Boundary problems for fractional Laplacians and other mu-transmission operators

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Introduction

Consider P_a equal to $(-\Delta)^a$ or to A^a for some symmetric, strongly elliptic second-order operator A on $\mathbb{R}^n$ with real C^∞-coefficients, $0 < a < 1$.
The Dirichlet-to-Neumann operator is principally of this type, with $a = \frac{1}{2}$.
Assume for simplicity $(P_a u, u) \geq 0$ for all $u \in C_0^\infty(\mathbb{R}^n)$.
For Ω open $\subset \mathbb{R}^n$, the Dirichlet realization $P_{a,\mathrm{Dir}}$ is defined by a variational construction, as the Friedrichs extension of $P_{a,C_0^\infty(\Omega)}$ in $L_2(\Omega)$.
Assume Ω smooth bounded. We shall use the notation

$$H_p^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L_p(\mathbb{R}^n)\},$$
$$\dot{H}_p^s(\overline{\Omega}) = \{u \in H_p^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}, \quad \overline{H}_p^s(\Omega) = r^+ H_p^s(\mathbb{R}^n).$$

Here $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$; r^+ restricts to Ω, e^+ extends by zero on $\complement\Omega$. (The notation with $\dot{H}$ and $\overline{H}$ stems from Hörmander's books '63 and '85.)

The operator $r^+ P_a$ maps continuously from $\dot{H}_2^a(\overline{\Omega})$ to $\overline{H}_2^{-a}(\Omega)$, and $P_{a,\mathrm{Dir}}$ is the restriction with domain

$$D(P_{a,\mathrm{Dir}}) = \{u \in \dot{H}_2^a(\overline{\Omega}) \mid r^+ P_a u \in L_2(\Omega)\}.$$

The operator is selfadjoint ≥ 0 with compact resolvent; for $P_a = (-\Delta)^a$ the operator is known to have a positive lower bound.

Q1. What is the domain? It is relatively easy to show that for $a < \frac{1}{2}$, $D(P_{a,\mathrm{Dir}}) = \dot{H}_2^{2a}(\overline{\Omega})$. But for $a \geq \frac{1}{2}$?

Q2. What is the spectrum? Is there a Weyl-type asymptotic formula for the eigenvalues, $\lambda_k \sim C\, k^{n/2a}$? The answer yes is known for $P_a = (-\Delta)^a$, but for general A?

Q3. Are eigenfunctions smooth in some sense? To my knowledge, very little is known.

We shall deal with all three questions, and some more.
Consider the problem

$$r^+ P_a u = f \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega},$$

called the homogeneous Dirichlet problem. Unique solvability in $L_\infty(\Omega)$ is known for $P_a = (-\Delta)^a$. But the results on the regularity of the solutions have been sparse.

- Vishik, Eskin, Shamir 1960's: $u \in \dot{H}_2^{\frac{1}{2}+a-\varepsilon}(\overline{\Omega})$ if $\frac{1}{2} < a < 1$.
- Some analysis of the behavior of solutions at $\partial\Omega$ when data are C^∞, Eskin '81, Bennish '93, Chkadua and Duduchava '01.

Q4. When $f \in \overline{H}_p^s(\Omega)$ for some $s \geq 0$, $1 < p < \infty$, how regular is u?

Q5. Same question in Hölder spaces C^t, $t \geq 0$ (continuously differentiable functions when t is integer).

Recent activity:

• Ros-Oton and Serra (arXiv 2012) showed for $(-\Delta)^a$ by potential theoretic and integral operator methods, when Ω is $C^{1,1}$, that

$$f \in L_\infty(\Omega) \implies u \in d^a C^\alpha(\overline{\Omega}),$$

for a small $\alpha > 0$. Here $d(x) = \operatorname{dist}(x, \partial\Omega)$. Moreover, $u \in C^a(\overline{\Omega}) \cap C^{2a}(\Omega)$. Lifted to at most $\alpha \leq 1$ when f is more smooth. They stated that they did not know of other regularity results for $(-\Delta)^a$ in the literature.

P_a is a *pseudodifferential operator* (ψdo), why not ψdo methods? There is a calculus initiated by Boutet de Monvel 1971 for ψdo boundary value problems. But it does not cover P_a.

I have been asked about such operators through the years, but only recently found an answer. It is presented in arXiv '13 as a new systematic theory of pseudodifferential boundary problems covering the operators P_a. A consquence is:

$$f \in L_\infty(\Omega) \implies u \in d^a C^{a\,(-\varepsilon)}(\overline{\Omega}), \quad (1)$$

$$f \in C^t(\overline{\Omega}) \implies u \in d^a C^{a+t\,(-\varepsilon)}(\overline{\Omega}), \text{ all } t > 0; \quad (2)$$

$(-\varepsilon)$ is included when $a = \frac{1}{2}$ in (1), $a + t$ or $2a + t \in \mathbb{N}$ in (2).
This theory will be the subject of the talk.

1. Pseudodifferential operators

Pseudodifferential operators (ψdo's) were introduced in the 1960's as a generalization of singular integral operators (Calderòn, Zygmund, Seeley, Kohn, Nirenberg, Hörmander, Giraud, Mikhlin, ...) They systematize the use of the Fourier transform $\mathcal{F}u = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx$:

$$Pu = \sum\nolimits_{|\alpha|\le m} a_\alpha(x) D^\alpha u = \mathcal{F}^{-1}\big(p(x,\xi)\hat{u}(\xi)\big) = \mathrm{OP}(p)u,$$

$$\text{where } p(x,\xi) = \sum\nolimits_{|\alpha|\le m} a_\alpha(x)\xi^\alpha, \text{ the symbol.}$$

This extends to more general functions $p(x,\xi)$ as symbols. In the classical theory, symbols are taken *polyhomogeneous*:

$$p(x,\xi) \sim \sum\nolimits_{j\in\mathbb{N}_0} p_j(x,\xi), \text{ where } p_j(x,t\xi) = t^{m-j} p_j(x,\xi)$$

for $|\xi| \ge 1, t \ge 1$ (here the order $m \in \mathbb{C}$).
The elliptic case is when the principal symbol p_0 is invertible; then $Q = \mathrm{OP}(p_0^{-1})$ is a good approximation to an inverse of P. The theory extends to manifolds by use of local coordinates.

An example of a ψdo is $(-\Delta)^a = \mathrm{OP}(|\xi|^{2a})$, of order $2a$, but also variable coefficients are allowed. E.g. A^a, and much more general operators.

When Ω is a smooth open subset of $\mathbb{R}^n$, there is a need to consider boundary value problems

$$P_+ u = f \text{ on } \Omega, \quad Tu = \varphi \text{ on } \partial\Omega;$$

with $P_+ = r^+ P e^+$, the truncation of P to Ω, and T a trace operator. Boutet de Monvel in 1971 introduced a calculus treating this when P is of *integer order* and has the *transmission property*:

P_+ maps $C^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$.

Solution operators for the problem are typically of the form

$$\begin{pmatrix} Q_+ + G & K \end{pmatrix},$$

where $Q \sim P^{-1}$ and G is an auxiliary operator called a singular Green operator, and K is a Poisson operator (going from $\partial\Omega$ to Ω).

But there are many interesting ψdo's not having the transmission property, e.g. $(-\Delta)^{\frac{1}{2}}$ does not have it, although of order 1.

2. The μ-transmission property

Definition 1. *For* $\operatorname{Re}\mu > -1$, $\mathcal{E}_\mu(\overline{\Omega})$ *consists of the functions* u *of the form*

$$u(x) = \begin{cases} d(x)^\mu v(x) \text{ for } x \in \Omega, \text{ with } v \in C^\infty(\overline{\Omega}), \\ 0 \text{ for } x \in \complement\Omega; \end{cases}$$

where $d(x)$ *is* > 0 *on* Ω, *belongs to* $C^\infty(\overline{\Omega})$, *and is proportional to* $\operatorname{dist}(x, \partial\Omega)$ *near* $\partial\Omega$. *More generally for* $j \in \mathbb{N}$, $\mathcal{E}_{\mu-j}$ *is spanned by the distribution derivatives up to order* j *of functions in* $\mathcal{E}_\mu$.

In Hörmander's book '85 Th. 18.2.18, for a classical ψdo P of order m:

Theorem 2. r^+P *maps* $\mathcal{E}_\mu(\overline{\Omega})$ *into* $C^\infty(\overline{\Omega})$ **if and only if** *the symbol has the* μ*-transmission property for* $x \in \partial\Omega$, *with* N *denoting the interior normal:*

$$\partial_x^\beta \partial_\xi^\alpha p_j(x, -N) = e^{\pi i(m-2\mu-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x, N),$$

for all indices.

It is a, possibly twisted, parity along the normal to $\partial\Omega$. Simple parity is the case $m = 2\mu$; it holds for $|\xi|^{2a}$ with $m = 2a$, $\mu = a$. Boutet de Monvel's transmission property is the case $m \in \mathbb{Z}$, $\mu = 0$.
The operators are for short said to be of *type* μ.

3. Solvability with homogeneous boundary conditions

The μ-transmission property was actually introduced far earlier by Hörmander in a lecture note from IAS 1965-66, distributed by photocopying. I received it in 1980, and have only last year studied it in depth. It contains much more, namely a *solvability theory* in L_2 Sobolev spaces for operators of type μ, which in addition have a certain factorization property of the principal symbol.

Definition 3. *P (of order m) has the factorization index μ_0 when, in local coordinates where Ω is replaced by $\mathbb{R}^n_+$ with coordinates (x', x_n),*

$$p_0(x', 0, \xi', \xi_n) = p_-(x', \xi', \xi_n) p_+(x', \xi', \xi_n),$$

with $p_\pm$ homogeneous in ξ of degrees μ_0 resp. $m - \mu_0$, and $p_\pm$ extending to $\{\operatorname{Im} \xi_n \lessgtr 0\}$ analytically in ξ_n.

Here $OP(p_\pm(x', \xi))$ on $\mathbb{R}^n$ preserve support in $\overline{\mathbb{R}}^n_+$ resp. $\overline{\mathbb{R}}^n_-$.

Example: For $(-\Delta)^a$ on $\mathbb{R}^n$ we have

$$|\xi|^{2a} = (|\xi'|^2 + \xi_n^2)^a = (|\xi'| - i\xi_n)^a (|\xi'| + i\xi_n)^a,$$

so that $p_\pm = (|\xi'| \pm i\xi_n)^a$, and the factorization index is a.

The operators $\Xi_\pm^\mu = \mathrm{OP}((\langle\xi'\rangle \pm i\xi_n)^\mu)$ play a great role in the theory.

Based on the factorization, Vishik and Eskin showed in '64 (extension to L_p by Shargorodsky '95, $1 < p < \infty$, $1/p' = 1 - 1/p$):

Theorem 4. *When P is elliptic of order m and has the factorization index μ_0, then*

$$r^+ P\colon \dot{H}_p^s(\overline{\Omega}) \to \overline{H}_p^{s-\operatorname{Re} m}(\Omega)$$

is a Fredholm operator for $\operatorname{Re}\mu_0 - 1/p' < s < \operatorname{Re}\mu_0 + 1/p$.

Note that s runs in a small interval $\,]\operatorname{Re}\mu_0 - 1/p', \operatorname{Re}\mu_0 + 1/p[\,$. The problem is now to find the solution space for higher s.
For this, Hörmander introduced for $p = 2$ a particular space combining the $\dot{H}$ and the $\overline{H}$ definitions:

Definition 5. *For $\mu \in \mathbb{C}$ and $s > \operatorname{Re}\mu - 1/p'$, the space $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ is defined by*

$$H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) = \Xi_+^{-\mu} e^+ \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+).$$

Here $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \subset \mathcal{S}'(\mathbb{R}^n)$, supported in $\overline{\mathbb{R}}^n_+$. Note the jump at $x_n = 0$ in $e^+\overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+)$.

Proposition 6. *Let $s > \operatorname{Re}\mu - 1/p'$. Then*

$$\Xi_+^{-\mu} e^+ \colon \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+) \to H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \text{ has the inverse}$$
$$r^+ \Xi_+^{\mu} \colon H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \to \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+),$$

and $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ is a Banach space with the norm

$$\|u\|_{\mu(s)} = \|r^+ \Xi_+^{\mu} u\|_{\overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+)}.$$

One has that $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \supset \dot{H}_p^{s}(\overline{\mathbb{R}}^n_+)$, and elements of $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ are locally in H_p^s on $\mathbb{R}^n_+$, but they are not in general H_p^s up to the boundary. The definition generalizes to $\Omega \subset \mathbb{R}^n$ by use of local coordinates.

These are Hörmander's μ-spaces, very important since they turn out to be the correct solution spaces.

The spaces $H_p^{\mu(s)}$ replace the $\mathcal{E}_\mu$ in a Sobolev space context, in fact one has:

Proposition 7. *Let $\overline{\Omega}$ be compact, and let $s > \operatorname{Re}\mu - 1/p'$. Then*

$$\mathcal{E}_\mu(\overline{\Omega}) \subset H_p^{\mu(s)}(\overline{\Omega}) \text{ densely, and } \bigcap\nolimits_s H_p^{\mu(s)}(\overline{\Omega}) = \mathcal{E}_\mu(\overline{\Omega}).$$

We can now state the basic theorems:

Theorem 8. *When P is of order m and type μ, r^+P maps $H_p^{\mu(s)}(\overline{\Omega})$ continuously into $\overline{H}_p^{s-\operatorname{Re} m}(\Omega)$ for all $s > \operatorname{Re}\mu - 1/p'$.*

Theorem 9. *Let P be elliptic of order m, with factorization index μ_0, and of type μ_0 (mod 1). Let $s > \operatorname{Re}\mu_0 - 1/p'$. When $u \in \dot{H}_p^{\operatorname{Re}\mu_0 - 1/p' + \varepsilon}(\overline{\Omega})$, then*

$$r^+Pu = f \in \overline{H}_p^{s-\operatorname{Re} m}(\Omega) \implies u \in H_p^{\mu_0(s)}(\overline{\Omega}).$$

Moreover, the mapping

$$r^+P \colon H_p^{\mu_0(s)}(\overline{\Omega}) \to \overline{H}_p^{s-\operatorname{Re} m}(\Omega) \quad (4)$$

is Fredholm.

This answers **Q4** in a precise way, with $m = 2a$, $\mu_0 = a$.
The proofs in the old 1965 notes (for $p = 2$) are long and difficult. One of the difficulties is that the $\Xi_\pm^\mu$ are not truly ψdo's in n variables, the derivatives of the symbols $(\langle\xi'\rangle \pm i\xi_n)^\mu$ do not decrease for $|\xi| \to \infty$ in the required way.

More recently we have found (G '90) a modified choice of symbol that gives true ψdo's $\Lambda_\pm^{(\mu)}$ with the same holomorphic extension properties for $\operatorname{Im}\xi_n \lessgtr 0$; they can be used instead of $\Xi_\pm^\mu$, also for $p \neq 2$.
This allows a reduction of some of the considerations to cases where the Boutet de Monvel calculus (extended to H_p^s in G '90) can be applied.

In fact, when we for Theorem 9 introduce

$$Q = \Lambda_-^{(\mu_0 - m)} P \Lambda_+^{(-\mu_0)},$$

we get a ψdo of order 0 and type 0, with factorization index 0; then

$$r^+ P u = f, \text{ with } \operatorname{supp} u \subset \overline{\Omega},$$

can be transformed to the equation

$$r^+ Q v = g, \text{ where } v = \Lambda_+^{(\mu_0)} u,\, g = r^+ \Lambda_-^{(\mu_0 - m)} e^+ f.$$

Here the Boutet de Monvel calculus applies to Q and provides good solvability properties for all $s > \operatorname{Re}\mu_0 - 1/p'$.

Since $\bigcap_s H_p^{\mu(s)}(\overline{\Omega}) = \mathcal{E}_\mu(\overline{\Omega})$, and $\bigcap_s \overline{H}_p^{s-\operatorname{Re} m}(\Omega) = C^\infty(\overline{\Omega})$, one finds as a corollary when $s \to \infty$:

Corollary 10. *Let P be as in Theorem 9 and let u be a function supported in $\overline{\Omega}$. If $r^+Pu \in C^\infty(\overline{\Omega})$, then $u \in \mathcal{E}_{\mu_0}(\overline{\Omega})$. Moreover, the mapping*

$$r^+P\colon \mathcal{E}_{\mu_0}(\overline{\Omega}) \to C^\infty(\overline{\Omega})$$

is Fredholm.

One can furthermore show that the finite dimensional kernel and cokernel (a complement of the range) of the mapping in Corollary 10 serve as kernel and cokernel also in the mappings for finite s in Theorem 9.

Note the sharpness: The functions in $\mathcal{E}_{\mu_0}$ have the behavior $u(x) = d(x)^{\mu_0} v(x)$ at the boundary with $v \in C^\infty(\overline{\Omega})$; they are *not in C^∞ themselves*, when $\mu_0 \notin \mathbb{N}_0$!

Now, answers to the remaining **Q1, Q2, Q3, Q5**:

Q1. The domain of $P_{a,\mathrm{Dir}}$.

$$D(P_{a,\mathrm{Dir}}) = H_2^{a(2a)}(\overline{\Omega}) = \Lambda_+^{(-a)} e^+ \overline{H}_2^a(\Omega) \begin{cases} = \dot{H}_2^{2a}(\overline{\Omega}), \text{ if } 0 < a < \frac{1}{2}, \\ \subset \bigcap_{\varepsilon > 0} \dot{H}_2^{1-\varepsilon}(\overline{\Omega}), \text{ if } a = \frac{1}{2}, \\ \subset d^a \overline{H}_2^a(\Omega) + \dot{H}_2^{2a}(\overline{\Omega}), \text{ if } \frac{1}{2} < a < 1. \end{cases}$$

The last line shows the appearance of a factor d^a; the proof uses Poisson operators from the Boutet de Monvel calculus. (Recall also that $D(P_{a,\mathrm{Dir}}) \subset \dot H^{\frac12+a-\varepsilon}(\overline\Omega)$ when $\frac12 < a < 1$.)

Q2. $P_{a,\mathrm{Dir}}^{-1}$ has the structure $(\Lambda_+^{(-a)})_+(\widetilde Q_+ + G)(\Lambda_-^{(-a)})_+$, where $\widetilde Q$ is a ψdo and G a singular Green operator of order 0. This can be used to show that indeed, the eigenvalues λ_k of $P_{a,\mathrm{Dir}}$ satisfy

$$\lambda_k \sim C(P_a,\Omega)\, k^{n/2a}, \text{ for } k\to\infty.$$

Extends to any bounded open Ω, by approximation from smooth cases.

Q5. The theory extends, using Johnsen '96, to Besov-Triebel-Lizorkin spaces $F^s_{p,q}$ and $B^s_{p,q}$, in particular to the Hölder-Zygmund spaces $B^s_{\infty,\infty}$, that coincide with Hölder spaces C^s for $s\in\mathbb R_+\setminus\mathbb N$. This leads to:

$$f\in C^t(\overline\Omega) \implies u\in d^aC^{a+t}(\overline\Omega), \text{ all } t\ge 0;$$

with $a+t$ replaced by $a+t-\varepsilon$ if $a+t$ or $2a+t$ is integer. **Optimal** in the non-exceptional cases.

Q3. Smoothness of eigenfunctions u_k? If 0 is an eigenvalue, its eigenfunctions are in $\mathcal E_a(\overline\Omega)$. For $\lambda_k>0$, one finds by an iterative argument using the regularity theory:

$$u_k\in d^aC^{2a\,(-\varepsilon)}(\overline\Omega)\cap C^\infty(\Omega);\ \varepsilon \text{ active if } 2a\in\mathbb N.$$

For simplicity, consider just P_a. The Fredholm map from Th. 9, Cor. 10,

$$r^+P_a\colon H_p^{a(s)}(\overline{\Omega}) \to \overline{H}_p^{s-2a}(\Omega), \quad s > a - 1/p', \qquad (6)$$
$$r^+P_a\colon \mathcal{E}_a(\overline{\Omega}) \to C^\infty(\overline{\Omega}),$$

represents the **homogeneous Dirichlet problem** for P_a. Recall that

$$\mathcal{E}_{a-1}(\overline{\Omega}) = e^+\{u(x) = d(x)^{a-1}v(x) \mid v \in C^\infty(\overline{\Omega})\}.$$

Here we have:

Theorem 11. *The boundary mapping*

$$\mathcal{E}_{a-1}(\overline{\Omega}) \ni u \mapsto \gamma_{a-1,0}u = \gamma_0(d^{1-a}u) \in C^\infty(\partial\Omega)$$

extends for $s > a - 1/p'$ to a continuous surjective mapping

$$\gamma_{a-1,0}\colon H_p^{(a-1)(s)}(\overline{\Omega}) \to B_p^{s-a+1-1/p}(\partial\Omega),$$

with kernel $H_p^{a(s)}(\overline{\Omega})$; in other words,

$$\gamma_{a-1,0}\colon H_p^{(a-1)(s)}(\overline{\Omega})/H_p^{a(s)}(\overline{\Omega}) \xrightarrow{\sim} B_p^{s-a+1-1/p}(\partial\Omega). \quad (7)$$

Combination with (6) gives a **nonhomogeneous Dirichlet problem**:

Theorem 12. *The following maps are Fredholm, for $s > a - 1/p'$:*

$$\{r^+P_a, \gamma_{a-1,0}\} \colon H_p^{(a-1)(s)}(\overline{\Omega}) \to \overline{H}_p^{s-2a}(\Omega) \times B_p^{s-a+1-1/p}(\partial\Omega),$$
$$\{r^+P, \gamma_{a-1,0}\} \colon \mathcal{E}_{a-1}(\overline{\Omega}) \to C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega).$$

What is perhaps suprising is that this naturally defined nonhomogeneous Dirichlet problem involves a blow-up at $\partial\Omega$! Namely, even for the smoothest possible data, the solutions behave like d^{a-1} at $\partial\Omega$, with $a - 1 < 0$.

There is a general version in Hölder spaces:

Theorem 13. *For $t \geq 0$, the problem*

$$r^+P_a u = f \in C^t(\overline{\Omega}), \quad \gamma_{a-1,0}u = \varphi \in C^{a+1+t}(\partial\Omega);$$

is Fredholm solvable, with solution

$$u \in e^+d(x)^{a-1}C^{a+1+t}(\overline{\Omega}) + \dot{C}^{2a+t}(\overline{\Omega}), \quad (8)$$

(with t replaced by $t - \varepsilon$ in (8) if $a + t$ or $2a + t$ is integer).

The component $e^+d(x)^{a-1}C^{t+a+1}(\overline{\Omega})$ enters nontrivially. Whenever $\varphi \neq 0$, it creates a term in that space, which is unbounded at $\partial\Omega$, behaving like d^{a-1}.

In the studies by potential- and integral operator-methods, this phenomenon is called "large solutions" (blowing up at $\partial\Omega$), e.g. by Abatangelo arXiv '13 working in low-regularity spaces. It is rather precisely described by our methods.

Further remarks. Our study moreover allows treatments of other boundary conditions (also vector valued). For example, $r^+(-\Delta)^a u = f$ with a *Neumann condition* $\gamma_{a-1,1} u \equiv \partial_n(d(x)^{1-a} u)|_{\partial\Omega} = \psi$ can be shown to be define a Fredholm operator:

$$\{r^+(-\Delta)^a, \gamma_{a-1,1}\}\colon H_p^{(a-1)(s)}(\overline{\Omega}) \to \overline{H}_p^{s-2a}(\Omega) \times B_p^{s-a-1/p}(\partial\Omega),\ s > a+1/p.$$

The current efforts for problems involving the fractional Laplacian are often concerned with nonlinear equations where it enters, and there is an interest also in generalizations with low regularity of the domain or the coefficients.

For problems where Δ itself enters, one has a old and well-known background theory of boundary value problems in the smooth case. Such a background theory has been absent in the case of $(-\Delta)^a$, and we can say that the present results provide that missing link.

The methods used in the current literature on $(-\Delta)^a$ are often integral operator methods and potential theory. Here is one of the strange formulations used there:

Because of the nonlocal nature of $(-\Delta)^a$, when we consider a subset $\Omega \subset \mathbb{R}^n$, auxiliary conditions may be given as *exterior conditions*, where the value of the unknown function is prescribed on the complement of Ω.

Then the homogeneous Dirichlet problem is formulated as:

$$\begin{cases} r^+(-\Delta)^a U &= f \text{ on } \Omega, \\ U &= g \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{8}$$

The nonhomogeneous Dirichlet problem is then formulated (for more general U) as:

$$\begin{cases} r^+(-\Delta)^a U &= f \text{ on } \Omega, \\ U &= g \text{ on } \mathbb{R}^n \setminus \Omega, \\ d(x)^{1-a} U &= \varphi \text{ on } \partial\Omega. \end{cases} \tag{9}$$

(Abatangelo arXiv November '13.) We can show, when Ω is smooth:

Within the framework of our function spaces, (8) *and* (9) *can be reduced to problems with the unknown u supported in $\overline{\Omega}$ (i.e., $g = 0$), uniquely solved by our preceding theorems.*