Extension theory for nonsmooth boundary value problems

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Introduction

The lecture brings together two subjects: 1) Nonsmooth pseudodifferential boundary operators. 2) Extension theories and Krein resolvent formulas for elliptic operators.

1) The theory of pseudodifferential operators, ψdo's, originally developed for smooth symbols and manifolds, was extended to nonsmooth x-dependence through works of Kumano-go, Nagase, Taylor and others. Recently, also the theory of pseudodifferential boundary operators, ψdbo's (as initiated by Boutet de Monvel), was generalized by Abels to situations with nonsmooth x-dependence.

2) Characterizations of extensions of elliptic operators A on a set with boundary have been known for many years in smooth cases. Recently, efforts have been made to implement such theories in nonsmooth situations. We shall show how this can be done by use of a calculus of nonsmooth ψdbo's, on domains with regularity including $C^{\frac{3}{2}+\varepsilon}$; here we obtain a Krein resolvent formula for arbitrary closed realizations of A. Joint work with Helmut Abels and Ian Wood.

1. Pseudodifferential boundary operators

Let $\Omega \subset \mathbb{R}^n$ smooth, $\partial\Omega = \Sigma$. Denote $\partial_n^j u|_\Sigma = \gamma_j u$, $j \in \mathbb{N}_0$.
Recall the pseudodifferential boundary operator (ψdbo) calculus introduced by Boutet de Monvel '71:

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(\overline{\Omega})^N \\ \times \\ C^\infty(\Sigma)^M \end{matrix} \to \begin{matrix} C^\infty(\overline{\Omega})^{N'} \\ \times \\ C^\infty(\Sigma)^{M'} \end{matrix}, \text{ where}$$

- P is a pseudodifferential operator (ψdo) on $\mathbb{R}^n$ (or on a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$), satisfying the *transmission condition* at Σ (always true for operators stemming from elliptic PDE),
- $P_+ = r^+ P e^+$ (the transmission condition assures that P_+ maps $C^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$). P must be of integer order.
- T is a trace operator from Ω to Σ, K is a Poisson operator from Σ to Ω, S is a ψdo on Σ.
- G is a singular Green operator, e.g. of type KT.

In local coordinates near the boundary, the operators T, K and G can be regarded as ψdo's in the tangential variables, *with values in* trace, Poisson or singular Green operators in one variable (the normal variable x_n).
The mappings extend to Sobolev spaces: For $\mathcal{A}$ of order m,

$$\mathcal{A}\colon H^{s+m}(\Omega)^N \times H^{s+m-\frac{1}{2}}(\Sigma)^M \to H^s(\Omega)^{N'} \times H^{s-\frac{1}{2}}(\Sigma)^{M'},$$

for $s+m > d - \frac{1}{2}$, when T and G are of *class* d, i.e. contain $\{\gamma_0, \dots, \gamma_{d-1}\}$.
The ψdbo calculus defines an "algebra" of operators, where the composition of two systems leads to a third one (when the matrix dimensions match). It allows operators of all orders, both positive and negative. In particular, when a system is *elliptic* of order m, then there exists a *parametrix* (an approximate inverse) of order $-m$, which also belongs to the calculus.

The ψdo calculus (on $\mathbb{R}^n$ or open subsets) has been generalized to symbols with limited smoothness in the x-variable by Kumano-go and Nagase '78, Marschall '85, Taylor '91 and '00. The results were extended to ψdbo's (on $\mathbb{R}^n_+$ and coordinate transformed versions) by Abels '05. (All these works have purposes in nonlinear applications.)

The main principles for ψdo's with C^τ-smoothness ($\tau \in]0,1[$) are:

- The m-order space $C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ consists of functions $p(x,\xi)$ that are C^τ in x along with all their ξ-derivatives, satisfying

$$\|\partial_\xi^\alpha p(.,\xi)\|_{C^\tau} \le C_\alpha \langle \xi \rangle^{m-|\alpha|} \qquad \text{for all } \xi \in \mathbb{R}^n.$$

  The subset of *classical* symbols moreover have asymptotic expansions in terms homogeneous in ξ (for $|\xi| \ge 1$) of degree $m - j$, for $j \in \mathbb{N}_0$.

The symbols define operators “in x-form” as follows:

$$\mathrm{Op}(p(x,\xi))u(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, d\xi.$$

Rules of calculus:

- When $P = \mathrm{Op}(p)$ is of order m, one has for $|s| < \tau$.

$$P\colon H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n).$$

- When P_1, P_2 are of orders m_1, m_2, one has for $0 < \theta < \tau$, $-\tau + \theta < s < \tau$,

$$P_1 P_2 - \mathrm{Op}(p_1 p_2)\colon H^{s+m_1+m_2-\theta}(\mathbb{R}^n) \to H^s(\mathbb{R}^n).$$

- When $P = \mathrm{Op}(p)$ is elliptic of order 0, with invertible principal symbol p^0, the ψdo $Q^0 = \mathrm{Op}((p^0)^{-1})$ is a parametrix in the sense that $PQ^0 - I$ and $Q^0 P - I$ map $H^{s-\theta}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ for $0 < \theta < \tau$, $-\tau + \theta < s < \tau$.

In Abels '05, the rules are generalized to ψdbo's.

For the parametrix construction, one difficulty is that it is only really simple when the order is 0, otherwise one needs to reduce to the zero-order case by use of “order-reducing operators”, such as:

$$\Lambda^t = \mathrm{Op}(\langle\xi\rangle^t)\colon H^{s+t}(\mathbb{R}^n) \overset{\sim}{\to} H^s(\mathbb{R}^n), \quad s \in \mathbb{R},$$

and related variants for ψdbo's. But here, whereas composition of $\mathrm{Op}(p(x,\xi))$ with Λ^t to the right gives the simple operator $\mathrm{Op}(p(x,\xi)\langle\xi\rangle^t)\colon H^{s+m+t} \to H^s$, $|s| < \tau$, composition to the left gives a ψdo $\mathrm{Op}(p(x,\xi)\langle\xi\rangle^t)$ plus a qualitative remainder (not in x-form), where the sum of them maps $H^{s+m} \to H^{s-t}$, $|s| < \tau$.

A helpful tool can be *symbol-smoothing*:
For every $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and $0 < \delta < 1$ there is a decomposition

$$p = p^\sharp + p^b,\ \text{where } p^\sharp \in S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n),\ p^b \in C^\tau S^{m-r\delta}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n);$$

here $\mathrm{Op}(p^\sharp)\colon H^{s+m} \to H^s$ for all $s \in \mathbb{R}$. Polyhomogeneity is lost.

2. Extension theory and Krein resolvent formulas

Consider a strongly elliptic $2m$-order differential operator on Ω

$$A = \sum_{|\alpha|,|\beta|\le m} D^\alpha a_{\alpha,\beta}(x) D^\beta \text{ with } \operatorname{Re} \sum_{|\alpha|,|\beta|=m} a_{\alpha,\beta}(x)\xi^{\alpha+\beta} \ge c|\xi|^{2m},$$

$c > 0$ (smooth Ω and $a_{\alpha,\beta}$). For simplicity, take $m = 1$.
The maximal operator $A_{\max}$ acts like A in $H = L_2(\Omega)$ with domain

$$D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\};$$

the minimal operator $A_{\min}$ equals $\overline{A|_{C_0^\infty}}$, with $D(A_{\min}) = H_0^2(\Omega)$. The set $\mathcal{M}$ of *realizations* $\widetilde{A}$ of A are the operators with $A_{\min} \subset \widetilde{A} \subset A_{\max}$.
The Dirichlet realization A_γ has

$$D(A_\gamma) = H^2(\Omega) \cap H_0^1(\Omega) = \{u \in H^2(\Omega) \mid \gamma_0 u = 0\};$$

we can assume A_γ invertible with positive lower bound.

$A'_{\max}$, $A'_{\min}$ and A'_γ are defined similarly for the formal adjoint A'; here $A'_{\max} = A^*_{\min}$ and $A^*_\gamma = A'_\gamma$.
Assume $0 \in \varrho(A_\gamma)$ (or replace A by $A - \lambda$ where $\lambda \in \varrho(A_\gamma)$). Denote $Z = \ker A_{\max}$, $Z' = \ker A'_{\max}$.
By use of the mapping $pr_\zeta = I - A_\gamma^{-1} A_{\max} \colon D(A_{\max}) \to Z$ and its analogue for A', one can establish an "abstract" characterization of all closed realizations of A (Krein '47, Vishik '52, Grubb '68) in terms of operators in Z, Z':

$$\widetilde{A} \in \mathcal{M} \text{ closed } \leftrightarrow \begin{cases} V \subset Z, W \subset Z', \text{ closed subspaces} \\ T\colon V \to W \text{ closed, densely defined} \end{cases}$$

In this correspondence, $D(T) = \mathrm{pr}_\zeta\, D(\widetilde{A})$, and $Tu_\zeta = \mathrm{pr}_W(\widetilde{A}u)$. ($\mathrm{pr}_W$ is orthog. proj.) Here $\widetilde{A}^*$ corresponds similarly to $T^*\colon W \to V$, and many properties carry over between $\widetilde{A}$ and T. In particular, if $\widetilde{A}^{-1}$ exists, we have an abstract resolvent formula:

$$\widetilde{A}^{-1} = A_\gamma^{-1} + \mathrm{i}_V T^{-1} \mathrm{pr}_W . \quad (1)$$

To rewrite the formulas in terms of operators acting in the boundary, we consider the problem

$$Au = 0 \text{ in } \Omega,\ \gamma_0 u = \varphi \text{ on } \Sigma.$$

It has a solution operator K_γ, a Poisson operator, mapping $H^{s-\frac{1}{2}}(\Sigma) \to H^s(\Omega)$ for all $s \in \mathbb{R}$. In particular:
γ_0 defines a bijection from Z to $H^{-\frac{1}{2}}(\Sigma)$ with K_γ acting as inverse.
We denote the restrictions of γ_0 to V by γ_V, etc. Then

$$\gamma_V\colon V \overset{\sim}{\to} X = \gamma_0(V),\ \gamma_W\colon W \overset{\sim}{\to} Y = \gamma_0(W),$$

where X, Y closed $\subset H^{-\frac{1}{2}}(\Sigma)$. By use of these homeomorphisms, $T\colon V \to W$ is carried over to a map $L\colon X \to Y^*$:

$$\begin{tikzcd} V \arrow[r, "\sim", "\gamma_V"'] \arrow[d, "T"'] & X \arrow[d, "L"] \\ W & Y^* \arrow[l, "\sim"', "\gamma_W^*"] \end{tikzcd} \qquad D(L) = \gamma_0 D(T).$$

In other words,

$$L = (\gamma_W^*)^{-1} T \gamma_V^{-1}.$$

NB! When A is replaced by $A - \lambda$ in the whole construction, V, W are replaced by $V_\lambda, W_{\bar\lambda}$, but X and Y *remain fixed*. L is replaced by L^λ with $D(L^\lambda) = D(L)$, and $L - L^\lambda$ is bounded.
The abstract resolvent formula (1) now carries over to the formula:

$$\widetilde{A}^{-1} = A_\gamma^{-1} + K_{\gamma,X} L^{-1} (K'_{\gamma,Y})^*$$

where $K_{\gamma,X} = \mathrm{i}_V \gamma_V^{-1} \colon X \to V \subset H$, $(K'_{\gamma,Y})^* = (\gamma_W^*)^{-1} \operatorname{pr}_W \colon H \to Y^*$.
With explicit λ-dependence, the formula reads:

$$(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} + K^\lambda_{\gamma,X} (L^\lambda)^{-1} (K'^{\bar\lambda}_{\gamma,Y})^*, \quad (2)$$

when $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$; *a Kreĭn resolvent formula.* Observe that the formula (2) applies to any closed realization with $\varrho(\widetilde{A}) \cap \varrho(A_\gamma) \neq \emptyset$.

There is a different line of extension theories, based on boundary triples and relations, and first aimed at applications in ODE, by Russian mathematicians such as Kochubei '75, Gorbachuk and Gorbachuk book '84 (translated '91), Derkach and Malamud '87, Malamud and Mogilevski '97, '02.... When X and Y equal the full space $H^{-\frac{1}{2}}(\Sigma)$, the operator L^λ is the inverse of a certain Weyl-Titchmarsh operator family $M(\lambda)$ arising from these other theories. The connection between our results and theirs was clarified in a joint work Brown-G-Wood '09.

To show how L enters in the boundary condition that $\widetilde{A}$ represents, we recall that A has a Green's formula for $u, v \in H^2(\Omega)$:

$$(Au, v)_\Omega - (u, A'v)_\Omega = (\nu u, \gamma_0 v)_\Sigma - (\gamma_0 u, \nu' v)_\Sigma, \qquad (3)$$

where

$$\nu u = s(x)\gamma_1 u + \mathcal{A}\gamma_0 u, \quad \nu' v = \bar{s}(x)\gamma_1 v + \mathcal{A}'\gamma_0 v;$$

here $s(x) \neq 0$, smooth, and $\mathcal{A}$, $\mathcal{A}'$ are suitable first-order differential operators on Σ.
Define the *Dirichlet-to-Neumann operator*

$$P_{\gamma,\nu} = \nu K_\gamma \colon H^{s-\frac{1}{2}}(\Sigma) \to H^{s-\frac{3}{2}}(\Sigma);$$

it is an elliptic ψdo of order 1.
Introduction of the *reduced Neumann trace operator* Γ by

$$\Gamma = \nu - P_{\gamma,\nu}\gamma_0 \colon D(A_{\max}) \to H^{\frac{1}{2}}(\Sigma)$$

allows a generalized Green's formula, for $u \in D(A_{\max})$, $v \in D(A'_{\max})$:

$$(Au, v)_\Omega - (u, A'v)_\Omega = (\Gamma u, \gamma_0 v)_{\frac{1}{2},-\frac{1}{2}} - (\gamma_0 u, \Gamma' v)_{-\frac{1}{2},\frac{1}{2}}. \qquad (4)$$

Then in fact $D(\widetilde{A})$ consists of the functions $u \in D(A_{\max})$ that satisfy the boundary condition

$$\gamma_0 u \in D(L), \quad (\Gamma u, \varphi)_{\frac{1}{2},-\frac{1}{2}} = (L\gamma_0 u, \varphi)_{Y^*,Y} \text{ for all } \varphi \in Y.$$

The second condition may be rewritten as $\mathrm{i}_Y^* \Gamma u = L\gamma_0 u$, or

$$\mathrm{i}_Y^* \nu u = (L + \mathrm{i}_Y^* P_{\gamma,\nu})\gamma_0 u.$$

The analysis covers *all closed realizations*.
In the case $X = Y = H^{-\frac{1}{2}}(\Sigma)$, this is a *Neumann-type condition*

$$\nu u = C\gamma_0 u, \text{ with } C = L + P_{\gamma,\nu}. \quad (5)$$

In recent other works based on boundary triples theory (Amrein-Pearson, Behrndt-Langer,...) the tendency has been to avoid "negative Sobolev spaces" by assuming that the realizations have domains where $\gamma_0 u, \nu u \in L_2(\Sigma)$; then not all closed realizations are covered.

3. Generalizations to nonsmooth domains

Can we extend the Krein resolvent formula

$$(\widetilde{A}-\lambda)^{-1} = (A_\gamma - \lambda)^{-1} + K^\lambda_{\gamma,X}(L^\lambda)^{-1}(K'^{\bar\lambda}_{\gamma,Y})^*. \qquad (2)$$

to nonsmooth cases? Note that there are three universal ingredients:

$$\begin{aligned}(A_\gamma-\lambda)^{-1}\colon H^s(\Omega) &\to H^{s+2}(\Omega), \quad s>-\tfrac12,\\ K^\lambda_\gamma\colon H^{s-\frac12}(\Sigma) &\to H^s(\Omega), \quad s\in\mathbb{R},\\ (K'^{\bar\lambda}_\gamma)^*\colon H^s(\Omega) &\to H^{s+\frac12}(\Sigma), \quad s>-\tfrac12,\end{aligned}$$

all belonging to the ψdbo calculus; $(K'^{\bar\lambda}_\gamma)^*$ is a trace operator of class 0. For (2), the mapping properties are especially important at $s=0$. Only X, Y and L^λ depend on which realization $\widetilde{A}$ we are considering.

There exist a few works dealing with extensions to nonsmooth domains:

- Gesztesy and Mitrea '09 have some results on *Robin problems* (essentially C is of order < 1) for $-\Delta$ on Lipschitz domains. Krein-type resolvent formulas are shown under the hypothesis that the domain is $C^{\frac{3}{2}+\varepsilon}$.
- Gesztesy and Mitrea in a paper to appear, have Krein resolvent formulas for selfadjoint realizations of $-\Delta$ on so-called *quasi-convex* Lipschitz domains (containing $C^{\frac{3}{2}+\varepsilon}$ and convex domains). They use G'68. The interesting new aspect is that $\{\gamma_0, \gamma_1\}$ maps $D(A_{\max})$ onto $(N^{\frac{1}{2}})^* \times (N^{\frac{3}{2}})^*$ that differs from $H^{-\frac{1}{2}}(\Sigma) \times H^{-\frac{3}{2}}(\Sigma)$ in the roughest cases (for $C^{\frac{3}{2}+\varepsilon}$-domains they are the same).
- Posilicano and Raimondi '09 have outlines of related results for selfadjoint realizations of more general second-order elliptic operators with nonsmooth coefficients, when the domain is $C^{1,1}$.

- In G '08, second-order nonselfadjoint operators with smooth coefficients on a $C^{1,1}$ domain are treated; here *Neumann-type boundary conditions* (5) are allowed.

  The $C^{1,1}$ hypothesis served to use results from Grisvard's '82 book.

- A very recent joint work Abels-G-Wood treats second-order strongly elliptic operators with nonsmooth coefficients on a scale of domains containing $C^{\frac{3}{2}+\varepsilon}$, obtaining a full extension theory and Krein resolvent formulas. In particular, Neumann type conditions with C of order 1 are included, with regularity in elliptic cases.

It is one of the main aims of G'08 and AGW'10 to allow C of order 1 and get ellipticity into play, dealing with operators having a principal part that governs the regularity results, plus a remainder with a minor effect. About AGW'10:

We work in a scale of spaces containing $C^{\frac{3}{2}+\varepsilon}(\Omega)$, but to profit from precise rules for products and traces, the considerations take place not only in Hölder spaces but also in some slightly larger Besov and Bessel-potential spaces. We consider

$$Au = -\sum \partial_j(a_{jk}\partial_k u) + \sum a_j \partial_j u + a_0 u \text{ in } \Omega,$$

where Ω is $B^{\frac{3}{2}}_{p,2}$, and $a_{jk}, a_j \in H^1_q(\Omega)$, $a_0 \in L_q(\Omega)$; with $p, q \geq 2$ and

$$\tau := \tfrac{1}{2} - \tfrac{n-1}{p} > 0, \quad 1 - \tfrac{n}{q} \geq \tau. \qquad (6)$$

Here $C^{\frac{3}{2}+\varepsilon} \subset B^{\frac{3}{2}}_{p,2} \subset C^{1+\tau}$, so indeed Ω being $C^{\frac{3}{2}+\varepsilon}$ is allowed.

Proposition 1 *Each boundary point x_0 has a neighborhood U and a C^1-diffeomorphism $F\colon U \to B(0,1)$ with ∇F in C^τ such that $F(U \cap \Omega) = B(0,1) \cap \mathbb{R}^n_+$, and the coordinate change defined by F preserves H^s for $-\frac{1}{2} < s \leq 2$.*
Moreover, the induced coordinate change from $U \cap \Sigma$ to $\{x \in B(0,1) \mid x_n = 0\}$ preserves H^s for $-\frac{1}{2} \leq s \leq \frac{3}{2}$.

Theorem 2 *Green's formula* (3) *is valid for* $u, v \in H^2(\Omega)$*; here* ν *and* ν' *have coefficients in* $H_p^{\frac{1}{2}}(\Sigma)$

The hypothesis (6) assures that $H_p^{\frac{1}{2}}(\Sigma) \subset C^\tau(\Sigma)$.
Now, using the ψdbo calculus with C^τ-smoothness of coefficients, in a λ-dependent version, we can show:

Theorem 3 *When* $\lambda \in \varrho(A_\gamma)$*, the operators* $(A_\gamma - \lambda)^{-1}$*,* K_γ^λ *and* $P_{\gamma,\nu}^\lambda$ *are defined as continuous maps:*

$$\begin{aligned}(A_\gamma - \lambda)^{-1} &\colon H^{s-2}(\Omega) \to H^s(\Omega), \ \text{for } s \in]2-\tau, 2]\\ K_\gamma^\lambda &\colon H^{s-\frac{1}{2}}(\Sigma) \to H^s(\Omega), \ \text{for } s \in]2-\tau, 2]\\ P_{\gamma,\nu}^\lambda &\colon H^{s-\frac{1}{2}}(\Sigma) \to H^{s-\frac{3}{2}}(\Sigma), \ \text{for } s \in]2-\tau, 2],\end{aligned}$$

with a ψ*dbo principal part and a remainder of lower order.*

The result for $(A_\gamma - \lambda)^{-1}$ is satisfactory since it includes the map from $L_2(\Omega)$ to $H^2(\Omega)$, but the other results are insufficient for the extension analysis since they do not allow $s = 0$.

This requires another effort. We can define an extension of K_γ^λ by

$$K_\gamma^\lambda = (\nu'(A_\gamma^* - \bar\lambda)^{-1})^*;$$

It has the desired continuity for $s \in [0, \tau[$, and we use interpolation to get the full interval $s \in [0, 2]$. But for $s \leq 2 - \tau$, the operator is not of the standard type; however, a symbol smoothing argument allows to write it as the sum of an operator with symbol of an $S_{1,\delta}$-type plus a remainder of lower order, when $s > 0$.
Also for the Dirichlet-to-Neumann operator $P_{\gamma,\nu}^\lambda$, we can get a continuous extension to $s \in [0, 2]$, but with a nonstandard structure for $s \leq 2 - \tau$. Symbol smoothing can be used for $s > 0$, and there is an additional construction leading to inclusion of the case $s = 0$.

Theorem 4. *The whole extension theory for the smooth case remains valid in the considered nonsmooth case, including the diagrams that define $L^\lambda : X \to Y^*$ for a general realization $\widetilde{A}$, and its Krein resolvent formula when $\varrho(\widetilde{A}) \cap \varrho(A_\gamma) \neq \emptyset$.*

As in the smooth case, we have that when $X = Y = H^{-\frac{1}{2}}(\Sigma)$, the boundary condition for $\widetilde{A}$ is of Neumann-type:

$$\nu u = C\gamma_0 u, \text{ where } C = L + P_{\gamma,\nu}. \qquad (7)$$

Here we can single out elliptic cases: Let $\widetilde{A}$ be defined by the boundary condition (7), where C is a first-order differential operator on Σ with coefficients in $H_p^{\frac{1}{2}}(\Sigma)$. If the ψdo $L^\lambda = C - P^\lambda_{\gamma,\nu}$ is parameter-elliptic on a ray in $\varrho(A_\gamma)$, then $\widetilde{A}$ has domain in $H^2(\Omega)$ (and $D(L) = H^{\frac{3}{2}}(\Sigma)$). Moreover, $\widetilde{A} - \lambda$ is invertible for large λ on the ray.