Resolvent studies on nonsmooth domains

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Introduction

Recent interest in so-called M-functions connected with boundary value problems for elliptic operators have lead to a revival of old theories, with modern generalizations. We shall explain a so-called Krein resolvent formula — different from the standard decomposition of the resolvent into an interior pseudodifferential term and a singular Green operator term — and give an account of how it may be established in non-smooth situations.

1. Pseudodifferential boundary operators

Consider a strongly elliptic $2m$-order differential operator A on a bounded smooth open subset Ω of $\mathbb{R}^n$ with coefficients in $C^\infty(\overline{\Omega})$.
Denote $\partial\Omega = \Sigma$, and $\gamma_j u = \partial_n^j u|_\Sigma$; then $\varrho u = \{\gamma_0 u, \dots, \gamma_{2m-1} u\}$ are the Cauchy data.
The maximal operator $A_{\max}$ acts like A in $L_2(\Omega)$ with domain

$$D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\},$$

and the minimal operator $A_{\min} = \overline{A|_{C_0^\infty}}$, with $D(A_{\min}) = H_0^{2m}(\Omega)$. The operators $\widetilde{A}$ in $L_2(\Omega)$ with $A_{\min} \subset \widetilde{A} \subset A_{\max}$ are the *realizations* of A.
For suitable matrices B of differential operators on Σ, the condition $B\varrho u = 0$ is an *elliptic* boundary condition for A, and the realization $\widetilde{A}$ of A with domain

$$D(\widetilde{A}) = \{u \in H^{2m}(\Omega) \mid B\varrho u = 0\}$$

has a resolvent $(\widetilde{A} - \lambda)^{-1}$ existing for λ in a sectorial region of $\mathbb{C}$.

We shall regard this from the point of view of the *pseudodifferential boundary operator* (ψdbo) calculus of Boutet de Monvel '71, extended to general parameter-dependent situations in the book G '86 (2nd ed. '96). The matrix-formed operator

$$\begin{pmatrix} A-\lambda \\ B\varrho \end{pmatrix} \text{ has the inverse } \begin{pmatrix} R(\lambda) & K(\lambda) \end{pmatrix}.$$

Here $R(\lambda)$ and $K(\lambda)$ solve the semi-homogeneous problems:

$$\begin{cases} (A-\lambda)u &= f \text{ in } \Omega, \\ B\varrho u &= 0 \text{ on } \Sigma; \end{cases} \quad \text{resp.} \quad \begin{cases} (A-\lambda)u &= 0 \text{ in } \Omega, \\ B\varrho u &= g \text{ on } \Sigma. \end{cases}$$

$K(\lambda)$ is a so-called *Poisson operator*.

$R(\lambda)$ is a sum of two terms:

$$R(\lambda) = Q(\lambda)_+ + G(\lambda) \qquad (1)$$

where $Q(\lambda)$ is the pseudodifferential operator $Q(\lambda) = (A - \lambda)^{-1}$ on $\mathbb{R}^n$ (defined in an approximate sense), $Q(\lambda)_+$ is its truncation to Ω,

$$Q(\lambda)_+ = r^+ Q(\lambda) e^+$$

(where e^+ extends by 0, r^+ restricts to Ω), and $G(\lambda)$ is a *singular Green operator*.

Boutet de Monvel defined pseudodifferential boundary operators (ψdbo's) in general as systems (called Green operators):

$$\begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(\overline{\Omega})^N \\ \times \\ C^\infty(\Sigma)^M \end{matrix} \to \begin{matrix} C^\infty(\overline{\Omega})^{N'} \\ \times \\ C^\infty(\Sigma)^{M'} \end{matrix}, \text{ where}$$

- P is a pseudodifferential operator (ψdo) on $\mathbb{R}^n$ (or on a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$), satisfying the *transmission condition* at Σ (always true for operators stemming from elliptic PDE),
- $P_+ = r^+ P e^+$ (the transmission condition assures that P_+ maps $C^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$).
- T is a trace operator from Ω to Σ, K is a Poisson operator from Σ to Ω, S is a ψdo on Σ.
- G is a singular Green operator, e.g. of type KT. (All can be matrix-formed.)

In local coordinates near the boundary, the operators T, K and G can be regarded as ψdo's in the tangential variables, *with values in* Poisson or singular Green operators in one variable (the normal variable x_n).

The ψdbo calculus defines an "algebra" of operators, where the composition of two systems leads to a third one (when the dimensions $N, M, \dots$ match). It has the advantage that when a system is *elliptic*, then there exists a *parametrix* (an inverse in an approximate sense) which also belongs to the calculus.

2. A Krein resolvent formula

The realization A_γ defined by the Dirichlet condition

$$D(A_\gamma)=\{u\in H^{2m}(\Omega)\mid \gamma u=0\},$$

where $\gamma=\{\gamma_0,\dots,\gamma_{m-1}\}$, can be assumed to have its spectrum in a sectorial region of $\{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda>0\}$; then it is bijective.
We denote by pr_X the orthogonal projection onto a closed subspace X of $H=L_2(\Omega)$. The (non-orthogonal) projections

$$\mathrm{pr}_\gamma\, u = A_\gamma^{-1}A_{\max}u,\quad \mathrm{pr}_\zeta = I-\mathrm{pr}_\gamma,$$

define a decomposition of $D(A_{\max})$ into a direct sum:

$$D(A_{\max})=D(A_\gamma)\dot{+}Z,\quad Z=\{u\in L_2(\Omega)\mid Au=0\}.$$

With the analogous decomposition for the formal adjoint A' indicated by primes everywhere, we have:

Theorem 1. [G '68] *There is a 1–1 corrspondence between the closed realizations $\widetilde{A}$ and the closed densely defined operators $T : V \to W$, where $V \subset Z$, $W \subset Z'$ (closed subspaces), such that $\widetilde{A}$ corresponds to $T : V \to W$ if and only if*

$$D(\widetilde{A}) = \{u \in D(A_{\max}) \mid \operatorname{pr}_\zeta u \in D(T),\ \operatorname{pr}_W Au = T \operatorname{pr}_\zeta u\}.$$

In this correspondence,

- $\widetilde{A}^*$ *corresponds analogously to* $T^* : W \to V$.
- $\ker \widetilde{A} = \ker T$; $\quad \operatorname{ran} \widetilde{A} = \operatorname{ran} T + (H \ominus W)$.
- *When* $\widetilde{A}$ *is bijective,*

$$\widetilde{A}^{-1} = A_\gamma^{-1} + \mathrm{i}_{V \to H} T^{-1} \operatorname{pr}_W . \qquad (2)$$

The result builds on previous work by Krein and Vishik.

The condition $\operatorname{pr}_W Au = T \operatorname{pr}_\zeta u$ is an "abstract boundary condition".

From now on, let $m = 1$, then the abstract condition is interpreted by means of the bijective maps

$$\gamma_0 : Z \xrightarrow{\sim} H^{-\frac{1}{2}}(\Sigma), \quad \gamma_0 : Z' \xrightarrow{\sim} H^{-\frac{1}{2}}(\Sigma),$$

that have the Poisson operators K_γ resp. K'_γ as inverses. (For $2m$-order operators, Z is mapped to an m-tuple.)
Consider the case $V = Z$, $W = Z'$. Here we carry $T : Z \to Z'$ over to the closed, densely defined operator $L : H^{-\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\Sigma)$ by the diagram

$$\begin{array}{ccc} Z & \xleftarrow{K_\gamma} & H^{-\frac{1}{2}}(\Sigma) \\ T \downarrow & & \downarrow L \\ Z' & \xrightarrow[(K'_\gamma)^*]{} & H^{\frac{1}{2}}(\Sigma) \end{array} \qquad D(L) = \gamma D(T),$$

where the horizontal maps are homeomorphisms.

We explain later how this turns $\mathrm{pr}_{Z'} Au = T\,\mathrm{pr}_\zeta\, u$ into a concrete boundary condition.
Formula (2) becomes

$$\widetilde{A}^{-1} = A_\gamma^{-1} + K_\gamma L^{-1} (K'_\gamma)^*. \qquad (3)$$

A replacement of A by $A - \lambda$ gives for $\lambda \in \varrho(A_\gamma) \cap \varrho(\widetilde{A})$,

$$(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} + K_\gamma^\lambda (L^\lambda)^{-1} (K_\gamma'^{\bar\lambda})^*, \qquad (4)$$

where the dependence on λ is indicated by an upper index. This is a so-called *Krein resolvent formula*.

The present version is proved in G-Brown-Wood Math. Nachr. '09, with generalizations to cases where V and W are subspaces of Z, Z'; many of the ingredients come from papers G '68 - '74.

When $\widetilde{A}$ is determined from an elliptic differential boundary condition, the term $K^{\lambda}_{\gamma}(L^{\lambda})^{-1}(K'^{\bar\lambda}_{\gamma})^{*}$ is a singular Green operator, but it differs from $G(\lambda)$ in (1) by carrying more precise eigenvalue information. Moreover, (4) is valid for general closed realizations, allowing more general operators L^{λ}.

Meanwhile, there has been a development in the treatment of boundary value problems, mostly aimed towards ODE. A key word is *boundary triplets*, where the Weyl-Titchmarsh *m*-function plays an important role. Many Russian mathematicians have taken part in this, e.g. Kochubei '75, Gorbachuk and Gorbachuk book '84 (translated '91), Derkach and Malamud '87, Malamud and Mogilevski '97, '02, ... They characterize realizations not just by *operators* defined over the nullspaces, but also *relations* (subsets of graphs).

For PDE cases, the concept of *m*-function has been generalized to *M*-functions in recent years (Amrein, Pearson, Behrndt, Langer, Brown, Marletta, Naboko, Wood, Ryshov...). The function $M(\lambda)$ for $\widetilde{A}$ is a family of operators in spaces over the boundary, holomorphic in $\lambda \in \varrho(\widetilde{A})$ and encoding spectral information on $\widetilde{A}$. We found (as explained in G-Brown-Wood '09) that

$$M(\lambda) = -(L^{\lambda})^{-1}, \text{ when } \lambda \in \varrho(A_\gamma) \cap \varrho(\widetilde{A}).$$

Then the preceding analysis sheds light on the *M*-function too.

3. Neumann-type boundary conditions

Now let us explain the boundary condition, still assuming $m = 1$, $V = Z$, $W = Z'$. A has a Green's formula (for smooth u, v)

$$(Au, v)_\Omega - (u, A'v)_\Omega = (\nu_1 u, \gamma_0 v)_\Sigma - (\gamma_0 u, \nu_1' v)_\Sigma,$$

where

$$\nu_1 = s\gamma_1 + \mathcal{A}\gamma_0, \quad \nu_1' = \bar{s}\gamma_1 + \mathcal{A}'\gamma_0,$$

with a nonvanishing smooth function s and suitable first-order differential operators $\mathcal{A}$, $\mathcal{A}'$ on Σ.

Let $\lambda \in \varrho(A_\gamma)$. In addition to the Poisson operators K_γ^λ resp. $K_\gamma'^{\bar\lambda}$ solving the Dirichlet problems for $A - \lambda$ resp. $A' - \bar\lambda$, we introduce the *Dirichlet-to-Neumann operators*

$$P^\lambda_{\gamma_0,\nu_1} = \nu_1 K^\lambda_\gamma, \quad P'^{\bar\lambda}_{\gamma_0,\nu_1'} = \nu_1' K'^{\bar\lambda}_\gamma,$$

that map the Dirichlet boundary value into the Neumann boundary value for null-solutions. They are elliptic ψdo's of order 1 on Σ.

Theorem 2. *Let $\Gamma = \nu_1 - P^0_{\gamma_0,\nu_1}\gamma_0$; it maps $D(A_{\max})$ into $H^{\frac12}(\Sigma)$. Let $\widetilde{A}$ correspond to the closed densely defined operator $L : H^{-\frac12}(\Sigma) \to H^{\frac12}(\Sigma)$ as described above; then*

$$D(\widetilde{A}) = \{u \in D(A_{\max}) \mid \gamma_0 u \in D(L),\ \Gamma u = L\gamma_0 u\}.$$

In other words, if we define

$$C = L + P^0_{\gamma_0,\nu_1},$$

then $D(\widetilde{A})$ is defined by the **Neumann-type** *boundary condition:*

$$\nu_1 u = C\gamma_0 u, \qquad (5)$$

(with $\gamma_0 u \in D(L)$).

Here C can be a quite general operator.

How do elliptic boundary conditions fit in?

If there is *given* a first-order differential operator (or ψdo) C on Σ such that (5) is an *elliptic boundary condition* for A, then it can be shown that the realization it defines has domain in $H^2(\Omega)$ and corresponds to L described by:

$$L = C - P^0_{\gamma_0,\nu_1}, \quad D(L) = H^{\frac{3}{2}}(\Sigma).$$

In fact, L is an elliptic ψdo then.

For the λ-dependent situation we have, for $\lambda \in \varrho(A_\gamma)$.

$$L^\lambda = C - P^\lambda_{\gamma_0,\nu_1} = L + P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1},$$

$D(L^\lambda) = D(L)$; again $D(L^\lambda) = H^{\frac{3}{2}}(\Sigma)$ in elliptic cases.
Under the hypotheses for Theorem 2, we have moreover:

Theorem 3. *For any $\lambda \in \varrho(A_\gamma)$,*

$$\dim\ker(\widetilde{A} - \lambda) = \dim\ker L^\lambda$$

$$\dim\operatorname{coker}(\widetilde{A} - \lambda) = \dim\operatorname{coker} L^\lambda.$$

The associated M-function satisfies, for $\lambda \in \varrho(A_\gamma) \cap \varrho(\widetilde{A})$,

$$M^\lambda = -(L^\lambda)^{-1} \in \mathcal{L}(H^{\frac{1}{2}}(\Sigma), H^{-\frac{1}{2}}(\Sigma)),$$

and extends to a holomorphic function of $\lambda \in \varrho(\widetilde{A})$. For elliptic boundary conditions, the poles of M^λ are the eigenvalues of $\widetilde{A}$.

Note that L^λ lives on $\varrho(A_\gamma)$ and M^λ lives on $\varrho(\widetilde{A})$; they supplement each other.

Consider again the Krein resolvent formula

$$(\widetilde{A}-\lambda)^{-1}=(A_\gamma-\lambda)^{-1}+K_\gamma^\lambda(L^\lambda)^{-1}(K_\gamma'^{\bar\lambda})^*. \qquad (4)$$

Note that it has three universal ingredients:

$$\begin{aligned}(A_\gamma-\lambda)^{-1}&: H^s(\Omega)\to H^{s+2}(\Omega), \quad s>-\tfrac12,\\ K_\gamma^\lambda&: H^{s-\frac12}(\Sigma)\to H^s(\Omega), \quad s\in\mathbb{R},\\ (K_\gamma'^{\bar\lambda})^*&: H^s(\Omega)\to H^{s+\frac12}(\Sigma), \quad s>-\tfrac12,\end{aligned}$$

all belonging to the ψdbo calculus; $(K_\gamma'^{\bar\lambda})^*$ is a trace operator of class 0. In (4), the mapping properties are especially important for $s=0$. Only L^λ depends on the choice of $\widetilde{A}$ (but is linked to C by the subtraction of $P^\lambda_{\gamma_0,\nu_1}$).

4. Extensions to nonsmooth domains

There exist a few works from '08 dealing with extensions to nonsmooth domains:

- Gesztesy and Mitrea have some results on Robin problems (essentially C is of order < 1) for the Laplacian on Lipschitz domains. To establish a Krein resolvent formula, they assume a little more smoothness, namely $C^{\frac{3}{2}+\varepsilon}$.
- Posilicano and Raimondi have announced related results on selfadjoint realizations of more general second-order elliptic operators with nonsmooth coefficients, provided the domain is $C^{1,1}$.
- In G '08 (Rendiconti Torino), second-order nonselfadjoint operators with smooth coefficients on a $C^{1,1}$ domain are treated; here Neumann-type boundary conditions (5) are allowed.

The $C^{1,1}$ hypothesis is made to use results from Grisvard's '82 book.

The main aim in G '08 was to allow C of order 1 and get ellipticity into play, dealing with operators having a principal part that governs the regularity results plus a remainder with a minor effect. We present this below, but mention already now that we have work in progress (jointly with Abels and Wood) on lowering the regularity hypotheses, essentially down to $C^{\frac{3}{2}+\varepsilon}$, and including $2m$-order operators. The work G '08 is a pilot project.

Our general strategy is to get the results by use of a "machine" such as the ψdbo calculus. The ψdo calculus (on $\mathbb{R}^n$ or open subsets) was generalized to symbols with limited smoothness in the x-variable by Kumano-go and Nagase '78, Marschall '85, Taylor '91 and 2000. The results were extended to ψdbo's (on $\mathbb{R}^n_+$ and coordinate transformed versions) by Abels '05. (All these works have purposes in nonlinear applications.)

There are of course other methods, especially if the differential operator has a simple form, but we think the ψdbo method has an interest in principle.

5. Application of the nonsmooth ψdbo calculus

We denote as usual

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \mid x_n > 0\}, \text{ with } (x_1, \dots, x_{n-1}) = x'.$$

Consider ψdbo's on $\mathbb{R}^n_+$ with $C^{k,\sigma}$-smoothness, $k \in \mathbb{N}_0$ and $\sigma \in]0,1]$, i.e., their symbols satisfy the usual estimates in the cotangent variables ξ', ξ_n, η_n, but only with respect to $C^{k,\sigma}$ Hölder norms in the x'-variable. (Assume for simplicity that the ψdo term P has symbol independent of x_n.) From Abels '05 we have:

Theorem I. *If P, G, T, K and S are of order m and Hölder smoothness $C^{k,\sigma}$, T of class r, then*

- P_+ *and* $G : H^{s+m}(\mathbb{R}^n_+) \to H^s(\mathbb{R}^n_+)$ *for* $|s| < k + \sigma$,
- $T : H^{s+m}(\mathbb{R}^n_+) \to H^{s-\frac12}(\mathbb{R}^{n-1})$ *for* $|s - \frac12| < k + \sigma$, $s + m > r - \frac12$,
- $K : H^{s+m-\frac12}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^n_+)$ *for* $|s| < k + \sigma$
- $S : H^{s+m-\frac12}(\mathbb{R}^{n-1}) \to H^{s-\frac12}(\mathbb{R}^{n-1})$ *for* $|s - \frac12| < k + \sigma$.

Theorem II. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are systems of order m_1 resp. m_2 and Hölder smoothness C^{k_1,σ_1} resp. C^{k_2,σ_2},*

$$\mathcal{A}_1 = \begin{pmatrix} P_{1+} + G_1 & K_1 \\ T_1 & S_1 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} P_{2+} + G_2 & K_2 \\ T_2 & S_2 \end{pmatrix},$$

that can be composed (the dimensions match), then $\mathcal{A}_3 = \mathcal{A}_1\mathcal{A}_2$ is the sum of a term $\mathrm{Op}(a_1 \circ_n a_2)$ with entries as in Th. I, of order $m_3 = m_1 + m_2$ and Hölder smoothness C^{k_3,σ_3}, $k_3 = \min\{k_1, k_2\}$ and $\sigma_3 = \min\{\sigma_1, \sigma_2\}$, plus a remainder with the Sobolev space mapping properties improved by θ for small $\theta > 0$.

Theorem III. *If $\mathcal{A}$ has entries as in Th. I and is polyhomogeneous and uniformly elliptic, with principal symbol a^0, then there is a Green operator $\mathcal{B}^0$ (with symbol $(a^0)^{-1}$ if $m = 0$) of order $-m$ and Hölder smoothness $C^{k,\sigma}$, continuous in the opposite direction, such that the remainder $\mathcal{R} = \mathcal{A}\mathcal{B}^0 - I$ has Sobolev space mapping properties impoved by θ for small $\theta > 0$.*

This all looks very promising for an "automatic" construction of solution operators in elliptic cases. But there are some difficulties in the application to our resolvent formula: We need the mapping properties to be valid in a certain range of Sobolev spaces, both with high and low values of s, but when $k+\sigma$ is small, the above results set severe limitations on the values s that are allowed.
A difficulty in applying Th. III is that the parametrix $\mathcal{B}^0$ is only really simple when $m=0$, otherwise one needs to reduce to the zero-order case by use of "order-reducing operators":

$$\Lambda_0^r : H^{s+r}(\mathbb{R}^{n-1}) \overset{\sim}{\to} H^s(\mathbb{R}^{n-1}), \quad \Lambda_-^r : H^{s+r}(\mathbb{R}^n_+) \overset{\sim}{\to} H^s(\mathbb{R}^n_+).$$

These operators have smooth coefficients, but compositions of nonsmooth operators with them of course gives nonsmooth operators.

There is also the issue of operators "in x-form" or "in y-form":

$$(Pu)(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, đ\xi \text{ or } (Pu)(x) = \int e^{i(x-y)\cdot\xi} p(y,\xi)u(y)\, đ\xi dy.$$

The theorems above describe operators in x-form, but their adjoints will be in y-form (with Sobolev mapping properties in different intervals), and compositions lead to operators of more general types.

Let us describe the construction of the operators

$$(A_\gamma - \lambda)^{-1}, \quad K_\gamma^\lambda, \quad P^\lambda_{\gamma_0,\nu_1},$$

in the case where Ω is $C^{1,1}$ and A has smooth coefficients (for simplicity); then in local coordinates near the boundary, we are dealing with operators with $C^{0,1}$-smoothness (since the normal vector field is $C^{0,1}$).

It is found by application of Theorems I–III in local coordinates that

$$\mathcal{A}(\lambda) = \begin{pmatrix} A - \lambda \\ \gamma_0 \end{pmatrix} : H^{s+2}(\Omega) \to \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix},$$

continuous for $-\frac{3}{2} < s \leq 0$, has a parametrix

$$\mathcal{B}^0(\lambda) = \begin{pmatrix} R^0(\lambda) & K^0(\lambda) \end{pmatrix} : \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix} \to H^{s+2}(\Omega),$$

continuous for $-\frac{1}{2} < s \leq 0$. The remainder

$$\mathcal{R}(\lambda) = \mathcal{A}(\lambda)\mathcal{B}^0(\lambda) - I : \begin{matrix} H^{s-\theta}(\Omega) \\ \times \\ H^{s-\theta+\frac{3}{2}}(\Sigma) \end{matrix} \to \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix}$$

maps continuously for $-\frac{1}{2} + \theta < s \leq 0$, small θ.

Using the parameter-dependent calculus (actually it here suffices to appeal to an old trick of Agmon), one finds that for large λ on rays with argument in $[\frac{\pi}{2}-\varepsilon, \frac{3\pi}{2}+\varepsilon]$, the remainder has norm $\le \frac{1}{2}$, so that an inverse $\mathcal{B}(\lambda)$ can be constructed by a Neumann series; here

$$\mathcal{B}(\lambda) = \begin{pmatrix} (A_\gamma - \lambda)^{-1} & K_\gamma^\lambda \end{pmatrix} : \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix} \to H^{s+2}(\Omega),$$

for $-\frac{1}{2} < s \le 0$.
The same properties hold for the Dirichlet problem for the formal adjoint A'.
The mapping properties of $(A_\gamma - \lambda)^{-1}$ are satisfactory, but those of K_γ^λ do not include Sobolev exponents near 0, which are needed for the Krein resolvent formula.

Here there is another device, namely the observation that

$$K_\gamma^\lambda = (\nu_1'(A_\gamma' - \bar\lambda)^{-1})^*;$$

(derived from Green's formula); it allows a proof that

$$K_\gamma^\lambda : H^{s-\frac12}(\Sigma) \to H^s(\Omega), \qquad (6)$$

for $0 \le s < \frac12$. By interpolation, we get the mapping property (6) for all $0 \le s \le 2$.
For $s \in]\frac32, 2]$, K_γ^λ is the sum of a $C^{0,1}$ Poisson operator and a lower order term.
We can furthermore deduce that

$$P_{\gamma_0,\nu_1}^\lambda : H^{s-\frac12}(\Sigma) \to H^{s-\frac32}(\Sigma) \text{ for } 0 \le s \le 2.$$

For $s \in]\frac32, 2]$, it is the sum of a $C^{0,1}$ ψdo and a lower order term.

Based on these mapping properties, we can conclude:

Theorem 4. *The whole set-up for the smooth case extends to this nonsmooth case, including the diagram that defines L (resp. L^λ) for the general realization $\widetilde{A}$ (resp. $\widetilde{A}-\lambda$), and the Krein resolvent formula.*
Also the interpretation in terms of boundary conditions remains valid: When $V = Z$, $W = Z'$, the boundary condition for $\widetilde{A}$ is

$$\gamma_0 u \in D(L), \quad \nu_1 u - P^0_{\gamma_0,\nu_1}\gamma_0 u = L\gamma_0 u$$

With $C = L + P^0_{\gamma_0,\nu_1}$, this is a Neumann-type boundary condition

$$\nu_1 = C\gamma_0 u.$$

Moreover, we have for the elliptic case:

Theorem 5. *If C is a given first-order differential operator on Σ such that the ψdo $L^\lambda = C - P^\lambda_{\gamma_0,\nu_1}$ is parameter-elliptic on a ray in $\{\operatorname{Im}\lambda \le 0\}$, then the realization $\widetilde{A} - \lambda$ determined by the boundary condition*

$$\nu_1 u = C\gamma_0 u$$

has domain in $H^2(\Omega)$ and corresponds to L^λ with domain $H^{\frac{3}{2}}(\Sigma)$, and it is invertible for large λ on the ray.

Here the Krein resolvent formula holds with all terms belonging to the nonsmooth ψdbo calculus, modulo lower order remainders.

In the mentioned recent studies of Gesztesy and Mitrea, Posilicano and Raimondi, essentially only operators that are compact relative to first-order operators are allowed in the place of C.

About the ongoing project with Helmut Abels and Ian Wood: We shall include higher-order operators and hence more general boundary conditions, and expect to improve the smoothness assumptions, adapted more carefully to what is minimally needed.

Nonsmooth ψdbo's have been applied before, to Navier-Stokes problems, by Abels '03—'09 and Abels-Terasawa '08. Probably useful in many other contexts. . . .