Spectral asymptotics for resolvent differences with rough coefficients

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Plan

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1. Spectral asymptotic estimates in smooth cases

Let Ω be smooth open $\subset \mathbb{R}^n$, with bounded boundary $\partial\Omega = \Sigma$. Denote $\partial_n^j u|_\Sigma = \gamma_j u$, $j \in \mathbb{N}_0$. Consider a strongly elliptic second-order differential operator on Ω with complex C^∞-coefficients

$$Au = -\sum\nolimits_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \sum\nolimits_{j=1}^{n} a_j\partial_j u + a_0 u, \text{ with}$$
$$\operatorname{Re}\sum\nolimits_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2 \text{ for } x \in \overline{\Omega}, \xi \in \mathbb{R}^n;$$

$c_0 > 0$. Set $\nu u = \sum n_j\gamma_0(a_{jk}\partial_k u)$ ($= \gamma_1 u$ when $A = -\Delta$), the conormal derivative ($(n_1, \dots, n_n)$ is the normal to Σ).

Consider the *Dirichlet realization* A_γ, acting like A and with $D(A_\gamma) = \{u \in H^2(\Omega) \mid \gamma_0 u = 0\}$, and a *Neumann-type realization* $A_{\nu,C}$ with domain $D(A_{\nu,C}) = \{u \in H^2(\Omega) \mid \nu u = C\gamma_0 u\}$; here C is first-order tangential, and the system $\{A - \lambda, \nu - C\gamma_0\}$ is assumed parameter-elliptic for $\lambda \in \mathbb{R}_-$. We can assume both A_γ and $A_{\nu,C}$ invertible.

The "weak Schatten class" $\mathfrak{S}_{p,\infty}$ consists of compact operators B such that $s_j(B) = \mu_j(B^*B)^{\frac{1}{2}}$ is $O(j^{-1/p})$ for $j \to \infty$; with quasi-norm $\mathbf{N}_p(B) \equiv \sup_j s_j(B) j^{1/p}$.

It is well-known (starting with Weyl 1912,...) that each of the operators A_γ and $A_{\nu,C}$ has a spectral asymptotics behavior

$$s_j(A_\gamma^{-1}) \text{ and } s_j(A_{\nu,C}^{-1}) \text{ equal } c_A j^{-2/n} + o(j^{-2/n}) \text{ for } j \to \infty, \quad (1)$$

with a constant c_A determined from A, and remainders improved to $O(j^{-3/n})$ or more precision (Hörmander '69, Ivrii '80s,...). Note that A_γ^{-1} and $A_{\nu,C}^{-1}$ are in $\mathfrak{S}_{n/2,\infty}$ (concerned with upper estimates).

It is also well-known (Birman 1962, Birman and Solomyak, Grubb in '70s and '80s) that

$$s_j(A_{\nu,C}^{-1} - A_\gamma^{-1}) \text{ equals } c\, j^{-2/(n-1)} + o(j^{-2/(n-1)}) \text{ for } j \to \infty. \quad (2)$$

Also here, the remainder can be refined using Hörmander and Ivrii results. In particular, $A_{\nu,C}^{-1} - A_\gamma^{-1} \in \mathfrak{S}_{(n-1)/2,\infty}$.

The dimension $n-1$ comes in because the resolvent difference has its essential effect in the neighborhood of the boundary $\partial\Omega$. Let us recall a general result of this type: The resolvent difference belongs to the so-called *singular Green operators* in the calculus of Boutet de Monvel '71 of *pseudodifferential boundary operators* (ψdbo's). For these it was shown in G'84;

When G is a singular Green operator on Ω of order $-t<0$ (and class zero), then

$$s_j(G) = c_G\, j^{-t/(n-1)} + o(j^{-t/(n-1)}) \text{ for } j\to\infty. \quad (3)$$

Our aim in this talk is to extend asymptotic estimates like (2) and (3) to problems where the coefficients are nonsmooth. Upper estimates allowing some nonsmoothness are known from Birman '62.

Reference: G '12, arXiv:1205.0094.

A basic ingredient in the study is the upper estimate:

Lemma A. *If Ξ bounded open $\subset \mathbb{R}^m$ and $B \in \mathcal{L}(L_2(\Xi), H^t(\Xi))$ with $t > 0$, then $B \in \mathfrak{S}_{m/t,\infty}$; indeed,*

$$\mathbf{N}_{m/t}(B) = \sup_j s_j(B) j^{t/m} \le C \|B\|_{\mathcal{L}(L_2, H^t)}.$$

For *asymptotic estimates*, perturbation results are useful:

Lemma B. 1° *If $B = B_0 + R$, where* $\lim_j s_j(B_0) j^{1/p} = C_0$ *and* $\lim_j s_j(R) j^{1/p} = 0$, *then* $\lim_j s_j(B) j^{1/p} = C_0$.
2° *If $B = B_M + B'_M$ for all $M \in \mathbb{N}_0$, with* $\lim_j s_j(B_M) j^{1/p} = C_M$, $\lim_M C_M = C_0$ *and* $\lim_M \mathbf{N}_p(B'_M) = 0$, *then* $\lim_j s_j(B) j^{1/p} = C_0$.

Asymptotic estimates require a structural knowledge of B. If B is a classical *pseudodifferential operator* (ψdo) cut down to Ξ, of order $-t$, it has a principal symbol b^0 defining the principal part B^0, such that $B - B^0$ is of lower order; then one can show that $\lim_j s_j(B) j^{t/m} = C(b^0)$.

This is useful for inverses of elliptic differential operators.

2. The rough ψdbo calculus

To get similar results for nonsmooth elliptic operators, we turn to ψdo's with nonsmooth coefficients, as treated by J. Marschall '87 and M. Taylor '91. They study ψdo symbols $p(x,\xi)$ with full estimates in ξ but only C^τ-smoothness in x ($\tau > 0$).

For $0 < \tau < 1$, composition rules work for such operators with a principal part and a less structured lower-order part.
Marschall's work is particularly useful, since it shows which *finitely many* symbol estimates are needed for operator norm estimates in Sobolev spaces. This allows us to show:

Theorem 1. *If P is a classical C^τ-smooth ψdo of order $-t < 0$, defined on a compact m-dimensional C^∞-manifold Ξ without boundary, then*

$$s_j(P)j^{t/m} \to C(p^0), \text{ for } j \to \infty.$$

The proof uses approximation by operators with smooth symbols.

Birman and Solomyak '77 have results of this kind requiring less smoothness in ξ, but more smoothness in x depending on the dimension.

Now we search for results on singular Green operators on manifolds with boundary, in particular resolvent differences, where the *boundary dimension* comes in. This needs an extension of the nonsmooth ψdo calculus to boundary value problems.

An extension was worked out by H. Abels in 2005, namely a generalization of the pseudodifferential boundary operator (ψdbo) calculus originating from Boutet de Monvel '71, to nonsmooth x-dependence, e.g. in C^τ.

Abels has applied it to the Navier-Stokes problem, and there is a joint work Abels-G-Wood '12 showing Krein resolvent formulas in nonsmooth situations.

The ψdbo calculus deals with matrices:

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} H^{s+d}(\Omega)^N \\ \times \\ H^{s+d}(\Sigma)^M \end{matrix} \rightarrow \begin{matrix} H^{s}(\Omega)^{N'} \\ \times \\ H^{s}(\Sigma)^{M'} \end{matrix}, \text{ where}$$

- P is a pseudodifferential operator (ψdo) on $\mathbb{R}^n$ of order d, and $P_+ = r^+ P e^+$ is its truncation to Ω (r^+ restricts to Ω and e^+ extends by zero).
- T is a trace operator from Ω to Σ of order $d - \frac{1}{2}$, K is a Poisson operator from Σ to Ω of order $d + \frac{1}{2}$, S is a ψdo on Σ of order d.
- G is a singular Green operator of order d, e.g. of type KT.

In the nonsmooth calculus, the continuity holds for $|s| < \tau$, when the operators are merely C^τ-smooth (and of class 0).

There are rules of calculus, e.g.

$$\mathcal{A}_1\mathcal{A}_2 - \mathrm{OP}(a_1 \circ a_2) : \begin{matrix} H^{s+d_1+d_2-\theta}(\Omega)^N \\ \times \\ H^{s+d_1+d_2-\theta}(\Sigma)^M \end{matrix} \to \begin{matrix} H^{s}(\Omega)^{N'} \\ \times \\ H^{s}(\Sigma)^{M'} \end{matrix}, \text{ for } -\tau+\theta < s < \tau.$$

For the 2. order elliptic operator A we have the examples of s.g.o.s:

- $A_\gamma^{-1} = Q_+ - G_\gamma$, where G_γ is the s.g.o. $K_\gamma\gamma_0 Q_+$; here $Q = A^{-1}$ on $\mathbb{R}^n$, and K_γ is the Poisson solution operator.
- $A_{\nu,C}^{-1} - A_\gamma^{-1} = K_\gamma L^{-1}(K'_\gamma)^*$, where $L = C - P_{\gamma,\nu}$, and $P_{\gamma,\nu}$ is the Dirichlet-to-Neumann operator νK_γ.

The formulas extend to the case where the coefficients of A are in $W^1_p(\Omega)$ with $p > n$, by Abels-G-Wood '12, as C^τ-smooth s.g.o.s of order -2, $\tau = 1 - n/p$.
For spectral estimates the calculus must be sharpened to operator norms depending on specific *finite* sets of symbol seminorms, as in Marschall's ψdo treatment.

3. Spectral asymptotics for rough singular Green operators

One difficulty in the application of spectral theory to rough singular Green operators is the handling of remainders:
If $G = G_0 + R$ on Ω, where G_0 has the expected asymptotic behavior

$$s_j(G_0) \sim c(G_0) j^{-t/(n-1)},$$

and $R\colon H^{-t-\theta}(\Omega) \to H^0(\Omega)$, then Lemma A for operators on Ω only gives that

$$s_j(R) \leq C j^{-(t+\theta)/n}.$$

This goes to zero faster than the estimate for $s_j(G_0)$ only if

$$(t+\theta)/n > t/(n-1), \text{ i.e., } \theta > t/(n-1),$$

and we usually do not have such large values of θ available.

So the remainders arising in compositions and approximations are a big problem. One has to involve the boundary dimension directly.

Let G be a C^τ-smooth s.g.o. of order $-t$ and class 0 on $\mathbb{R}^n_+$,

$$(Gu)(x) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} \int_0^\infty \tilde{g}(x', x_n, y_n, \xi') \acute{u}(\xi', y_n)\, dy_n d\xi',$$

where $\acute{u}(\xi', y_n) = \mathcal{F}_{y'\to\xi'} u(y', y_n)$, $y' = (y_1, \dots, y_{n-1})$.
We use that the symbol-kernel $\tilde{g}$ has a rapidly convergent double expansion in Laguerre functions $\varphi_m(x_n, \sigma)$,

$$\tilde{g}(x', x_n, y_n, \xi') = \sum\nolimits_{l,m\in\mathbb{N}_0} c_{lm}(x', \xi')\varphi_l(x_n, \langle\xi'\rangle)\varphi_m(y_n, \langle\xi'\rangle),$$
$$\varphi_m(x_n, \langle\xi'\rangle) = \mathrm{pol}_m(x_n) e^{-x_n\langle\xi'\rangle}, \quad x_n \geq 0,$$

with polynomials of degree m in x_n with coefficients depending on $\langle\xi'\rangle = (1 + |\xi'|^2)^{\frac{1}{2}}$. The $\varphi_m(x_n, \sigma)$, $m \geq 0$, are a basis of $L_2(\mathbb{R}_+)$.
Let Φ_m denote the Poisson operator with symbol-kernel $\varphi_m(x_n, \langle\xi'\rangle)$, then we can write, with C^τ-smooth ψdo's C_{lm} of order $-t$ on $\mathbb{R}^{n-1}$,

$$G = \sum\nolimits_{l,m\in\mathbb{N}_0} \Phi_l C_{lm} \Phi_m^*.$$

Theorem 2. *Let G be selfadjoint ≥ 0 on $\mathbb{R}^n_+$, and let $\psi(x) = \psi_0(x')\psi_n(x_n)$ with $\psi_0 \in C_0^\infty(\mathbb{R}^{n-1}, \mathbb{R})$ and $\psi_n \in C_0^\infty(\mathbb{R}, \mathbb{R})$ equal to 1 near 0. Then the positive eigenvalues of $\psi G \psi$ satisfy*

$$\mu_j(\psi G \psi) j^{t/(n-1)} \to c(\psi_0^2 g^0)^{t/(n-1)} \text{ for } j \to \infty.$$

Here

$$c(\psi_0^2 g^0) = \tfrac{1}{(n-1)(2\pi)^{(n-1)}} \int_\Sigma \int_{|\xi'|=1} \operatorname{tr}((\psi_0^2 g^0(x', \xi', D_n))^{(n-1)/t})\, d\omega dx'. \quad (4)$$

The proof uses a decomposition

$$G = G_M + G_M^\dagger, \text{ where}$$

$$G_M = \sum\nolimits_{l,m<M} \Phi_l C_{lm} \Phi_m^*, \quad G_M^\dagger = \sum\nolimits_{l \text{ or } m \geq M} \Phi_l C_{lm} \Phi_m^*.$$

For the first sum, define

$$\mathcal{K}_M = \begin{pmatrix} \Phi_0 & \Phi_1 & \cdots & \Phi_{M-1} \end{pmatrix}, \quad \mathcal{C}_{M,\psi_0} = \begin{pmatrix} \psi_0 C_{lm} \psi_0 \end{pmatrix}_{l,m=0,\dots,M-1},$$

and note that $\mathcal{K}_M^* \mathcal{K}_M = I_M$. Then $G_M = \mathcal{K}_M \mathcal{C}_{M,1} \mathcal{K}_M^*$.

Moreover,

$$\mu_j(\psi G_M\psi) \sim \mu_j(\textstyle\sum_{l,m<M}\Phi_l\psi_0 C_{lm}\psi_0\Phi_m^*) = \mu_j(\mathcal{K}_M\mathcal{C}_{M,\psi_0}\mathcal{K}_M^*)$$
$$= \mu_j(\mathcal{C}_{M,\psi_0}\mathcal{K}_M^*\mathcal{K}_M) = \mu_j(\mathcal{C}_{M,\psi_0}).$$

This is an $M\times M$-matrix-formed ψdo of order $-t$ on $\mathbb{R}^{n-1}$, to which Theorem 1 applies to give a spectral asymptotic estimate.

For the second sum, $\psi G_M^\dagger\psi = \sum_{l \text{ or } m\geq M}\psi\Phi_l C_{lm}\Phi_m^*\psi$, one can use the rapid decrease of the C_{lm}, combined with the control of $\mathfrak{S}_{p,\infty}$-quasinorms in terms of finite sets of symbol seminorms, to show that $\psi G_M^\dagger\psi$ goes to zero in the quasi-norm on $\mathfrak{S}_{(n-1)/t,\infty}$ for $M\to\infty$.

Now Lemma B is applied to the decomposition $\psi G\psi = \psi G_M\psi + \psi G_M^\dagger\psi$ for $M\to\infty$, to complete the proof.

There is an extension of the theorem to selfadjoint C^τ-smooth s.g.o.s on bounded open smooth sets $\Omega\subset\mathbb{R}^n$.

4. Resolvent differences in rough cases

Now consider the Krein resolvent formula

$$A_{\nu,C}^{-1} - A_{\gamma}^{-1} = K_{\gamma} L^{-1} (K'_{\gamma})^* \equiv G_C,$$

for A with W^1_p-coefficients. We want to find spectral asymptotics of G_C; recall that K_γ is the Dirichlet Poisson operator, and $L = C - P_{\gamma,\nu}$.
In the selfadjoint case, G_C the sum of a s.g.o. of order -2 (as treated above) and a lower-order term. However, perturbation methods fail, since the lower-order term is linked with dimension n.
Instead we shall use that G_C is here already in a product form passing via the boundary, and we can even allow nonselfadjointness.
In the original boundary problems for
$A = -\sum_{j,k=1}^n \partial_j a_{jk} \partial_k + \sum_{j=1}^n a_j \partial_j + a_0$ we approximate the coefficients by C^∞-functions a^ε_{jk}, a^ε_j (by convolution with an approximate identity), and we likewise approximate C by smoothed out versions C^ε.

Following the construction of A_γ^{-1}, K_γ and L^{-1} in Abels-G-Wood '12, we can show that for $\varepsilon \to 0$,

$$\begin{aligned} \|K_\gamma^\varepsilon - K_\gamma\|_{\mathcal{L}(H^{s-\frac12}(\Sigma),H^s(\Omega))} &\to 0, \text{ each } s \in [0,2], \\ \|K_\gamma'^\varepsilon - K_\gamma'\|_{\mathcal{L}(H^{s-\frac12}(\Sigma),H^s(\Omega))} &\to 0, \text{ each } s \in [0,2], \qquad (5) \\ \|(L^\varepsilon)^{-1} - L^{-1}\|_{\mathcal{L}(H^{\frac12}(\Sigma),H^{\frac32}(\Sigma))} &\to 0. \end{aligned}$$

It follows by use of Lemma A that

$$G_C^\varepsilon - G_C = K_\gamma^\varepsilon (L^\varepsilon)^{-1} (K_\gamma'^\varepsilon)^* - K_\gamma L^{-1} (K_\gamma')^* \to 0 \text{ in } \mathfrak{S}_{(n-1)/2,\infty},$$

for $\varepsilon \to 0$.
Then, since the result is known in the smooth case, we conclude by use of Lemma B:

Theorem 3. *For the resolvent difference $G_C = K_\gamma L^{-1}(K'_\gamma)^*$ defined from the Dirichlet realization and a Neumann-type realization of a strongly elliptic operator A with W^1_p-smooth coefficients, $p > n$, the s-numbers satisfy*

$$s_j(G_C)j^{2/(n-1)} \to c(g^0_C)^{2/(n-1)} \text{ for } j \to \infty,$$

where $c(g^0_C)$ is defined similarly to (4).

Moreover, a further analysis shows that there is the following formula for the constant:

Theorem 4. *With $l^0(x',\xi')$ denoting the principal symbol of L and $\lambda^\pm(x',\xi')$ denoting the root in $\mathbb{C}_\pm$ of the principal symbol $a^0(x',0,\xi',\xi_n)$ of A (as a polynomial in ξ_n, in local coordinates)*

$$c(g^0_C) = \tfrac{1}{(n-1)(2\pi)^{(n-1)}} \int_\Sigma \int_{|\xi'|=1} |4(l^0)^2 \operatorname{Im}\lambda^+ \operatorname{Im}\lambda^-|^{-(n-1)/4}\, d\omega dx'. \quad (6)$$