New residue definitions arising from zeta values for boundary value problems

Gerd Grubb
Copenhagen University

September 16, 2008

The noncommutative residue has been defined for classical pseudodifferential operators (ψdo's) on closed manifolds, for operators in the Boutet de Monvel calculus on manifolds with boundary, for logarithms of ψdo's on closed manifolds, and in other cases.

We shall show how the analysis of the basic zeta coefficient — the zeta-regularized trace — for operators on manifolds with boundary leads to still other constants, involving logarithms of boundary operators, with tracial properties that justify seeing them as new cases of noncommutative residues.

- $\widetilde{X}$ — n-dimensional compact boundaryless C^∞ manifold,
- $X \subset \widetilde{X}$, smoothly imbedded, with boundary X',
- $\widetilde{E}$ — hermitian C^∞ vector bundle over $\widetilde{X}$, $E = \widetilde{E}|_X$, $E' = \widetilde{E}|_{X'}$.

On $\widetilde{X}$:

- A — classical (1-step polyhomogeneous) pseudodifferential operator (ψdo) of order $\sigma \in \mathbb{R}$ acting in $\widetilde{E}$,
- P_1 — classical *elliptic* ψdo of order $m > 0$ acting in $\widetilde{E}$, with no eigenvalues on $\overline{\mathbb{R}}_-$, such that $P_1 - \lambda$ is invertible for λ in a sector V around $\mathbb{R}_-$.

Simple example of boundary operators

On X:
Simple example when $X \subset \mathbb{R}^n$:

$$\begin{pmatrix} 1-\Delta \\ \gamma_0 \end{pmatrix} \text{ and its inverse } \begin{pmatrix} R & K \end{pmatrix}, \; R = Q_+ + G;$$

Here R and K solve the semi-homogeneous problems:

$$\begin{cases} (1-\Delta)u &= f \text{ in } X, \\ \gamma_0 u &= 0 \text{ on } X'; \end{cases} \quad \text{resp.} \quad \begin{cases} (1-\Delta)u &= 0 \text{ in } X, \\ \gamma_0 u &= g \text{ on } X'. \end{cases}$$

γ_0 is the trace operator $u \mapsto u|_{X'}$,
$Q = (1-\Delta)^{-1} = \mathrm{Op}(\frac{1}{1+|\xi|^2})$ on $\mathbb{R}^n$,
$Q_+ = r^+ Q e^+$ (e^+ extends by 0, r^+ restricts to X),
G is a singular Green operator (the “boundary correction”),
K is a Poisson operator.

General case

Boutet de Monvel '71 defined pseudodifferential boundary operators (ψdbo's) in general as systems (Green operators):

$$\begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(X, E) \\ \times \\ C^\infty(X', F) \end{matrix} \to \begin{matrix} C^\infty(X, E_1) \\ \times \\ C^\infty(X', F_1) \end{matrix},$$

where:

P is a ψdo on a closed manifold $\widetilde{X} \supset X$, satisfying the *transmission condition* at X' (always true for operators stemming from elliptic PDE),

$P_+ = r^+ P e^+$.

G is a singular Green operator. NB! The "leftover operator" $L(P, Q) = (PQ)_+ - P_+ Q_+$ is such one.

T is a trace operator from X to X', K is a Poisson operator from X' to X, S is a ψdo on X'.

Traces apply when $E = E_1$, $F = F_1$; the new object is $B = P_+ + G$.

The transmission condition requires integer order. For G alone one can study all real orders. We consider *classical* symbols (polyhomogeneous). The setup consists of

- $B = P_+ + G$ — of order $\sigma \in \mathbb{Z}$ acting in E,
- An elliptic differential boundary operator $\begin{pmatrix} P_1 \\ T \end{pmatrix}$ defining the *realization* $P_{1,T}$ acting like P_1 in $L_2(X, E)$ with domain

$$D(P_{1,T}) = \{u \in H^m(X, E) \mid Tu = 0\},$$

  having no eigenvalues on $\overline{\mathbb{R}}_-$, such that $R_\lambda = (P_{1,T} - \lambda)^{-1}$ exists for $\lambda \in V$.

Then we study the traces of $B(P_{1,T} - \lambda)^{-1}$, $BP_{1,T}^{-s}$, etc.

II. Some definitions of noncommutative residues

1. res *A, by Wodzicki and Guillemin '84.*

Let A be a ψdo on $\widetilde{X}$ with symbol $a \sim \sum_{j\geq 0} a_{\sigma-j}$; $a_{\sigma-j}(x,\xi)$ homogeneous of degree $\sigma - j$ in ξ. Define in local coordinates

$$\mathrm{res}_x(A) = \int_{|\xi|=1} \mathrm{tr}\, a_{-n}(x,\xi)\, đS(\xi).$$

Here $đ = (2\pi)^{-n} d$. Then

$$\mathrm{res}(A) = \int_{\widetilde{X}} \int_{|\xi|=1} \mathrm{tr}\, a_{-n}(x,\xi)\, đS(\xi) dx = \int_{\widetilde{X}} \mathrm{res}_x(A)\, dx$$

has a meaning independent of the choice of local coordinates. It is *tracial*, i.e., vanishes on commutators:

$$\mathrm{res}([A, A']) = 0.$$

It vanishes if $\sigma < -n$, if σ is not integer, and if A is a diff. op.

2. $\operatorname{res} B = \operatorname{res}(P_+ + G)$, *by Fedosov-Golse-Leichtnam-Schrohe '96.*
Let $B = P_+ + G$ on X, with symbols $p(x,\xi)$ resp. $g(x',\xi',\xi_n,\eta_n)$,.
From P we define $\operatorname{res}_x P$ as above. From G we first define the *normal trace*, in local coordinates;

$$\operatorname{tr}_n g = s(x',\xi') = \int^+ g(x',\xi',\xi_n,\xi_n) đ\xi_n, \quad \operatorname{tr}_n G = \operatorname{Op}(s(x',\xi')),$$

it is a ψdo on X'. We define as above, in $n-1$ dimensions,

$$\operatorname{res}_{x'}(\operatorname{tr}_n G) = \int_{|\xi'|=1} \operatorname{tr} s_{1-n}(x',\xi')\, đS(\xi').$$

Then

$$\operatorname{res}(B) = \operatorname{res}_+(P) + \operatorname{res}(\operatorname{tr}_n G), \text{ where}$$

$$\operatorname{res}_+(P) = \int_X \operatorname{res}_x(P)\, dx, \quad \operatorname{res}(\operatorname{tr}_n G) = \int_{X'} \operatorname{res}_{x'}(\operatorname{tr}_n G)\, dx'.$$

It has an invariant meaning on X, and is tracial, $\operatorname{res}([B,B']) = 0$.

3. res$(\log P_1)$, *by Okikiolu '95.* $\log P_1$ has symbol $\log[\xi]^m + \ell(x,\xi)$ with $\ell(x,\xi)$ classical of order 0 ($[\xi]$ is smooth positive, equal to $|\xi|$ for $|\xi| \geq 1$), and the residue has an invariant meaning:

$$\operatorname{res}(\log P_1) = \int_{\widetilde{X}} \int_{|\xi|=1} \operatorname{tr} \ell_{-n}(x,\xi)\, đS(\xi) dx.$$

Tracial? Yes and no. Yes in the sense that $[A, \log P_1]$ is classical, and $\operatorname{res}([A, \log P_1]) = 0$. But the symbol $r(x,\xi)$ of $R = A \log P_1$ is of the form

$$r(x,\xi) \sim \sum_{j\geq 0} (r_{\sigma-j,1}(x,\xi) \log[\xi] + r_{\sigma-j,0}(x,\xi)),$$

with $r_{\sigma-j,l}$ homogeneous in ξ of degree $\sigma - j$. Integration over $|\xi| = 1$ gives

$$\operatorname{res}_{x,0}(A \log P_1) = \int_{|\xi|=1} \operatorname{tr} r_{-n,0}(x,\xi) đS(\xi).$$

Only locally defined, does not integrate to a functional. (Lesch '99 defined a “higher trace” res_1 from $r_{-n,1}$.)

III. Occurrence as invariants

Case 1. Define the zeta function $\zeta(A, P_1, s) =$ meromorphic extension of $\mathrm{Tr}(AP_1^{-s})$, then

$$\frac{1}{m} \operatorname{res} A = \operatorname{Res}_{s=0} \zeta(A, P_1, s).$$

Equivalent resolvent formulation (take $m > n + \sigma$):

$$\mathrm{Tr}(A(P_1 - \lambda)^{-1}) \sim \sum_{0 \le j < \sigma + n} a_j (-\lambda)^{\frac{\sigma+n-j}{m} - 1}$$
$$+ C_{-1}(A, P_1)(-\lambda)^{-1} \log(-\lambda) + C_0(A, P_1)(-\lambda)^{-1} + O(\lambda^{-1-\varepsilon}),$$

for $\lambda \to \infty$ in V. Here $\frac{1}{m} \operatorname{res} A = C_{-1}(A, P_1)$, *local*. The *regular value* of $\zeta(A, P_1, s)$ at 0 (the constant behind the residue) is equal to $C_0(A, P_1)$ and is *global*.

Case 2. Define the zeta function $\zeta(B, P_{1,T}, s) =$ meromorphic extension of $\operatorname{Tr}(BP_{1,T}^{-s})$, then

$$\tfrac{1}{m}\operatorname{res} B = \operatorname{Res}_{s=0} \zeta(B, P_{1,T}, s).$$

This was shown via the resolvent formulation by G.-Schrohe '01:

$$\operatorname{Tr}(B(P_{1,T} - \lambda)^{-1}) \sim \sum_{0 \le j < \sigma + n} b_j(-\lambda)^{\frac{\sigma+n-j}{m} - 1}$$
$$+ C_{-1}(B, P_{1,T})(-\lambda)^{-1} \log(-\lambda) + C_0(B, P_{1,T})(-\lambda)^{-1} + O(\lambda^{-1-\varepsilon}),$$

for $\lambda \to \infty$ in V. Here $\frac{1}{m}\operatorname{res} B = C_{-1}(B, P_{1,T})$, *local*.

The regular value of $\zeta(B, P_{1,T}, s)$ at 0 is equal to $C_0(B, P_{1,T})$ and is *global*.

Case 3. The zeta function $\zeta(P_1, s)$ $(= \zeta(I, P_1, s) = \operatorname{Tr} P_1^{-s})$ is regular at 0, and the value is

$$-\tfrac{1}{m}\operatorname{res}(\log P_1) = \zeta(P_1, 0),$$

local. Shown by Scott '05, reworked in resolvent setup by G. '05.

Remarks. The logarithm of P_1 is defined by

$$\log P_1 = \lim_{s \searrow 0} \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^{-s} \log \lambda (P_1 - \lambda)^{-1}\, d\lambda$$

strongly.

Scott and Paycha (GAFA '07) have analysed all the poles of $\zeta(A, P_1, -s) = \operatorname{Tr}(AP_1^{-s})$; here $(\log P_1)^k$ come into the picture.

IV. Hadamard finite-part integrals.

Another trace-like functional is the *canonical trace* of Kontsevich-Vishik '95, see also Lesch '99. Let

$$\mathrm{TR}_x(A) = ⨍ \mathrm{tr}\, a(x,\xi)\, đ\xi,$$

where $⨍ f(x,\xi)\, đ\xi$ is a *partie finie* integral:
When $f(x,\xi)$ is a classical symbol of order σ, then

$$\int_{|\xi|\le R} f(x,\xi)\, đ\xi \sim \sum_{j\ge 0, j\ne\sigma+n} c_j(x) R^{\sigma+n-j} + C_{-1}(x)\log R + C_0(x)$$

for $R\to\infty$, and one defines $⨍ f(x,\xi)\, đ\xi = C_0(x)$; *global.*

A symbol is said to have *even-even (resp. even-odd) parity* when the even/odd-numbered symbols are even/odd in ξ (resp. odd/even).

When

$$1)\ \sigma < -n \text{ or } \notin \mathbb{Z}, \text{ or}$$
$$2)\ a(x,\xi) \text{ has even-even parity, } n \text{ is odd, or}$$
$$3)\ a(x,\xi) \text{ has even-odd parity, } n \text{ is even,}$$

then $\operatorname{TR} A = \int_{\widetilde{X}} \operatorname{TR}_x A\, dx$ has an invariant meaning and is tracial; it is the *canonical trace*. Then (when P_1 is even-even)

$$\operatorname{TR} A = C_0(A, P_1), \quad \operatorname{res} A = 0.$$

What happens outside these cases?

General rules: $C_0(A, P_1)$ is quasi-tracial, in the sense that the *trace-defects*

$$C_0(A, P_1) - C_0(A, P_2) \text{ and } C_0([A, A'], P_1) \text{ are } \textit{local}$$

(equal residues of $-\frac{1}{m}A(\log P_1 - \log P_2)$ resp. $-\frac{1}{m}A[A', \log P_1]$, Okikiolu, Kontsevich-Vishik, Melrose-Nistor, '95-'96.)

But what is the value of $C_0(A, P_1)$ itself? Must be a mix of local and global terms. Paycha-Scott showed (GAFA '07):

$$C_0(A, P_1) = \int_{\widetilde{X}} [\mathrm{TR}_x\, A - \tfrac{1}{m}\, \mathrm{res}_{x,0}(A \log P_1)]\, dx.$$

Both "almost-trace functionals" appear here!

The proof of Paycha and Scott relies heavily on studies of holomorphic families of ψdo's $A(z)$ of order $\alpha z + \beta$ ($\alpha \neq 0$). The proof can also be based directly on resolvent considerations, G. '06.

Remark. C_0 is important in index formulas: For an elliptic A with parametrix $A^{(-1)}$,

$$\operatorname{ind} A = C_0([A, A^{(-1)}], P_1) = -\tfrac{1}{m}\operatorname{res}(A[A^{(-1)}, \log P_1]).$$

(These expressions are independent of P_1 and of the choice of parametrix.)

V. C_0 for cases with boundary

In the search for a similar result for manifolds with boundary, one can expect the formulas to involve an operator $\log P_{1,T}$. This operator can be defined by functional calculus applied to $(P_{1,T} - \lambda)^{-1} = Q_{\lambda,+} + G_\lambda$,

$$\log P_{1,T} = (\log P_1)_+ + \tfrac{i}{2\pi} \int_{\mathcal{C}} \log \lambda G_\lambda \, d\lambda = (\log P_1)_+ + G_1^{\log},$$

formed of a ψdo term and a singular Green-like term.
When m is even, the classical part of $\log P_1$ has the transmission property (required in the Boutet de Monvel calculus). The symbol-kernel of $G_1^{\log}$ will contain terms of the type

$$\frac{1}{x_n + y_n} e^{-[\xi'](x_n + y_n)}$$

with a singularity at $x_n = y_n = 0$. (More on $\log P_{1,T}$ in Gaarde-G., to appear in Math. Scand.)

The main result is the following (G., Math. Ann. '08):

Theorem 1. *When P_1 is of even order, $C_0(B, P_{1,T})$ is a sum of terms, calculated in local coordinates (recall $B = P_+ + G$):*

$$
\begin{aligned}
C_0(B, P_{1,T}) = &\int_X [\mathrm{TR}_x\, P - \tfrac{1}{m}\operatorname{res}_{x,0}(P \log P_1)]\, dx && (1)\\
&+ \tfrac{1}{m}\operatorname{res}(L(P, \log P_1)) && (2)\\
&+ \int_{X'} [\mathrm{TR}_{x'}\operatorname{tr}_n G - \tfrac{1}{m}\operatorname{res}_{x',0}\operatorname{tr}'_n(G(\log P_1)_+)]\, dx' && (3)\\
&- \tfrac{1}{m}\operatorname{res}(P_+ G_1^{\log}) && (4)\\
&- \tfrac{1}{m}\operatorname{res}(G G_1^{\log}). && (5)
\end{aligned}
$$

Each line is an invariant defined from P, G, $\log P_1$ and $G_1^{\log}$.

Here, (1):

$$\int_X [\mathrm{TR}_x P - \tfrac{1}{m} \operatorname{res}_{x,0}(P \log P_1)]\, dx$$

is like the formula of Paycha-Scott for $\widetilde{X}$, but integrated only over X.

The contribution (3):

$$\int_{X'} [\mathrm{TR}_{x'} \operatorname{tr}_n G - \tfrac{1}{m} \operatorname{res}_{x',0} \operatorname{tr}'_n(G(\log P_1)_+)]\, dx'$$

was the hardest to pin down; it also has both global and local components, and tr'_n is a special variant of tr_n.

The contributions (2), (4) and (5) are local and appear as various generalizations of the FGLS residue:

$$\tfrac{1}{m} \operatorname{res}(L(P, \log P_1)),\ -\tfrac{1}{m} \operatorname{res}(P_+ G_1^{\log}),\ -\tfrac{1}{m} \operatorname{res}(G G_1^{\log}).$$

Background: If M_λ is a family of ψdbo's such that for $\lambda \to \infty$ in V,

$$\operatorname{Tr} M_\lambda = \sum_{\text{finite set of } \sigma_j > -1} c_j(-\lambda)^{\sigma_j} + C_{-1}(-\lambda)^{-1}\log(-\lambda) + C_0(-\lambda)^{-1} + O(\lambda^{-1-\varepsilon}), \quad (6)$$

denote $C_{-1} = \ell_{-1}(M_\lambda)$, $C_0 = \ell_0(M_\lambda)$. With $(P_{1,T} - \lambda)^{-1} = Q_{\lambda,+} + G_\lambda$, we have

$$\begin{aligned} BQ_{\lambda,+} &= P_+Q_{\lambda,+} + GQ_{\lambda,+} = (PQ_\lambda)_+ - L(P,Q_\lambda) + GQ_{\lambda,+} \\ BG_\lambda &= P_+G_\lambda + GG_\lambda. \end{aligned}$$

Here each function has an expansion (6), with ℓ_0-coefficient

$$\begin{aligned} \ell_0(BQ_{\lambda,+}) &= \ell_0((PQ_\lambda)_+) + \ell_0(-L(P,Q_\lambda)) + \ell_0(GQ_{\lambda,+}) \\ &= (1) + (2) + (3); \\ \ell_0(BG_\lambda) &= \ell_0(P_+G_\lambda) + \ell_0(GG_\lambda) \\ &= (4) + (5) = -\tfrac{1}{m}\operatorname{res}(BG_1^{\log}). \end{aligned}$$

Theorem 1 is shown using the "positive regularity" concept from G-book '96, assuring convergent integrals of strictly homogeneous symbols, this gives the local terms. For the terms without positive regularity, one compares with special operators using defect formulas, to get the global terms. A difficulty in (3) is that homogeneity properties of G and Q_λ do not match well.

Are the new residues *tracial?* Yes, to some extent:

When B is of order and class 0, we may as well place it to the right, studying $Q_{\lambda,+}B$ and $G_\lambda B$; here

$$\mathrm{Tr}(Q_{\lambda,+}B) = \mathrm{Tr}(BQ_{\lambda,+}), \quad \mathrm{Tr}(G_\lambda B) = \mathrm{Tr}(BG_\lambda),$$

so the ℓ_0 values are the same:

$$\ell_0(Q_{\lambda,+}B) = \ell_0(BQ_{\lambda,+}), \quad \ell_0(G_\lambda B) = \ell_0(BG_\lambda).$$

This leads to:

Theorem 2. *When B is of order and class 0,*

$$\operatorname{res}(BG_1^{\log}) = \operatorname{res}(G_1^{\log}B).$$

Moreover, $[B, (\log P_1)_+]$ has a residue, defined as

$$\operatorname{res}([P, \log P_1]_+) - \operatorname{res}(L(P, \log P_1) - L(\log P_1, P)) + \operatorname{res}([G, (\log P_1)_+]),$$

and it equals zero. Altogether:

$$\operatorname{res}([B, \log P_{1,T}]) = 0.$$

However, $B \log P_{1,T}$ itself is — like $A \log P_1$ on $\widetilde{X}$ — not assigned a residue generalizing FGLS.

VI. Sectorial projections.

If the spectrum of $P_{1,T}$ lies in two sectors of $\mathbb{C}$, separated by two rays $e^{i\theta}\mathbb{R}_+$ and $e^{i\varphi}\mathbb{R}_+$ free of eigenvalues ($\theta < \varphi < \theta + 2\pi$), then, with $\Gamma_{\theta,\varphi} = e^{i\theta}\mathbb{R}_+ \cup e^{i\varphi}\mathbb{R}_+$, the operator (studied in Gaarde-G.)

$$\Pi_{\theta,\varphi}(P_{1,T}) = \tfrac{i}{2\pi} \int_{\Gamma_{\theta,\varphi}} \lambda^{-1} P_1 (P_{1,T} - \lambda)^{-1}\, d\lambda$$

is a projection in L_2, essentially projecting onto the generalized eigenspace for eigenvalues with argument in $]\theta, \varphi[$, and vanishing on the generalized eigenspace for the other evs. Moreover,

$$\Pi_{\theta,\varphi}(P_{1,T}) = \tfrac{i}{2\pi}(\log_\theta P_{1,T} - \log_\varphi P_{1,T}),$$

where $\log_\theta P_{1,T}$ is defined using a logarithm with cut at $e^{i\theta}\mathbb{R}_+$. For $\Pi_{\theta,\varphi}(P_{1,T})$, the ψdo part belongs to the Boutet de Monvel calculus if m is even, and the s.g.o. part may or may not do so.

We can show using Theorems 1–2 (m even):

Theorem 3. *When B is of order and class 0, $B\,\Pi_{\theta,\varphi}(P_{1,T})$ and $\Pi_{\theta,\varphi}(P_{1,T})B$ have residues, and*

$$\operatorname{res}([B, \Pi_{\theta,\varphi}(P_{1,T})]) = 0.$$

Recall the result for closed manifolds that $\operatorname{res}\Pi_{\theta,\varphi}(P_1)$ vanishes (Atiyah-Patodi-Singer, Gilkey, Wodzicki). For manifolds with boundary, Gaarde has recently obtained by use of K-theoretic results for operators in the Boutet de Monvel calculus (by Nest, Schrohe et al.):

Theorem 4. *If $\Pi_{\theta,\varphi}(P_{1,T})$ belongs to the Boutet de Monvel calculus, then*

$$\operatorname{res}\Pi_{\theta,\varphi}(P_{1,T}) = 0.$$

It is still an open question whether this holds in general.