Chapter 12
Unbounded linear operators

12.1 Unbounded operators in Banach spaces

In the elementary theory of Hilbert and Banach spaces, the linear operators that are considered acting on such spaces — or from one such space to another — are taken to be *bounded*, i.e., when T goes from X to Y, it is assumed to satisfy

$$\|Tx\|_Y \le C\|x\|_X, \text{ for all } x \in X; \tag{12.1}$$

this is the same as being continuous. We denote the space of these operators $\mathbf{B}(X,Y)$; $\mathbf{B}(X,X)$ is also denoted $\mathbf{B}(X)$. We generally consider complex vector spaces; most of the theory holds word-to-word also for real spaces.

But when one deals with differential operators, one discovers the need to consider also unbounded linear operators. Here $T : X \to Y$ need not be defined on all of X but may be so on a linear subset, $D(T)$, which is called the domain of T. Of special interest are the operators with *dense* domain in X (i.e., with $\overline{D(T)} = X$). When T is bounded and densely defined, it extends by continuity to an operator in $\mathbf{B}(X,Y)$, but when it is not bounded, there is no such extension. For such operators, another property of interest is the property of being *closed*:

Definition 12.1. A linear operator $T : X \to Y$ is said to be closed when the graph $G(T)$,

$$G(T) = \{\{x, Tx\} \mid x \in D(T)\} \tag{12.2}$$

is a closed subspace of $X \times Y$.

Since X and Y are metric spaces, we can reformulate the criterion for closedness as follows:

Lemma 12.2. *$T : X \to Y$ is closed if and only if the following holds: When $(x_n)_{n\in\mathbb{N}}$ is a sequence in $D(T)$ with $x_n \to x$ in X and $Tx_n \to y$ in Y, then $x \in D(T)$ with $y = Tx$.*

The closed graph theorem (recalled in Appendix B, Theorem B.16) implies that if $T : X \to Y$ is closed and has $D(T) = X$, then T is bounded. Thus for closed, densely defined operators, $D(T) \neq X$ is equivalent with unboundedness.

Note that a subspace G of $X \times Y$ is the graph of a linear operator $T : X \to Y$ if and only if the set $\operatorname{pr}_1 G$,

$$\operatorname{pr}_1 G = \{x \in X \mid \exists y \in Y \text{ so that } \{x, y\} \in G\},$$

has the property that for any $x \in \operatorname{pr}_1 G$ there is *at most one* y so that $\{x, y\} \in G$; then $y = Tx$ and $D(T) = \operatorname{pr}_1 G$. In view of the linearity we can also formulate the criterion for G being a graph as follows:

Lemma 12.3. *A subspace G of $X \times Y$ is a graph if and only if $\{0, y\} \in G$ implies $y = 0$.*

All operators in the following are assumed to be linear, this will not in general be repeated.

When S and T are operators from X to Y, and $D(S) \subset D(T)$ with $Sx = Tx$ for $x \in D(S)$, we say that T is an extension of S and S is a restriction of T, and we write $S \subset T$ (or $T \supset S$). One often wants to know whether a given operator T has a closed extension. If T is bounded, this always holds, since we can simply take the operator $\overline{T}$ with graph $\overline{G(T)}$; here $\overline{G(T)}$ is a graph since $x_n \to 0$ implies $Tx_n \to 0$. But when T is unbounded, one cannot be certain that it has a closed extension (cf. Exercise 12.1). But *if* T has a closed extension T_1, then $G(T_1)$ is a closed subspace of $X \times Y$ containing $G(T)$, hence also containing $\overline{G(T)}$. In that case $\overline{G(T)}$ is a graph (cf. Lemma 12.3). It is in fact the graph of the smallest closed extension of T (the one with the smallest domain); we call it *the closure of* T and denote it $\overline{T}$. (Observe that when T is unbounded, then $D(\overline{T})$ is a proper subset of $\overline{D(T)}$.)

When S and T are operators from X to Y, the sum $S + T$ is defined by

$$\begin{aligned} D(S+T) &= D(S) \cap D(T)\,, \\ (S+T)x &= Sx + Tx \text{ for } x \in D(S+T)\,; \end{aligned} \tag{12.3}$$

and when R is an operator from Y to Z, the product (or composition) RT is defined by

$$\begin{aligned} D(RT) &= \{x \in D(T) \mid Tx \in D(R)\}\,, \\ (RT)x &= R(Tx) \text{ for } x \in D(RT)\,. \end{aligned} \tag{12.4}$$

As shown in Exercise 12.4, $R(S + T)$ need not be the same as $RS + RT$. Concerning closures of products of operators, see Exercise 12.6. When S and T are invertible, one has $(ST)^{-1} = T^{-1}S^{-1}$.

Besides the norm topology we can provide $D(T)$ with the so-called *graph topology*. For Banach spaces it is usually defined by the following *graph norm*

$$\|x\|'_{D(T)} = \|x\|_X + \|Tx\|_Y \;, \tag{12.5}$$

and for Hilbert spaces by the equivalent norm (also called the graph norm)

$$\|x\|_{D(T)} = (\|x\|_X^2 + \|Tx\|_Y^2)^{\frac{1}{2}} \;, \tag{12.6}$$

which has the associated scalar product

$$(x,y)_{D(T)} = (x,y)_X + (Tx,Ty)_Y \;.$$

(These conventions are consistent with (A.10)–(A.12)). The graph norm on $D(T)$ is clearly stronger than the X-norm on $D(T)$; the norms are equivalent if and only if T is a bounded operator. Observe that the operator T is closed if and only if $D(T)$ is complete with respect to the graph norm (Exercise 12.3).

Recall that when X is a Banach space, the dual space $X^* = \mathbf{B}(X,\mathbb{C})$, consists of the bounded linear functionals x^* on X; it is a Banach space with the norm

$$\|x^*\|_{X*} = \inf\{\, |x^*(x)| \mid x \in X, \|x\| = 1\,\}.$$

When $T : X \to Y$ is *densely defined*, we can define *the adjoint operator* $T^* : Y^* \to X^*$ as follows: The domain $D(T^*)$ consists of the $y^* \in Y^*$ for which the linear functional

$$x \mapsto y^*(Tx) \;, \quad x \in D(T) \;, \tag{12.7}$$

is *continuous* (from X to $\mathbb{C}$). This means that there is a constant c (depending on y^*) such that

$$|y^*(Tx)| \le c\|x\|_X, \text{ for all } x \in D(T).$$

Since $D(T)$ is dense in X, the mapping extends by continuity to X, so there is a uniquely determined $x^* \in X^*$ so that

$$y^*(Tx) = x^*(x) \quad \text{for } x \in D(T) \;. \tag{12.8}$$

Since x^* is determined from y^*, we can define the operator T^* from Y^* to X^* by:

$$T^*y^* = x^* \;, \text{ for } y^* \in D(T^*). \tag{12.9}$$

Lemma 12.4. *Let T be densely defined. Then there is an adjoint operator $T^* : Y^* \to X^*$, uniquely defined by* (12.7)–(12.9). *Moreover, T^* is closed.*

Proof. The definition of T^* is accounted for above; it remains to show the closedness.

Let $y_n^* \in D(T^*)$ for $n \in \mathbb{N}$, with $y_n^* \to y^*$ and $T^*y_n^* \to z^*$ for $n \to \infty$; then we must show that $y^* \in D(T^*)$ with $T^*y^* = z^*$ (cf. Lemma 12.2). Now we have for all $x \in D(T)$:

$$y^*(Tx) = \lim_{n\to\infty} y_n^*(Tx) = \lim_{n\to\infty} (T^* y_n^*)(x) = z^*(x)\ .$$

This shows that $y^* \in D(T^*)$ with $T^*y^* = z^*$. □

Here is some more notation: We denote the range of T by $R(T)$, and we denote the kernel of T (i.e. the nullspace) by $Z(T)$,

$$Z(T) = \{x \in D(T) \mid Tx = 0\}\ .$$

When $X = Y$, it is of interest to consider the operators $T - \lambda I$ where $\lambda \in \mathbb{C}$ and I is the identity operator (here $D(T - \lambda I) = D(T)$). The *resolvent set* $\varrho(T)$ is defined as the set of $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is a bijection of $D(T)$ onto X with bounded inverse $(T - \lambda I)^{-1}$; the *spectrum* $\sigma(T)$ is defined as the complement $\mathbb{C} \setminus \varrho(T)$. $T - \lambda I$ is also written $T - \lambda$.

12.2 Unbounded operators in Hilbert spaces

We now consider the case where X and Y are complex Hilbert spaces. Here the norm on the dual space X^* of X is a Hilbert space norm, and the Riesz representation theorem assures that for any element $x^* \in X^*$ there is a unique element $v \in X$ such that

$$x^*(x) = (x, v) \text{ for all } x \in X;$$

and here $\|x^*\|_{X^*} = \|v\|_X$. In fact, the mapping $x^* \mapsto v$ is a bijective isometry, and one usually *identifies* X^* with X by this mapping.

With this identification, the adjoint operator T^* of a densely defined operator $T : X \to Y$ is defined as the operator from Y to X for which

$$(Tx, y)_Y = (x, T^*y)_X \text{ for all } x \in D(T)\ , \tag{12.10}$$

with $D(T^*)$ equal to the set of all $y \in Y$ for which *there exists* a $z \in X$ such that z can play the role of T^*y in (12.10).

Observe in particular that $y \in Z(T^*)$ if and only if $y \perp R(T)$, so we always have:

$$Y = \overline{R(T)} \oplus Z(T^*)\ . \tag{12.11}$$

It is not hard to show that when $S : X \to Y$, $T : X \to Y$ and $R : Y \to Z$ are densely defined, with $D(S + T)$ and $D(RT)$ dense in X, then

$$S^* + T^* \subset (S + T)^* \text{ and } T^*R^* \subset (RT)^*\ ; \tag{12.12}$$

these inclusions can be sharp (cf. Exercise 12.7). Note in particular that for $\alpha \in \mathbb{C} \setminus \{0\}$,

$$(T + \alpha I)^* = T^* + \overline{\alpha} I\ , \text{ and } (\alpha T)^* = \overline{\alpha} T^*\ . \tag{12.13}$$

(But $(0T)^* = 0 \in \mathbf{B}(Y, X)$ is different from $0T^*$ when $D(T^*) \neq Y$.)

The following theorem gives an efficient tool to prove some important facts on closedness and denseness, in connection with taking adjoints.

Theorem 12.5. *Let $T : X \to Y$ be a densely defined operator between two Hilbert spaces X and Y. Then*

$$X \oplus Y = \overline{G(T)} \oplus UG(T^*) , \tag{12.14}$$

where U is the operator from $Y \oplus X$ to $X \oplus Y$ given by $U\{v, w\} = \{-w, v\}$. If in addition T is closed, then T^* is densely defined and $T^{**} = T$.

Proof. Let $\{v, w\} \in X \oplus Y$. The following statements are equivalent:

$$\begin{aligned} \{v, w\} \in UG(T^*) &\iff \{w, -v\} \in G(T^*) \\ &\iff (Tx, w)_Y = -(x, v)_X \quad \forall x \in D(T) \\ &\iff (\{x, Tx\}, \{v, w\})_{X \oplus Y} = 0 \quad \forall x \in D(T) \\ &\iff \{v, w\} \perp G(T) . \end{aligned}$$

Since $\overline{G(T)} = G(T)^{\perp\perp}$ (by a standard rule for subspaces), this shows the identity (12.14). U is clearly an isometry of $Y \oplus X$ onto $X \oplus Y$, and preserves orthogonality, so we have moreover:

$$Y \oplus X = U^{-1}(X \oplus Y) = U^{-1}\overline{G(T)} \oplus G(T^*) . \tag{12.15}$$

Now assume that T is closed, i.e., $G(T) = \overline{G(T)}$. We can then show that $D(T^*)$ is dense in Y: If $y \in Y \ominus \overline{D(T^*)}$, then $\{y, 0\} \perp G(T^*)$, hence $\{y, 0\} \in U^{-1}\overline{G(T)}$ by (12.15) and then also $\{0, y\} \in G(T)$. By Lemma 12.3 we must have $y = 0$, which shows that $Y \ominus \overline{D(T^*)} = \{0\}$.

In this case, $T^* : Y \to X$ has an adjoint operator T^{**}, and we have from what was already shown:

$$Y \oplus X = \overline{G(T^*)} \oplus U^{-1}G(T^{**}) = G(T^*) \oplus U^{-1}G(T^{**}) ,$$

since T^* is closed according to Lemma 12.4. This implies

$$X \oplus Y = U(Y \oplus X) = UG(T^*) \oplus G(T^{**}) ,$$

which gives, by comparison with (12.14), that $G(T^{**}) = \overline{G(T)} = G(T)$ (since T was closed). Thus $T^{**} = T$. □

Note that when S is densely defined, then the following holds:

$$S \subset T \text{ implies } S^* \supset T^* . \tag{12.16}$$

Corollary 12.6. *Let $T : X \to Y$ be densely defined. Then T has a closed extension if and only if T^* is densely defined, and in the affirmative case,*

$$T^* = (\overline{T})^* \quad and \quad T^{**} = \overline{T} \ .$$

Proof. If T has a closure, then in particular $\overline{G(T)} = G(\overline{T})$. Then $T^* = (\overline{T})^*$ by (12.14), and $(\overline{T})^*$ is densely defined according to Theorem 12.5, with $T^{**} = (\overline{T})^{**} = \overline{T}$. Conversely, it is clear that if T^* is densely defined, then T^{**} is a closed extension of T. □

We can also show an important theorem on the relation between adjoints and inverses:

Theorem 12.7. *Assume that $T : X \to Y$ has the properties:*

(1) *T is densely defined,*
(2) *T is closed,*
(3) *T is injective,*
(4) *T has range dense in Y.*

Then T^* and T^{-1} also have these properties, and

$$(T^*)^{-1} = (T^{-1})^* \ . \tag{12.17}$$

Proof. T^{-1} is clearly injective, densely defined and closed (cf. Lemma 12.2) with dense range, and the same holds for T^* by Theorem 12.5, Corollary 12.6 and (12.11) (applied to T and T^*). It then follows moreover that $(T^*)^{-1}$ and $(T^{-1})^*$ have the same properties. Using the linearity of the operators we find, with notation as in Theorem 12.5, that

$$X \oplus Y = G(-T) \oplus UG(-T^*)$$

implies

$$\begin{aligned} Y \oplus X &= U^{-1}(X \oplus Y) = U^{-1}G(-T) \oplus G(-T^*) \\ &= G(T^{-1}) \oplus G(-T^*) = G(T^{-1}) \oplus U^{-1}G((T^*)^{-1}) \ . \end{aligned}$$

An application of Theorem 12.5 to $T^{-1} : Y \to X$ then shows that $(T^{-1})^* = (T^*)^{-1}$. □

We end this section with a remark that is useful when discussing various Hilbert space norms:

It follows from the open mapping principle (recalled e.g. in Theorem B.15), that if a linear space X is a Hilbert space with respect to two norms $\|x\|$ and $\|x\|'$, and there is a constant $c > 0$ such that $\|x\| \le c\|x\|'$ for all $x \in X$, then the two norms are equivalent:

$$\|x\| \le c\|x\|' \le C\|x\| \text{ for all } x \in X,$$

for some $C > 0$. In particular, if the domain $D(T)$ of a closed operator $T : X \to Y$ is a Hilbert space with respect to a norm $\|x\|'$ such that $\|x\|_X +$

$\|Tx\|_Y \le c\|x\|'$ for all $x \in D(T)$, then $\|x\|'$ is equivalent with the graph norm on $D(T)$. (There is a similar result for Banach spaces.)

12.3 Symmetric, selfadjoint and semibounded operators

When X and Y equal the same Hilbert space H, and T is a linear operator in H, we say that T is *symmetric* if

$$(Tx, y) = (x, Ty) \text{ for } x \text{ and } y \in D(T)\ . \tag{12.18}$$

We say that T is *selfadjoint*, when T is densely defined (so that the adjoint T^* exists) and $T^* = T$. (It is a matter of taste whether the assumption $\overline{D(T)} = H$ should also be included in the definitionen of symmetric operators — we do not do it here, but the operators we consider will usually have this property.)

Lemma 12.8. *Let T be an operator in the complex Hilbert space H.*

1° *T is symmetric if and only if (Tx, x) is real for all x.*

2° *When T is densely defined, T is symmetric if and only if $T \subset T^*$. In the affirmative case, T has a closure $\overline{T}$, and*

$$T \subset \overline{T} \subset \overline{T}^* = T^*\ . \tag{12.19}$$

3° *When T is densely defined, T is selfadjoint if and only if T is closed and both T and T^* are symmetric.*

Proof. When T is symmetric,

$$(Tx, x) = (x, Tx) = \overline{(Tx, x)} \quad \text{for } x \in D(T)\ ,$$

whereby $(Tx, x) \in \mathbb{R}$. Conversely, when $(Tx, x) \in \mathbb{R}$ for all $x \in D(T)$, then we first conclude that $(Tx, x) = (x, Tx)$ for $x \in D(T)$; next, we obtain for x and $y \in D(T)$:

$$\begin{aligned} 4(Tx, y) &= \textstyle\sum_{\nu=0}^3 i^\nu (T(x + i^\nu y), x + i^\nu y) \\ &= \textstyle\sum_{\nu=0}^3 i^\nu (x + i^\nu y, T(x + i^\nu y)) = 4(x, Ty). \end{aligned}$$

This shows 1°.

The first assertion in 2° is seen from the definition; the second assertion follows from Corollary 12.6.

For 3° we observe that we have according to Corollary 12.6 for a densely defined operator T with densely defined adjoint T^* that T is closed if and only if $T = T^{**}$. When $T = T^*$, T is of course closed. When T is closed and T and T^* are symmetric, then $T \subset T^*$ and $T^* \subset T^{**} = T$; hence $T = T^*$. □

An operator T for which $\overline{T}$ exists and is selfadjoint, is said to be *essentially selfadjoint* (sometimes in the physics literature such operators are simply called selfadjoint).

A symmetric operator T is called *maximal symmetric* if $S \supset T$ with S symmetric implies $S = T$. Selfadjoint operators are maximal symmetric, but the converse does not hold, cf. Exercise 12.12.

It is useful to know that when S is symmetric and $\lambda = \alpha + i\beta$ with α and $\beta \in \mathbb{R}$, then

$$\begin{aligned}\|(S-\lambda I)x\|^2 &= (Sx - \alpha x - i\beta x, Sx - \alpha x - i\beta x) \\ &= \|(S-\alpha I)x\|^2 + \beta^2\|x\|^2 \quad \text{for } x \in D(S)\ .\end{aligned} \tag{12.20}$$

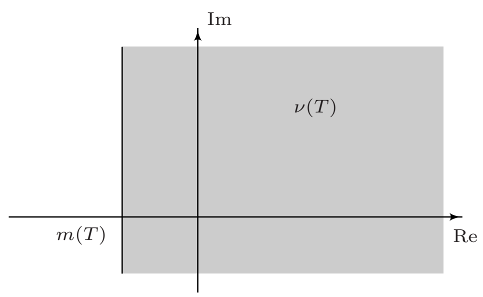
For an arbitrary linear operator T in H we define the *numerical range* $\nu(T)$ by

$$\nu(T) = \{(Tx, x) \mid x \in D(T)\ ,\ \|x\| = 1\} \subset \mathbb{C}\ , \tag{12.21}$$

and the *lower bound* $m(T)$ by

$$m(T) = \inf\{\operatorname{Re}(Tx, x) \mid x \in D(T)\ ,\ \|x\| = 1\} \geq -\infty\ . \tag{12.22}$$

T is said to be *lower (semi)bounded* when $m(T) > -\infty$, and *upper (semi)-bounded* when $m(-T) > -\infty$; $-m(-T)$ is called the upper bound for T.



From a geometric point of view, $m(T) > -\infty$ means that the numerical range $\nu(T)$ is contained in the halfspace $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq m(T)\}$.

Note in particular that the symmetric operators S are precisely those whose numerical range is contained in the real axis (Lemma 12.8.1°). They are also characterized by the property $m(iS) = m(-iS) = 0$.

Bounded operators are clearly upper and lower semibounded. As for a converse, see Theorem 12.12 later and Exercise 12.9. For symmetric operators T, we express $m(T) \geq \alpha$ and $m(-T) \geq -\beta$ briefly by saying that $T \geq \alpha$, resp. $T \leq \beta$. T is called positive (resp. nonnegative) when $m(T) > 0$ (resp. $m(T) \geq 0$).

Theorem 12.9. 1° *If $m(T) \geq \alpha > 0$, then T is injective, and T^{-1} (with $D(T^{-1}) = R(T)$) is a bounded operator in H with norm $\|T^{-1}\| \leq \alpha^{-1}$.*

2° *If furthermore T is closed, then $R(T)$ is closed.*

3° *If T is closed and densely defined, and both $m(T)$ and $m(T^*)$ are $\geq \beta$, then the halfspace $\{\lambda \mid \operatorname{Re}\lambda < \beta\}$ is contained in the resolvent set for T and for T^*.*

Proof. The basic observation is that $m(T) \geq \alpha$ implies the inequality

$$\|Tx\|\,\|x\| \geq |(Tx,x)| \geq \operatorname{Re}(Tx,x) \geq \alpha\|x\|^2 \text{ for } x \in D(T)\,, \tag{12.23}$$

from which we obtain (by division by $\|x\|$ if $x \neq 0$) that

$$\|Tx\| \geq \alpha\|x\| \text{ for } x \in D(T)\,.$$

If $\alpha > 0$, T is then injective. Inserting $Tx = y \in R(T) = D(T^{-1})$, we see that

$$\|T^{-1}y\| = \|x\| \leq \alpha^{-1}\|y\|\,,$$

which shows 1°.

When T is closed, T^{-1} is then a closed bounded operator, so $D(T^{-1})$ $(= R(T))$ is closed; this shows 2°.

For 3° we observe that when $\operatorname{Re}\lambda = \beta - \alpha$ for some $\alpha > 0$, then $m(T - \lambda I) \geq \alpha$ and $m(T^* - \overline{\lambda}I) \geq \alpha$, and $T^* - \overline{\lambda}I$ and $T - \lambda I$ are surjective, by 2° and (12.11), which shows 3°. □

Operators L such that $m(-L) \geq 0$ are called dissipative in works of R. S. Phillips, see e.g. [Phi59], where it is shown that L is maximal dissipative (maximal with respect to the property of being dissipative) precisely when $-L$ satisfies Theorem 12.9 3° with $\beta = 0$; also the problem of finding maximal dissipative extensions of a given dissipative operator is treated there. The maximal dissipative operators are of interest in applications to partial differential equations, because they are the generators of contraction semigroups e^{tL}, used to solve Cauchy problems; an extensive book on semigroups is Hille and Phillips [HP57]. There is more on semigroups in Chapter **??**.

In the book of Kato [Kat66], the T such that $m(T) \geq 0$ are called accretive (i.e., $-T$ is dissipative), and the maximal such operators are called m-accretive.

We can in particular apply Theorem 12.9 to $i(S - \lambda I)$ and $-i(S - \lambda I)$ where S is a symmetric operator. This gives that $S - \lambda I$ is injective with

$$\|(S - \lambda I)^{-1}\| \leq |\operatorname{Im}\lambda|^{-1}\,, \text{ when } \operatorname{Im}\lambda \neq 0 \tag{12.24}$$

(which could also have been derived directly from (12.20)).

Theorem 12.10. *Let S be densely defined and symmetric. Then S is selfadjoint if and only if*

$$R(S + iI) = R(S - iI) = H\,; \tag{12.25}$$

and in the affirmative case, $\mathbb{C} \setminus \mathbb{R} \subset \varrho(S)$.

Proof. Let S be selfadjoint. Then iS and $-iS$ satisfy the hypotheses of Theorem 12.9.3° with $\beta = 0$, and hence the halfspaces

$$\mathbb{C}_\pm = \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \gtrless 0\} \tag{12.26}$$

are contained in the resolvent set; in particular, we have (12.25).

Conversely, if (12.25) holds, we see from (12.11) that the operators $S^* \pm iI$ are injective. Here $S^* + iI$ is an injective extension of $S + iI$, which is a bijection of $D(S)$ onto H; this can only happen if $S = S^*$. □

See also Exercise 12.32.

What a symmetric, densely defined operator "lacks" in being selfadjoint can be seen from what $S+iI$ and $S-iI$ lack in being surjective. The *deficiency indices* of S are defined by

$$\operatorname{def}_+(S) = \dim R(S + iI)^\perp \text{ and } \operatorname{def}_-(S) = \dim R(S - iI)^\perp \, . \tag{12.27}$$

It is very interesting to study the possible selfadjoint extensions of a symmetric, densely defined operator S. It can be shown that S *has* a selfadjoint extension if and only if the two deficiency indices in (12.27) are equal (with suitable interpretations of infinite dimensions, cf. Exercise 12.26); and in the case of equal indices the family of selfadjoint extensions may be parametrized by the linear isometries of $R(S + iI)^\perp$ onto $R(S - iI)^\perp$; cf. Exercise 12.19. One can also use the Cayley transformation $S \mapsto U = (S+iI)(S-iI)^{-1}$ (see e.g. Rudin [Rud74]), carrying the study over to a study of isometries U. We shall later (in Chapter 13 consider cases where S in addition is injective or positive, by different methods).

The next theorem gives a type of examples of selfadjoint unbounded operators, using the technique of Theorem 12.5 in a clever way.

Theorem 12.11. *Let H and H_1 be Hilbert spaces, and let $T : H \to H_1$ be densely defined and closed. Then $T^*T : H \to H$ is selfadjoint and ≥ 0. In particular, $T^*T + I \geq 1$ and is bijective from $D(T^*T)$ to H, and the inverse has norm ≤ 1 and lower bound ≥ 0. Moreover, $D(T^*T)$ is dense in $D(T)$ with respect to the graph norm on $D(T)$.*

Proof. The operator T^*T is clearly symmetric and ≥ 0, since

$$(T^*Tx, x)_H = (Tx, Tx)_{H_1} \geq 0 \text{ for } x \in D(T^*T) \, , \tag{12.28}$$

cf. Lemma 12.8.1°. Since T is densely defined and closed,

$$H \oplus H_1 = G(T) \oplus UG(T^*)$$

by Theorem 12.5. Every $\{x, 0\} \in H \times H_1$ then has a unique decomposition

$$\{x, 0\} = \{y, Ty\} + \{-T^*z, z\} \, ,$$

where y and z are determined from x. Since this decomposition is linear, it defines two bounded linear operators $S : H \to H$ and $P : H \to H_1$ such that $y = Sx$ and $z = Px$. Note that $R(S) \subset D(T)$ and $R(P) \subset D(T^*)$. We will show that S equals $(T^*T + 1)^{-1}$ and is selfadjoint, bounded and ≥ 0; this will imply the assertions on T^*T.

In view of the orthogonality,

$$\begin{aligned}\|x\|_H^2 &= \|\{y, Ty\}\|_{H\oplus H_1}^2 + \|\{-T^*z, z\}\|_{H\oplus H_1}^2 \\ &= \|Sx\|_H^2 + \|TSx\|_{H_1}^2 + \|T^*Px\|_H^2 + \|Px\|_{H_1}^2 \ ,\end{aligned}$$

which implies that S and P have norm ≤ 1. Since

$$x = Sx - T^*Px \ , \ 0 = TSx + Px \ ,$$

we see that $TSx = -Px \in D(T^*)$ and

$$x = (1 + T^*T)Sx \ ; \tag{12.29}$$

hence S maps the space H into $D(T^*T)$, and $(1 + T^*T)S = I$ on H. The bounded operator S^* is now seen to satisfy

$$(S^*x, x) = (S^*(1 + T^*T)Sx, x) = (Sx, Sx) + (TSx, TSx) \geq 0 \text{ for } x \in H \ ,$$

which implies that S^* is symmetric ≥ 0, and $S = S^{**} = S^*$ is likewise symmetric ≥ 0.

Since S is injective (cf. (12.29)), selfadjoint, closed and densely defined, and has dense range (since $Z(S^*) = \{0\}$, cf. (12.11)), Theorem 12.7 implies that S^{-1} has the same properties. According to (12.29), $1 + T^*T$ is a symmetric extension of S^{-1}. Since $1+T^*T$ is injective, $1+T^*T$ must equal S^{-1}. Hence $I + T^*T$ and then also T^*T is selfadjoint.

The denseness of $D(T^*T)$ in $D(T)$ with respect to the graph norm (cf. (12.6)) is seen as follows: Let $x \in D(T)$ be orthogonal to $D(T^*T)$ with respect to the graph norm, i.e.,

$$(x, y)_H + (Tx, Ty)_{H_1} = 0 \text{ for all } y \in D(T^*T) \ .$$

Since $(Tx, Ty)_{H_1} = (x, T^*Ty)_H$, we see that

$$(x, y + T^*Ty)_H = 0 \text{ for all } y \in D(T^*T) \ ,$$

from which it follows that $x = 0$, since $I + T^*T$ is surjective. □

We have the following connection between semiboundedness and boundedness.

Theorem 12.12. 1° *When S is symmetric ≥ 0, one has the following version of the Cauchy-Schwarz inequality:*

$$|(Sx,y)|^2 \le (Sx,x)(Sy,y) \text{ for } x,y \in D(S)\,. \tag{12.30}$$

2° *If S is a densely defined, symmetric operator with $0 \le S \le \alpha$, then S is bounded with $\|S\| \le \alpha$.*

Proof. 1°. When $t \in \mathbb{R}$, we have for x and $y \in D(S)$,

$$\begin{aligned} 0 &\le (S(x+ty), x+ty) = (Sx,x) + t(Sx,y) + t(Sy,x) + t^2(Sy,y) \\ &= (Sx,x) + 2\,\mathrm{Re}(Sx,y)t + (Sy,y)t^2\,. \end{aligned}$$

Since this polynomial in t is ≥ 0 for all t, the discriminant must be ≤ 0, i.e.,

$$|2\,\mathrm{Re}(Sx,y)|^2 \le 4(Sx,x)(Sy,y)\,.$$

When x is replaced by $e^{i\theta}x$ where $-\theta$ is the argument of (Sx,y), we get (12.30).

2°. It follows from 1° that

$$|(Sx,y)|^2 \le (Sx,x)(Sy,y) \le \alpha^2\|x\|^2\|y\|^2 \quad\text{for}\quad x,y \in D(S)\,.$$

Using that $D(S)$ is dense in H, we derive:

$$|(Sx,z)| \le \alpha\|x\|\,\|z\| \quad\text{for}\quad x \in D(S) \quad\text{and}\quad z \in H\,. \tag{12.31}$$

Recall that the Riesz representation theorem defines an *isometry* between H^* and H such that the norm of an element $y \in H$ satisfies

$$\|y\| = \sup\Big\{\frac{|(y,z)|}{\|z\|} \mid z \in H \setminus \{0\}\Big\}\,. \tag{12.32}$$

In particular, we can conclude from the information on S that $\|Sx\| \le \alpha\|x\|$ for $x \in D(S)$, which shows that S is bounded with norm $\le \alpha$. □

A similar result is obtained for slightly more general operators in Exercise 12.9.

An important case of a complex Hilbert space is the space $L_2(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$ and the functions are complex valued. The following gives a simple special case of not necessarily bounded operators in $L_2(\Omega)$, where one can find the adjoint in explicit form and there is a simple criterion for selfadjointness.

Theorem 12.13. *Let Ω be an open subset of $\mathbb{R}^n$, and let $p : \Omega \to \mathbb{C}$ be a measurable function. The multiplication operator M_p in $L_2(\Omega)$ defined by*

$$\begin{aligned} D(M_p) &= \{u \in L_2(\Omega) \mid pu \in L_2(\Omega)\}, \\ M_p u &= pu \quad \text{for } u \in D(M_p), \end{aligned}$$

is densely defined and closed, and the adjoint operator M_p^* *is precisely the multiplication operator* $M_{\overline{p}}$.

Here M_p *is selfadjoint if* p *is real.*

If $|p(x)| \le C$ *(for a constant* $C \in [0, \infty[$ *), then* M_p *is everywhere defined and bounded, with norm* $\le C$.

Proof. Clearly, the operator is linear. Observe that a measurable function f on Ω lies in $D(M_p)$ if and only if $(1 + |p|)f \in L_2(\Omega)$. Hence

$$D(M_p) = \Big\{ \frac{\varphi}{1+|p|} \;\Big|\; \varphi \in L_2(\Omega) \Big\}.$$

It follows that $D(M_p)$ is dense in $L_2(\Omega)$, for if $\overline{D(M_p)}$ were $\ne L_2(\Omega)$, then there would exist a $f \in L_2(\Omega) \setminus \{0\}$ such that $f \perp \overline{D(M_p)}$, and then one would have:

$$0 = \Big(\frac{\varphi}{1+|p|}, f \Big) = \Big(\varphi, \frac{f}{1+|p|} \Big), \text{ for all } \varphi \in L_2(\Omega),$$

and this would imply $\dfrac{f}{1+|p|} = 0$ in contradiction to $f \ne 0$.

By definition of the adjoint, we have that $f_1 \in D(M_p{}^*)$ if and only if there exists a $g \in L_2(\Omega)$ such that

$$(M_p f, f_1) = (f, g) \text{ for all } f \in D(M_p),$$

i.e.,

$$\Big(\frac{p\varphi}{1+|p|}, f_1 \Big) = \Big(\frac{\varphi}{1+|p|}, g \Big) \text{ for all } \varphi \in L_2(\Omega).$$

We can rewrite this as

$$\Big(\varphi, \frac{\overline{p} f_1}{1+|p|} \Big) = \Big(\varphi, \frac{g}{1+|p|} \Big), \text{ for all } \varphi \in L_2(\Omega); \tag{12.33}$$

here we see, since $\dfrac{\overline{p} f_1}{1+|p|}$ and $\dfrac{g}{1+|p|}$ belong to $L_2(\Omega)$, that (12.33) holds if and only if they are the same element, and then

$$\overline{p} f_1 = g.$$

This shows that $M_p{}^* = M_{\overline{p}}$. It is a closed operator; then so is $M_{\overline{p}}$.

It follows immediately that if $p = \overline{p}$, then M_p is selfadjoint.

Finally, it is clear that when $|p(\xi)| \le C$ for $\xi \in X$, then $D(M_p) = L_2(\Omega)$ and

$$\|M_p f\|_{L_2} \le C \|f\|_{L_2};$$

and then M_p is bounded with $\|M_p\| \le C$. □

Note that $D(M_p) = D(M_{\overline{p}}) = D(M_{|p|})$; here we have a case where the operator and its adjoint have the same domain (which is not true in general).

We can also observe that when p and q are bounded functions, then $M_{pq} = M_p M_q$. (For unbounded functions, the domains of M_{pq} and $M_p M_q$ may not be the same; consider e.g. $p(x) = x$ and $q(x) = 1/x$ on $\Omega = \mathbb{R}_+$.)

It is not hard to see that $M_p M_{\overline{p}} = M_{\overline{p}} M_p$, so that M_p is *normal* (i.e., commutes with its adjoint).

Since $M_p u$ and $M_{p_1} u$ define the same element of L_2 if p and p_1 differ on a null-set only, the definition of M_p is easily extended to almost everywhere defined functions p. We can also observe that if $p \in L_\infty(\Omega)$, then $M_p \in \mathbf{B}(L_2(\Omega))$, and

$$\|M_p\| = \operatorname{ess\,sup}_{x\in\Omega} |p(x)|. \tag{12.34}$$

Here the inequality $\|M_p\| \le \operatorname{ess\,sup} |p(x)|$ is easily seen by choosing a representative p_1 for p with $\sup |p_1| = \operatorname{ess\,sup} |p|$. On the other hand, one has for $\varepsilon > 0$ that the set $\omega_\varepsilon = \{x \in \Omega \mid |p(x)| \ge \operatorname{ess\,sup} |p| - \varepsilon\}$ has positive measure, so $\|M_p u\| \ge (\sup |p| - \varepsilon)\|u\|$ for $u = 1_K$ (cf. (A.27)), where K denotes a measurable subset of ω_ε with finite, positive measure; here $u \in L_2(\Omega)$ with $\|u\| > 0$.

It is seen in a similar way that $p = \overline{p}$ a.e. is necessary for the selfadjointness of M_p.

12.4 Operators associated with sesquilinear forms

A complex function $a : \{x, y\} \mapsto a(x, y) \in \mathbb{C}$, defined for x, y in a vector space V, is said to be *sesquilinear*, when it is linear in x and conjugate linear — also called antilinear or semilinear — in y:

$$\begin{aligned} a(\alpha x_1 + \beta x_2, y) &= \alpha a(x_1, y) + \beta a x_2, y) \\ a(x, \gamma y_1 + \delta y_2) &= \overline{\gamma} a(x, y_1) + \overline{\delta} a(x, y_2). \end{aligned}$$

(The word "sesqui" is Latin for $1\frac{1}{2}$.) For example, the scalar product on a complex Hilbert space is sesquilinear.

Let H be a complex Hilbert space, and let $s(x, y)$ be a sesquilinear form defined for x and y in a subspace $D(s)$ of H; $D(s)$ is called the domain of s, and we may say that s is a sesquilinear form on H (even if $D(s) \ne H$). The *adjoint* sesquilinear form s^* is defined to have $D(s^*) = D(s)$ and

$$s^*(x, y) = \overline{s(y, x)} \text{ for } x, y \in D(s)\ , \tag{12.35}$$

and we call s *symmetric* when $s = s^*$. A criterion for symmetry is that $s(x, x)$ is real for all x; this is shown just like in Lemma 12.8.1°.

s is said to be *bounded* (on H) when there is a constant C so that

$$|s(x,y)| \le C\|x\|_H\|y\|_H \ , \quad \text{for } \ x \ \text{ and } \ y \in D(s) \ .$$

Any (linear, as always) operator T in H gives rise to a sesquilinear form t_0 by the definition:

$$t_0(x,y) = (Tx,y)_H \ , \quad \text{with } \ D(t_0) = D(T) \ .$$

If T is bounded, so is t_0. The converse holds when $D(T)$ is dense in H, for then the boundedness of t_0 implies that when $x \in D(T)$,

$$|(Tx,y)_H| \le C\|x\|_H\|y\|_H$$

for y in a dense subset of H and hence for all y in H; then (cf. (12.32))

$$\|Tx\|_H = \sup\Big\{\frac{|(Tx,y)_H|}{\|y\|_H} \,|\, y \in H \setminus \{0\}\Big\} \le C\|x\|_H \ .$$

In this case, T and t_0 are extended in a trivial way to a bounded operator, resp. a bounded sesquilinear form, defined on all of H.

The unbounded case is more challenging.

Definition 12.14. Let $t(x,y)$ be a sesquilinear form on H (not assumed to be bounded), with $D(t)$ dense in H. **The associated operator** T in H is defined as follows:

$D(T)$ consists of the elements $x \in D(t)$ for which there exists $y \in H$ such that

$$t(x,v) = (y,v)_H \ \text{ for all } \ v \in D(t) \ .$$

When the latter equations hold, y is uniquely determined from x since $D(t)$ is dense in H, and we set

$$Tx = y \ .$$

When t is unbounded, $D(T)$ will usually be a proper subset of $D(t)$ and T an unbounded operator in H.

The construction leads to a useful class of operators in a special case we shall now describe.

Let V be a linear subspace of H, which is dense in H and which is a Hilbert space with a norm that is stronger than the norm in H:

$$\|v\|_V \ge c\|v\|_H \ \text{ for } \ v \in V \ , \tag{12.36}$$

with $c > 0$. We then say that $V \subset H$ *algebraically, topologically and densely.* Let $a(u,v)$ be a sesquilinear form with $D(a) = V$ and a bounded on V, i.e.,

$$|a(u,v)| \le C\|u\|_V\|v\|_V, \text{ for all } u,v \in V. \tag{12.37}$$

The form a induces the following two different operators: a bounded operator $\mathcal{A}$ in V and a (usually) unbounded operator A in H, obtained by applying Definition 12.14 to a as a form on V resp. H. Let us denote them $\mathcal{A}$ resp. A.

The form a is called V-*elliptic* if there is a constant $c_0 > 0$ so that

$$\operatorname{Re} a(v,v) \geq c_0 \|v\|_V^2 \quad \text{for} \quad v \in V\ . \tag{12.38}$$

It will be called V-*coercive* if an inequality as in (12.38) can be obtained by adding a multiple of $(u,v)_H$ to a, i.e., if there exists $c_0 > 0$ and $k \in \mathbb{R}$ so that

$$\operatorname{Re} a(v,v) + k\|v\|_H^2 \geq c_0 \|v\|_V^2 \quad \text{for } v \in V\ . \tag{12.39}$$

For $\mu \in \mathbb{C}$ we denote

$$a_\mu(u,v) = a(u,v) + \mu(u,v)_H, \text{ with } D(a_\mu) = D(a). \tag{12.40}$$

In view of (12.36), a_μ is bounded on V. When (12.39) holds, then a_μ is V-elliptic whenever $\operatorname{Re}\mu \geq k$.

Note that when a is V-elliptic or V-coercive, then the same holds for the adjoint form a^* (recall (12.35)), with the same constants.

The results in the following are known under the name of "the Lax-Milgram lemma."

First there is an easy variant:

Lemma 12.15. *Let a be a bounded everywhere defined sesquilinear form on V, and let $\mathcal{A}$ be the associated operator in V. Then $\mathcal{A} \in \mathbf{B}(V)$ with norm $\leq C$ (cf.* (12.37)*), and its adjoint $\mathcal{A}^*$ in V is the operator in V associated with a^*.*

Moreover, if a is V-elliptic, then $\mathcal{A}$ and $\mathcal{A}^*$ are homeomorphisms of V onto V, the inverses having norms $\leq c_0^{-1}$ (cf. (12.38)*).*

Proof. The boundedness of $\mathcal{A}$ was shown above. That the adjoint is the operator in V associated with a^* follows from (12.10) and (12.35). Now (12.38) implies that $m(\mathcal{A}) \geq c_0$ as well as $m(\mathcal{A}^*) \geq c_0$, where $c_0 > 0$. Then $\mathcal{A}$ and $\mathcal{A}^*$ are bijective from V to V with bounded inverses by Theorem 12.9.3°, which also gives the bound on the inverses. □

By the Riesz representation theorem, there is an identification of H with the dual space of continuous linear functionals on H, such that $x \in H$ is identified with the functional $y \mapsto (y,x)_H$. To avoid writing x to the right, we can instead identify H with the *antidual* space, where x corresponds to the *antilinear* (conjugate linear) functional $y \mapsto (x,y)_H$. (This follows from the usual Riesz theorem by conjugation.)

From now on, *we denote by H^* the antidual space of H*. Stated in details: H^* is the space of continuous antilinear functionals on H, and we identify $x \in H$ with the functional $y \mapsto (x,y)_H$.

In the situation of real Hilbert spaces, the antidual and dual spaces are of course the same.

The space V likewise has an antidual space V^* consisting of the continuous antilinear functionals on V. By the Riesz representation theorem, every element $w \in V$ corresponds to an element $Jw \in V^*$ such that

$$(Jw)(v) = (w, v)_V, \text{ all } v \in V, \tag{12.41}$$

and the map J is an isometry of V onto V^*. But rather than using this isometry to identify V and V^*, we want to focus on the imbedding $V \subset H$.

Since $V \subset H$ densely and (12.36) holds, we can define a map from H to V^* sending $f \in H$ over into the antilinear functional $\ell_f \in V^*$ for which

$$\ell_f(v) = (f, v)_H \text{ for all } v \in V. \tag{12.42}$$

(12.36) assures that ℓ_f is continuous on V. The mapping from f to ℓ_f is injective by the denseness of V in H (if ℓ_{f_1} acts like ℓ_{f_2}, $f_1 - f_2$ is H-orthogonal to V, hence is zero). Thus the map $f \mapsto \ell_f$ can be regarded as an imbedding of H into V^*, and we henceforth denote ℓ_f by f, writing $\ell_f(v)$ as $(f, v)_H$.

Lemma 12.16. *There are continuous injections*

$$V \hookrightarrow H \hookrightarrow V^*; \tag{12.43}$$

here, when $f \in H$,

$$\|f\|_{V^*} \le c^{-1}\|f\|_H. \tag{12.44}$$

Proof. The injections are accounted for above, and (12.44) follows from the calculation

$$\begin{aligned}\|f\|_{V^*} &= \sup\Big\{\frac{|\ell_f(v)|}{\|v\|_V} \mid v \in V \setminus \{0\}\Big\} = \sup\Big\{\frac{|(f,v)_H|}{\|v\|_V} \mid v \in V \setminus \{0\}\Big\} \\ &\le \sup\Big\{\frac{\|f\|_H\|v\|_H}{\|v\|_V} \mid v \in V \setminus \{0\}\Big\} \le c^{-1}\|f\|_H,\end{aligned}$$

using the definition of the norm of a functional, the denseness of V in H, (12.42) and (12.36). □

Note that in the identification of ℓ_f with f when $f \in H$, we have obtained that the duality between V^* and V *extends the scalar product in H*, cf. (12.42).

When $\mathcal{A}$ is defined as in Lemma 12.15, we let $\widetilde{\mathcal{A}} = J\mathcal{A}$ (recall (12.41)); it is the operator in $\mathbf{B}(V, V^*)$ that satisfies

$$(\widetilde{\mathcal{A}}u)(v) = (\mathcal{A}u, v)_V = a(u, v) \text{ for all } u, v \in V.$$

We also define $\widetilde{\mathcal{A}}' = J\mathcal{A}^*$. Lemma 12.15 implies immediately:

Corollary 12.17. *When a is bounded on V and V-elliptic (satisfying* (12.37), (12.38)*), then $\widetilde{\mathcal{A}} = J\mathcal{A}$ is a homeomorphism of V onto V^*, with $\|\widetilde{\mathcal{A}}\|_{\mathbf{B}(V,V^*)} \le C$ and $\|\widetilde{\mathcal{A}}^{-1}\|_{\mathbf{B}(V^*,V)} \le c_0^{-1}$. $\widetilde{\mathcal{A}}' = J\mathcal{A}^*$ is similar.*

Now we take A into the picture.

Theorem 12.18. *Consider a triple (H, V, a) where H and V are complex Hilbert spaces with $V \subset H$ algebraically, topologically and densely (satisfying* (12.36)*), and where a is a bounded sesquilinear form on V with $D(a) = V$ (satisfying* (12.37)*). Let A be the operator associated with a in H:*

$$\begin{aligned} D(A) &= \{\, u \in V \mid \exists f \in H \text{ so that } a(u,v) = (f,v)_H \text{ for all } v \in V \,\}, \\ Au &= f. \end{aligned} \tag{12.45}$$

When a is V-elliptic (satisfying (12.38)*), then A is a closed operator with $D(A)$ dense in H and in V, and with $m(A) \ge c_0 c^2 > 0$. It is a bijection of $D(A)$ onto H, and*

$$\{\lambda \mid \operatorname{Re}\lambda < c_0 c^2\} \subset \varrho(A)\,. \tag{12.46}$$

Moreover, the operator associated with a^* in H equals A^*; it likewise has the properties listed for A. In particular, if a is symmetric, A is selfadjoint > 0.

Proof. By Corollary 12.17, a and a^* give rise to bijections $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{A}}'$ from V to V^*. Now observe that, by the definition of A,

$$D(A) = \widetilde{\mathcal{A}}^{-1} H, \text{ with } Au = \widetilde{\mathcal{A}}u \text{ for } u \in D(A),$$

when H is considered as a subset of V^* as explained above. Denoting the operator associated with a^* in H by A', we have similarly:

$$D(A') = (\widetilde{\mathcal{A}}')^{-1} H, \text{ with } A'u = \widetilde{\mathcal{A}}'u \text{ for } u \in D(A').$$

Thus A and A' are *bijective from their domains onto* H, with inverses $T = A^{-1}$ and $T' = (A')^{-1}$, defined on all of H as the restrictions of $\widetilde{\mathcal{A}}^{-1}$ resp. $(\widetilde{\mathcal{A}}')^{-1}$ to H:

$$A^{-1} = T = \widetilde{\mathcal{A}}^{-1}|_H; \quad (A')^{-1} = T' = (\widetilde{\mathcal{A}}')^{-1}|_H.$$

Here T and T' are bounded from H to V and a fortiori from H to H, since

$$\|Tf\|_H \le c^{-1}\|Tf\|_V = c^{-1}\|\widetilde{\mathcal{A}}^{-1}f\|_V \le c^{-1}c_0^{-1}\|f\|_{V^*} \le c^{-2}c_0^{-1}\|f\|_H,$$

for $f \in H$, cf. (12.44); there is a similar calculation for T'.

Now we have for all $f, g \in H$, setting $u = Tf$, $v = T'g$ so that $f = Au$, $g = A'v$:

$$(f, T'g)_H = (f, v)_H = a(u,v) = \overline{a^*(v,u)} = \overline{(A'v, u)_H} = (u, A'v) = (Tu, g)_H;$$

this shows that the bounded operators T and T' in H are adjoints of one another. Their ranges are dense in H since their nullspaces are 0 (the operators are injective), cf. (12.11).

Since T and $T' = T^*$ are closed densely defined injective operators in H with dense ranges, we can apply Theorem 12.7 to conclude that their inverses are also each other's adjoints. So $A = T^{-1}$ and $A' = (T^*)^{-1}$ are each other's adjoints, as unbounded operators in H, and they are closed and densely defined there.

From (12.38) and (12.36) follows moreover:

$$\operatorname{Re}(Au, u)_H = \operatorname{Re} a(u, u) \geq c_0 \|u\|_V^2 \geq c_0 c^2 \|u\|_H^2$$

for all $u \in D(A)$, so $m(A) \geq c_0 c^2$. We likewise find that $m(A^*) \geq c_0 c^2$, and (12.46) then follows from Theorem 12.9.3°.

The set $D(A)$ is dense in V, for if $v_0 \in V$ is such that $(u, v_0)_V = 0$ for all $u \in D(A)$, then we can let $w_0 = (\mathcal{A}^*)^{-1} v_0$ (using Lemma 12.15) and calculate:

$$0 = (u, v_0)_V = (u, \mathcal{A}^* w_0)_V = a(u, w_0) = (Au, w_0)_H, \text{ for all } u \in D(A),$$

which implies $w_0 = 0$ and hence $v_0 = 0$. Similarly, $D(A^*)$ is dense in V. □

Corollary 12.19. *Hypotheses as in Theorem* 12.18*, except that V-ellipticity is replaced by V-coercivity* (12.39)*. Then A is a closed operator with $D(A)$ dense in H and in V, and with $m(A) \geq c_0 c^2 - k$. Moreover,*

$$\{\lambda \mid \operatorname{Re} \lambda < c_0 c^2 - k\} \subset \varrho(A) \,. \tag{12.47}$$

The operator associated with a^* in H equals A^* and has the same properties.

Proof. Note that for $\mu \in \mathbb{C}$, $A + \mu I$ (with $D(A + \mu I) = D(A)$) is the operator in H associated with the sesquilinear form a_μ defined in (12.40). When (12.39) holds, we replace a by a_k. Theorem 12.18 applies to this form and shows that the associated operator $A + kI$ and its adjoint $A^* + kI$ have the properties decribed there. This gives for A itself the properties we have listed in the corollary. □

We shall call the operator A defined from the triple (H, V, a) by Theorem 12.18 or Corollary 12.19 the *variational operator* associated with (H, V, a) (it can also be called the Lax-Milgram operator). The above construction has close links with the calculus of variations, see the remarks at the end of this section.

The construction and most of the terminology here is based on works of J.-L. Lions, as presented e.g. in his book [Lio63] and subsequent papers and books. The operators are also studied in the book of T. Kato [Kat66], where they are called m-sectorial.

Example 12.20. Variational operators are studied in many places in this book; abstract versions enter in Chapter 13 and concrete versions enter in Chapter 4 (and **??**), see in particular Section 4.4 where the Dirichlet and Neumann realizations of the Laplace operator are studied. The distribution theory, or at least the definition of Sobolev spaces, is needed to give a satisfactory interpretation of the operators that arise from the construction. In fact, H is then ususally a space $L_2(\Omega)$ over a subset $\Omega \subset \mathbb{R}^n$, and V is typically a Sobolev space such as $H^1(\Omega)$ or $H^1_0(\Omega)$. (There is then also an interpretation of V^* as a Sobolev space with exponent -1 — for the case $\Omega = \mathbb{R}^n$, we explain such spaces in Section 6.3.)

Let us at present just point to the one-dimensional example taken up in Exercise 12.25. Here V is the closure of $C^1(\overline{I})$ in the norm $\|u\|_1 = (\|u\|^2_{L_2} + \|u'\|^2_{L_2})^{\frac{1}{2}}$ (identified with $H^1(I)$ in Section 4.3), and $\frac{d}{dt}$ can be defined by extension as a continuous operator from V to H. Let $q(t)$ be real ≥ 1, then $a(u,u) \geq \|u\|_1^2$, and Theorem 12.18 applies, defining a selfadjoint operator A in H. When $u \in C^2(\overline{I})$, $v \in C^1(\overline{I})$, we have by integration by parts that

$$a(u,v) = (-u'' + qu, v) + u'(\beta)\overline{v}(\beta) - u'(\alpha)\overline{v}(\alpha).$$

Then if $u \in C^2(\overline{I})$ with $u'(\beta) = u'(\alpha) = 0$, it satisfies the requirement $a(u,v) = (f,v)$ for $v \in V$, *with* $f = -u'' + qu$. Hence it is in $D(A)$ with $Au = -u'' + qu$.

The typical information here is that A acts as a *differential operator*, namely $-\frac{d^2}{dt^2} + q$ (of order 2, while $a(u,v)$ is of order 1), and the domain $D(A)$ involves a *boundary condition*, namely $u'(\beta) = u'(\alpha) = 0$.

The example, and many more, can be completely worked out with the tools from distribution theory established in other parts of this book.

We note that $D(A)$ and $D(A^*)$ need not be the same set even though $D(a) = D(a^*)$ (cf. e.g. Exercise 12.37).

We can use the boundedness of a on (12.37) to show that when a satisfies (12.39), then

$$|\operatorname{Im} a(u,u)| \leq |a(u,u)| \leq C\|u\|_V^2 \leq Cc_0^{-1}(\operatorname{Re} a(u,u) + k\|u\|_H^2)\ ,$$

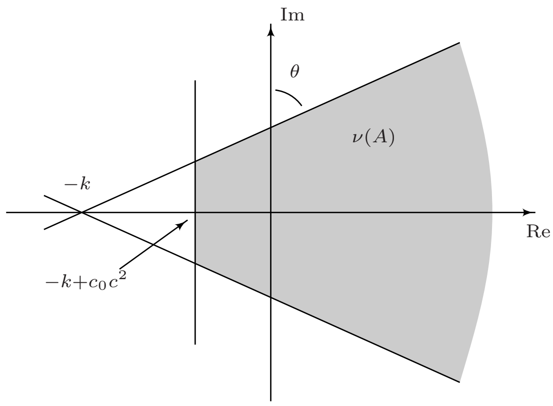
and hence

$$|\operatorname{Im}(Au,u)_H| \leq Cc_0^{-1}(\operatorname{Re}(Au,u)_H + k\|u\|_H^2)\ , \tag{12.48}$$

when $u \in D(A)$. This shows that the numerical range $\nu(A)$ for A — and correspondingly the numerical range $\nu(A^*)$ for A^* — satisfy

$$\nu(A) \text{ and } \nu(A^*) \subset M' = \{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| \leq Cc_0^{-1}(\operatorname{Re}\lambda + k)\}\ . \tag{12.49}$$

But this means that certain *rotations* of A and A^*, namely $e^{\pm i\theta}A$ and $e^{\pm i\theta}A^*$ for a suitable θ (see the figure), are semibounded below, which implies by Theorem 12.9.3° that the spectra $\sigma(A)$ and $\sigma(A^*)$ are likewise contained in M'.



Thus we have:

Corollary 12.21. *When A and A^* are defined from the triple (H, V, a) as in Corollary 12.19, then the spectra $\sigma(A)$ and $\sigma(A^*)$ and the numerical ranges $\nu(A)$ and $\nu(A^*)$ are contained in the angular set with opening $< \pi$:*

$$M = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq -k + c_0 c^2\ ,\ |\operatorname{Im}\lambda| \leq C c_0^{-1}(\operatorname{Re}\lambda + k)\}\ , \qquad (12.50)$$

where the constants are taken from (12.36), (12.39), (12.37).

Finally we observe that when A is a variational operator in a Hilbert space H, then V and a are *uniquely determined by* A. For if A is the variational operator associated with two triples (H, V_1, a_1) and (H, V_2, a_2), then we have for u and $v \in D(A)$:

$$(Au, v)_H = a_1(u, v) = a_2(u, v)\ ;$$

When k is sufficiently large, $[\operatorname{Re}((A+k)v, v)_H]^{\frac{1}{2}}$ is a norm on $D(A)$ which is equivalent with the V_1-norm and with the V_2-norm. Since $D(A)$ is dense in V_1 and in V_2 (with respect to their norms), we get by completion an identification between V_1 and V_2. Since a_1 and a_2 coincide on the dense subset $D(A)$, they must be equal. This shows:

Corollary 12.22. *When A is a variational operator in H, then A stems from one and only one triple (H, V, a); here V is determined as the completion of $D(A)$ under the norm $[\operatorname{Re}((A + k)v, v)_H]^{\frac{1}{2}}$ for a suitably large k, and a is defined on V by closure of $(Au, v)_H$.*

In this result it is assumed that A stems from a triple (H, V, a). One can show that such a triple exists, when A is closed densely defined and there exists a sector M' as in (12.49) such that A is *maximal* with regards to the property $\nu(A) \subset M'$. We return to this alternative description in Section 12.6.

The variational operators are a useful generalization of selfadjoint lower bounded operators, which enter for example in the study of partial differential equations; they have the advantage in comparison with normal operators (closed, densely defined operators N with $NN^* = N^*N$) that the class of variational operators is more stable under the perturbations that occur naturally in the theory.

Remark 12.23. In the case of a symmetric sesquilinear form $a(u, v)$, the connection between Theorem 12.18 and variational calculus is as follows: Assume that a is V-elliptic, and define A by Definition 12.14. Then the problem of solving $Au = f$ is equivalent with the following variational problem:

$$\begin{gathered}\text{For a given } f \in H, \text{ minimize the functional}\\ J(u) = a(u,u) - 2\operatorname{Re}(f,u)_H \text{ for } u \in V.\end{gathered} \tag{12.51}$$

For, the equation $Au = f$ is (in this abstract setting) the Euler equation associated with J. More precisely, we have: Let $u, v \in V$ and let $\varepsilon \in \mathbb{R}$. Then

$$\begin{aligned}I(\varepsilon) \equiv J(u+\varepsilon v) = a(u,u) + 2\varepsilon \operatorname{Re} a(u,v) + \varepsilon^2 a(v,v)\\ -2\operatorname{Re}(f,u)_H - 2\varepsilon \operatorname{Re}(f,v)_H,\end{aligned} \tag{12.52}$$

so in particular $\frac{d}{d\varepsilon}I(0)$ (the so-called *first variation*) satisfies

$$\tfrac{d}{d\varepsilon}I(0) = 2\operatorname{Re}(a(u,v) - (f,v)_H).$$

If u solves (12.51), then $\frac{d}{d\varepsilon}I(0) = 0$ for all $v \in V$, so since we can insert αv instead of v for any complex α, it follows that

$$a(u,v) - (f,v)_H = 0 \text{ for all } v \in V. \tag{12.53}$$

By definition of A, this means precisely that

$$u \in D(A) \text{ with } Au = f. \tag{12.54}$$

Conversely, if u satisfies (12.54), (12.53) holds, so $\frac{d}{d\varepsilon}I(0) = 0$ for all $v \in V$. Then

$$J(u+\varepsilon v) = a(u,u) + \varepsilon^2 a(v,v) - 2\operatorname{Re}(f,u)_H \geq a(u,u) - 2\operatorname{Re}(f,u)_H,$$

for all $v \in V$ and $\varepsilon \in \mathbb{R}$, so u solves (12.51) (uniquely).

12.5 The Friedrichs extension

Let S be a symmetric, densely defined operator. When $S \geq c > 0$, it is easy to find a selfadjoint extension of S.

Let us first consider $\overline{S}$, the closure of S, which is also symmetric, cf. (12.19). It is easily seen that $m(S) = m(\overline{S})$, which is then $\geq c > 0$. According to Theorem 12.9, $\overline{S}$ has a bounded, symmetric inverse (the closure of S^{-1}), so if $R(S)$ and hence $R(\overline{S})$ is dense in H, $R(\overline{S})$ must equal H, so that $\overline{S}^{-1}$ is selfadjoint and $\overline{S}$ is a selfadjoint extension of S by Theorem 12.7. (This is the case where S is essentially selfadjoint.)

If, on the other hand, $Z(S^*) = R(S)^\perp$ is $\neq \{0\}$, we can introduce the operator P with

$$\begin{aligned} D(P) &= D(\overline{S}) \dot{+} Z(S^*) \\ P(v+z) &= \overline{S}v \quad \text{ for } v \in D(\overline{S})\ ,\ z \in Z(S^*)\ ; \end{aligned} \tag{12.55}$$

it is a selfadjoint extension (Exercise 12.23). This extension has $m(P) = 0$ in contrast to $m(S) > 0$. It is J. von Neumann's solution [N 1929] of the problem of finding a selfadjoint semibounded extension of S. A more refined (and useful) extension was found by K. Friedrichs [Fri34]; it has the same lower bound as S, and we shall explain its construction in the following.

M. G. Krein later showed [Kre47] that the full set of selfadjoint extensions $\widetilde{T} \geq 0$, via the associated sesquilinear forms, can be characterized at the operators "lying between" T and P in a certain way. Here T ("the hard extension") is closest to S, whereas P ("the soft extension") is farthest from S. In practical applications, T is of great interest whereas P is somewhat exotic. More about the role of P in Section 13.2, see in particular Corollary 13.21 and the following text. Concrete interpretations are given in Chapter **??** in a manageable case, see Example **??**.

Theorem 12.24 (Friedrichs). *Let S be densely defined, symmetric and lower bounded in H. There exists a selfadjoint extension T with $m(T) = m(S)$.*

Proof. Assume first that $m(S) = c > 0$. The sesquilinear form

$$s_0(u,v) = (Su, v)$$

is then a scalar product on $D(S)$ (cf. Theorem 12.12), and we denote the completion of $D(S)$ with respect to this scalar product by V. Hereby $s_0(u,v)$ is extended to a sesquilinear form $s(u,v)$ with $D(s) = V$ (s is the scalar product itself on V).

We would like to use the Lax-Milgram lemma (Theorem 12.18), but this requires that we show that there is an injection of V into H. The inequality

$$\|v\|_V^2 = s_0(v,v) \geq c\|v\|_H^2 \ \text{ for } v \in D(S),$$

implies that the injection $J_0 : D(S) \hookrightarrow H$ extends to a continuous map J from V to H, but does not assure that the extended map J is *injective.* This point (which is sometimes overlooked in other texts) can be treated as follows: From $(Su, v)_H = (u, v)_V$ for $u, v \in D(S)$ it follows by passage to the limit that

$$(Su, Jv)_H = (u, v)_V \quad \text{for } u \in D(S),\ v \in V.$$

If $Jv = 0$, v is orthogonal to the dense subspace $D(S)$ of V, and then $v = 0$. Hence J is injective, and we may identify V with the subspace $J(V)$ of H.

Since clearly

$$s(v, v) = \|v\|_V^2 \geq c\|v\|_H^2 \ \text{ for } v \in V\ ,$$

the conditions for the Lax-Milgram lemma are fulfilled by the triple (H, V, s) with $c_0 = 1$, and we obtain a selfadjoint operator T in H associated with s, satisfying $m(T) = c = m(S)$. It is clear that $T \supset S$.

When $m(S)$ is arbitrary, we define a selfadjoint extension T' of $S' = S + (1-m(S))I$ by the procedure above; then $T = T' - (1-m(S))I$ is the desired extension of S. □

The proof of the Lax-Milgram lemma can of course be simplified in the case where the sesquilinear form equals the scalar product on V, but the basic principles are the same.

In view of Corollary 12.22 we also have:

Theorem 12.25. *Let S be a densely defined, symmetric, lower bounded operator in H. For $k \geq -m(S)$, the completion V of $D(S)$ with respect to the scalar product*

$$(u, v)_V = (Su + (1 + k)u, v)_H \tag{12.56}$$

can be identified with a subspace of H independent of k, and the Friedrichs extension T of S is characterized by being the only *lower bounded selfadjoint extension of S which has its domain $D(T)$ contained in V.*

In general, there exist many more selfadjoint extensions of S; this will be taken up in Chapter 13.

12.6 More on variational operators

In this section, we take up a converse question in connection with Corollary 12.19ff.: When is an operator with numerical range in a sector (12.49) variational? This discussion is not needed for the immediate applications of the variational construction. (Theorem 12.26 is used in the proof of Corollary 13.15 below.) We shall show:

Theorem 12.26. *When A is a closed densely defined operator in H such that A and A^* both have their numerical ranges in a sector:*

$$\nu(A) \text{ and } \nu(A^*) \subset M' = \{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| \leq c_1(\operatorname{Re}\lambda + k)\}, \tag{12.57}$$

for some $c_1 \geq 0$, $k \in \mathbb{R}$, *then* A *is variational with spectrum in* M' *(and so is* A^**). The associated sesquilinear form* $a(u,v)$ *is the closure of* $a_0(u,v) = (Au,v)$, *defined on the completion of* $D(A)$ *with respect to the norm* $(\operatorname{Re} a_0(u,u) + (1+k)\|u\|_H^2)^{\frac{1}{2}}$.

Before giving the proof we make some preparations. When s is a given sesquilinear form, we define the sesquilinear forms s_{Re} and s_{Im}, with domain $D(s)$, by

$$\begin{aligned} s_{\mathrm{Re}}(u,v) &= \tfrac{1}{2}(s(u,v) + s^*(u,v)), \\ s_{\mathrm{Im}}(u,v) &= \tfrac{1}{2i}(s(u,v) - s^*(u,v)) \ ; \end{aligned} \tag{12.58}$$

note that they are both symmetric (and can take complex values), and that

$$s(u,v) = s_{\mathrm{Re}}(u,v) + i s_{\mathrm{Im}}(u,v),\ u,v \in D(s). \tag{12.59}$$

We define the numerical range and lower bound:

$$\begin{aligned} \nu(s) &= \{s(u,u) \in \mathbb{C} \mid u \in D(s),\ \|u\|_H = 1\}, \\ m(s) &= \inf\{\operatorname{Re} s(u,u) \mid u \in D(s),\ \|u\|_H = 1\}; \end{aligned} \tag{12.60}$$

then s is said to be lower bounded, positive, resp. nonnegative, when $m(s)$ is $> -\infty$, > 0, resp. ≥ 0.

Lemma 12.27. *When* b *and* b' *are symmetric sesquilinear forms with the same domain* $D(b)$, *satisfying*

$$|b(u,u)| \leq b'(u,u),\ \text{all } u \in D(b), \tag{12.61}$$

then

$$|b(u,v)| \leq b'(u,u)^{\frac{1}{2}} b'(v,v)^{\frac{1}{2}},\ \text{all } u,v \in D(b). \tag{12.62}$$

Proof. Note that b' is a nonnegative sesquilinear form. Let $u,v \in D(b)$. If $b'(u,u)$ or $b'(v,v)$ is 0, so is $b(u,u)$ resp. $b(v,v)$ according to (12.61), so (12.62) is valid then. We now assume that $b'(u,u)$ and $b'(v,v) \neq 0$. By multiplication of u by $e^i\theta$ for a suitable θ, we can obtain that $b(u,v)$ is real, equal to $b(v,u)$. Then

$$b(u,v) = \tfrac{1}{4}(b(u+v,u+v) - b(u-v,u-v)), \tag{12.63}$$

as is easily checked. It follows by use of (12.61) that

$$|b(u,v)| \leq \tfrac{1}{4}(b'(u+v,u+v) + b'(u-v,u-v)) = \tfrac{1}{2}(b'(u,u) + b'(v,v)).$$

Then for any $\alpha > 0$,

$$|b(\alpha u, \alpha^{-1} v)| \leq \tfrac{1}{2}(\alpha^2 b'(u,u) + \alpha^{-2} b'(v,v)).$$

Taking $\alpha^2 = b'(v,v)^{\frac{1}{2}} b'(u,u)^{-\frac{1}{2}}$, we obtain (12.62). □

Proof (Proof of Theorem 12.26). By adding a constant to A, we reduce to the case where (12.57) holds with $k = -1$, so that

$$\operatorname{Re} a_0(u,u) \geq \|u\|_H^2. \tag{12.64}$$

Let us define $a_0(u,v) = (Au, v)$, with $D(a_0) = D(A)$. Then $a_{0,\mathrm{Re}}$ is a positive symmetric sesquilinear form defining a scalar product and norm on $D(a_0)$; we denote the completion of $D(a_0)$ in this norm by V.

The crucial part of the proof is to show that V identifies with a subspace of H and that a_0 extends to a bounded positive sesquilinear form a on V. (This resembles a step in the proof of Theorem 12.24, but demands more effort.) In view of (12.57) with $k = -1$,

$$\begin{aligned} |a_0(u,u)| &\leq |\operatorname{Re} a_0(u,u)| + |\operatorname{Im} a_0(u,u)| \\ &\leq (1+c_1)\operatorname{Re} a_0(u,u) = (1+c_1)a_{0,\mathrm{Re}}(u,u). \end{aligned} \tag{12.65}$$

Moreover, by an application of Lemma 12.27 with $b = a_{0,\mathrm{Im}}$, $b' = (1+c_1)a_{0,\mathrm{Re}}$,

$$\begin{aligned} |a_0(u,v)| &\leq |a_{0,\mathrm{Re}}(u,v)| + |a_{0,\mathrm{Im}}(u,v)| \\ &\leq a_{0,\mathrm{Re}}(u,u)^{\frac{1}{2}} a_{0,\mathrm{Re}}(v,v)^{\frac{1}{2}} + (1+c_1)a_{0,\mathrm{Re}}(u,u)^{\frac{1}{2}} a_{0,\mathrm{Re}}(v,v)^{\frac{1}{2}} \\ &= (2+c_1)a_{0,\mathrm{Re}}(u,u)^{\frac{1}{2}} a_{0,\mathrm{Re}}(v,v)^{\frac{1}{2}}. \end{aligned} \tag{12.66}$$

In view of (12.64), the map $J : D(a_0) \hookrightarrow H$ extends to a continuous map $J : V \to H$ with $\|Jv\|_H \leq \|v\|_V$. To show that J is injective, let $v \in V$ with $Jv = 0$; we must show that $v = 0$. There exists a sequence $v_k \in D(a_0)$ such that v_k converges to v in V and v_k converges to 0 in H; then we are through if we can show that $\|v_k\|_V \to 0$, i.e., $\operatorname{Re} a_0(v_k, v_k) \to 0$, for $k \to \infty$. We know that $\operatorname{Re} a_0(v_k - v_l, v_k - v_l) \to 0$ for $k, l \to \infty$, and that $\operatorname{Re} a_0(v_l, v_l) \leq C$ for some $C > 0$, all l. Now

$$\begin{aligned} |\operatorname{Re} a_0(v_k,v_k)| \leq |a_0(v_k,v_k)| &\leq |a_0(v_k, v_k - v_l)| + |a_0(v_k, v_l)| \\ &\leq |a_0(v_k, v_k - v_l)| + |(Av_k, v_l)_H|, \end{aligned}$$

where

$$|a_0(v_k, v_k - v_l)| \leq (2+c_1)\|v_k\|_V \|v_k - v_l\|_V$$

in view of (12.66). For any $\varepsilon > 0$ we can find N such that $\|v_k - v_l\|_V \leq \varepsilon$ for $k, l \geq N$. Let $l \to \infty$, then since $\|v_l\|_H \to 0$, we get the inequality

$$|\operatorname{Re} a_0(v_k, v_k)| \leq (2+c_1)C\varepsilon, \text{ for } k \geq N.$$

Since the constant $(2+c_1)C$ is independent of k, this shows the desired fact, that $\|v_k\|_V \to 0$. Thus J is injective from V to H, so that V can be identified with $JV \subset H$. Here $V \subset H$ algebraically, topologically and densely.

Next, since $a_{0,\mathrm{Re}}$ and $a_{0,\mathrm{Im}}$ are bounded in the V-norm, they extend uniquely to bounded sesquilinear forms on V, so $a_0(u,v)$ does so too, and we can denote the extension by $a(u,v)$. The information in (12.57) with $k=-1$ implies

$$|\operatorname{Im} a(v,v)| \le c_1(\operatorname{Re} a(v,v) - \|v\|_H^2). \tag{12.67}$$

So a is V-elliptic, and hence defines a variational operator A_1; clearly

$$A \subset A_1, \tag{12.68}$$

and $\nu(A_1) \subset M'$.

We can do the whole procedure for A^* too. It is easily checked that the space V is the same as for A, and that the extended sesquilinear form is a^*. Then the variational operator defined from (H, V, a^*) is equal to A_1^* and extends A^*; in view of (12.68) this implies $A = A_1$. So A is the variational operator defined from (H, V, a). □

Remark 12.28. An equivalent way to formulate the conditions on A for being variational, is to say that it is closed, densely defined, and *maximal* with respect to the property $\nu(A) \subset M'$. Then the construction of V and a go through as in the above proof, leading to a variational operator A_1 such that (12.68) holds. Here A_1 is an extension of A with $\nu(A_1) \subset M'$, so in view of the maximality, A_1 must equal A.

The result is due to Schechter and Kato, see e.g. [Kat66, Th. VI.1.27].

Exercises for Chapter 12

12.1. Let H be a separable Hilbert space, with the orthonormal basis $(e_j)_{j\in\mathbb{N}}$. Let V denote the subspace of (finite) linear combinations of the basis-vectors. Define the operator T in H with $D(T) = V$ by:

$$T\Big(\sum_{j=1}^{n} c_j e_j\Big) = \sum_{j=1}^{n} c_j e_1 \ .$$

Show that T has no closure. Find T^*, and find the closure of $G(T)$.

12.2. With H and T as in the preceding exercise, let T_1 be the restriction of T with $D(T_1) = V_1$, where V_1 is the subspace of linear combinations of the basis-vectors e_j with $j \ge 2$. Show that T_1 is a symmetric operator which has no closure.
Find an isometry U of a subspace of H into H such that $U - I$ is injective, but $\overline{U} - I$ is not injective.

12.3. Show that an operator $T : X \to Y$ is closed if and only if $D(T)$ is complete with respect to the graph-norm.

12.4. Show that when R, S and T are operators in a Banach space X, then $RS + RT \subset R(S + T)$. Investigate the example

$$S = \frac{d}{dx}\,, \quad T = -\frac{d}{dx}\,, \quad R = \frac{d}{dx}\,,$$

for the Banach space $C^0([0,1])$.

12.5. Let H be a Hilbert space, and let B and T be operators in H, with $B \in \mathbf{B}(H)$. Show the following assertions:

(a) If T is closed, then TB is closed.

(b) If T is densely defined, T^*B^* is closed but not necessarily densely defined; and BT is densely defined but not necessarity closable. Moreover, $T^*B^* = (BT)^*$.

(c) If T and T^* are densely defined and BT is closed, then T is closed and T^*B^* is densely defined, and $BT = (T^*B^*)^*$.

(d) If T is densely defined and closed, and TB is densely defined, then $(TB)^* = \overline{B^*T^*}$.

12.6. Find, for example for $H = L_2(\,]0,1[\,)$, selfadjoint operators $B \in \mathbf{B}(H)$ and T in H such that TB is not densely defined and BT is not closable. (*Hint:* Using results from Chapter 4, one can take as T the realization $A_\#$ defined in Theorem 4.16 and take a suitable B with $\dim R(B) = 1$.)

12.7. Investigate (12.12) by use of the examples in Exercises 12.4 and 12.6.

12.8. Show that if the operator T in H is densely defined, and $(Tx, x) = 0$ for all $x \in D(T)$, then $T = 0$. Does this hold without the hypothesis $\overline{D(T)} = H$?

12.9. Let T be a densely defined operator in H with $D(T) \subset D(T^*)$. Show that if $\nu(T)$ is bounded, then T is bounded.
(*Hint:* One can consider the symmetric operators $\operatorname{Re} T = \frac{1}{2}(T + T^*)$ and $\operatorname{Im} T = \frac{1}{2i}(T - T^*)$.)

12.10. With H equal to the complex Hilbert space $L_2(\,]0,1[\,)$, consider the operator T defined by

$$D(T) = \{u \in H \mid (u, 1) = 0\},$$
$$T : u(t) \mapsto f(t) = \int_0^t u(s)\,ds.$$

(a) Show that T is bounded.

(b) Show that T is skew-symmetric (i.e., iT is symmetric). (*Hint.* The illustration in Section 4.3 may be helpful.)

(c) Is iT selfadjoint?

12.11. Consider the intervals

$$I_1 =]0,1[, \quad I_2 =]0,\infty[= \mathbb{R}_+, \quad I_3 = \mathbb{R},$$

and the Hilbert spaces $H_j = L_2(I_j)$, $j = 1,2,3$. Let T_j be the multiplication operator M_{t^3} (the mapping $u(t) \mapsto t^3 u(t)$) defined for functions in H_j, $j = 1,2,3$, respectively.
Find out in each case whether T_j is

(1) bounded,
(2) lower bounded,
(3) selfadjoint.

12.12. Let $I =]0,\infty[$ and consider the Hilbert space $H = L_2(I)$ (where $C_0^\infty(I)$ is a dense subset). Let T be the operator acting like $D = \frac{1}{i}\frac{d}{dx}$ with $D(T) = C_0^\infty(I)$.

(a) Show that T is symmetric and has a closure $\overline{T}$ (as an operator in H).

(b) Show that the equation $u'(t) + u(t) = f(t)$ has a solution in H for every $f \in C_0^\infty(I)$. (The solution method is known from elementary calculus.)

(c) Show that the function e^{-t} is orthogonal to $R(T + iI)$.

(d) Show that $\overline{T}$ is maximal symmetric, but does not have a selfadjoint extension.

12.13. Let $J = [\alpha, \beta]$. Show that the operator $\frac{d}{dx}$ in the Banach space $C^0(J)$, with domain $C^1(J)$, is closed.

12.14. Show the assertions in Corollary 12.21 on the spectra $\sigma(A)$ and $\sigma(A^*)$ in details.

12.15. Let H be a separable Hilbert space, with the orthonormal basis $(e_j)_{j\in\mathbb{Z}}$. For each $j > 0$, let $f_j = e_{-j} + je_j$. Let V and W be the closures of the spaces of linear combinations of, respectively, the vectors $(e_j)_{j\geq 0}$ and the vectors $(f_j)_{j>0}$. Show that $V + W$ is dense in H but not closed. (*Hint:* Consider for example the vector $x = \sum_{j>0} \frac{1}{j} e_{-j}$.)

12.16. Let H be a Hilbert space over $\mathbb{C}$ or $\mathbb{R}$, and let V be a dense subspace.

(a) For each $x \in H$, $\{x\}^\perp \cap V$ is dense in $\{x\}^\perp$. (*Hint*: Choose for example sequences x_n and y_n from V which converge to x resp. $y \in \{x\}^\perp$ and consider $(x_n, x)y_n - (y_n, x)x_n$ for $\|x\| = 1$.)

(b) For each finite dimensional subspace K of H, $K^\perp \cap V$ is dense in $K^\perp$.

(c) To x and y in H with $x \perp y$ there exist sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in V such that $x_n \to x$, $y_n \to y$, and

$$(x_n, y_m) = 0 \quad \text{for all } n \text{ and } m\,.$$

(*Hint*: Choose successively x_{n+1} such that $\|x - x_{n+1}\| \leq 2^{-n-1}$ and $x_{n+1} \perp \{y_1, y_2, \ldots, y_n, y\}$, with y_{n+1} chosen similarly.)

12.17. Let X and Y be Banach spaces and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of operators in $\mathbf{B}(X,Y)$. Assume that there is a constant $c>0$ so that $\|T_n\|\le c$ for all n, and that for x in a dense subspace V of X, T_nx is convergent in Y. Show that there is a uniquely determined operator $T\in\mathbf{B}(X,Y)$ such that $T_nx\to Tx$ for all $x\in X$. (One can use an $\varepsilon/3$-argument.)

12.18. Let A be an operator in a Hilbert space H such that $D(A^2)=D(A)$ and $A^2x=-x$ for $x\in D(A)$. Show that $D(A)$ is a direct sum of eigenspaces for A corresponding to the eigenvalues $+i$ and $-i$, that $V_+\perp V_-$, and that

$$G(A)=V_+\dotplus V_-\ ,$$

where

$$V_\pm=\{\,\{\,x,y\,\}\in G(A)\mid y=\pm ix\,\}\ .$$

Show that the spaces $V_\pm$ are closed when A is closed.

12.19. Let T be a densely defined, closed symmetric operator in a Hilbert space H.

(a) Show that

$$G(T^*)=G(T)\oplus W_+\oplus W_-\ ,$$

where $W_\pm=\{\,\{x,y\}\in G(T^*)\mid y=\pm ix\,\}$, and that

$$D(T^*)=D(T)\dotplus Z(T^*-iI)\dotplus Z(T^*+iI)\ .$$

(One can use Exercise 12.18.)

(b) Let S be a closed, symmetric operator which extends T, i.e., $T\subset S$. Show that if u and $v\in D(T^*)$ with $T^*u=iu$, $T^*v=-iv$ and $u+v\in D(S)$, then $\|u\|=\|v\|$. Show that there exists an isometry U of a closed subspace K of $Z(T^*-iI)$ into $Z(T^*+iI)$ such that

$$D(S)=\{\,x{+}u{+}Uu\mid x\in D(T)\,,\ u\in K\,\}\quad\text{with}\quad S(x{+}u{+}Uu)=Tx{+}iu{-}iUu\ .$$

(*Hint*: For K one can take $\{u\in Z(T^*{-}i)\mid\exists\, v\in Z(T^*{+}i)\text{ s. t. }u{+}v\in D(S)\}$, and let U map u to v.)

Conversely, every operator defined in this way is a closed, symmetric extension of T.

(c) Show that S is a selfadjoint extension of T if and only if $K=Z(T^*-iI)$ and U is an isometry of $Z(T^*-iI)$ onto $Z(T^*+iI)$.

(d) Show that there exists a selfadjoint extension S of T if and only if $Z(T^*-iI)$ and $Z(T^*+iI)$ have the same Hilbert dimension (cf. Exercise 12.26).

(*Comment.* If Exercise 12.26 has not been covered, assume H separable; then any closed infinite dimensional subspace has a countable orthonormal basis. Applications of this result are found e.g. in Naimark's book [**?**], and works of Everitt and Markus, cf. [EM99].)

12.20. An operator N in a Hilbert space H is called *normal* if N is closed and densely defined, and $NN^* = N^*N$. Show that N is normal if and only if N is a densely defined operator and N and N^* are metrically alike, i.e., $D(N) = D(N^*)$ and $\|Nx\| = \|N^*x\|$ for all $x \in D(N)$.

12.21. Let H be a separable Hilbert space. Show that there exists a densely defined, closed unbounded operator A in H satisfying $A^2 = I|_{D(A)}$. (Cf. Exercises 12.15 and 12.18.) Show that such an A cannot be selfadjoint, nor symmetric.

12.22. Let X and Y be vector spaces, and let $A : X \to Y$ and $B : Y \to X$ be operators with $R(A) \subset D(B)$ and $R(B) \subset D(A)$. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and let $k \in \mathbb{N}$. Show that λ is an eigenvalue of AB with multiplicity k if and only if λ is an eigenvalue for BA with multiplicity k.

12.23. Let S be a densely defined, closed symmetric operator in a Hilbert space H, with $m(S) > 0$. Show that $D(S) \cap Z(S^*) = \{0\}$. Show that the operator P defined by

$$\begin{aligned} &D(P) = D(S) \dot{+} Z(S^*) \\ &P(v+z) = Sv\,, \quad \text{when } v \in D(S) \text{ and } z \in Z(S^*), \end{aligned}$$

is a selfadjoint extension of S with $m(P) \geq 0$. Show that P is different from the Friedrichs extension T of S if and only if $Z(S^*) \neq \{0\}$.

12.24. Consider S, T and P as in Exercise 12.23, and let $t(u, v)$ and $p(u, v)$ denote the sesquilinear forms associated with T and P. Here $D(t) = V$ as described in Theorem 12.25.
Show that $D(p)$ equals the direct sum

$$D(p) = V \dot{+} Z(S^*),$$

and that

$$p(v + z, v' + z') = t(v, v'), \text{ when } v, v' \in V,\ z, z' \in Z(S^*).$$

12.25. (This exercise uses definitions from Chapter 4, see in particular Exercise 4.14.) Let $H = L_2(I)$, where $I =]\alpha, \beta[$, and let $V = H^1(I)$. Let $a(u, v)$ be the sesquilinear form on V,

$$a(u, v) = \int_\alpha^\beta (u'(t)\overline{v'(t)} + q(t)u(t)\overline{v}(t))dt$$

where q is a function in $C^0(\overline{I})$. Show that a is bounded on V and V-coercive. Show that the operator A associated with (H, V, a) by the Lax-Milgram lemma is the operator

$$Au = -\frac{d^2}{dt^2}u + qu\ ,$$

with domain $D(A)$ consisting of the functions $u \in H^2(I)$ with

$$u'(\alpha) = u'(\beta) = 0\,.$$

Show that if one replaces V by $V_0 = H_0^1(I)$ and a by $a_0 = a|_{V_0}$, one obtains an operator A_0 acting like A but with domain consisting of the functions $H^2(I)$ with

$$u(\alpha) = u(\beta) = 0.$$

12.26. Let H be a pre-Hilbert space. The set of orthonormal systems in H is inductively ordered and therefore has maximal elements. Let $(e_i)_{i\in I}$ and $(f_j)_{j\in j}$ be two maximal orthonormal systems. Show that I and J have the same cardinality. (*Hint*: If I is finite, H is a finite dimensional vector space with the basis $(e_i)_{i\in I}$; since the vectors f_j, $j \in J$, are linearly independent, card $J \leq$ card I. If I is infinite, let $J_i = \{\, j \in J \mid (f_j, e_i) \neq 0 \,\}$ for each $i \in I$; since all J_i are denumerable and $J = \bigcup_{i\in I} J_i$, one finds that card $J \leq$ card $(I \times \mathbb{N}) =$ card I.)
When H is a Hilbert space, the subspace H_0 of linear combinations of a maximal orthonormal system $(e_i)_{i\in I}$ is dense in H, since $\overline{H_0} = \{\, e_i \mid i \in I \,\}^{\perp\perp} = \{\, 0 \,\}^{\perp}$. Here card I is called the Hilbert dimension of H.

12.27. Let H be a Hilbert space and let E and $F \in \mathbf{B}(H)$ be two orthogonal projections (i.e., $E = E^* = E^2$ and $F = F^* = F^2$). Show that if EH has a larger Hilbert dimension (cf. Exercise 12.26) than FH, then EH contains a unit vector which is orthogonal to FH. Show that if $\|E - F\| < 1$, then EH and FH have the same Hilbert dimension.

12.28. Let T be an operator in a complex Hilbert space H. Let $\Omega_e = \Omega_e(T) = \{\, \lambda \in \mathbb{C} \mid \exists c_\lambda > 0\ \forall x \in D(T) : \|(T-\lambda)x\| \geq c_\lambda \|x\| \,\}$.

(a) If $S \subset T$, then $\Omega_e(T) \subset \Omega_e(S)$. If T is closable, then $\Omega_e(\overline{T}) = \Omega_e(T)$.

(b) If T is closed, then: $\lambda \in \Omega_e$ if and only if $T - \lambda$ is injective and $R(T-\lambda)$ is closed.

(c) If $\lambda \notin \Omega_e(S)$, and T is an extension of S, then $T - \lambda$ does not have a bounded inverse.

(d) For $\lambda \in \Omega_e$ and $|\mu - \lambda| < c_\lambda$ one has: $\mu \in \Omega_e$, and

$$R(T-\lambda) \cap R(T-\mu)^{\perp} = R(T-\lambda)^{\perp} \cap R(T-\mu) = \{\, 0 \,\}.$$

(*Hint:* For $(T-\lambda)z \perp (T-\mu)z$ one can show that $c_\lambda \|z\|\, \|(T-\lambda)z\| \leq \|(T-\lambda)z\|^2 = |\mu - \lambda|\, |(z, (T-\lambda)z)| \leq |\mu-\lambda|\, \|z\|\, \|(T-\lambda)z\|$.)

(e) Assume that T is closed. The Hilbert dimension of $R(T-\lambda)^{\perp}$ is constant on each connected component of Ω_e. (Use for example Exercise 12.27.)

(f) If T is symmetric, then $\mathbb{C} \setminus \mathbb{R} \subset \Omega_e$.

(g) If T is lower bounded with lower bound c, then $\left]-\infty, c\right[\subset \Omega_e$.

(h) Show that if T is densely defined and symmetric, and $\mathbb{R} \cap \Omega_e \neq \emptyset$, then T can be extended to a selfadjoint operator. (The deficiency indices for $\overline{T}$ are equal, by (a), (e) and (f).)

12.29. Let H be a Hilbert space, F a closed subspace. Let S and T be closed injective operators in H with $R(S)$ and $R(T)$ closed.

(a) ST is closed.

(b) Show that $R(S|_{F\cap D(S)})$ is closed. Define $Q \in \mathbf{B}(H)$ by $Qx = S^{-1}x$ for $x \in R(S)$, $Qx = 0$ for $x \perp R(S)$. Show that $R(S|_{F\cap D(S)})$ and $R(S)^{\perp}$ and $\overline{Q^*(F^{\perp})}$ are pairwise orthogonal subspaces of H, with sum H.

(c) Assume that S is densely defined. Show that the Hilbert dimension of $R(S|_{F\cap D(S)})^{\perp}$ is the sum of the Hilbert dimensions of $R(S)^{\perp}$ and $F^{\perp}$.

(d) Assume that S and T are densely defined, and that $R(T)^{\perp}$ has finite dimension. Show that ST is a closed densely defined operator (use for example Exercise 12.16). Show that $R(ST)$ is closed and that the Hilbert dimension of $R(ST)^{\perp}$ is the sum of the Hilbert dimensions of $R(S)^{\perp}$ and $R(T)^{\perp}$.

(e) Let A be a closed, densely defined, symmetric operator on H with deficiency indices m and n. If m or n is finite, then A^2 is a closed, densely defined, symmetric, lower bounded operator with defeciency indices $m+n$ and $m+n$. (*Hint*: Use for example $A^2 + I = (A+i)(A-i) = (A-i)(A+i)$, and Exercise 12.28.)

12.30. Let there be given two Hilbert spaces H_1 and H_2. Let φ_i denote the natural injection of H_i in $H_1 \oplus H_2$, $i = 1, 2$. Find φ_i^*. An operator $A \in \mathbf{B}(H_1 \oplus H_2)$ is described naturally by a matrix $(A_{ij})_{i,j=1,2}$, $A_{ij} = \varphi_i^* A \varphi_j$. Find conversely A expressed by the matrix elements, and find the matrix for A^*.

12.31. Let H be a Hilbert space. Let T be a densely defined closed operator in H, and let $P(T)$ denote the orthogonal projection ($\in \mathbf{B}(H \oplus H)$) onto the graph $G(T)$.

(a) Show that $P(T^*) = 1 + VP(T)V$, where $V \in \mathbf{B}(H \oplus H)$ has the matrix (cf. Exercise 12.30) $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

(b) Show that $P(T)$ has the matrix

$$\begin{pmatrix} rr(1+T^*T)^{-1} & T^*(1+TT^*)^{-1} \\ T(1+T^*T)^{-1} & TT^*(1+TT^*)^{-1} \end{pmatrix} .$$

12.32. Show that when S is a symmetric operator in H with $R(S+i) = R(S-i) = H$, then S is densely defined. (Hence the requirement on dense domain in Theorem 12.10 is superfluous.)

12.33. Let S be a densely defined, closed symmetric operator in H. Show that S is maximally symmetric if and only if either $S+i$ or $S-i$ is surjective. (One can use Exercise 12.19.)

12.34. Let K denote the set of complex selfadjoint 2×2 matrices $A \geq 0$ with trace $\operatorname{Tr} A = 1$. (The trace of a matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ equals $\alpha + \delta$.) Define $\partial K = \{ A \in K \mid A^2 = A \}$. Set $P_u v = (v, u)u$ for u and $v \in \mathbb{C}^2$.

(a) Show that $\partial K = \{ P_u \mid u \in \mathbb{C}^2 \, , \; \|u\| = 1 \}$.

(b) Show that

$$\varphi(a, b, c) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} a & b - ic \\ b + ic & -a \end{pmatrix}$$

defines an affine homeomorphism of $\mathbb{R}^3$ onto the set of selfadjoint 2×2 matrices with trace 1; show that

$$\varphi(\{ (a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 \leq \tfrac{1}{4} \}) = K;$$
$$\varphi(\{ (a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = \tfrac{1}{4} \}) = \partial K.$$

(c) Let T be a 2×2 matrix; show that $A \mapsto \operatorname{Tr}(TA)$ is a continuous affine map ψ of K onto a compact convex set $N \subset \mathbb{C}$; and show that $\psi(\partial K) = N$.

(d) Show that $\operatorname{Tr}(TP_u) = (Tu, u)$ for $u \in \mathbb{C}^2$; show that $N = \nu(T)$.

Now let H be an arbitrary complex Hilbert space and S an operator in H.

(e) Show that $\nu(S)$ is convex (the Toeplitz-Hausdorff theorem). (For $x, y \in D(S)$, consider $P_K S|_K$, where P_K is the projection onto the subspace K spanned by x and y.)

(f) Assume that S is closed and densely defined with $D(S^*) = D(S)$. Show that $\nu(S^*) = \overline{\nu(S)}$ $(= \{ \overline{\lambda} \mid \lambda \in \nu(S) \})$; show that the spectrum $\sigma(S)$ is contained in any closed halfplane that contains $\nu(S)$; show that $\sigma(S)$ is contained in the closure of $\nu(S)$.

(g) Show that for $H = l^2(\mathbb{N})$ and a suitably chosen $S \in \mathbf{B}(H)$, $\nu(S)$ is not closed (use for example $S(x_1, x_2, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \dots)$, where $\lambda_n > 0$ for $n \in \mathbb{N}$, and $\lambda_n \to 0$).

12.35. Let Ω be an open subset of $\mathbb{R}^n$ and let M_p be the multiplication operator defined from a continuous function p on Ω by Theorem 12.13.

(a) Show that $m(M_p) \geq \alpha$ if and only if $\operatorname{Re} p(x) \geq \alpha$ for all x.

(b) Show that $\nu(M_p)$ and $\sigma(M_p)$ are contained in the intersection of all closed halfplanes which contain the range of p (and hence $\nu(M_p)$ and $\sigma(M_p)$ are contained in the closed convex hull of the range of p).

12.36. Let M_p be as in Exercise 12.35, and assume that the range of p is contained in a sector

$$\{ z \in \mathbb{C} \mid |\operatorname{Im} z| \leq c \operatorname{Re} z \}$$

where $c > 0$. Show that M_p is the variational operator defined from the triple $(L_2(\Omega), V, s)$, where

$$s(u,v) = \int pu\overline{v}\,dx \quad \text{on} \quad V = D(M_{|p|^{\frac{1}{2}}})\ ,$$

which is a Hilbert space with norm

$$(\||p|^{\frac{1}{2}}u\|_{L_2}^2 + \|u\|_{L_2}^2)^{\frac{1}{2}}\ .$$

12.37. (This exercise uses definitions from Chapter 4.) Let $H = L_2(\mathbb{R}_+)$, and let $V = H^1(\mathbb{R}_+)$. Let $a(u,v)$ be the sesquilinear form on V,

$$a(u,v) = \int_0^\infty (u'(t)\overline{v}'(t) + u'(t)\overline{v}(t) + u(t)\overline{v}(t))dt.$$

Show that a is continuous on V and V-elliptic. Show that the operator A associated with (H,V,a) by the Lax-Milgram lemma is of the form

$$Au = -u'' + u' + u,$$

with domain $D(A)$ consisting of the functions $u \in H^2(\mathbb{R}_+)$ with

$$u'(0) = 0.$$

Show that A^* is of the form

$$A^*v = -v'' - v' + v,$$

with domain $D(A^*)$ consisting of the functions $v \in H^2(\mathbb{R}_+)$ with

$$v'(0) + v(0) = 0.$$

So $D(A^*) \neq D(A)$.

12.38. (Communicated by Peter Lax, May 2006.) Let H be a Hilbert space provided with the norm $\|x\|$, and let $|x|$ be another norm on H such that

$$\|x\| \le C|x| \text{ for all } x \in H,$$

with $C > 0$. Let $T : H \to H$ be a symmetric operator with $D(T) = H$ such that

$$|Tx| \le k|x| \text{ for all } x \in H,$$

for some $k > 0$. Show that $T \in \mathbf{B}(H)$.
(*Hint*: Use calculations such as

$$\|Tx\|^2 = (Tx,Tx) = (x,T^2x) \le \|x\|\|T^2x\|,$$

to show successively that

$$\|Tx\|^4 \leq \|x^2\|\|T^2x\|^2 \leq \|x\|^3\|T^4\|,$$
$$\vdots$$
$$\|Tx\|^m \leq \|x\|^{m-1}\|T^m x\| \quad \text{for } m = 2^j, j = 1, 2, \ldots$$

Conclude that with $m = 2^j$,

$$\|Tx\| \leq \|x\|^{\frac{m-1}{m}} C^{\frac{1}{m}} k|x|^{\frac{1}{m}} \to \|x\|k$$

for $j \to \infty$.)