_1 be a given real number. There is a closed subspace V of finite codimension, invariant under A in the sense that $A(D(A) \cap V) \subset V$, such that when f is given with*

$$e^{\sigma t} f \in L^2(\mathbf{R}_+; H^r(\Omega, E) \cap V) \cap H_0^{r/d}(\mathbf{R}_+; V) \tag{7.25}$$

for some $r \geqslant 0$, $\sigma \leqslant a_1$, then the solution u of (7.19) *satisfies*

$$e^{\sigma t} u \in L^2(\mathbf{R}_+; H^{r+d}(\Omega, E) \cap V) \cap H_0^{r/d+1}(\mathbf{R}_+; V), \tag{7.26}$$

and the estimate (7.24) *holds. In particular, it holds with $\sigma = 0$ if $a_1 \geqslant 0$.*

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