Chapter 10
Pseudodifferential boundary operators

10.1 The real formulation

In this chapter we present some essential ingredients from the calculus of pseudodifferential boundary operators (ψdbo's, Green operators) introduced by Boutet de Monvel [B66], [B71], with further developments e.g. from [G84], [G90] and [G96]; see also Rempel and Schulze [RS82]. The basic notions are defined relative to $\mathbb{R}^n$ and the subset $\overline{\mathbb{R}}^n_+ \subset \mathbb{R}^n$; then they are carried over to manifold situations by use of local coordinate systems.

In the case of a differential operator A, the analysis of the relevant boundary conditions is usually based on the polynomial structure of the symbol of A; in particular the roots in ξ_n of the principal symbol polynomial $a^0(x', 0, \xi', \xi_n)$ (in the situation where the domain is $\mathbb{R}^n_+$) play a role. When pseudodifferential operators P are considered, the principal symbol p^0 is generally not a polynomial. It may be a rational function (this happens naturally when one makes reductions in a system of differential operators), in which case one can consider the roots and poles with respect to ξ_n. But then, even when P is elliptic, there is much less control over how these behave than when a^0 is a polynomial; roots and poles may cancel each other or reappear, as the coordinate ξ' varies. For a workable theory, a more universal point of view is needed.

Vishik and Eskin (see [VE67] and [E81]) based a theory on a factorization of a symbol in two factors with different domains of holomorphy in ξ_n. This works well in the scalar case but can be problematic in the case of matrix-formed operators (since the factorization here is generally only piecewise continuous in ξ'). They mainly consider ψdo's of a general kind, with less restrictions on the behavior in ξ than our standard symbol spaces require.

The calculus introduced by Boutet de Monvel takes a special class of ψdo's; one of the advantages of that theory is that it replaces the factorization by a *projection* procedure that works equally well for scalar and matrix-formed operators (depends smoothly on ξ'). This is linked in a natural way with the projections $e^{\pm}r^{\pm}$ of $L_2(\mathbb{R})$ onto $e^{\pm}L_2(\mathbb{R}_\pm)$ (cf. (9.3)). The description that now follows is given in relation to the latter projections, and the Fourier-transformed version (used in symbol calculations) will be taken up in Section 10.3.

Pseudodifferential operators satisfying the transmission condition

When P is a ψdo on $\mathbb{R}^n$, its truncation (or "restriction") to the subset $\Omega = \mathbb{R}^n_+$ is defined by

$$P_+u = r^+Pe^+u, \text{ also denoted } P_{\mathbb{R}^n_+}u \text{ or } P_\Omega u, \tag{10.1}$$

where, as in (9.2) and (9.3), r^+ restricts $\mathscr{D}'(\mathbb{R}^n)$ to $\mathscr{D}'(\mathbb{R}^n_+)$, and e^+ extends locally integrable functions on $\mathbb{R}^n_+$ by zero on $\mathbb{R}^n_-$. We underline that r^+ restricts to the interior of $\overline{\mathbb{R}}^n_+$ so that singularities supported at $x_n = 0$ disappear. As usual, $C^0(\overline{\mathbb{R}}^n_+)$ identifies with a subspace of $\mathscr{D}'(\mathbb{R}^n_+)$.

When P is properly supported and of order d, the operator P_+ is continuous from $L_{2,\mathrm{comp}}(\mathbb{R}^n_+)$ to $H^{-d}_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+)$ (defined in (10.148) below), but in general does not map $H^m_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+)$ into $H^{m-d}_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+)$ for $m > 0$; the discontinuity of e^+u at $x_n = 0$ causes a singularity. Boutet de Monvel singled out a class of ψdo's where one does get these mapping properties for P_+, namely, the ψdo's having the transmission property.

P is said to have *the transmission property* with respect to $\mathbb{R}^n_+$ when $P_{\mathbb{R}^n_+}$ "preserves C^∞ up to the boundary", i.e., $P_{\mathbb{R}^n_+}$ maps $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ into $C^\infty(\overline{\mathbb{R}}^n_+)$. Boutet de Monvel showed in [B66] a necessary and sufficient condition for a polyhomogeneous symbol $p(x,\xi) \sim \sum_{l\in\mathbb{N}_0} p_{d-l}(x,\xi)$ to define an operator $\mathrm{OP}(p)$ with the transmission property w.r.t. $\mathbb{R}^n_+$. It states that the homogeneous terms p_{d-l} have the symmetry property

$$D^\beta_x D^\alpha_\xi p_{d-l}(x',0,0,-\xi_n) = e^{i\pi(d-l-|\alpha|)} D^\beta_x D^\alpha_\xi p_{d-l}(x',0,0,\xi_n), \tag{10.2}$$

for $|\xi_n| \geq 1$ and all indices α, β, l.

We write "OP" instead of "Op" from now on, to allow generalizations to OPT, OPK and OPG definitions below. (The minus on ξ_n should be placed in the left-hand side as in (10.2). Only when d is integer can it equally well be placed in the right-hand side, as done in many texts, e.g. in [G96, (1.2.7)].)

Example 10.1. A parametrix symbol $q(x,\xi)$ for an elliptic *differential operator* P of order d certainly has the transmission property, since its symbol terms are rational functions of ξ. In fact, it has the stronger property

$$q_{-d-l}(x,-\xi) = (-1)^{d-l} q_{-d-1}(x,\xi) \text{ for } |\xi| \geq 1, \text{ all } l, \tag{10.3}$$

guaranteeing the transmission property for whichever direction taken as x_n. Polyhomogeneous symbols having the property (10.3) are in some texts said to have *even-even* alternating parity (the even-order symbols are even), or just to be even-even, for short. The opposite parity

$$q_{-d-l}(x, -\xi) = (-1)^{d-l+1} q_{-d-l}(x, \xi) \text{ for } |\xi| \geq 1, \text{ all } l, \tag{10.4}$$

is then called *even-odd* alternating parity. An example with the latter property is $q(\xi) = \langle\xi\rangle$, the symbol of the square root of $1 - \Delta$. Note that the symbol $\langle\xi\rangle^s$ has the transmission property (and is even-even) if and only if s is an even integer.

When d is *integer* and the equations (10.2) hold, then they also hold for the symbol $p(x, -\xi)$; this implies that also $P_{\mathbb{R}^n_-}$ preserves C^∞ up to the boundary in $\overline{\mathbb{R}}^n_-$.

When d is not integer, (10.2) implies a "wrong" kind of symmetry for $p(x, -\xi)$, and P will not in general preserve C^∞ in $\overline{\mathbb{R}}^n_-$, but maps the functions in $C^\infty_{(0)}(\overline{\mathbb{R}}^n_-)$ into functions with a specific singular behavior at $x_n = 0$.

There is a complete discussion of necessary and sufficient conditions for the transmission property, extending the analysis to $S^d_{\varrho,\delta}$ symbols with $\varrho > \delta$, in Grubb and Hörmander [GH90]. Besides in [B66], [B71], studies related to the transmission property are found in [VE67], [E81] and in [H85, Sect. 18.2]. There is also an introductory explanation in [G91].

It is shown in [B71] that the properties (10.2) in the case $d \in \mathbb{Z}$ may be rewritten as an expansion property of $p(x, \xi)$ and its derivatives at $x_n = 0$:

$$D^\beta_x D^\alpha_\xi p(x', 0, \xi) \sim \sum_{-\infty < l \leq d - |\alpha|} s_{l,\alpha,\beta}(x', \xi')\xi^l_n, \tag{10.5}$$

where the $s_{l,\alpha,\beta}(x', \xi')$ are *polynomials* in ξ' of degree $d - |\alpha| - l$, for all α and $\beta \in \mathbb{N}^n_0$. We denote $s_{l,0,0} = s_l$. More precisely, let us introduce (with X representing x', y' or (x', y')):

Definition 10.2. A symbol $p(X, x_n, y_n, \xi)$ of order $d \in \mathbb{Z}$ satisfies the transmission condition (at $x_n = y_n = 0$), when there exist symbols $s_l(X, \xi')$, polynomial in ξ' of order $d - l$, such that for all indices,

$$\begin{aligned} |D^\beta_X D^\alpha_\xi [\xi^m_n p(X, 0, 0, \xi) - \sum_{-m \leq l \leq d - |\alpha|} s_l(X, \xi')\xi^{l+m}_n]| \\ \leq c(X)\langle\xi'\rangle^{d+1+m-|\alpha|}\langle\xi\rangle^{-1}, \end{aligned} \tag{10.6}$$

with continuous functions $c(X)$, and there are similar expansions of $\partial^j_{x_n}\partial^{j'}_{y_n} p$.

For polyhomogeneous symbols of integer order, (10.2) holds if and only if Definition 10.2 is satisfied by p (and by each term p_{d-j} in its symbol). But (10.5) and (10.6) have the advantage that they make sense for $S^d_{1,0}$ symbols also and guarantee their C^∞ preserving properties, as noted in [B71]. For such symbols it is a sufficient condition, which is why we call it the transmision *condition*.

In the following, we formulate the principles for symbols in x-form (cf. Remark 7.4). Formula (10.5) means that $p(x', 0, \xi)$ (and in a similar way the derivatives $D_x^\beta D_\xi^\alpha p(x',0,\xi)$) has an expansion in integer powers of ξ_n such that at each (x',ξ') one has for any $m \in \mathbb{N}_0$ that

$$\xi_n^m p(\xi_n) - \sum_{-m<l\le d} s_l \xi_n^{l+m} \tag{10.7}$$

has the same limit (namely, s_{-m}) for $\xi_n \to +\infty$ as for $\xi_n \to -\infty$. The C^∞-functions of ξ_n with this property form the important space $\mathcal{H}$ that we study in Section 10.2 below. In the present section we do not want to go into details with the complex analysis involved in studying $\mathcal{H}$; instead we shall explain the point of view one gets by studying the inverse Fourier transforms in ξ_n (the "real" point of view).

Note in particular that the definition implies

$$p(x',0,\xi) = \sum_{0\le l\le d} s_l(x',\xi')\xi_n^l + p'(x',\xi), \tag{10.8}$$

where p' is $O(\langle\xi\rangle^{-1}\langle\xi'\rangle^{d+1})$, the sum over l is *polynomial* in ξ of order d and the top coefficient s_d is a *function of* x',

$$s_d(x',\xi') = s_d(x'). \tag{10.9}$$

Thus, *if p is constant in x_n*, P is the sum of a differential operator and an operator that preserves L_2 with respect to x_n. More generally, p could be much less well-behaved for $x_n \ne 0$, but here a Taylor expansion in x_n gives a number of good terms $\frac{1}{j!}x_n^j \partial_{x_n}^j p(x',0,\xi)$ where $\partial_{x_n}^j p$ behaves similarly to (10.8), plus a term with a factor x_n^M that can make it harmless when M is large.

The partial inverse Fourier transform $\tilde p(x',0,\xi',z_n) = \mathscr{F}^{-1}_{\xi_n\to z_n} p(x',0,\xi)$ is always rapidly decreasing for $z_n \to \pm\infty$, since $D_{\xi_n}^k(\xi_n^m p)$ is integrable in ξ_n when $k \ge m+d+2$. But in case of general symbols, it has a singularity at $z_n = 0$. However, as shown in [B66] and [GH90], symbols of integer order d satisfy the transmission condition *if and only if,* for any $|\xi'| \ge 1$, any α, β, the functions

$$\tilde p_{\alpha,\beta}(x',0,\xi',z_n) = \mathscr{F}^{-1}_{\xi_n\to z_n} D_x^\beta D_\xi^\alpha p(x',0,\xi), \text{ considered for } (x',z_n) \in \mathbb{R}^n_\pm,$$
$$\text{extend to } C^\infty\text{-functions of } (x',z_n) \in \overline{\mathbb{R}}^n_\pm. \tag{10.10}$$

The limits for $z_n \to 0+$ and $z_n \to 0-$ are in general different, and the condition does not exclude singularities supported in $\{z_n = 0\}$. (See also Exercises 10.2 and 10.3.)

Let us show that the transmission condition implies (10.10). In view of (10.8), $\tilde p$ is the sum of a distribution $\sum_{0\le l\le d} s_l(x',\xi')D_{z_n}^l \delta(z_n)$ supported in $\{z_n = 0\}$ and a *function* $\tilde p'(x',\xi',z_n)$ in L_2 with respect to $z_n \in \mathbb{R}$. Here

$$\|\tilde{p}'(x',\xi',z_n)\|_{L_{2,z_n}(\mathbb{R})} = (2\pi)^{-\frac{1}{2}}\|p'(x',\xi)\|_{L_{2,\xi_n}(\mathbb{R})}$$
$$\le c(x')\langle\xi'\rangle^{d+1}\|\langle\xi\rangle^{-1}\| \le c'(x')\langle\xi'\rangle^{d+\frac{1}{2}}$$

(recall (9.19)); so in particular, the functions $r^{\pm}_{z_n}\tilde{p}'$ $(= r^{\pm}_{z_n}\tilde{p})$ satisfy such estimates in $L_2(\mathbb{R}_\pm)$-norm, respectively:

$$\|r^+\tilde{p}(x',\xi',z_n)\|_{L_{2,z_n}(\mathbb{R}_+)} = \|r^+\tilde{p}'(x',\xi',z_n)\|_{L_{2,z_n}(\mathbb{R}_+)} \le c(x')\langle\xi'\rangle^{d+\frac{1}{2}},$$
$$\|r^-\tilde{p}(x',\xi',z_n)\|_{L_{2,z_n}(\mathbb{R}_-)} = \|r^-\tilde{p}'(x',\xi',z_n)\|_{L_{2,z_n}(\mathbb{R}_-)} \le c(x')\langle\xi'\rangle^{d+\frac{1}{2}},$$

with $c(x')$ continuous. For any k, k', α', β', the derived distribution

$$z_n^k D_{z_n}^{k'} D_{x'}^{\beta'} D_{\xi'}^{\alpha'} \tilde{p}(x',0,\xi',z_n) = \mathscr{F}^{-1}_{\xi_n\to z_n} D_{\xi_n}^k \xi_n^{k'} D_{\xi'}^{\alpha'} D_{x'}^{\beta'} p(x',0,\xi)$$

comes from a symbol satisfying Definition 10.2 with d replaced by $d-k+k'-|\alpha'|$, hence we likewise find

$$\|z_n^k D_{z_n}^{k'} D_{x'}^{\beta'} D_{\xi'}^{\alpha'} r^{\pm}\tilde{p}'\|_{L_{2,z_n}(\mathbb{R}_\pm)} = \|z_n^k D_{z_n}^{k'} D_{x'}^{\beta'} D_{\xi'}^{\alpha'} r^{\pm}\tilde{p}\|_{L_{2,z_n}(\mathbb{R}_\pm)}$$
$$\le c(x')\langle\xi'\rangle^{d+\frac{1}{2}-k+k'-|\alpha'|}, \qquad (10.11)$$

using again that $\tilde{p}-\tilde{p}'$ is supported in $\{z_n = 0\}$. Similar arguments apply to $\partial^j_{x_n} p(x',0,\xi)$. Since all derivatives have bounded L_2-norms, they have bounded sup-norms (as in Theorem 4.18), locally uniformly in (x',ξ'), so it follows that the functions extend to C^∞-functions of $(x',z_n,\xi') \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}$ resp. $\overline{\mathbb{R}}^n_- \times \mathbb{R}^{n-1}$.

Function spaces defined by estimates as in (10.11) will now be introduced systematically. For this we first introduce some abbreviations: We use the notation $\mathbb{R}^2_{++} = \mathbb{R}_+ \times \mathbb{R}_+$, $\overline{\mathbb{R}}^2_{++} = \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$, and $\mathscr{S}(\overline{\mathbb{R}}^2_{++}) = r^+_{x_n} r^+_{y_n}\mathscr{S}(\mathbb{R}^2)$ (where r^+_z indicates restriction to $\{z > 0\}$); recall also (9.4). Here we write for short:

$$\mathscr{S}(\overline{\mathbb{R}}_+) = \mathscr{S}_+, \quad \mathscr{S}(\overline{\mathbb{R}}^2_{++}) = \mathscr{S}_{++}. \qquad (10.12)$$

Definition 10.3. Let $d \in \mathbb{R}$ and let $\varXi$ be open $\subset \mathbb{R}^{n'}$.

1° The space $S^d_{1,0}(\varXi,\mathbb{R}^{n-1},\mathscr{S}_+)$ consists of the functions $\tilde{f}(X,x_n,\xi') \in C^\infty(\varXi\times\overline{\mathbb{R}}_+\times\mathbb{R}^{n-1})$, lying in $\mathscr{S}_+$ with respect to x_n, such that for all α,β,k,k',

$$\|x_n^k D_{x_n}^{k'} D_X^\beta D_{\xi'}^\alpha \tilde{f}(X,x_n,\xi')\|_{L_2(\mathbb{R}_+)} \le c(X)\langle\xi'\rangle^{d+\frac{1}{2}-k+k'-|\alpha|}, \qquad (10.13)$$

with continuous functions $c(X)$. Moreover, $\tilde{f}$ is said to have the asymptotic expansion $\tilde{f} \sim \sum_{l\in\mathbb{N}_0} \tilde{f}_{d-l}$ in $S^d_{1,0}(\varXi,\mathbb{R}^{n-1},\mathscr{S}_+)$, when there is a sequence of functions $\tilde{f}_{d-l}$ lying in $S^{d-l}_{1,0}(\varXi,\mathbb{R}^{n-1},\mathscr{S}_+)$ such that $\tilde{f} - \sum_{l<M}\tilde{f}_{d-l} \in S^{d-M}_{1,0}(\varXi,\mathbb{R}^{n-1},\mathscr{S}_+)$ for any $M \in \mathbb{N}_0$.

2° The subspace $S^d(\varXi,\mathbb{R}^{n-1},\mathscr{S}_+)$ of polyhomogeneous elements consists of the functions $\tilde{f} \in S^d_{1,0}(\varXi,\mathbb{R}^{n-1},\mathscr{S}_+)$ that have asymptotic expansions $\tilde{f} \sim \sum_{l\in\mathbb{N}_0}\tilde{f}_{d-l}$ where the functions $\tilde{f}_{d-l}$ have the quasi-homogeneity property

$$\tilde f_{d-l}(X, \tfrac{1}{\lambda}x_n, \lambda\xi') = \lambda^{d+1-l}\, \tilde f_{d-l}(X, x_n, \xi') \text{ for } \lambda \ge 1 \text{ and } |\xi'| \ge 1. \quad (10.14)$$

3° The space $S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_{++})$ consists of the functions $\tilde g(X, x_n, y_n, \xi') \in C^\infty(\Xi \times \overline{\mathbb{R}}^2_{++} \times \mathbb{R}^{n-1})$, lying in $\mathscr{S}_{++}$ with respect to (x_n, y_n), such that for all $\alpha, \beta, k, k', m, m'$,

$$\|x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} D_X^\beta D_{\xi'}^\alpha \tilde g(x', x_n, y_n, \xi')\|_{L_2(\mathbb{R}^2_{++})} \le c(X)\langle\xi'\rangle^{d+1-k+k'-m+m'-|\alpha|}. \quad (10.15)$$

Here $\tilde g$ is said to have the asymptotic expansion $\tilde g \sim \sum_{l\in\mathbb{N}_0} \tilde g_{d-l}$ in $S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_{++})$, when there are functions $\tilde g_{d-l} \in S^{d-l}_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_{++})$ such that $\tilde g - \sum_{l<M} \tilde g_{d-l} \in S^{d-M}_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_{++})$ for any $M \in \mathbb{N}_0$.

4° The subspace $S^d(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_{++})$ of polyhomogeneous elements consists of the functions $\tilde g \in S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_{++})$ that have asymptotic expansions $\tilde g \sim \sum_{l\in\mathbb{N}_0} \tilde g_{d-l}$ where the functions $\tilde g_{d-l}$ have the quasi-homogeneity property

$$\tilde g_{d-l}(X, \tfrac{1}{\lambda}x_n, \tfrac{1}{\lambda}y_n, \lambda\xi') = \lambda^{d+2-l}\, \tilde g_{d-l}(X, x_n, y_n, \xi') \text{ for } \lambda \ge 1 \text{ and } |\xi'| \ge 1. \quad (10.16)$$

What we showed in (10.11) is that when p satisfies Definition 10.2, then $r^+\tilde p$ belongs to $S^d_{1,0}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, \mathscr{S}_+)$. This is taken up again in Theorem 10.21 below. When p is polyhomogeneous, $r^+\tilde p$ is in $S^d(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, \mathscr{S}_+)$. Similar statements hold for $(r^-\tilde p)|_{x_n=-z_n}$.

Functions with the properties in Definition 10.3 are called *symbol-kernels*. Just like for pseudodifferential symbols, one can turn a series of symbol-kernels of decreasing orders into an asymptotic series for a suitable symbol-kernel of the highest order (cf. Lemma 7.3). 3° and 4° will be used in the description of singular Green operators below.

The properties of the functions could equivalently be formulated in terms of sup-norms: $\tilde f$ is in $S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_+)$ if and only if

$$\sup_{x_n>0} |x_n^k D_{x_n}^{k'} D_X^\beta D_{\xi'}^\alpha \tilde f(x', x_n, \xi')| \le c(X)\langle\xi'\rangle^{d+1-k+k'-|\alpha|} \quad (10.17)$$

(one can use estimates as in the proof of Theorem 4.18 to go from L_2-norms to sup-norms, and insert factors like $(1+ix_n\langle\xi'\rangle)(1+ix_n\langle\xi'\rangle)^{-1}$ for the other direction). We use the L_2-norms for convenience in Fourier transformation.

The quasi-homogeneity properties correspond to homogeneity of the Fourier-transformed functions

$$f(X, \xi', \xi_n) = \mathscr{F}_{x_n\to\xi_n} e^+ \tilde f(X, x_n, \xi'),$$
$$g(X, \xi', \xi_n, \eta_n) = \mathscr{F}_{x_n\to\xi_n} \overline{\mathscr{F}}_{y_n\to\eta_n} e^+_{x_n} e^+_{y_n} \tilde g(X, x_n, y_n, \xi');$$

namely,

$$f_{d-l}(X, \lambda\xi) = \lambda^{d-l}\, f_{d-l}(X, \xi) \quad \text{for } \lambda \ge 1,\ |\xi'| \ge 1,$$

$$g_{d-l}(X, \lambda\xi', \lambda\xi_n, \lambda\eta_n) = \lambda^{d-l} g_{d-l}(X, \xi', \xi_n, \eta_n),\ \lambda \geq 1, |\xi'| \geq 1;$$

as is easily checked from the definition of Fourier transformation.

In the following, all ψdo symbols on $\mathbb{R}^n$ will tacitly be assumed to be of integer order satisfying the transmission condition. We denote by $p(x', \xi', D_n)$ or $\mathrm{OP}_n(p(x, \xi))$ the operator on $\mathbb{R}$ where the ψdo definition (7.2) is applied with respect to (x_n, ξ_n) only, and by $\mathrm{OP}'(p(x, \xi))$ the operator on $\mathbb{R}^{n-1}$ where (7.2) is applied with respect to (x', ξ') only.

Systems (Green operators)

We shall now introduce the other ingredients in the Boutet de Monvel calculus. Let P be $N' \times N$-matrix formed. Along with P_+, which operates on $\mathbb{R}^n_+$, we shall consider operators going to and from the boundary $\mathbb{R}^{n-1}$, forming together with P a system

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ \\ T & S \end{pmatrix} : \begin{matrix} C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)^N \\ \times \\ C^\infty_0(\mathbb{R}^{n-1})^M \end{matrix} \to \begin{matrix} C^\infty(\overline{\mathbb{R}}^n_+)^{N'} \\ \times \\ C^\infty(\mathbb{R}^{n-1})^{M'} \end{matrix}. \tag{10.18}$$

Here T is a so-called *trace operator*, going from $\mathbb{R}^n_+$ to $\mathbb{R}^{n-1}$; K is a so-called *Poisson operator* (called a potential operator or coboundary operator in some other texts), going from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n_+$; S is a *pseudodifferential operator on* $\mathbb{R}^{n-1}$; and G is an operator on $\mathbb{R}^n_+$ called a *singular Green operator*, a non-pseudodifferential term that has to be included in order to have adequate composition rules. The full system $\mathcal{A}$ was called a *Green operator* by Boutet de Monvel, or a *pseudodifferential boundary operator* (that we can write ψdbo). Since some authors later used the name "Green operator" for a generalization of singular Green operators, we shall mainly write ψdbo's. We shall usually take $N = N'$, whereas the dimensions M and M' can have all values, including zero.

When P is a differential operator, it is classical to study systems of the form

$$\mathcal{A} = \begin{pmatrix} P_+ \\ T \end{pmatrix}; \tag{10.19}$$

here $M = 0$ and $M' > 0$. When this $\mathcal{A}$ has an inverse in the calculus, the inverse will be of the form

$$\mathcal{A}^{-1} = \begin{pmatrix} Q_+ + G & K \end{pmatrix}; \tag{10.20}$$

where $M > 0$ and $M' = 0$. Simple examples are found in Chapter 9, with $P = 1 - \Delta$, $T = \gamma_0$ or γ_1, the inverses described in Theorems 9.18 and 9.23.

The entries in (10.18) will now be explained.

Trace operators

The trace operators include the usual differential trace operators $\gamma_j : u \mapsto (D^j_{x_n} u)|_{x_n=0}$ composed with ψdo's on $\mathbb{R}^{n-1}$, and moreover some integral operator types, governed by the fact that

$$T = \gamma_0 P_+ \tag{10.21}$$

should be a trace operator whenever P is a ψdo satisfying the transmission condition. Here, it is found for the right-hand side that when $P = \mathrm{OP}(p(x,\xi))$, then $\gamma_0 P_+ u = \gamma_0 r^+ \mathrm{OP}(p(x',0,\xi))e^+ u$, where we can insert (10.8). This gives a sum of differential trace operators plus a term where the ψdo symbol is p'; here $r^\pm \tilde{p}'$ satisfy estimates as in (10.11). The term is (using the notation $\mathscr{F}_{x'\to\xi'} u = \acute{u}$, cf. (9.5))

$$\begin{aligned}\gamma_0 \,\mathrm{OP}(p')_+ u(x) &= \gamma_0 \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} p'(x',0,\xi) e^+ u(y)\,dy\,đ\xi \\ &= \gamma_0 \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_{\mathbb{R}} \tilde{p}'(x',0,\xi',x_n - y_n) e^+ \acute{u}(\xi',y_n)\,dy_n\,đ\xi' \\ &= \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{p}'(x',0,\xi',-y_n) \acute{u}(\xi',y_n)\,dy_n\,đ\xi'.\end{aligned} \tag{10.22}$$

As noted above, $\tilde{p}(x',0,\xi',-y_n)|_{y_n>0} \in S^d_{1,0}(\mathbb{R}^{n-1},\mathbb{R}^{n-1},\mathscr{S}_+)$.

The general definition goes as follows:

Definition 10.4. A trace operator of order d ($\in \mathbb{R}$) and class r ($\in \mathbb{N}_0$) is an operator of the form

$$Tu = \sum_{0\le j\le r-1} S_j \gamma_j + T', \tag{10.23}$$

where γ_j denotes the standard trace operator $(\gamma_j u)(x') = D^j_{x_n} u(x',0)$, the S_j are ψdo's in $\mathbb{R}^{n-1}$ of order $d-j$, and T' is an operator of the form

$$(T'u)(x') = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{t}'(x',x_n,\xi')\acute{u}(\xi',x_n)\,dx_n\,đ\xi', \tag{10.24}$$

with $\tilde{t}' \in S^d_{1,0}(\mathbb{R}^{n-1},\mathbb{R}^{n-1},\mathscr{S}_+)$. $\tilde{t}'$ is called the *symbol-kernel* of T'. See also (10.25), (10.26), (10.27).

When $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$, $\|\acute{u}(\xi',x_n)\|_{L_2(\mathbb{R}_+)}$ is $O(\langle\xi'\rangle^{-N})$ for all N, so $T'u$ is C^∞ in x'.

Observe the meaning of the *class* number r; it counts the number of standard trace operators γ_j that enter into T. *Class zero* means that there are no such terms; we shall see that T is then well-defined on $L_2(\mathbb{R}^n_+)$.

Symbol-kernels depending on (x',y') instead of x' can also be allowed; then the defining formula (10.24) should be written

$$(T'u)(x') = \int_{\mathbb{R}^{2(n-1)}} e^{i(x'-y')\cdot\xi'} \int_0^\infty \tilde{t}'(x',y',y_n,\xi')u(y',y_n)\,dy\,đ\xi', \tag{10.25}$$

and an interpretation via oscillatory integrals is understood. To keep down the volume of the formulas, we mostly write x'-forms in the following, leaving some generalizations to (x',y')-form to the reader (also for the other operator types to come). Symbol-kernels in (x', y')-form are reduced to x'-form or y'-form by the formulas in Theorem 7.13 1°, applied in the primed variables.

Continuity properties of the operators will be systematically investigated in Section 10.5.

One can show (by use of the Seeley extension [S64]) that for any T' with symbol-kernel in $S^d_{1,0}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, \mathscr{S}_+)$, there exists a ψdo Q of $S_{1,0}$ type, satisfying the transmission condition at $x_n = 0$, such that $T' = \gamma_0 Q_+$.

The subclass of *polyhomogeneous* trace operators are those where the S_j are polyhomogeneous, and $\tilde{t}'$ lies in the subspace $S^d(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, \mathscr{S}_+)$. Note that, in contrast with polyhomogeneous ψdo's, the homogeneity in ξ of the symbol terms is only required for $|\xi'| \ge 1$, not $|\xi| \ge 1$.

The function (distribution when $r > 0$)

$$\tilde{t}(x', x_n, \xi') = \sum_{0 \le j < r} s_j(x', \xi') D^j_{x_n} \delta(x_n) + \tilde{t}'(x', x_n, \xi'), \tag{10.26}$$

understood as extended by 0 on $\mathbb{R}_-$ if needed, is called the *symbol-kernel* of T; its conjugate Fourier transform

$$t(x', \xi) = \overline{\mathscr{F}}_{x_n \to \xi_n} e^+ \tilde{t}(x, \xi') = \sum_{0 \le j < r} s_j(x', \xi') \xi_n^j + t'(x', \xi) \tag{10.27}$$

(where $t'(x', \xi) = \overline{\mathscr{F}}_{x_n \to \xi_n} e^+ \tilde{t}'(x, \xi')$) is the *symbol* of T.

In the polyhomogeneous case, we often denote t_d and $\tilde{t}_d$ by t^0 resp. $\tilde{t}^0$, the *principal* symbol and symbol-kernel.

Application of the operator definition with respect to only the x_n-variable gives the *boundary symbol operator* $t(x', \xi', D_n)$ (resp. *principal* boundary symbol operator $t^0(x', \xi, D_n)$) from $\mathscr{S}(\overline{\mathbb{R}}_+)$ to $\mathbb{C}$,

$$t(x', \xi', D_n) u = \sum_{0 \le j < r} s_j(x', \xi') \gamma_j u + \int_0^\infty \tilde{t}'(x', x_n, \xi') u(x_n) dx_n; \tag{10.28}$$

it is also denoted $\mathrm{OPT}_n(t)$ or $\mathrm{OPT}_n(\tilde{t})$. We can then write

$$\begin{aligned} Tu &= \mathrm{OP}'(t(x', \xi', D_n))u = \mathrm{OP}' \,\mathrm{OPT}_n(t(x', \xi))u, \text{ also denoted} \\ &\quad \mathrm{OPT}(t(x', \xi))u \text{ or } \mathrm{OPT}(\tilde{t}(x, \xi'))u. \end{aligned}$$

Poisson operators

We use the same symbol-kernel spaces from Definition 10.3 1° and 2° to define Poisson operators, but now doing a multiplication instead of taking a scalar product in the x_n-variable.

Definition 10.5. A Poisson operator of order d is an operator defined by a formula

$$(Kv)(x', x_n) = \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \tilde{k}(x', x_n, \xi') \hat{v}(\xi') \,đ\xi' \tag{10.29}$$

where the *symbol-kernel* $\tilde{k}$ belongs to $S^{d-1}_{1,0}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, \mathscr{S}_+)$. See also (10.30), (10.31).

Again, symbol-kernels depending on (x', y') can be allowed:

$$(Kv)(x', x_n) = \int_{\mathbb{R}^{2(n-1)}} e^{i(x'-y')\cdot\xi'} \tilde{k}(x', y', x_n, \xi') v(y')\, dy' đ\xi'. \tag{10.30}$$

The *symbol* corresponding to $\tilde{k}(x, \xi')$ is

$$k(x', \xi) = \mathscr{F}_{x_n \to \xi_n} e^+ \tilde{k}(x, \xi'). \tag{10.31}$$

In the polyhomogeneous case, it has an expansion in homogeneous terms in ξ (for $|\xi'| \geq 1$) of degree $d - 1 - l$. In this case we often denote $\tilde{k}_{d-1} = \tilde{k}^0$ and $k_{d-1} = k^0$, the *principal* symbol-kernel or symbol. Again, one can view K defined in (10.29) (also denoted $\mathrm{OPK}(k)$ or $\mathrm{OPK}(\tilde{k})$) as an operator $K = \mathrm{OP}'(k(x', \xi', D_n))$, where $k(x', \xi', D_n)$ is the *boundary symbol operator* from $\mathbb{C}$ to $\mathscr{S}(\overline{\mathbb{R}}_+)$:

$$k(x', \xi', D_n)a = \tilde{k}(x', x_n, \xi') \cdot a \text{ for } a \in \mathbb{C}, \tag{10.32}$$

also denoted $\mathrm{OPK}_n(k)$ or $\mathrm{OPK}_n(\tilde{k})$.

Remark 10.6. The above order convention, introduced originally in [B71], may seem a bit strange: Polyhomogeneous Poisson operators of order d have principal symbols homogeneous of degree $d - 1$. But the convention will fit the purpose that the composition of two operators of order d resp. d' will be of order $d + d'$ (valid, e.g., for the ψdo TK on $\mathbb{R}^{n-1}$).

The trace operators T' of class 0 (and order d) have as adjoints precisely the Poisson operators (of order $d+1$), and vice versa. This is obvious on the boundary-symbol-operator level:

$$t(D_n) : u \in L_2(\mathbb{R}_+) \mapsto (u, \overline{\tilde{f}}) \in \mathbb{C} \text{ and } k(D_n) : v \in \mathbb{C} \mapsto v \cdot \overline{\tilde{f}} \in L_2(\mathbb{R}_+)$$

are adjoints of one another. On the full operator level it is easy to show for symbols depending on (x', y') instead of x'; here if T' has symbol-kernel $\tilde{f}(x', y', x_n, \xi')$, the Poisson operator T'^* has symbol-kernel $\overline{\tilde{f}}(y', x', x_n, \xi')$. Details will be given in Theorem 10.29 later.

Trace operators of class $r > 0$ do not have adjoints within the calculus.

Example 10.7. The operator K_γ introduced in Theorem 9.3 is the Poisson operator with symbol-kernel $\tilde{k}(x_n, \xi') = e^{-\langle \xi' \rangle x_n}$; it is of order 0. Its symbol is

$$k(\xi', \xi_n) = \frac{1}{\langle \xi' \rangle + i\xi_n}, \tag{10.33}$$

cf. Exercise 5.3. Inserting the expansion (7.11) of $\langle \xi' \rangle$ in (10.33) one can expand in homogeneous terms of falling degree (beginning with degree -1), showing that the symbol and symbol-kernel are polyhomogeneous of degree -1.

The adjoint of K_γ is the trace operator T of class 0 with symbol-kernel $\tilde{t}(x_n, \xi') = e^{-\langle \xi' \rangle x_n}$ and symbol $t(\xi', \xi_n) = \frac{1}{\langle \xi' \rangle - i\xi_n}$; it is of degree and order -1. Furthemore, a calculation shows that for $Q = \mathrm{OP}(\langle \xi \rangle^{-2})$ as in (9.36)ff., $\gamma_0 Q_+$ is the trace operator with symbol-kernel $\frac{1}{2\langle \xi' \rangle} e^{-\langle \xi' \rangle x_n}$.

Poisson operators also arise from the following situation: Let $v(x') \in \mathscr{S}(\mathbb{R}^{n-1})$, and consider the distribution $v(x') \otimes \delta(x_n)$ (the product of $v(x')$ and $\delta(x_n)$). When P is a ψdo satisfying the transmission condition, one can show that $r^+ P(v \otimes \delta)$ makes sense as a function in $C^\infty(\overline{\mathbb{R}}^n_+)$, and the mapping $K : v \mapsto r^+ P(v \otimes \delta)$ is a Poisson operator. See Theorem 10.25 later.

Singular Green operators

We now get to the most unfamiliar element G of $\mathcal{A}$ in (10.18). A singular Green operator (s.g.o.) G arises, for instance, when we compose a Poisson operator K with a trace operator T as $G = KT$; this operator acts in $\mathbb{R}^n_+$ but is not a P_+. Another situation where s.g.o.s enter is when we compose two ψdo's P_+ and Q_+ (satisfying the transmission condition); then the " leftover operator"

$$\begin{aligned} L(P, Q) \equiv (PQ)_+ - P_+ Q_+ &= r^+ PQe^+ - r^+ Pe^+ r^+ Qe^+ \\ &= r^+ P(I - e^+ r^+) Qe^+ \end{aligned} \tag{10.34}$$

is an operator acting in $\mathbb{R}^n_+$, that is not a ψdo (more about $L(P, Q)$ in Section 10.4). It turns out that these cases are covered by operators of the following form (they are in fact convergent series of products of Poisson and trace operators, cf. (10.107) later):

Definition 10.8. A singular Green operator G of order d ($\in \mathbb{R}$) and class r ($\in \mathbb{N}_0$) is an operator

$$G = \sum_{0 \le j \le r-1} K_j \gamma_j + G', \tag{10.35}$$

where the K_j are Poisson operators of order $d - j$, the γ_j are standard trace operators and G' is an operator of the form

$$(G'u)(x) = \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \int_0^\infty \tilde{g}'(x', x_n, y_n, \xi') \acute{u}(\xi', y_n) dy_n đ\xi', \tag{10.36}$$

where $\tilde{g}'$, the *symbol-kernel* of G', is in $S^{d-1}_{1,0}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, \mathscr{S}_{++})$, cf. Definition 10.3 3°. There is a corresponding *symbol* g', defined by

$$g'(x', \xi', \xi_n, \eta_n) = \mathscr{F}_{x_n \to \xi_n} \overline{\mathscr{F}}_{y_n \to \xi_n} e^+_{x_n} e^+_{y_n} \tilde{g}'(x', x_n, y_n, \xi'). \tag{10.37}$$

The symbol-kernel and symbol of G itself are

$$\begin{aligned}\tilde g(x',x_n,y_n,\xi') &= \sum_{0\le j<r} \tilde k_j(x',x_n,\xi')D^j_{y_n}\delta(y_n) + \tilde g'(x',x_n,y_n,\xi'),\\ g(x',\xi',\xi_n,\eta_n) &= \sum_{0\le j<r} k_j(x',\xi)\eta_n^j + g'(x',\xi',\xi_n,\eta_n). \end{aligned} \tag{10.38}$$

In the polyhomogeneous case, the *principal* symbol-kernel and symbol are $\tilde g^0 = \tilde g_{d-1}$ resp. $g^0 = g_{d-1}$. (Both for singular Green symbols and for trace symbols, the definition can be refined further to allow the notion of negative class $r<0$, see [G96, Sect. 2.8].) In some recent works, it has been practical to replace the enumeration $d-1-l$ by $d-l$, but we here stick to the notation of [G96].

We define the *boundary symbol operator* $g(x',\xi',D_n)$ from $\tilde g$ by

$$g(x',\xi',D_n)u(x_n) = \sum_{0\le j<r} \tilde k_j(x',x_n,\xi')\gamma_j u + \int_0^\infty \tilde g'(x',x_n,y_n,\xi')u(y_n)dy_n, \tag{10.39}$$

also called $\mathrm{OPG}_n(g)$ or $\mathrm{OPG}_n(\tilde g)$; then G (also called $\mathrm{OPG}(g)$ or $\mathrm{OPG}(\tilde g)$) can be viewed as $G = \mathrm{OP}'(g(x',\xi',D_n)) = \mathrm{OP}'\,\mathrm{OPG}_n(g)$.

Example 10.9. As a simple example of a singular Green symbol-kernel, let us take $\tilde g(x_n,y_n,\xi') = e^{-\langle\xi'\rangle(x_n+y_n)}$. Its symbol is

$$g(\xi',\xi_n,\eta_n) = \frac{1}{(\langle\xi'\rangle + i\xi_n)(\langle\xi'\rangle - i\eta_n)},$$

it is of degree -2 and order -1.

In view of the last remark in Example 10.7, $-\frac{1}{2\langle\xi'\rangle}\tilde g$ is the symbol-kernel of the last term $-K_\gamma\gamma_0 Q_+$ in (9.38); it is the singular Green operator part of the solution operator for the Dirichlet problem considered there.

Singular Green operators of class 0 have *adjoints* of the same kind: Allowing symbols depending on (x',y') we have that

$$\begin{aligned} G &= \mathrm{OPG}(\tilde g(x',y',x_n,y_n,\xi')) \quad \text{implies}\\ G^* &= \mathrm{OPG}(\overline{\tilde g}(y',x',y_n,x_n,\xi')), \end{aligned} \tag{10.40}$$

when G is of class 0. More on this in Remark 10.35. Singular Green operators of class $r>0$ do not have adjoints within the calculus.

Negligible operators

To the above operators defined by Fourier integral formulas, one adds the *negligible operators of each type*, defined as operators of the form (10.23), (10.29), (10.35) with S_j, T', K, K_j and G' replaced by integral operators with C^∞-kernels (up to the boundary) over the respective domains:

$$S_j\gamma_j u = \int_{\mathbb{R}^{n-1}} \mathcal{K}_{S_j}(x',y')(\gamma_j u)(y')\,dy', \quad T'u = \int_{\mathbb{R}^n} \mathcal{K}_{T'}(x',y)u(y)\,dy,$$
$$Kv = \int_{\mathbb{R}^{n-1}} \mathcal{K}_K(x,y')v(y')\,dy', \quad G'u = \int_{\mathbb{R}^n} \mathcal{K}_{G'}(x,y)u(y)\,dy, \tag{10.41}$$

with $\mathcal{K}_{S_j} \in C^\infty(\mathbb{R}^{2n-2})$, $\mathcal{K}_{T'} \in C^\infty(\mathbb{R}^{n-1}\times\overline{\mathbb{R}}^n_+)$, $\mathcal{K}_K \in C^\infty(\overline{\mathbb{R}}^n_+\times\mathbb{R}^{n-1})$, $\mathcal{K}_{G'} \in C^\infty(\overline{\mathbb{R}}^n_+\times\overline{\mathbb{R}}^n_+)$. They are of class r when they contain trace operators γ_j for $j \le r-1$. It will be seen from the mapping properties that we show in detail in Section 10.5 that the negligible operators include the operators defined above with symbol-kernel of order $-\infty$ in (x',y')-form. However, the kernels of these are decreasing in x_n and y_n, and we observe that more has been included now: r^+Pe^+ is included as a negligible G when P is negligible.

We shall show the mapping property indicated in (10.18) as well as mapping properties in Sobolev spaces in Section 10.5 below.

The various operator classes defined above are *invariant* under coordinate changes in $\overline{\mathbb{R}}^n_+$ preserving the boundary $\{x_n = 0\}$; this holds both for the polyhomogeneous classes and the $S_{1,0}$ classes. This is stated in [B71], with an indication of how to conclude the invariance for $S_{1,0}$ Poisson operators once it is shown for $S_{1,0}$ ψdo's satisfying the transmission condition. [RS82] proves the invariance under coordinate changes in x' alone, where the rules for ψdo's in x' apply. A complete proof, with formulas for the symbols of the transformed operators, is found in [G96, Sect. 2.4 and Th. 2.2.13], covering also parameter-dependent symbols.

Thanks to the invariance, one can also define the operators as acting in vector bundles over manifolds, by use of local coordinates. The book [G96] allows noncompact manifolds, but we at present just consider the compact case: X is an n-dimensional compact C^∞-manifold with boundary $\partial X = X'$, smoothly embedded in a neighboring n-dimensional manifold $\widetilde{X}$, and $\widetilde{E}$, $E = \widetilde{E}\big|_X$, $\widetilde{E}'$ and $E' = \widetilde{E}'\big|_X$, resp. F and F', are vector bundles of dimension N and N', resp. M and M', over $\widetilde{X}$ and X, resp. X' (described by local coordinates and trivializations). P is a ψdo in $\widetilde{E}$ over $\widetilde{X}$ satisfying the transmission condition at X' and the ψdbo's considered in connection with P are of the form

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(E) \\ \times \\ C^\infty(F) \end{matrix} \to \begin{matrix} C^\infty(E') \\ \times \\ C^\infty(F') \end{matrix}; \tag{10.42}$$

here $P_+ = r_{X^\circ}Pe_{X^\circ}$ (defined similarly to (10.1) for $X^\circ \subset \widetilde{X}$). All the operators are assumed to be such that they in local trivializations act in the way we have described above for $\mathbb{R}^n_+$. The terms T, K and S (and sometimes even $P_+ + G$) are often given as block matrices with different orders for different entries (fitting together in a suitable way); see the remarks on multi-order systems around (11.13).

Having defined the ingredients in $\mathcal{A}$ in (10.18), we shall now look at composition rules. When $\mathcal{A}'$ is another such system, going from $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)^{N'} \times C^\infty_0(\mathbb{R}^{n-1})^{M'}$ to $C^\infty(\overline{\mathbb{R}}^n_+)^{N''} \times C^\infty(\mathbb{R}^{n-1})^{M''}$, and one of the operators is properly supported (with a suitable generalization of Definition 7.6), the composition may be written

$$\mathcal{A}'' = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} \begin{pmatrix} P'_+ + G' & K' \\ T' & S' \end{pmatrix} = \begin{pmatrix} P''_+ - L + G''' & K'' \\ T'' & S'' \end{pmatrix}. \tag{10.43}$$

The point is to show that $\mathcal{A}''$ again has the structure of a ψdbo, which really amounts to showing 14 different composition rules:

$$\begin{aligned}
&\text{(i)} && P'' = PP' \text{ is a } \psi\text{do with transm. cond.,} \\
&\text{(ii)} && L(P,P') = (PP')_+ - P_+P'_+ \text{ is an s.g.o.,} \\
&\text{(iii)} && G''' = P_+G' + GP'_+ + GG' + KT' \text{ is an s.g.o.,} \\
&\text{(iv)} && T'' = TP'_+ + TG' + ST' \quad \text{is a trace operator,} \\
&\text{(v)} && K'' = P_+K' + GK' + KS' \quad \text{is a Poisson operator,} \\
&\text{(vi)} && S'' = TK' + SS' \quad \text{is a } \psi\text{do on } \mathbb{R}^{n-1}.
\end{aligned} \tag{10.44}$$

We observe right away that the first rule (i) is easy to check from Definition 10.2, in view of the standard composition rule Theorem 7.13 for ψdo's (recall that we only consider P's of integer order). For the other rules, there will be some information in Proposition 10.10 below and a full treatment in Section 10.4. In this sense, the ψdbo's $\mathcal{A}$ form an "algebra".

It is also of interest to see whether $\mathcal{A}$ has an adjoint within the ψdbo calculus. In view of the preceding information, this occurs for general K, and for G and T when they are of class 0. For P_+, the adjoint $(P_+)^*$ equals $(P^*)_+$ when P is of order ≤ 0. More on adjoints in Theorem 10.29 and Remark 10.35.

There is a technical device worth mentioning here, which can transform the system $\mathcal{A}$ into one that does have an adjoint, namely, that there exists a family of ψdo's Λ^r_- of orders $r \in \mathbb{Z}$, such that $(\Lambda^r_-)_+$ maps $H^s(\mathbb{R}^n_+)$ homeomorphically onto $H^{s-r}(\mathbb{R}^n_+)$ for all s. They are called order-reducing operators. (See Exercise 10.11.)

Now some details on compositions. The new operators are introduced in such a way that they have a special definition with respect to the x_n-variable (defined by OPT_n, OPK_n, OPG_n), whereas the definition with respect to the x'-variable is the standard ψdo definition OP'. In compositions, the new thing to deal with is therefore just what happens in the x_n-direction, whereas the rules in the x'-direction are as in Chapter 7. So let us now study x_n-compositions (denoted $\circ_n$ for the symbols).

From the real formulation given above we have easily:

Proposition 10.10. *Consider a ψdo symbol $p(x',0,\xi',\xi_n)$ and trace, Poisson and singular Green symbol-kernels $\tilde t(x',x_n,\xi')$, $\tilde k(x',x_n,\xi')$, $\tilde g(x',x_n,y_n,\xi')$ of order d, and $\tilde t'(x',x_n,\xi')$, $\tilde k'(x',x_n,\xi')$, $\tilde g'(x',x_n,y_n,\xi')$ of order d', all of class 0. Let $d''=d+d'$. We have the rules:*

(i) *If p is $O(\langle\xi_n\rangle^{-1})$, then $\gamma_0\operatorname{OP}_n(p)_+=\operatorname{OPT}_n(\tilde t'')$, where $\tilde t''(x',x_n,\xi')=\tilde p(x',0,\xi',-x_n)|_{x_n>0}$, a trace symbol-kernel of order d and class 0.*

(ii) *$\gamma_0\operatorname{OPK}_n(\tilde k)=s''$, where $s''(x',\xi')=\tilde k(x',0,\xi')$, a ψdo symbol of order d.*

(iii) *$\gamma_0\operatorname{OPG}_n(\tilde g)=\operatorname{OPT}_n(\tilde t'')$, where $\tilde t''(x',y_n,\xi')=\tilde g(x',0,y_n,\xi')$, a trace symbol-kernel of order d and class 0.*

(iv) *$\operatorname{OPK}_n(\tilde k)\operatorname{OPT}_n(\tilde t')=\operatorname{OPG}_n(\tilde g'')$, where*

$$\tilde g''(x',x_n,y_n,\xi')=\tilde k(x',x_n,\xi')\tilde t'(x',y_n,\xi'),$$

an s.g.o. symbol-kernel of order d'' and class 0.

(v) *$\operatorname{OPT}_n(\tilde t)\operatorname{OPK}_n(\tilde k')=s''$, a ψdo symbol of order d'' and defined by*

$$s''(x',\xi')=\int_0^\infty \tilde t(x',x_n,\xi')\tilde k'(x',x_n,\xi')\,dx_n.$$

(vi) *$\operatorname{OPG}_n(\tilde g)\operatorname{OPK}_n(\tilde k')=\operatorname{OPK}_n(\tilde k'')$, of order d'' and defined by*

$$\tilde k''(x',x_n,\xi')=\int_0^\infty \tilde g(x',x_n,y_n,\xi')\tilde k'(x',y_n,\xi')\,dy_n.$$

(vii) *$\operatorname{OPT}_n(\tilde t)\operatorname{OPG}_n(\tilde g')=\operatorname{OPT}_n(\tilde t'')$, of order d'' and class 0, defined by*

$$\tilde t''(x',y_n,\xi')=\int_0^\infty \tilde t(x',x_n,\xi')\tilde g'(x',x_n,y_n,\xi')\,dx_n.$$

(viii) *$\operatorname{OPG}_n(\tilde g)\operatorname{OPG}_n(\tilde g')=\operatorname{OPG}_n(\tilde g'')$, of order d'' and class 0, defined by*

$$\tilde g''(x',x_n,y_n,\xi')=\int_0^\infty \tilde g(x',x_n,z_n,\xi')\tilde g''(x',z_n,y_n,\xi')\,dz_n.$$

(ix) *One has for all $l,m\in\mathbb N_0$, with resulting operators of order $d-l+m$:*

$$x_n^lD_{x_n}^m\operatorname{OPK}_n(\tilde k(x',x_n,\xi'))\varphi=\operatorname{OPK}_n(x_n^lD_{x_n}^m\tilde k(x',x_n,\xi'))\varphi,$$
$$x_n^lD_{x_n}^m\operatorname{OPG}_n(\tilde g(x',x_n,y_n,\xi'))u=\operatorname{OPG}_n(x_n^lD_{x_n}^m\tilde g(x',x_n,y_n,\xi'))u.$$

Proof. The first rule follows from (10.22), in view of the estimates (10.11). For the second rule, we have in view of (10.32) that for $a\in\mathbb C$,

$$\gamma_0\operatorname{OPK}_n(k)a=\gamma_0\tilde k(x',x_n,\xi')a=s''(x',\xi')a$$

with $s''(x',\xi')=\tilde k(x',0,\xi')$. The third rule follows similarly. The other rules are likewise verified immediately from the defining formulas. The symbol-kernel estimates for (v)–(viii) are shown using the Cauchy-Schwarz inequality; the detailed proofs can be left as an exercise for the reader (Exercise 10.4). The rules in (ix) follow from the definitions of the symbol-kernel spaces. □

As an important corollary we observe that the "singular" ingredients in $\mathcal{A}$ — the operators G, T and K — are negligible at a distance from the boundary:

Proposition 10.11. *Let $\zeta \in C^\infty(\overline{\mathbb{R}}^n_+)$ be such that $\zeta(x) = 0$ for $x_n \le \varepsilon$, some $\varepsilon > 0$ (e.g., $\zeta(x) = 1 - \chi(x_n/\varepsilon)$ on $\overline{\mathbb{R}}^n_+$). Let G, T and K be operators as in* (10.18) *of order $d \in \mathbb{R}$, and class $r \ge 0$ when relevant. Then*
1° ζG is negligible of class r, $G\zeta$ is negligible of class 0.
2° ζK is negligible, $T\zeta$ is negligible of class 0.

Proof. For any $N \in \mathbb{N}_0$, $\zeta_N(x) = \zeta(x)/x_n^N$ is in $C^\infty(\overline{\mathbb{R}}^n_+)$, supported in $\{x_n \ge \varepsilon\}$. Then

$$\zeta K = \zeta_N x_n^N K, \quad \zeta G = \zeta_N x_n^N G,$$

where it is seen from Proposition 10.10 (ix) that $x_n^N K$ and $x_n^N G$ are of order $d - N$ (the class of G remains the same), hence so are $\zeta_N x_n^N K$ and $\zeta_N x_n^N G$. Since N can be arbitrarily large, the orders are $-\infty$, so the operators are negligible.

For $T = \sum_{j<r} S_j \gamma_j + T'$, $T\zeta$ equals $T'\zeta$, of class 0. This is the adjoint of the Poisson operator $\bar{\zeta} T'^*$, to which the preceding result applies; so it is negligible. There is a similar proof for $G\zeta$. □

The rules in Proposition 10.10 are quite straightforward and unsophisticated. However, when we get to compositions of $\mathrm{OP}_n(p)_+$ with the general boundary operators, the real formulation requires convolutions of $\tilde{p}'$, in a suitably truncated version, with the other symbol-kernels. Here the usual ψdo experience tells us that it should be an advantage to work with *symbols*, after Fourier transformation from x_n to ξ_n, where convolutions are replaced by products. But then the cut-off operator $e^+ r^+$ must be replaced by its effect in the ξ_n-variable, and here the transmission condition will be important.

There is a little piece of function theory that takes care of all this, which we shall now explain.

10.2 Fourier transform and Laguerre expansion of $\mathscr{S}_+$

The treatment of ψdo symbols satisfying the transmission condition (10.5) takes place in the following spaces of functions of a real variable t (playing the role of ξ_n).

Definition 10.12. For each integer $d \in \mathbb{Z}$, the space $\mathcal{H}_d$ is defined as the space of C^∞-functions $f(t)$ on $\mathbb{R}$ with the asymptotic property: There exist complex numbers $s_d, s_{d-1}, \dots$ such that for all indices k, l and $N \in \mathbb{N}_0$,

$$\partial_t^l[t^k f(t) - \sum_{d-N\le j\le d} s_j t^{j+k}] \text{ is } O(|t|^{d-N-1+k-l}) \text{ for } |t|\to\infty. \tag{10.45}$$

Clearly, the s_j are uniquely determined from f. We denote

$$\mathcal{H} = \bigcup_{d\in\mathbb{Z}} \mathcal{H}_d \tag{10.46}$$

and observe also the decomposition in a direct sum

$$\mathcal{H} = \mathcal{H}_{-1} \dot{+} \mathbb{C}[t], \tag{10.47}$$

where $\mathbb{C}[t]$ is the space of polynomials in t. The corresponding projection of $\mathcal{H}$ onto $\mathcal{H}_{-1}$ is denoted h_{-1}, and $(I-h_{-1})f$ is called the polynomial part of f. Occasionally, we also use the projector h_0 of $\mathcal{H}$ onto $\mathcal{H}_0$, which removes $\sum_{1\le j\le d} s_j t^j$ from f. The spaces $\mathcal{H}_d$ have Fréchet topologies (defined by families of seminorms in relation to (10.45)), and $\mathcal{H}$ is an inductive limit of such spaces.

Lemma 10.13. *Let $\sigma>0$ and let $d\in\mathbb{Z}$. Let $f(t)\in C^\infty(\mathbb{R})$, and define*

$$\begin{aligned} &\tau = t^{-1}\,,\ k(\tau) = \tau^d f(\tau^{-1}) \ \text{ for } \tau\in\mathbb{R}\setminus\{0\}\,;\\ &z = \frac{\sigma - it}{\sigma + it} \quad (\text{hence } t = \frac{\sigma}{i}\frac{1-z}{1+z}\,,\ 1+z = \frac{2\sigma}{\sigma+it})\,,\\ &g(z) = (1+z)^d f(\frac{\sigma}{i}\frac{1-z}{1+z}) \ \text{ for } z\in S^1 = \{z\in\mathbb{C}\mid |z|=1\}\,,\ z\ne -1\,. \end{aligned} \tag{10.48}$$

The following statements (i)–(iii) *are equivalent:*

(i) $f\in\mathcal{H}_d$.
(ii) k *extends to a function in* $C^\infty(\mathbb{R})$.
(iii) g *extends to a function in* $C^\infty(S^1)$.

Proof. Consider first the case $d=0$. Assume that f satisfies the conditions (10.45) (which then also hold with $\partial_t^l t^k$ replaced by $t^k\partial_t^l$). Since $f(t)-s_0$ is $O(t^{-1})$ for $t\to\pm\infty$, $k(\tau)$ extends to a continuous function k on $\mathbb{R}$ (it is the point $\tau=0$ that needs checking). Now

$$\partial_\tau k(\tau) = -t^2\partial_t f(t)\big|_{t=\tau^{-1}}\,, \tag{10.49}$$

so since $-t^2\partial_t(f(t)-s_0-s_{-1}t^{-1}) = -t^2\partial_t f(t) - s_{-1}$ is $O(t^{-1})$ for $t\to\pm\infty$, $\partial_\tau k(\tau) - s_{-1}$ is $O(\tau)$ for $\tau\to 0\pm$. Similarly, for each m,

$$(-t^2\partial_t)^m(f(t) - \sum_{0\le l\le m} s_{-l}t^{-l}) = (-t^2\partial_t)^m f(t) - m!s_{-m} \tag{10.50}$$

is $O(t^{-1})$ for $t\to\pm\infty$, showing that $\partial_\tau^m k(\tau) - m!s_{-m}$ is $O(\tau)$ for $\tau\to 0\pm$. Thus (i) implies (ii) with

$$\partial_\tau^m k(0) = m!s_{-m} \ \text{ for } m\in\mathbb{N}_0. \tag{10.51}$$

Conversely, the formulas (10.49), (10.50) allow us to conclude from (ii) to (i) with the coefficients s_{-m} determined successively from (10.51) (the particular family of estimates $(-t^2\partial_t)^m(f(t) - \sum_{-m\le j\le 0} s_j t^j) = O(t^{-1})$, $m \in \mathbb{N}_0$, implies the full family (10.45)).

For the transition between (ii) and (iii) we now just observe that

$$1 + z = 1 + \frac{\sigma - i/\tau}{\sigma + i/\tau} = \frac{2\sigma\tau}{\sigma\tau + i};$$

so smoothness of k at $\tau = 0$ is equivalent with smoothness of g at $z = -1$.

When $d < 0$, one reduces to the case $d = 0$ by replacing $f(t)$ by $f^*(t) = t^{-d} f(t)$, corresponding to the same $k(\tau)$ and to $g^*(z) = (i/\sigma(1-z))^d g(z)$; and when $d > 0$, one makes a slight generalization to the above proof. □

Note that the coefficients s_j in (10.45) are proportional to the Taylor coefficients of $k(\tau)$ at $\tau = 0$; in this sense, they are "Taylor coefficients of f at ∞".

The function $k(\tau)$ here is just an auxiliary function, whereas $g(z)$ has a particular interest, in an analysis of $\mathcal{H}_{-1}$ that we shall now describe. Let $f(t) \in \mathcal{H}_{-1}$, let $g(z) = (1+z)^{-1} f(-i\sigma(1-z)/(1+z))$ (by (10.48) with $d = -1$), and consider its Fourier series expansion (for $z = e^{i\theta}$), with the convention

$$\begin{aligned} &g(z) = (2\sigma)^{-\frac12} \sum_{k\in\mathbb{Z}} b_k z^k, \quad \text{decomposed in} \\ &g^+(z) = (2\sigma)^{-\frac12} \sum_{k\ge 0} b_k z^k \quad \text{and} \quad g^-(z) = (2\sigma)^{-\frac12} \sum_{k<0} b_k z^k. \end{aligned} \tag{10.52}$$

This decomposition gives rise to a decomposition of $\mathcal{H}_{-1}$,

$$\mathcal{H}_{-1} = \mathcal{H}^+_{-1} \dotplus \mathcal{H}^-_{-1}, \tag{10.53}$$

where f is decomposed in the sum of $f^\pm(t) \in \mathcal{H}^\pm_{-1}$ corresponding to the functions $g^\pm(z)$ as in (10.48). We shall analyze the spaces $\mathcal{H}^\pm_{-1}$, showing in particular that they are the Fourier transforms of the spaces $e^\pm \mathscr{S}(\overline{\mathbb{R}}_\pm)$. This will be done in an elementary way based on orthogonal expansions.

We know from the theory of trigonometric series that the function g on the circle $\{|z| = 1\}$ is C^∞ if and only if the sequence $(b_k)_{k\in\mathbb{Z}}$ is *rapidly decreasing* for $|k| \to \infty$, i.e., the sequences $(k^N b_k)_{k\in\mathbb{Z}}$ are bounded, for all N. Equivalently, the series $\sum_{k\in\mathbb{Z}} |(1+|k|)^N b_k|^2$ are convergent for all N. The space of rapidly decreasing sequences (b_k) will be denoted by $\mathfrak{s}(\mathbb{Z})$, and we shall use

$$\|(b_k)_{k\in\mathbb{Z}}\|_{\ell_2^N} = (\sum_{k\in\mathbb{Z}} |\,(1+|k|)^N b_k|^2)^{1/2}, \tag{10.54}$$

as a norm on the Hilbert space $\ell_2^N(\mathbb{Z})$; it is equivalent with the norm (8.11). (One can replace $\mathbb{Z}$ by $\mathbb{N}_0$ or other index sets.) So $\mathfrak{s}(\mathbb{Z}) = \bigcap_{N\ge 0} \ell_2^N(\mathbb{Z})$.

For $f(t) = (1+z)g(z)$, the expansion (10.52) of g gives an expansion of f in terms of the functions

$$\widehat{\varphi}_k(t,\sigma) = (2\sigma)^{\frac{1}{2}} \frac{(\sigma - it)^k}{(\sigma + it)^{k+1}}, \text{ corresponding to } (2\sigma)^{-\frac{1}{2}}(1+z)z^k, \quad (10.55)$$

cf. (10.48). They are easily checked to be *orthogonal in* $L_2(\mathbb{R})$ (with norms $(2\pi)^{\frac{1}{2}}$); and the completeness of the trigonometric system $(z^k)_{k\in\mathbb{Z}}$ implies the completeness of the system $(\widehat{\varphi}_k)_{k\in\mathbb{Z}}$.

Now the inverse Fourier transform carries the $\widehat{\varphi}_k(t,\sigma)$ over to the functions $\varphi_k(x,\sigma)$ defined by

$$\begin{aligned} \varphi_k(x,\sigma) &= \begin{cases} (2\sigma)^{\frac{1}{2}}(\sigma - \partial_x)^k (x^k e^{-x\sigma})/k! & \text{for } k \geq 0,\ x \geq 0, \\ 0 & \text{for } k \geq 0,\ x < 0; \end{cases} \\ \varphi_k(x,\sigma) &= \varphi_{-k-1}(-x,\sigma) \text{ for } k < 0; \end{aligned} \quad (10.56)$$

they are a variant of the *Laguerre functions*. By the Parseval-Plancherel theorem, the functions $(\varphi_k)_{k\in\mathbb{Z}}$ form a complete orthonormal system in $L_2(\mathbb{R})$, and hence the functions with $k \geq 0$, resp. $k < 0$, span $L_2(\mathbb{R}_+)$, resp. $L_2(\mathbb{R}_-)$. (We often write φ_k for $r^+\varphi_k$ when $k \geq 0$, and φ_k for $r^-\varphi_k$ when $k < 0$.)

One can check that the φ_k with $k \geq 0$ are the eigenfunctions of the (variant of a) Laguerre operator

$$\mathcal{L}_{\sigma,+} = \sigma^{-1}(\sigma + \partial_x)x(\sigma - \partial_x) = -\sigma^{-1}\partial_x x \partial_x + \sigma x + 1 \quad (10.57)$$

in $L_2(\mathbb{R}_+)$, with simple eigenvalues $2(k+1)$; and the φ_k with $k < 0$ are similarly the eigenfunctions for $\mathcal{L}_{\sigma,-}$ defined by the same expression on $\mathbb{R}_-$.

A property of expansions in the Laguerre system that is of particular interest here is the fact that rapidly decreasing coefficient series correspond to functions in $\mathscr{S}(\overline{\mathbb{R}}_+)$.

Lemma 10.14. *Let $u \in L_2(\mathbb{R}_+)$, expanded in the Laguerre system $(\varphi_k)_{k\in\mathbb{N}_0}$, by*

$$u(x) = \sum_{k\in\mathbb{N}_0} b_k \varphi_k(x,\sigma).$$

Then $u \in \mathscr{S}(\overline{\mathbb{R}}_+)$ if and only if $(b_k)_{k\in\mathbb{N}_0}$ is rapidly decreasing. More precisely, one has the identity

$$\|u\|_{L_2(\mathbb{R}_+)} = \|(b_k(u))_{k\in\mathbb{N}_0}\|_{\ell_2^0}, \quad (10.58)$$

and there are estimates of $(b_k)_{k\in\mathbb{N}_0}$ in terms of u:

$$\begin{aligned} \|(b_k(u))_{k\in\mathbb{N}_0}\|_{\ell_2^N} &= 2^{-N}\|(b_k(\mathcal{L}_{\sigma,+}^N u))\|_{\ell_2} \\ &= 2^{-N}\|\mathcal{L}_{\sigma,+}^N u\|_{L_2} \leq c_N \max_{j+l\leq N} \sigma^{j-l}\|x^{j+l}\partial_x{}^{2l} u\|_{L_2}, \end{aligned} \quad (10.59)$$

and estimates of u in terms of $(b_k)_{k\in\mathbb{N}_0}$ (with any $\varepsilon > 0$):

$$\|x^j\partial_x^l u\|_{L_2} \le c_\varepsilon\, \sigma^{-j+l}\|(b_k(u))\|_{\ell_2^{j+(1+\varepsilon)l}}. \tag{10.60}$$

Proof. The identity (10.58) follows from the orthonormality and completeness of the system φ_k in $L_2(\mathbb{R}_+)$. (10.59) then follows easily from the eigenvalue property of the φ_k:

$$\begin{aligned}\|(b_k)_{k\in\mathbb{N}_0}\|^2_{\ell_2^N} &= \textstyle\sum_k |(1+k)^N b_k|^2 = 2^{-2N}\|\sum_k b_k\, \mathcal{L}^N_{\sigma,+}\varphi_k\|^2_{L_2}\\ &= 2^{-2N}\|\mathcal{L}^N_{\sigma,+}u\|^2_{L_2} = 2^{-2N}\|(-\sigma^{-1}\partial_x x\partial_x + \sigma x + 1)^N u\|^2_{L_2}\\ &\le c_N \max_{j+l\le N} \sigma^{2(j-l)}\|x^{j+l}\partial_x^{2l}u\|^2_{L_2},\end{aligned}$$

since $\partial_x(xu) = x\partial_x u + u$. For the estimates (10.60) one calculates the expansion coefficients of $xu(x)$ and $\partial_x u(x)$ in terms of those of u; details are found in [G96, Lemma 2.2.1]. We refer to the proof given there, and shall just quote the formulas that give the general idea:

$$\begin{aligned}x\varphi_k(x,\sigma) &= \tfrac{1}{2\sigma}(k\varphi_{k-1} + (2k+1)\varphi_k + (k+1)\varphi_{k+1})\\ \partial_x\varphi_k(x,\sigma) &= -\sigma\varphi_k + 2\sigma \textstyle\sum_{0\le j<k}(-1)^{k-1-j}\varphi_j + (-1)^k(2\sigma)^{\frac12}\delta,\end{aligned} \tag{10.61}$$

for $k \ge 0$ (easily proved using the $\hat\varphi_k$). □

For certain combinations of x^j and ∂_x^l there are better estimates than (10.60), see [G96, Lemma 2.2.1].

We now return to the decomposition (10.52), which has the counterpart for $f(t)$, in view of (10.55),

$$\begin{gathered} f(t) = f^+(t) + f^-(t), \text{ where}\\ f^+(t) = \sum_{k\ge 0} b_k\hat\varphi_k(t,\sigma) \text{ and } f^-(t) = \sum_{k<0} b_k\hat\varphi_k(t,\sigma);\end{gathered} \tag{10.62}$$

here the sequences $(b_k)_{k\ge 0}$ and $(b_k)_{k<0}$ can be arbitrary rapidly decreasing sequences. The hereby defined decomposition of the space $\mathcal{H}_{-1}$ is denoted $\mathcal{H}^+_{-1} \dotplus \mathcal{H}^-_{-1}$ as already stated in (10.53); we shall denote the corresponding projections h^+_{-1} resp. h^-_{-1}. Note that they are *orthogonal projections* with respect to $L_2(\mathbb{R})$-norm (since the $\hat\varphi_k$ are mutually orthogonal), so that

$$\|f^+\|_{L_2} \le \|f\|_{L_2};\quad \|f^-\|_{L_2} \le \|f\|_{L_2} \text{ for } f \in \mathcal{H}_{-1}. \tag{10.63}$$

Observe that

$$\overline{\hat\varphi}_k = \hat\varphi_{-k-1} \text{ for all } k, \tag{10.64}$$

which implies that

$$f \in \mathcal{H}^+_{-1} \iff \overline{f} \in \mathcal{H}^-_{-1}. \tag{10.65}$$

By Lemma 10.14 we now see that $\mathcal{H}^+_{-1}$ *is precisely the space of Fourier transforms of functions* $e^+u(x)$*, where* $u \in \mathscr{S}(\overline{\mathbb{R}}_+)$*,*

$$\mathcal{H}^+_{-1} = \mathscr{F}(e^+\mathscr{S}(\overline{\mathbb{R}}_+)). \tag{10.66}$$

Similarly (cf. (10.65))

$$\mathcal{H}^-_{-1} = \mathscr{F}(e^-\mathscr{S}(\overline{\mathbb{R}}_-)). \tag{10.67}$$

So, when $f = f^+ + f^-$ is such that $f^\pm = \mathscr{F}(e^\pm u^\pm)$, then the projections $h^\pm_{-1}$ in $\mathcal{H}_{-1}$ correspond to the projections $e^\pm r^\pm$ applied to $u = e^+u^+ + e^-u^- \in e^+\mathscr{S}(\overline{\mathbb{R}}_+)\dot{+}e^-\mathscr{S}(\overline{\mathbb{R}}_-)$.

The above analysis is concerned with $\mathcal{H}_{-1}$; for the complete description of $\mathcal{H}$ one has to adjoin $\mathbb{C}[t]$ (cf. (10.47)), and it is customary to define (with a slight asymmetry)

$$\begin{aligned} \mathcal{H}^+ &= \mathcal{H}^+_{-1}, \\ \mathcal{H}^- &= \mathcal{H}^-_{-1} \dot{+} \mathbb{C}[t]. \end{aligned} \tag{10.68}$$

Then $\mathcal{H} = \mathcal{H}^+\dot{+}\mathcal{H}^-$, and the corresponding projections are denoted h^+ and h^-, extending h^+_{-1} resp. h^-_{-1}.

Note that $\mathbb{C}[t]$ is the space of Fourier transforms of the "polynomials" $\sum c_k\delta^{(k)}$ where $\delta^{(k)} = D^k_x\delta$; we call the latter space $\mathbb{C}[\delta']$. Then the analysis can be summed up in the following statement.

Theorem 10.15. 1° *The space* $\mathcal{H} = \bigcup_{d\in\mathbb{Z}}\mathcal{H}_d$ *admits a decomposition in a direct sum*

$$\mathcal{H} = \mathcal{H}^+ \dot{+} \mathcal{H}^-, \tag{10.69}$$

with projections denoted h^+ *and* h^-*; moreover*

$$\mathcal{H}^- = \mathcal{H}^-_{-1} \dot{+} \mathbb{C}[t], \quad \mathcal{H}^+ = \mathcal{H}^+_{-1}.$$

The decompositions are defined in such a way that the space $\mathcal{H}$ *by inverse Fourier transformation is mapped onto the space*

$$\dot{\mathscr{S}}(\mathbb{R}) \equiv e^+\mathscr{S}(\overline{\mathbb{R}}_+) \dot{+} e^-\mathscr{S}(\overline{\mathbb{R}}_-) \dot{+} \mathbb{C}[\delta'], \textit{where} \tag{10.70}$$

$$\mathscr{F}^{-1}\mathcal{H}^\pm_{-1} = e^\pm\mathscr{S}(\overline{\mathbb{R}}_+), \quad \mathscr{F}^{-1}\mathcal{H}^- = e^-\mathscr{S}(\overline{\mathbb{R}}_-) \dot{+} \mathbb{C}[\delta']. \tag{10.71}$$

The projectors h^+ *and* h^- *carry over to the projectors* e^+r^+ *and* $I - e^+r^+$ *by* $\mathscr{F}^{-1}$*; here* $(I - e^+r^+)v = e^-r^-v$ *when* $v \in e^+\mathscr{S}(\overline{\mathbb{R}}_+) \dot{+} e^-\mathscr{S}(\overline{\mathbb{R}}_-)$*.*

2° *The spaces* $\mathscr{S}(\overline{\mathbb{R}}_+)$ *and* $\mathcal{H}^+$ *can be described as the spaces of functions*

$$u(x) = \sum_{k\in\mathbb{N}_0} b_k\varphi_k(x,\sigma) \textit{ resp. } f(t) = \sum_{k\in\mathbb{N}_0} b_k\hat{\varphi}_k(t,\sigma), \tag{10.72}$$

expanded in the orthonormal Laguerre system $(\varphi_k)_{k\in\mathbb{N}_0}$ *on* $\mathbb{R}_+$ *(cf.* (10.56)*), resp. the Fourier-transformed Laguerre system* $(\hat{\varphi}_k)_{k\in\mathbb{N}_0}$ *in* $L_2(\mathbb{R})$ *(cf.* (10.55)*),* *with rapidly decreasing coefficient series* $(b_k)_{k\in\mathbb{N}_0}$. *There are similar statements for* $\mathscr{S}(\overline{\mathbb{R}}_-)$ *and* $\mathcal{H}^-_{-1}$ *using* $(\varphi_k)_{k<0}$ *on* $\mathbb{R}_-$, *resp.* $(\hat\varphi_k)_{k<0}$ *in* $L_2(\mathbb{R})$ *(the latter system is the same as* $(\overline{\hat\varphi}_l)_{l\geq 0}$*).*

3° *The general element* $f \in \mathcal{H}_d$ *has an expansion with uniquely determined coefficients*

$$f(t) = \sum_{0\leq j\leq d} s_j t^j + \sum_{k\in\mathbb{Z}} b_k \hat\varphi_k(t,\sigma), \tag{10.73}$$

where the last sum equals $h_{-1}f$, *and*

$$h^+ f(t) = \sum_{k\geq 0} b_k \hat\varphi_k(t,\sigma),$$
$$h^- f(t) = \sum_{0\leq j\leq d} s_j t^j + \sum_{k<0} b_k \hat\varphi_k(t,\sigma) = \sum_{0\leq j\leq d} s_j t^j + \sum_{l\geq 0} b_l \overline{\hat\varphi}_l(t,\sigma);$$

here

$$\sum_{k\in\mathbb{Z}} |b_k|^2 = (2\pi)^{-1}\|h_{-1}f\|^2_{L_2(\mathbb{R})}.$$

For later reference we define

$$\mathcal{H}^+_d = \mathcal{H}^+ \cap \mathcal{H}_d, \quad \mathcal{H}^-_d = \mathcal{H}^- \cap \mathcal{H}_d, \quad \text{any } d\in\mathbb{Z}. \tag{10.74}$$

Remark 10.16. There is another decomposition related to (10.73) that is sometimes useful. Define the functions, for $k \in \mathbb{Z}$,

$$\hat\psi_k(t,\sigma) = \frac{(\sigma - it)^k}{(\sigma + it)^k} \quad \big[= (2\sigma)^{-\frac12}(\sigma + it)\hat\varphi_k(t,\sigma)\big], \tag{10.75}$$

and note that

$$\begin{aligned} \hat\varphi_k(t,\sigma) &= (2\sigma)^{\frac12}\frac{(\sigma - it)^k}{(\sigma + it)^{k+1}}\,\frac{(\sigma - it + \sigma + it)}{2\sigma} \\ &= (2\sigma)^{-\frac12}(\hat\psi_{k+1}(t,\sigma) + \hat\psi_k(t,\sigma)). \end{aligned} \tag{10.76}$$

Inserting this in (10.73), we find the expansion

$$f(t) = \sum_{1\leq j\leq d} s_j t^j + \sum_{k\in\mathbb{Z}} a_k \hat\psi_k(t,\sigma), \tag{10.77}$$

where the s_j for $j \geq 1$ are the same as in (10.73) and the other coefficients are determined by the formulas

$$a_0 = (2\sigma)^{-\frac12}(b_0 + b_{-1}) + s_0, \quad a_k = (2\sigma)^{-\frac12}(b_k + b_{k-1}).$$

The system $\hat\psi_k$ is a complete orthogonal system in the weighted L_2-space over $\mathbb{R}$ with weight $(\sigma^2 + t^2)^{-1}$. Their inverse Fourier transforms are the distributions (cf. (10.61))

$$\psi_k(x,\sigma) = (2\sigma)^{\frac{1}{2}} \sum_{0\le j\le k-1} (-1)^{k-1-j}\varphi_j + (-1)^k\delta \text{ for } k \ge 0,$$
$$\psi_k(x,\sigma) = \psi_{-k}(-x,\sigma) \qquad \text{for } k \le 0. \tag{10.78}$$

Remark 10.17. Since (b_k) is rapidly decreasing, $g^+(z)$ in (10.52) extends to a C^∞-function on the closed unit disk $\{|z| \le 1\}$ that is holomorphic on the open disk $\{|z| < 1\}$. It follows (cf. (10.48)) that $f^+(t) \in \mathcal{H}^+$ has a C^∞ extension to $\overline{\mathbb{C}}_-$ which is holomorphic in $\mathbb{C}_-$; cf. (A.3). One can moreover show that $f^+(t)$ satisfies estimates (10.45) on $\overline{\mathbb{C}}_-$ with $d = -1$. The complex conjugates in $\mathcal{H}^-_{-1}$ have the analogous behavior with respect to $\mathbb{C}_+$, and for $d \ge -1$, the functions in $\mathcal{H}^-_d$ satisfy estimates (10.45) on $\overline{\mathbb{C}}_+$.

Also the Paley-Wiener theorem, linking the holomorphy of h^+f in $\mathbb{C}_-$ with the fact that $\operatorname{supp}\mathscr{F}^{-1}f \subset \overline{\mathbb{R}}_+$, could be used. Moreover, h^+f and h^-f can be defined from f by Cauchy integrals; this point of view has a prominent role in [B71].

Note that the (Fréchet) topology on the space $\mathcal{H}^+$ can be defined by either of the following three systems of norms (where $f = \mathscr{F}e^+u$, with expansions (10.72)):

$$\begin{aligned} &\|(b_k)_{k\in\mathbb{N}_0}\|_{\ell_2^N}, \quad N \in \mathbb{N}_0,\\ &\|x^j D_x^m u(x)\|_{L_2(\mathbb{R}_+)} \\ = {}&(2\pi)^{-\frac{1}{2}}\|h^+(D_t^j[t^m f(t)])\|_{L_2(\mathbb{R})}, \quad j,m \in \mathbb{N}_0. \end{aligned} \tag{10.79}$$

The equivalence of the first and second norm systems was shown in Lemma 10.14, and for the second and third norm we use the Parseval-Plancherel theorem. The application of h^+ here just removes polynomial terms; it corresponds before Fourier transformation to removing those singularities supported in $\{0\}$ that arise from differentiating e^+u.

The mapping that assigns the coefficients s_j in (10.45) to a function $f \in \mathcal{H}$ will now be investigated. As mentioned before, they are linked with the Taylor coefficients of $k(\tau)$ at $\tau = 0$; cf. (10.48). When we consider $\mathscr{F}^{-1}f \in \dot{\mathscr{S}}(\mathbb{R})$, we see that the coefficients s_j with $j \ge 0$ appear here as coefficients in a "polynomial" $\sum s_j\delta^{(j)}$. For the coefficients with $j < 0$, the most important step is the analysis of the case where $f \in \mathcal{H}^+$:

Let $f \in \mathcal{H}^+$, having the asymptotic expansion

$$f(t) \sim s_{-1}t^{-1} + s_{-2}t^{-2} + \cdots, \tag{10.80}$$

and let $u \in \mathscr{S}_+$ be such that $\mathscr{F}e^+u = f$. We shall see that the s_j are linked with the boundary values of u. The Lagrange (or Green) formula (4.53)

$$(D_x^N u, v)_{\mathbb{R}_+} - (u, D_x^N v)_{\mathbb{R}_+} = i \sum_{0 \le k \le N-1} \gamma_{N-k-1} u \cdot \overline{\gamma_k v}, \; u, v \in \mathscr{S}_+ \quad (10.81)$$

(recall that $\gamma_j u = D_x^j u(0)$), can be written in distribution form when we insert $v = r^+ \varphi$ for some $\varphi \in \mathscr{S}(\mathbb{R})$ and observe that

$$\begin{aligned}
(D_x^N u, v)_{\mathbb{R}_+} &= (e^+ D_x^N u, \varphi)_{\mathbb{R}} = \langle e^+ D_x^N u, \overline{\varphi} \rangle_{\mathbb{R}}, \\
(u, D_x^N v)_{\mathbb{R}_+} &= \langle e^+ u, \overline{D_x^N \varphi} \rangle_{\mathbb{R}} = \langle D_x^N e^+ u, \overline{\varphi} \rangle_{\mathbb{R}}, \\
\gamma_j u \cdot \overline{\gamma_k v} &= \langle (\gamma_j u) \delta, \overline{D_x^k \varphi} \rangle_{\mathbb{R}} = \langle (\gamma_j u) D_x^k \delta, \overline{\varphi} \rangle_{\mathbb{R}}.
\end{aligned}$$

Then (10.81) becomes

$$D_x^N e^+ u = -i \sum_{0 \le k \le N-1} (\gamma_{N-k-1} u) D_x^k \delta + e^+ D_x^N u, \text{ for any } N \in \mathbb{N}_0. \quad (10.82)$$

This gives by Fourier transformation

$$t^N f(t) = -i \sum_{0 \le k < N} (\gamma_{N-k-1} u) t^k + g_N(t), \quad (10.83)$$

where $g_N(t) \in \mathcal{H}^+ \subset \mathcal{H}_{-1}$. Comparing (10.83) with (10.80), we conclude from the uniqueness of the coefficients that

$$\gamma_j u = i s_{-1-j} \text{ for all } j \ge 0. \quad (10.84)$$

Another interpretation of the coefficients can be given by use of the so-called *plus-integral* $\int^+ f(t)\, dt$. It is defined on $\mathcal{H}$ by linear extension of the two cases:

$$\int^+ f(t)\, dt = \begin{cases} \int_{\mathbb{R}} f(t)\, dt \text{ when } f \in L_1(\mathbb{R}) \text{ (i.e., } f \in \mathcal{H}_{-2}), \\ \int_C f(t)\, dt \text{ when } f \text{ is meromorphic in } \mathbb{C}_+, \end{cases} \quad (10.85)$$

C denoting a contour around the poles in $\mathbb{C}_+$. This covers all $f \in \mathcal{H}$, since, when $f \in \mathcal{H}_{-1}$ with the expansion (10.45), $f - s_{-1}(t+i)^{-1} \in \mathcal{H}_{-2}$. In particular, in view of Remark 10.17,

$$\int^+ f(t)\, dt = 0 \text{ if } f \in \mathcal{H}^- \text{ or } f \in \mathcal{H}^+ \cap \mathcal{H}_{-2}; \quad (10.86)$$

for in the latter case, the integration can be carried over to a contour in $\mathbb{C}_-$.

Then when $f \in \mathcal{H}^+$ and has the expansion (10.80), we can write, for any $k \in \mathbb{N}_0$,

$$t^k f(t) = \sum_{-k \le j \le -1} s_j t^{j+k} + s_{-1-k} \frac{1}{t-i} + g_k(t) \text{ with } g_k(t) \in \mathcal{H}^+ \cap \mathcal{H}_{-2},$$

and conclude by the residue theorem that $\frac{1}{2\pi i}\int^+ t^k f(t)\,dt = s_{-1-k}$. Recalling (10.84), we have obtained:

Lemma 10.18. *When $f = \mathscr{F}e^+u$ ($u \in \mathscr{S}_+$), with the expansion* (10.80), *then*

$$\frac{1}{2\pi}\int^+ t^k f(t)\,dt = is_{-1-k} = \gamma_k u. \tag{10.87}$$

The coefficient s_{-1-k} can be *estimated* by use of the standard trace estimates

$$\begin{aligned} |s_{-1-k}|^2 &= |\gamma_k u|^2 = -\int_0^\infty \partial_x[u^{(k)}\overline{u}^{(k)}]\,dx \\ &\le 2\|D_x^k u\|_{L_2(\mathbb{R}_+)}\|D_x^{k+1}u\|_{L_2(\mathbb{R}_+)} \\ &= \tfrac{1}{\pi}\|h^+(t^k f)\|_{L_2(\mathbb{R})}\|h^+(t^{k+1}f)\|_{L_2(\mathbb{R})},\ \text{when } f \in \mathcal{H}^+. \end{aligned} \tag{10.88}$$

In the general case where $f \in \mathcal{H}_{-1}$, one can obtain similar results by applying the above to the components $f^+ = h^+f$ and $f^- = h^-f$.

10.3 The complex formulation

The considerations in Section 10.2 are relevant not only for the ψdo's satisfying the transmission property, but also for the appropriate definition of symbol spaces for trace operators, Poisson operators and singular Green operators. The symbols were mentioned in Section 10.1 along with the symbol-kernels (as their Fourier transforms in the normal variable(s)), but without a systematic definition of symbol spaces. We shall give this now using the notions from Section 10.2.

Definition 10.19. Let $d \in \mathbb{R}$, let $r \in \mathbb{N}_0$, let Ξ be open $\subset \mathbb{R}^{n'}$ and let $\mathcal{K}$ be one of the spaces $\mathcal{H}^-_{r-1}$ or $\mathcal{H}^+$.

1° The space $S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathcal{K})$ consists of the functions $f(X, \xi', \xi_n) \in C^\infty(\Xi \times \mathbb{R}^n)$, lying in $\mathcal{K}$ with respect to ξ_n, such that when f is written in the form

$$f(X, \xi', \xi_n) = \sum_{0 \le j \le r-1} s_j(X, \xi')\xi_n^j + f'(X, \xi', \xi_n) \tag{10.89}$$

with $f' = h_{-1}f$, then $s_j(X, \xi') \in S^{d-j}_{1,0}(\Xi, \mathbb{R}^{n-1})$ and f' satisfies

$$\|D_X^\beta D_{\xi'}^\alpha h_{-1}(D_{\xi_n}^k \xi_n^{k'} f')\|_{L_{2,\xi_n}} \le c(X)\langle\xi'\rangle^{d+\frac{1}{2}-k+k'-|\alpha|-j},\ \text{all } \xi', \tag{10.90}$$

for all indices $\beta \in \mathbb{N}_0^{n'}$, $\alpha \in \mathbb{N}_0^{n-1}$, k and $k' \in \mathbb{N}_0$, with a continuous function $c(X)$ depending on the indices.

2° The space $S^d(\Xi, \mathbb{R}^{n-1}, \mathcal{K})$ of polyhomogeneous symbols (in $S^d_{1,0}$) consists of those symbols $f \in S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathcal{K})$ that furthermore have asymptotic expansions

$$f \sim \sum_{l \in \mathbb{N}_0} f_{d-l}, \tag{10.91}$$

where $f - \sum_{l<M} f_{d-l} \in S^{d-M}_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathcal{K})$ for any $M \in \mathbb{N}_0$, and the symbols f_{d-l} are homogeneous of degree $d-l$ in ξ on the set where $|\xi'| \geq 1$:

$$f_{d-l}(X, t\xi) = t^{d-l} f_{d-l}(X, \xi) \text{ for } |\xi'| \geq 1,\ t \geq 1. \tag{10.92}$$

In the decomposition (10.89), the sum over j is empty when $r \leq 0$. Note that f' is in the space with r replaced by 0; it is often called "the part of f of class 0". The first term f_d is also denoted f^0.

Definition 10.20. Let $d \in \mathbb{R}$ and $r \in \mathbb{N}_0$.

1° $S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathcal{H}^-_{r-1})$ and $S^d(\Xi, \mathbb{R}^{n-1}, \mathcal{H}^-_{r-1})$ are called, respectively, the space of $S_{1,0}$ or polyhomogeneous **trace symbols** of degree d, class r and order d.

When $r = 0$, the inverse co-Fourier transform gives (after application of $\overline{\mathscr{F}}^{-1}_{\xi_n \to x_n}$ and restriction to $\mathbb{R}_+$) the spaces $S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_+)$ resp. $S^d(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_+)$ of $S_{1,0}$ resp. polyhomogeneous **trace symbol-kernels** of degree d, class 0 and order d, defined in Definition 10.3.

2° $S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathcal{H}^+)$ and $S^d(\Xi, \mathbb{R}^{n-1}, \mathcal{H}^+)$ are called, respectively, the space of $S_{1,0}$ or polyhomogeneous **Poisson symbols** of degree d and **order** $d+1$.

The inverse Fourier transform gives (after application of $\mathscr{F}^{-1}_{\xi_n \to x_n}$ and restriction to $\mathbb{R}_+$) the spaces $S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_+)$ resp. $S^d(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_+)$, here called the $S_{1,0}$ resp. polyhomogeneous **Poisson symbol-kernels** of degree d and order $d+1$.

For the order convention, cf. Remark 10.6. The fact that the inverse Fourier-transformed spaces match the ones introduced in Definition 10.3 follows from Theorem 10.15 and the Parseval-Plancherel theorem. The application of h_{-1} in (10.90) serves to remove polynomials (that may arise when $k' > k$); this corresponds to the fact that we are in (10.13), (10.14) considering functions of $x_n \in \mathbb{R}_+$ (extendable to smooth functions on $\overline{\mathbb{R}}_+$), disregarding δ-derivatives that would arise from the discontinuity at 0 if one takes derivatives on $\mathbb{R}$.

For the ψdo symbols satisfying the transmission condition, the $h^{\pm}$ projections map into these spaces:

Theorem 10.21. *When $p(X, x_n, y_n, \xi)$ satisfies Definition* 10.2*, then, with $r = \max\{d+1, 0\}$,*

$$\begin{aligned} h^+p(X,0,0,\xi) &\in S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathcal{H}^+), \\ h^-p(X,0,0,\xi) &\in S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathcal{H}^-_{r-1}), \\ r^\pm \tilde{p}(X,0,0,\xi',\pm z_n) &\in S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_+). \end{aligned} \tag{10.93}$$

Proof. From Definition 10.2 follows that $h_{-1}[D^\beta_X D^\alpha_\xi(\xi^m_n p(X,0,0,\xi))]$ satisfies estimates

$$\|h_{-1}[D^\beta_X D^\alpha_\xi(\xi^m_n p(X,0,0,\xi))]\|_{L_{2,\xi_n}(\mathbb{R})} \le c(X)\langle\xi'\rangle^{d+\frac{1}{2}+m-|\alpha|};$$

this implies the estimates for h^+p required in the first line of (10.93) since $\|h^+p\| \le \|h_{-1}p\|$ (cf. (10.63)). Similar estimates are valid for $h^-_{-1}p$. By Theorem 10.15 this translates to the estimates for $r^\pm\tilde{p}$ in the last line. Since h^-p is the sum of $h^-_{-1}p$ and the polynomial part (as in (10.8)), the assertion in the second line follows easily. □

Example 10.22. With $\sigma = \langle\xi'\rangle$, the nonnormalized Fourier-transformed Laguerre functions

$$(2\sigma)^{-\frac{1}{2}}\widehat{\varphi}_l(\xi_n,\sigma) = \frac{(\sigma - i\xi_n)^l}{(\sigma + i\xi_n)^{l+1}}$$

(cf. (10.55)) with $l \ge 0$ lie in $S^{-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, \mathcal{H}^+)$, so they are Poisson symbols of order 0, degree -1. Their conjugates $(2\sigma)^{-\frac{1}{2}}\overline{\widehat{\varphi}}_l(\xi_n,\sigma)$ lie in $S^{-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, \mathcal{H}^-_{-1})$, so they are trace symbols of order and degree -1 and class 0.

In order to describe the symbol spaces for singular Green operators we need to describe the space of Fourier transforms of functions in $\mathscr{S}_{++}$ extended by 0 to $\mathbb{R}^2$.

The tensor product of $\mathscr{S}(\overline{\mathbb{R}}_+)$ with itself is the linear space of functions on $\overline{\mathbb{R}}^2_{++}$ spanned by the products $a(x_n)b(y_n)$, $a, b \in \mathscr{S}(\overline{\mathbb{R}}_+)$. One can define completions of such tensor product spaces in several topologies, but it is known e.g. from Treves [T67] that the space $\mathscr{S}(\overline{\mathbb{R}}_+)$ is *nuclear*, and hence that the various completions coincide and identify with $\mathscr{S}(\overline{\mathbb{R}}^2_{++})$, the restriction of $\mathscr{S}(\mathbb{R}^2)$ to $\overline{\mathbb{R}}^2_{++}$. We simply write

$$\mathscr{S}(\overline{\mathbb{R}}_+)\hat{\otimes}\mathscr{S}(\overline{\mathbb{R}}_+) = \mathscr{S}(\overline{\mathbb{R}}^2_{++}).$$

The (nuclear Fréchet) topology on $\mathscr{S}(\overline{\mathbb{R}}^2_{++})$ is described e.g. by the system of seminorms

$$\|x^k_n D^{k'}_{x_n} y^m_n D^{m'}_{y_n} \tilde{g}(x_n,y_n)\|_{L(\mathbb{R}^2_{++})} \text{ for } k,k',m,m' \in \mathbb{N}_0. \tag{10.94}$$

By Fourier transformation in x_n and co-Fourier transformation in y_n (i.e., by sesqui-Fourier transformation) of $e^+_{x_n}e^+_{y_n}\mathscr{S}(\overline{\mathbb{R}}^2_{++})$, we obtain (in view of (10.65)–(10.67)) the completed tensor product

$$\begin{aligned}&\mathscr{F}_{x_n\to\xi_n}\overline{\mathscr{F}}_{y_n\to\eta_n}(e^+_{x_n}e^+_{y_n}\mathscr{S}(\overline{\mathbb{R}}^2_{++}))\\&\qquad=\mathscr{F}_{x_n\to\xi_n}\overline{\mathscr{F}}_{y_n\to\eta_n}(e^+\mathscr{S}(\overline{\mathbb{R}}_+)\hat{\otimes}e^+\mathscr{S}(\overline{\mathbb{R}}_+))=\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1};\end{aligned}\tag{10.95}$$

here the sesqui-Fourier transform acts as a *homeomorphism.* In particular, the (semi)norm (10.94) on $\tilde{g}(x_n,y_n)$ carries over to the (semi)norm

$$\tfrac{1}{2\pi}\|h^+_{\xi_n}h^-_{-1,\eta_n}(D^k_{\xi_n}\xi_n^{k'}D^m_{\eta_n}\eta_n^{m'}g(\xi_n,\eta_n))\|_{L_2(\mathbb{R}^2)}\ \text{for}\ g\in\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1},\tag{10.96}$$

where

$$g(\xi_n,\eta_n)=\int_{\mathbb{R}^2_{++}}e^{-ix_n\xi_n+iy_n\eta_n}\tilde{g}(x_n,y_n)\,dx_ndy_n;\tag{10.97}$$

the system of seminorms (10.96) defines the (nuclear Fréchet) topology on $\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}$. (Again, the projections h^+ and h^-_{-1} serve to remove polynomials, which would correspond to δ-derivatives at 0 in the (x_n,y_n)-formulation.)

By use of the direct sum decomposition

$$\mathcal{H}^-_{r-1}=\mathcal{H}^-_{-1}\dot{+}\mathbb{C}_{r-1}[t],\ \text{for}\ r\geq 0,$$

where $\mathbb{C}_{r-1}[t]$ denotes the (r-dimensional) space of polynomials of degree $< r$, we can likewise define $\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1}$, with elements

$$g(\xi_n,\eta_n)=\sum_{0\leq j\leq r-1}s_j\eta_n^j+g'(\xi_n,\eta_n),$$

where the s_j are constants and $g'\in\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}$. Then we can formulate the appropriate definition:

Definition 10.23. Let $d\in R$, let $r\in\mathbb{N}_0$ and let Ξ be open $\subset\mathbb{R}^{n'}$.

1° The space $S^d_{1,0}(\Xi,\mathbb{R}^{n-1},\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1})$ consists of the functions $g(X,\xi',\xi_n,\eta_n)\in C^\infty(\Xi\times\mathbb{R}^{n+1})$, lying in $\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1}$ with respect to (ξ_n,η_n), such that when g is written in the form

$$g(X,\xi',\xi_n,\eta_n)=\sum_{0\leq j\leq r-1}k_j(X,\xi',\xi_n)\eta_n^j+g'(X,\xi',\xi_n,\eta_n)\tag{10.98}$$

with $g'=h_{-1,\eta_n}g$, then $k_j\in S^{d-j}_{1,0}(\Xi,\mathbb{R}^{n-1},\mathcal{H}^+)$ for each j, and g' satisfies the estimates

$$\begin{aligned}&\|D^\beta_XD^\alpha_{\xi'}h^+_{-1,\xi_n}h^-_{-1,\eta_n}(D^k_{\xi_n}\xi_n^{k'}D^m_{\eta_n}\eta_n^{m'}g')\|_{L_2(\mathbb{R}^2)}\\&\qquad\qquad\leq c(X)\langle\xi'\rangle^{d+1-k+k'-m+m'-|\alpha|-j},\end{aligned}\tag{10.99}$$

for all $\beta\in\mathbb{N}_0^{n'},\alpha\in\mathbb{N}_0^{n-1},k,k',m$ and $m'\in\mathbb{N}_0$.

2° The polyhomogeneous subspace $S^d(\Xi,\mathbb{R}^{n-1},\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1})$ consists of those symbols $g\in S^d_{1,0}(\Xi,\mathbb{R}^{n-1},\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1})$ that furthermore have asymptotic expansions

$$g \sim \sum_{l \in \mathbb{N}_0} g_{d-l},$$

where $g - \sum_{l<M} g_{d-l} \in S_{1,0}^{d-M}(\Xi, \mathbb{R}^{n-1}, \mathcal{H}^+ \hat{\otimes} \mathcal{H}_{r-1}^-)$ for any $M \in \mathbb{N}_0$, and the symbols g_{d-l} are homogeneous of degree $d-l$ in (ξ', ξ_n, η_n):

$$g_{d-l}(X, t\xi', t\xi_n, t\eta_n) = t^{d-l} g_{d-l}(X, \xi', \xi_n, \eta_n) \text{ for } t \text{ and } |\xi'| \geq 1. \quad (10.100)$$

3° The spaces are called, respectively, the space of $S_{1,0}$ or polyhomogeneous **singular Green symbols** of degree d and class r, and of **order** $d+1$.

When $r = 0$, the inverse Fourier transform from ξ_n to x_n together with the inverse co-Fourier transform from η_n to y_n gives (after restriction to $\mathbb{R}^2_{++}$) the spaces $S_{1,0}^d(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_{++})$ resp. $S^d(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_{++})$ of $S_{1,0}$ resp. polyhomogeneous **singular Green symbol-kernels** of degree d, class 0 and order $d+1$; they are defined as in Definition 10.3 3° and 4°.

Also for singular Green symbols, the Laguerre expansions are highly relevant. Functions in $\mathscr{S}_{++}$ can be expanded in the orthonormal double sequence $(\varphi_l(x_n, \sigma)\varphi_m(y_n, \sigma))_{l,m \in \mathbb{N}_0}$ formed of the Laguerre functions; it is a complete orthonormal basis for $L_2(\mathbb{R}^2_{++})$, and its elements lie in $\mathscr{S}_{++}$. Here, when $\tilde{g}$ and g are expanded in double Laguerre series:

$$\begin{aligned} \tilde{g}(x_n, y_n) &= \sum_{l,m \in \mathbb{N}_0} c_{lm} \varphi_l(x_n, \sigma) \varphi_m(y_n, \sigma), \\ g(\xi_n, \eta_n) &= \sum_{l,m \in \mathbb{N}_0} c_{lm} \hat{\varphi}_l(\xi_n, \sigma) \overline{\hat{\varphi}}_m(\eta_n, \sigma), \end{aligned} \quad (10.101)$$

one has that

$$\tilde{g} \in \mathscr{S}_{++} \Leftrightarrow g \in \mathcal{H}^+ \hat{\otimes} \mathcal{H}_{-1}^- \Leftrightarrow (c_{lm})_{l,m \in \mathbb{N}_0} \in \mathfrak{s}(\mathbb{N}_0 \times \mathbb{N}_0), \quad (10.102)$$

where $\mathfrak{s}(\mathbb{N}_0 \times \mathbb{N}_0)$ is the space of rapidly decreasing sequences indexed by $(l, m) \in \mathbb{N}_0 \times \mathbb{N}_0$. The system of (semi)norms where N and N' run through $\mathbb{N}_0$,

$$\|(c_{lm})_{l,m \in \mathbb{N}_0}\|_{\ell_2^{N,N'}} \equiv \Big(\sum_{l,m \in \mathbb{N}_0} |(1+l)^N (1+m)^{N'} c_{lm}|^2 \Big)^{\frac{1}{2}}, \quad (10.103)$$

defines the topology, equivalently with the systems (10.94) and (10.96).

For the functions in $S_{1,0}^{d-1}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_{++})$ it is natural to take $\sigma = \langle \xi' \rangle$, then when

$$\tilde{g}(X, x_n, y_n, \xi') = \sum_{l,m \in \mathbb{N}_0} c_{lm}(X, \xi') \varphi_l(x_n, \langle \xi' \rangle) \varphi_m(y_n, \langle \xi' \rangle), \quad (10.104)$$

the $c_{lm}(X, \xi')$ are in $S_{1,0}^d(\Xi, \mathbb{R}^{n-1})$ (ψdo symbols), and the topology of the space $S_{1,0}^{d-1}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_{++})$ is equivalently defined by the system of seminorms

$$\sup_{\xi' \in \mathbb{R}^{n-1}, X \in K} \langle \xi' \rangle^{-d+|\alpha|} \Big(\sum_{l,m \in \mathbb{N}_0} |(1+l)^N (1+m)^{N'} D_X^\beta D_{\xi'}^\alpha c_{lm}(X,\xi')|^2 \Big)^{\frac{1}{2}}, \tag{10.105}$$

where $\alpha \in \mathbb{N}_0^{n-1}$, $\beta \in \mathbb{N}_0^{n'}$, N and $N' \in \mathbb{N}_0$ and K runs through compact subsets of Ξ. A more refined choice of σ is to take $\sigma = [\xi']$, where

$$[\xi'] = |\xi'| \text{ for } |\xi'| \geq 1, \quad [\xi'] \text{ is } C^\infty \text{ and } > 0; \tag{10.106}$$

the modified length. This can be useful in the consideration of polyhomogeneous symbols.

Note that when for example $X = x'$, and $\tilde{g}$ is written in the form (10.104) with $\sigma = \langle \xi' \rangle$, then

$$\tilde{g}(x', x_n, y_n, \xi') = \sum_{m \in \mathbb{N}_0} \tilde{k}_m(x', x_n, \xi') \tilde{t}_m(y_n, \xi'), \text{ where} \tag{10.107}$$

$$\tilde{k}_m = \sum_{l \in \mathbb{N}_0} c_{lm}(x', \xi')(2\sigma)^{-\frac{1}{2}} \varphi_l(x_n, \sigma), \quad \tilde{t}_m = (2\sigma)^{\frac{1}{2}} \varphi_m(y_n, \sigma);$$

cf. Example 10.22. This shows in view of Proposition 10.10 (iv) that any s.g.o. of order d and class 0 can be written as a series

$$G = \sum_{m \in \mathbb{N}_0} K_m T_m \tag{10.108}$$

of compositions of Poisson and trace operators of orders d resp. 0. The series converges rapidly in the symbol space seminorms.

For brevity, we shall often omit the indications $\Xi, \mathbb{R}^{n-1}$ from the notation for the various symbol and symbol-kernel spaces, when they are understood from the context.

We now describe how the operators are defined from the symbols. As noted in Section 10.1, the definition with respect to the x'-variable or (x', y')-variable is the standard ψdo definition, so it suffices to describe the definition w.r.t. x_n. Here we have, consistently with (10.24), (10.28), (10.29), (10.36), (10.39), when the plus-integral (10.85) $\int^+$ is used to include the polynomial parts of the trace and s.g.o. symbols in the integrals:

$$\begin{aligned} \mathrm{OPK}_n(k)v &= \tilde{k}(X, x_n, \xi') \cdot v = \int e^{ix_n\xi_n} k(X,\xi) v \, đ\xi_n \\ &= [k(X, \xi', D_n)v](x_n), \end{aligned} \tag{10.109}$$

$$\mathrm{OPT}_n(t)u = \sum_{0 \leq j < r} s_j(X,\xi')\gamma_j u + \int_0^\infty \tilde{t}'(X, y_n, \xi') u(y_n)\, dy_n$$

$$= \int^+ t(X,\xi)\widehat{e^+u}(\xi_n)\,đ\xi_n = t(X,\xi',D_n)u,$$

$$\begin{aligned}\mathrm{OPG}_n(g)u &= \sum_{0\le j<r} \tilde{k}_j(X,x_n,\xi')\gamma_j u + \int_0^\infty \tilde{g}'(X,x_n,y_n,\xi')u(y_n)dy_n \\ &= \int e^{ix_n\xi_n}\int^+ g(X,\xi,\eta_n)\widehat{e^+u}(\eta_n)đ\eta_n đ\xi_n \\ &= [g(X,\xi',D_n)u](x_n),\end{aligned}$$

for $v \in \mathbb{C}$, $u \in \mathscr{S}_+$. The effect of $p(X,\xi)$ with truncation is

$$\begin{aligned}\mathrm{OP}_n(p)_+u &= r^+\int e^{ix_n\xi_n}p(X,\xi)\widehat{e^+u}(\xi_n)\,đ\xi_n \\ &= r^+\mathscr{F}^{-1}h^+(p\,\widehat{e^+u}) = p(X,\xi',D_n)_+u.\end{aligned} \tag{10.110}$$

All these operators are called boundary symbol operators. The full operator definitions are obtained by combining the above with OP′.

10.4 Composition rules

As noted earlier, the new elements in the proof of the statements (10.44) lie in the compositions with respect to the x_n-variable. We shall write $a\circ_n b = c$, when c is the symbol (in one of our symbol spaces) arising from composing the boundary symbol operators with symbol a resp. b. The rules for (iii)–(vi) in (10.44) are treated in the following theorem.

Theorem 10.24. *Let d and $d' \in \mathbb{R}$, and let r and $r' \in \mathbb{N}_0$. Let*

$$\begin{aligned}&\text{(i)} && p(X,\xi) \in S^d_{1,0}(\Xi,\mathbb{R}^n), \text{ with transm. cond.,}\\ &\text{(ii)} && g(X,\xi',\xi_n,\eta_n) \in S^{d-1}_{1,0}(\Xi,\mathbb{R}^{n-1},\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1}),\\ &\text{(iii)} && t(X,\xi) \in S^d_{1,0}(\Xi,\mathbb{R}^{n-1},\mathcal{H}^-_{r-1}),\\ &\text{(iv)} && k(X,\xi) \in S^{d-1}_{1,0}(\Xi,\mathbb{R}^{n-1},\mathcal{H}^+),\\ &\text{(v)} && s(X,\xi') \in S^d_{1,0}(\Xi,\mathbb{R}^{n-1}),\end{aligned} \tag{10.111}$$

and let p', g', t', k' and s' be given similarly with symbols in the spaces with d and r replaced by d' and r'. In the formulas where p (resp. p') occur, we assume that d (resp. d') is integer. Define

$$d'' = d + d', \quad r'' = \max\{r + d', 0\}. \tag{10.112}$$

Then the $\circ_n$-compositions give rise to ψdbo's whose symbols are determined by the following formulas (where $S^d_{1,0}(\Xi,\mathbb{R}^{n-1},\mathcal{K})$ is written as $S^d_{1,0}(\mathcal{K})$):

$$
\begin{aligned}
&1^\circ \quad p_+ \circ_n k' = h^+_{\xi_n}[p(X,\xi)k'(X,\xi)] \in S^{d''-1}_{1,0}(\mathcal{H}^+), &(10.113)\\
&2^\circ \quad g \circ_n k' = \int^+ g(X,\xi,\eta_n)k'(X,\xi',\eta_n)đ\eta_n \in S^{d''-1}_{1,0}(\mathcal{H}^+),\\
&3^\circ \quad k \circ_n s' = k(X,\xi)s'(X,\xi') \in S^{d''-1}_{1,0}(\mathcal{H}^+),\\
&4^\circ \quad t \circ_n p'_+ = h^-_{\xi_n}[t(X,\xi)p'(X,\xi)] \in S^{d''}_{1,0}(\mathcal{H}^-_{r''-1}),\\
&5^\circ \quad t \circ_n g' = \int^+ t(X,\xi)g'(X,\xi,\eta_n)\, đ\xi_n \in S^{d''}_{1,0}(\mathcal{H}^-_{r'-1}),\\
&6^\circ \quad s \circ_n t' = s(X,\xi')t'(X,\xi) \in S^{d''}_{1,0}(\mathcal{H}^-_{r'-1}),\\
&7^\circ \quad t \circ_n k' = \int^+ t(X,\xi)k'(X,\xi)\, đ\xi_n \in S^{d''}_{1,0}(\Xi,\mathbb{R}^{n-1}),\\
&8^\circ \quad s \circ_n s' = s(X,\xi')s'(X,\xi') \in S^{d''}_{1,0}(\Xi,\mathbb{R}^{n-1}),\\
&9^\circ \quad p_+ \circ_n g' = h^+_{\xi_n}[p(X,\xi)g'(X,\xi,\eta_n)] \in S^{d''-1}_{1,0}(\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r'-1}),\\
&10^\circ \quad g \circ_n p'_+ = h^-_{\eta_n}[g(X,\xi,\eta_n)p'(X,\xi',\eta_n)]\\
&\qquad\qquad \in S^{d''-1}_{1,0}(\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r''-1}),\\
&11^\circ \quad g \circ_n g' = \int^+ g(X,\xi',\xi_n,\zeta_n)g'(X,\xi,\zeta_n,\eta_n)đ\zeta_n\\
&\qquad\qquad \in S^{d''-1}_{1,0}(\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r'-1}),\\
&12^\circ \quad k \circ_n t' = k(X,\xi',\xi_n)t'(X,\xi',\eta_n) \in S^{d''-1}_{1,0}(\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r'-1}).
\end{aligned}
$$

Here composition of polyhomogeneous symbols gives polyhomogeneous symbols.

Proof. The formulas follow rather naturally from the calculus described in Section 10.2. Consider 1°. Here, for $v \in \mathbb{C}$, $k'v$ is as in (10.109), so

$$
\begin{aligned}
p_+(D_n) \circ_n k'(D_n)v &= r^+ \int e^{ix_n\xi_n} p(X,\xi)\mathscr{F}[e^+\tilde{k}(X,x_n,\xi')]v\, đ\xi_n\\
&= r^+ \int e^{ix_n\xi_n} p(X,\xi)k(X,\xi)v\, đ\xi_n\\
&= r^+\mathscr{F}^{-1}[p(X,\xi)k(X,\xi)]v = \mathrm{OPK}_n(h^+(pk))v,
\end{aligned}
$$

since h^+ corresponds to e^+r^+ by Fourier transformation (its effect is the restriction to $\mathbb{R}_+$ in the real formulation). There is a slight abuse of notation when we do not always mention the extension by 0 on $\mathbb{R}_-$.

For 4° one has that for $u \in \mathscr{S}(\overline{\mathbb{R}}_+)$ with $\mathscr{F}e^+u = f$, the operator $t(D_n) \circ_n p'(D_n)_+$ is defined by

$$
t(D_n) \circ_n p'(D_n)_+u = t(D_n)\mathscr{F}^{-1}[h^+(p'f)]
$$

$$
\begin{aligned}
&= \int^+ t \cdot h^+(p'f)\, d\!\!\bar{}\,\xi_n \\
&= \int^+ t\, p' f\, d\!\!\bar{}\,\xi_n \quad (\text{since } \int^+ t\, h^-(p'f)\, d\!\!\bar{}\,\xi_n = 0) \\
&= \int^+ h^-(tp') f\, d\!\!\bar{}\,\xi_n \quad (\text{since } \int^+ h^+(tp') f\, d\!\!\bar{}\,\xi_n = 0) \\
&= \mathrm{OPT}_n(h^-(tp'))u;
\end{aligned}
$$

here we have used (10.109), (10.110) and (10.86).

The rules 9° and 10° follow the same pattern, except that there is an extra integration (in η_n resp. ξ_n) to carry along.

In rules 3°, 6° and 8°, the effect of s or s' is purely multiplicative.

Rule 7° is the Fourier-transformed version of Proposition 10.10 (v) if $r = 0$. When $r > 0$, we must also deal with compositions $s_j \gamma_j k(D_n)v$; here

$$
\gamma_j \tilde{k}(X, x_n, \xi') = \int^+ \xi_n^j k(X, \xi', \xi_n)\, d\!\!\bar{}\,\xi_n
$$

in view of Lemma 10.18, so these terms contribute in the stated way. The rules 2°, 5° and 11° are elaborations of this observation, based on Proposition 10.10 (vi)–(viii) and carrying other variables along. 12° is an obvious extension of Proposition 10.10 (iv).

This shows the formulas, and the estimates required for the indicated spaces are easily checked. □

In preparation for the treatment of the leftover operator $L(P, P')$, we introduce a new rule. When $v \in \mathscr{S}(\mathbb{R}^{n-1})$, it can be multiplied by the distribution $\delta(x_n)$ to define a temperate distribution on $\mathbb{R}^n$; it is traditionally denoted $v(x') \otimes \delta(x_n)$, and acts as follows:

$$
\langle v(x') \otimes \delta(x_n), \varphi(x) \rangle_{\mathbb{R}^n} = \langle v(x'), \varphi(x', 0) \rangle_{\mathbb{R}^{n-1}}, \text{ when } \varphi \in \mathscr{S}(\mathbb{R}^n). \quad (10.114)
$$

We shall show that when P is a ψdo of order d (satisfying the transmission condition, as always), then the operator K defined by

$$
(Kv)(x') = r^+ P(v(x') \otimes \delta(x_n)) \quad (10.115)
$$

is a Poisson operator of order $d + 1$.

Theorem 10.25. *Let $P = \mathrm{OP}(p(x, y, \xi))$ be of order d, and define K by (10.115). Then K is a Poisson operator of order $d + 1$. The symbol-kernel $\tilde{k}$ and symbol k satisfy (with $\tilde{p} = \mathscr{F}^{-1}_{\xi_n \to z} p$):*

$$
\begin{aligned}
\tilde{k}(x', y', x_n, \xi') &= r^+ \tilde{p}(x', 0, y', 0, \xi', z)|_{z = x_n}, \\
k(x', y', \xi) &= h^+ p(x', 0, y', 0, \xi), \text{ if } p \text{ is independent of } x_n;
\end{aligned} \quad (10.116)
$$

$$\tilde{k}(x', y', x_n, \xi') \sim r^+ \sum_{j\in\mathbb{N}_0} \tfrac{1}{j!} x_n^j \partial_{x_n}^j \tilde{p}(x', 0, y', 0, \xi', z)|_{z=x_n},$$
$$k(x', y', \xi) \sim \sum_{j\in\mathbb{N}_0} \tfrac{1}{j!} h^+\big(\overline{D}_{\xi_n}^j \partial_{x_n}^j p(x', 0, y', 0, \xi)\big) \text{ in general.}$$

If P is a differential operator, $K = 0$.

Proof. The last statement is obvious since $P(v(x') \otimes \delta(x_n))$ is supported in $\{x_n = 0\}$ when P is a differential operator.

To show the formula, let first p be independent of x_n. Then

$$\begin{aligned} r^+ \operatorname{OP}(p)(v\otimes\delta) &= r^+ \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} p(x', y', y_n, \xi) v(y')\delta(y_n)\, dy đ\xi \\ &= r^+ \int_{\mathbb{R}^{2n-1}} e^{i(x'-y')\cdot\xi' + ix_n\xi_n} p(x', y', 0, \xi) v(y')\, dy' đ\xi \\ &= \int_{\mathbb{R}^{2n-2}} e^{i(x'-y')\cdot\xi'} r^+ \tilde{p}(x', y', 0, \xi', x_n) v(y')\, dy' đ\xi' \\ &= \operatorname{OPK}(\tilde{k}(x', y', x_n, \xi'))v, \end{aligned}$$

where $\tilde{k} = r^+\tilde{p} = r^+ \mathscr{F}^{-1}_{\xi_n\to x_n} p = r^+ \mathscr{F}^{-1}_{\xi_n\to x_n}(h^+p)$; the corresponding symbol is $k = h^+p$. In view of Theorem 10.21, $\tilde{k}(x', y', x_n, \xi')$ is a Poisson symbol-kernel of degree d (order $d+1$). (The application of $\delta(y_n)$ in the first line can be justified by writing it as the limit of an approximate identity.)

When p depends on x_n, it is natural to do a Taylor expansion of p in x_n and apply the preceding result to each term; this gives the third formula in (10.116), except that we have not yet accounted for the justification of the asymptotic series. But this is easy on the symbol level: We can use Theorem 7.13 1° with respect to the variables (x_n, y_n) to replace p by an equivalent symbol p' depending on y_n instead of (x_n, y_n); it has the form

$$p'(x', y', y_n, \xi) \sim \sum_{j\in\mathbb{N}_0} \tfrac{1}{j!} \overline{D}_{\xi_n}^j \partial_{x_n}^j p(x', x_n, y', y_n, \xi)\big|_{x_n=y_n},$$

and is likewise of order d and satisfies the transmission condition at $y_n = 0$. Then we apply the second formula in (10.116) to this symbol, obtaining the fourth formula, and the third formula follows by inverse Fourier transformation in ξ_n. □

We finally treat rule (ii) in (10.44) for the leftover operator $L(P, P')$. Assume that P and P' are given by symbols $p(X, \xi)$, $p'(X, \xi)$, where $X = x'$, y' or (x', y'), so they are independent of x_n (or y_n). The decomposition of $L(P, P')$ that we shall now explain, was first introduced in [G84]. (In that paper it was useful in obtaining the first complete treatment of $L(P, P')$ in the general case with x_n-dependent symbols.)

According to (10.18) we can write

$$P = \sum_{0\le l\le d} S_l D_n^l + Q, \quad P' = \sum_{0\le l\le d'} S_l' D_n^l + Q', \tag{10.117}$$

where the S_l and S_l' are differential operators on $\mathbb{R}^{n-1}$ (with symbols polynomial in ξ') and Q and Q' have symbols that are $O(\langle\xi_n\rangle^{-1})$. One finds the following rules: Since differential operators are local,

$$L(\textstyle\sum_{0\le l\le d} S_l D_n^l, P') = 0. \tag{10.118}$$

By Green's formula (10.82),

$$\begin{aligned} L(P, \textstyle\sum_{0\le l\le d'} S_l' D_n^l)u &= r^+ \textstyle\sum_{l\le d'} P S_l'(D_n^l e^+ u - e^+ D_n^l u) \\ = -ir^+ P \textstyle\sum_{1\le l\le d'} S_l' \sum_{k<l} (\gamma_{l-k-1} u \otimes D_n^k \delta) &= \textstyle\sum_{m\le d'-1} K_m \gamma_m u, \end{aligned} \tag{10.119}$$

where K_m is the Poisson operator of order $d+d'-m$ (as in Theorem 10.25),

$$K_m v = -i \textstyle\sum_{l=m+1}^{d'} r^+ P S_l' D_{x_n}^{l-1-m}(v(x') \otimes \delta(x_n)); \tag{10.120}$$

so (10.119) defines an s.g.o. of class d'. For the analysis of the remaining term we introduce the reflection operator J,

$$J : u(x', x_n) \mapsto u(x', -x_n), \tag{10.121}$$

it sends spaces over $\mathbb{R}^n_\pm$ into spaces over $\mathbb{R}^n_\mp$. Then we can write, for $u \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$:

$$\begin{aligned} L(Q, Q')u &= r^+ Q Q' e^+ u - r^+ Q e^+ r^+ Q' e^+ u = r^+ Q(I - e^+ r^+) Q' e^+ u \\ &= r^+ Q e^- r^- Q' e^+ u = (r^+ Q e^- J)(J r^- Q' e^+) u \\ &= G^+(Q) G^-(Q') u, \text{ with} \\ G^+(Q) &= r^+ Q e^- J = r^+ P e^- J = G^+(P), \\ G^-(Q') &= J r^- Q' e^+ = J r^- P' e^+ = G^-(P') = [G^+(P'^*)]^*. \end{aligned} \tag{10.122}$$

We shall show that the latter are singular Green operators of class 0 and orders d resp. d'. Similar formulas hold on the boundary symbol level. Altogether,

$$L(P, P') = \textstyle\sum_{0\le m<d'} K_m \gamma_m + G^+(Q) G^-(Q'). \tag{10.123}$$

The resulting symbols will now be analyzed on the one-dimensional level.

Theorem 10.26. *Let $d \in \mathbb{Z}$ and $d' \in \mathbb{N}_0$, let $p(X, \xi) \in S^d_{1,0}(\Xi, \mathbb{R}^n)$ and let $s_l'(X, \xi')$ be polynomial in ξ' of degree $d' - l$ for $0 \le l \le d'$.*

1° *One has for the singular Green symbol resulting from the formation of $L(p(X, 0, \xi', D_n), \sum_{0\le l\le d'} s_l'(X, 0, \xi') D_n^l)$,*

$$L(p, \sum s'_l \xi_n^l) = \sum_{0\le m<d'} k_m(X,\xi)\eta_n^m \in S_{1,0}^{d+d'-1}(\varXi, \mathbb{R}^{n-1}, \mathcal{H}^+ \hat{\otimes} \mathcal{H}^-_{d'-1}), \quad (10.124)$$

where the k_m are Poisson symbols

$$k_m(X,\xi) = -ih^+\big(\sum_{l=m+1}^{d'} p(X,\xi)s'_l(X,\xi')\xi_n^{l-1-m}\big) \in S_{1,0}^{d+d'-1-m}(\varXi, \mathbb{R}^{n-1}, \mathcal{H}^+). \quad (10.125)$$

2° *For the operators introduced in* (10.122) *one has*

$$g^+(p) = r^+p(X,0,\xi',D_n)e^-J, \quad g^-(p) = Jr^-p(X,0,\xi',D_n)e^+, \quad (10.126)$$

are singular Green operators with symbol-kernels (where $\tilde p = \mathscr{F}^{-1}_{\xi_n\to z_n}p$), resp. symbols:

$$\begin{aligned} \tilde g^+(p)(X,x_n,y_n,\xi') &= r^+\,\tilde p(X,\xi',z_n)|_{z_n=x_n+y_n} \quad \text{and} \\ \tilde g^-(p)(X,x_n,y_n,\xi') &= r^-\,\tilde p(X,\xi',z_n)|_{z_n=-x_n-y_n} \\ &\in S_{1,0}^{d-1}(\varXi,\mathbb{R}^{n-1},\mathscr{S}_{++}), \quad (10.127)\\ g^+(p)(X,\xi,\eta_n) &= \mathscr{F}_{x_n\to\xi_n}\overline{\mathscr{F}}_{y_n\to\eta_n}e^+_{x_n}e^+_{y_n}\tilde g^+(p) \quad \text{and} \\ g^-(p)(X,\xi,\eta_n) &= \mathscr{F}_{x_n\to\xi_n}\overline{\mathscr{F}}_{y_n\to\eta_n}e^+_{x_n}e^+_{y_n}\tilde g^-(p) \\ &\in S_{1,0}^{d-1}(\varXi,\mathbb{R}^{n-1},\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}), \end{aligned}$$

of order d and class 0*.*

3° *If $p(X,\xi',D_n)$ is a differential operator, these symbols vanish.*

Proof. 1°. The statement on k_m follows from Theorem 10.25 1°. The resulting symbol (10.124) is an s.g.o. symbol of the stated type in view of Definition 10.23. Note that only h^+p enters (cf. also (10.118)).

2°. Denote $h^\pm p = p^\pm$, and omit the variable X. We have, reading the integrals as Fourier transforms and using that the distribution kernel of p equals $\tilde p(x_n - y_n)$ where $\tilde p \in \dot{\mathscr{S}}'(\mathbb{R})$,

$$\begin{aligned} r^+p(\xi',D_n)e^-Ju &= r^+\int e^{ix_n\xi_n}p(\xi',\xi_n)\int_{-\infty}^0 e^{-iy_n\xi_n}u(-y_n)\,dy_n đ\xi_n \\ &= r^+\int e^{ix_n\xi_n}p(\xi',\xi_n)\int_0^\infty e^{iy_n\xi_n}u(y_n)\,dy_n đ\xi_n \\ &= r^+\int_0^\infty \tilde p(\xi',x_n+y_n)u(y_n)\,dy_n = r^+\int_0^\infty \tilde p^+(\xi',x_n+y_n)u(y_n)\,dy_n, \end{aligned}$$

since $\tilde p(\xi',z_n) = \tilde p^+(\xi',z_n)$ for $z_n > 0$. Thus $g^+(p)$ is the integral operator on $\mathbb{R}_+$ with kernel

$$\tilde g^+(p)(x_n,y_n,\xi') = r^+\,\tilde p(\xi',x_n+y_n) = \tilde p^+(\xi',x_n+y_n).$$

We have from Theorem 10.21 that $\tilde{p}^+ \in S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_+)$. The kernel is estimated by use of coordinate changes $z = x_n - y_n$, $w = x_n + y_n$:

$$\begin{aligned}
\|\tilde{g}^+(p)(x_n, y_n, \xi')\|^2_{L_{2,x_n,y_n}(\mathbb{R}^2_{++})} &= \int_0^\infty \int_0^\infty |\tilde{p}^+(\xi', x_n + y_n)|^2 dx_n dy_n \\
&= \tfrac{1}{2} \int_0^\infty dw \int_{-w}^{w} |\tilde{p}^+(\xi', w)|^2 dz dw = \int_0^\infty w |\tilde{p}^+(\xi', w)|^2 dw \\
&\le \|w\tilde{p}^+(w, \xi)\| \|\tilde{p}^+(\xi', w)\| \le c\langle \xi' \rangle^{d-\frac{1}{2}} \langle \xi' \rangle^{d+\frac{1}{2}} = c\langle \xi' \rangle^{2d}.
\end{aligned}$$

This is the basic estimate, which easily generalizes to derivatives in X and ξ', and to lower-order parts. For the symbol-kernels resulting from application of $x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'}$, we observe that for $w = x_n + y_n$,

$$x_n^k y_n^m \le w^{k+m} \quad \text{when} \quad x_n, y_n \ge 0,$$

and

$$D_{x_n}^{k'} D_{y_n}^{m'} \tilde{p}(\xi', x_n + y_n) = D_w^{k'+m'} \tilde{p}(\xi', w),$$

so that

$$\begin{aligned}
&\|x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} \tilde{g}^+(p)\|^2_{L_{2,x_n,y_n}(\mathbb{R}^2_{++})} \\
&\quad \le \int_0^\infty w^{2k+2m+1} |D_w^{k'+m'} \tilde{p}^+(\xi', w)|^2 dw \\
&\quad = \int_0^\infty w |w^{k+m} D_w^{k'+m'} \tilde{p}^+(\xi', w)|^2 dw \le c\langle \xi' \rangle^{2(d-k-m+k'+m')}.
\end{aligned}$$

It is altogether seen that $\tilde{g}^+(p)$ satisfies all the estimates required of a symbol-kernel in $S^d_{1,0}(\Xi, \mathbb{R}^{n-1}, \mathscr{S}_{++})$. The proof for $\tilde{g}^-(p)$ is similar, based on the identities

$$\begin{aligned}
Jr^- p e^+ u &= Jr^- \int_{\mathbb{R}} e^{ix_n\xi_n} p(\xi', \xi_n) \int_0^\infty e^{-iy_n\xi_n} u(y_n)\, dy_n đ\xi_n \\
&= r^+ \int_{\mathbb{R}} e^{-ix_n\xi_n} p(\xi', \xi_n) \int_0^\infty e^{-iy_n\xi_n} u(y_n)\, dy_n đ\xi_n \\
&= r^+ \int_0^\infty \tilde{p}(\xi', -x_n - y_n) u(y_n)\, dy_n.
\end{aligned}$$

For 3°, recall (10.118). □

From the above analysis we can conclude:

Corollary 10.27. *Let p and p' be as in Theorem* 10.24 *and write*

$$p' = \sum_{0 \le j \le d'} s'_j(X, \xi') \xi_n^j + h_{-1} p'.$$

Then $L(p,p') = (p(D_n)p'(D_n))_+ - p(D_n)_+ \circ_n p'(D_n)_+$ *is a singular Green boundary symbol operator of order* $d+d'$ *and class* $\max\{d',0\}$. *More precisely, the symbol is defined by*

$$L(p,p')(X,\xi,\eta_n) = \sum_{0\le m<d'} k_m(X,\xi)\eta_n^m + g^+(p)\circ_n g^-(p'), \tag{10.128}$$

where k_m, $g^+(p)$ *and* $g^-(p')$ *are as defined in Theorem* 10.26. *In particular,* $L(p,p')$ *depends only on* h^+p *and* h^-p'.

The full composition rules, where the action in x' is also taken into account, look as follows when the ψdo symbols are independent of the normal variable:

Theorem 10.28. *Let symbols be given as in Theorem* 10.24, *with* $\Xi = \mathbb{R}^{n-1}$ *with points* x'. *Let* $a(x',\xi',D_n)$ *stand for the boundary symbol operator, and let* A *stand for the full operator* $\mathrm{OP}'(a(x',\xi',D_n))$ *defined from one of the symbols* $a = p,t,k,g$ *or* s, *and let* $a'(x',\xi',D_n)$ *and* A' *be similarly defined from the primed symbols. (One can also consider symbols in* (x',y')*-form, in which case one can begin by reducing them to* x'*-form, by Theorem* 7.13 1° *applied in the tangential variables.)*

1° *Consider one of the compositions*

$$A'' = AA' = \mathrm{OP}'(a)\,\mathrm{OP}'(a')$$

listed in (10.44) (iii)–(vi), *with one factor properly supported w.r.t.* (x',y'). *Here* $A'' \sim \mathrm{OP}'(a'')$, *where* a'' *has the asymptotic expansion*

$$a''(x',\xi',D_n) \sim \sum_{\alpha\in{\mathbb{N}_0}^{n-1}} \tfrac{1}{\alpha!} D_{\xi'}^{\alpha} a(x',\xi',D_n)\circ_n \partial_{x'}^{\alpha} a'(x',\xi',D_n), \tag{10.129}$$

each term being determined by the appropriate composition rule in Theorem 10.24. *The expansion holds in the space of operators and symbols of order* $d+d'$, *and class* r' *resp.* $\max\{r+d',0\}$ *(in the relevant cases).*

2° *Consider the singular Green operator*

$$G = L(P,P') \equiv (PP')_+ - P_+P'_+$$

derived from P *and* P', *with one of the operators properly supported w.r.t.* (x',y'). *Here* $G = \mathrm{OP}'(g)$, *where*

$$g(x',\xi',D_n) \sim \sum_{\alpha\in{\mathbb{N}_0}^{n-1}} \tfrac{1}{\alpha!} L(D_{\xi'}^{\alpha} p(x,\xi',D_n), \partial_{x'}^{\alpha} p'(x,\xi',D_n)), \tag{10.130}$$

with the terms defined by Corollary 10.27. *The expansion holds in the space of operators and symbols of order* $d+d'$ *and class* $\max\{d',0\}$.

Proof. 1° If a' were in y'-form, the resulting operator would simply have the boundary symbol operator in (x',y')-form

$$a(x',\xi',D_n)\circ_n a'(y',\xi',D_n).$$

This can be reduced to x'-form as in the proof of Theorem 7.13 1°. The procedure of replacing a' by its y'-form and reducing the resulting product to x'-form gives altogether the formula (10.129). The proof of 2° is similar. □

When the symbol of P (or P') moreover depends on x_n, one considers each term in its Taylor expansion at $x_n = 0$ separately:

$$p(x',x_n,\xi) \text{ has the expansion } \sum_{j\geq 0} \tfrac{1}{j!} x_n^j \partial_{x_n}^j p(x',0,\xi); \tag{10.131}$$

the $\partial_{x_n}^j p$ satisfy the transmission condition. In the compositions in (10.44) (ii)–(v), a factor x_n^j decreases the order by j steps (cf. Proposition 10.10 (ix)), so it is indeed possible to collect the resulting symbols as expansions in homogeneous terms of decreasing order. The resulting formulas can e.g. be found in [G96], Section 2.7 (disregard the parameter).

10.5 Continuity

For the proof of continuity properties in Sobolev spaces it will be practical to introduce spaces of different order in the normal and the tangential direction, as in [H63]:

$$\begin{aligned} &H^{s,t}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) \mid \langle\xi\rangle^s\langle\xi'\rangle^t\hat{u}(\xi) \in L_2(\mathbb{R}^n)\} \text{ with norm}\\ &\quad \|u\|_{s,t} = (2\pi)^{-\frac{n}{2}}\|\langle\xi\rangle^s\langle\xi'\rangle^t\hat{u}(\xi)\|_{L_2(\mathbb{R}^n)}, \end{aligned} \tag{10.132}$$

for $s,t\in\mathbb{R}$. Note that

$$\|u\|_{s,t} \leq \|u\|_{s',t'} \text{ when } s\leq s',\ s+t\leq s'+t'. \tag{10.133}$$

From these spaces we can define the related Hilbert spaces over the half-space:

$$\begin{aligned} &H^{s,t}(\mathbb{R}^n_+) = \{u \in \mathscr{D}'(\mathbb{R}^n_+) \mid u = U|_{\mathbb{R}^n_+} \text{ for some } U \in H^{s,t}(\mathbb{R}^n)\} \text{ with norm}\\ &\quad \|u\|_{s,t} = \inf_{\text{such } U}\|U\|_{s,t},\\ &H_0^{s,t}(\overline{\mathbb{R}}^n_+) = \{u \in H^{s,t}(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\mathbb{R}}^n_+\}, \end{aligned} \tag{10.134}$$

as mentioned for the case $t=0$ in (9.18), (9.25). It is shown in [H63, Sect. 2.5] that for all s,t, $H^{s,t}(\mathbb{R}^n_+)$ and $H_0^{-s,-t}(\overline{\mathbb{R}}^n_+)$ are dual spaces of one another, with a duality extending the scalar product in $L_2(\mathbb{R}^n_+)$. $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ is dense in each space $H^{s,t}(\mathbb{R}^n_+)$, and $C_0^\infty(\mathbb{R}^n_+)$ is dense in each space $H_0^{s,t}(\overline{\mathbb{R}}^n_+)$. For $s=m\in\mathbb{N}_0$, one has a more elementary characterization of $H^{m,t}(\mathbb{R}^n_+)$ and

$H_0^{m,t}(\overline{\mathbb{R}}^n_+)$, as the closures of $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ resp. $C_0^\infty(\mathbb{R}^n_+)$ with respect to the norm

$$\begin{aligned}\|u\|_{m,t,\prime} &= \big((2\pi)^{1-n}\sum_{j=0}^{m}\|\langle\xi'\rangle^{(t+m-j)}D^j_{x_n}\acute{u}(\xi',x_n)\|^2_{L_2(\mathbb{R}^n_+)}\big)^{\frac12}\\ &= \big(\sum_{j=0}^{m}\|D^j_{x_n}u(x',x_n)\|^2_{0,t+m-j}\big)^{\frac12} \simeq \big(\sum_{|\alpha|\le m}\|D^\alpha u\|^2_{0,t}\big)^{\frac12},\end{aligned} \tag{10.135}$$

which is just a slight generalization of (9.6). We shall give proof details for this case, and refer to e.g. [G96] for the documentation that the various properties extend to suitable cases with noninteger first exponent.

For $m = 0$ it is easy to see (using the method of Theorem 6.15 in the tangential variables x') that

$$H^{0,t}(\mathbb{R}^n_+) \text{ and } H^{0,-t}(\mathbb{R}^n_+) \text{ are dual spaces of one another.} \tag{10.136}$$

In the following, we leave out the indications $\wedge$ and $\prime$ in the norms.

Theorem 10.29. 1° *Let K be a Poisson operator of order $d \in \mathbb{R}$, with symbol-kernel $\tilde{k}(x',y',x_n,\xi') \in S^{d-1}_{1,0}(\mathbb{R}^{2(n-1)},\mathbb{R}^{n-1},\mathscr{S}_+)$, compactly supported with respect to x' and y'. For all $m \in \mathbb{Z}$, all $t \in \mathbb{R}$, there are estimates for $v \in \mathscr{S}(\mathbb{R}^{n-1})$:*

$$\|Kv\|_{m,t-d+\frac12} \le c\|v\|_{m+t};\ \textit{in particular } \|Kv\|_m \le c\|v\|_{m+d-\frac12}. \tag{10.137}$$

2° *Let T be a trace operator of class 0 and order d, with symbol-kernel $\tilde{t}(x',y',x_n,\xi') \in S^d_{1,0}(\mathbb{R}^{2(n-1)},\mathbb{R}^{n-1},\mathscr{S}_+)$, compactly supported with respect to x' and y'. Then T is the adjoint of the Poisson operator K with symbol-kernel*

$$\tilde{k}(x',y',x_n,\xi') = \overline{\tilde{t}}(y',x',x_n,\xi'); \tag{10.138}$$

in the sense that

$$(Tu,v)_{L_2(\mathbb{R}^{n-1})} = (u,Kv)_{L_2(\mathbb{R}^n_+)} \textit{ for } u \in \mathscr{S}(\overline{\mathbb{R}}^n_+), v \in \mathscr{S}(\mathbb{R}^{n-1}). \tag{10.139}$$

All trace operators of order d and class 0 (with symbol-kernel compactly supported in x',y') are obtained from Poisson operators of order $d+1$ (with symbol-kernel compactly supported in x',y') in this way, and vice versa.

For all $m \in \mathbb{N}_0$, all $t \in \mathbb{R}$, there are estimates for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$:

$$\|Tu\|_{m+t-d-\frac12} \le c\|u\|_{0,m+t} \le c\|u\|_{m,t},\ \textit{in particular } \|Tu\|_{m-d-\frac12} \le c\|u\|_m. \tag{10.140}$$

3° *By extension by continuity, the operators define mappings from the full spaces, satisfying the estimates in* (10.137)–(10.140). *The estimates* (10.137) *are also valid with m replaced by any $s \in \mathbb{R}$, and the estimates* (10.140) *are valid with m replaced by any $s \in \overline{\mathbb{R}}_+$. In 2°,*

$T : L_2(\mathbb{R}^n_+) \to H^{-d-\frac{1}{2}}(\mathbb{R}^{n-1})$ *and* $K : H^{d+\frac{1}{2}}(\mathbb{R}^{n-1}) \to L_2(\mathbb{R}^n_+)$ *are adjoints.* (10.141)

Proof. We have for $v \in \mathscr{S}(\mathbb{R}^{n-1})$, $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$:

$$(Kv, u)_{L_2(\mathbb{R}^n_+)} = \int e^{i(x'-y')\cdot\xi'} \tilde{k}(x', y', x_n, \xi') v(y') \bar{u}(x', x_n)\, dy' đ\xi' dx' dx_n \tag{10.142}$$

(oscillatory integral w.r.t. ξ'). Here we can insert the Fourier transforms of v w.r.t. y' and u w.r.t. x', and reinterpret the integrations against the exponential functions to Fourier transforms w.r.t. x' and y' of $\tilde{k}$, just as in the proof of Theorem 7.5, obtaining the expression

$$\int \tilde{k}(\widehat{\theta' - \xi'}, \widehat{\xi' - \eta'}, x_n, \xi') \hat{v}(\eta') \overline{\hat{u}}(\theta', x_n)\, đ\xi' đ\eta' đ\theta' dx_n.$$

The $\tilde{k}$ factor is estimated with respect to the primed variables similarly as in the proof of Theorem 7.5, when we consider it as a function on $\mathbb{R}^{3(n-1)}$ valued in $L_{2,x_n}(\mathbb{R}_+)$. This gives

$$\|\tilde{k}(\widehat{\theta'}, \widehat{\eta'}, x_n, \xi')\|_{L_2(\mathbb{R}_+)} \le M_N \langle\xi'\rangle^{d-\frac{1}{2}} \langle\theta'\rangle^{-2N} \langle\eta'\rangle^{-2N},$$

where N can be taken arbitrarily large. Hence, using the Cauchy-Schwarz inequality w.r.t. x_n,

$$\begin{aligned} &|(Kv, u)| \le \\ &\quad c \int \langle\theta' - \xi'\rangle^{-2N} \langle\xi' - \eta'\rangle^{-2N} \langle\xi'\rangle^{d-\frac{1}{2}} |\hat{v}(\eta')| \|\acute{u}(\theta', x_n)\|_{L_2(\mathbb{R}_+)}\, đ\xi' đ\eta' đ\theta'. \end{aligned}$$

Finally, using the Peetre inequality as in Theorem 7.5 and the Cauchy-Schwarz inequality for the integrals over $\mathbb{R}^{3(n-1)}$, followed by integrating out superfluous variables, we arrive at

$$|(Kv, u)| \le c' \|v\|_s \|u\|_{0, d-\frac{1}{2}-s}.$$

Since this holds for all u, we conclude in view of (10.136) that

$$\|Kv\|_{0, s-d+\frac{1}{2}} \le c' \|v\|_s.$$

This shows (10.137) with $m = 0$. For $m \le 0$, it follows immediately in view of (10.133):

$$\|Kv\|_{m, s-d+\frac{1}{2}} \le \|Kv\|_{0, m+s-d+\frac{1}{2}} \le c' \|v\|_{m+s}, \text{ for } m \le 0.$$

To show it for $m > 0$, note that $D^\alpha K$ is a Poisson operator of order $d + |\alpha|$, so

$$\|Kv\|_{m,s-d+\frac{1}{2}} \le c\big(\textstyle\sum_{|\alpha|\le m} \|D^\alpha Kv\|_{0,s-d+\frac{1}{2}}\big)^{\frac{1}{2}}$$
$$\le c'\big(\textstyle\sum_{|\alpha|\le m} \|v\|_{s+|\alpha|}\big)^{\frac{1}{2}} \le c''\|v\|_{s+m}.$$

This implies (10.137).

For 2°, the formula (10.139) for T and K with symbol-kernels as in (10.138) follows by changing the order of integration in (10.142). Then we have by the preceding proof:

$$|(Tu, v)| = |(u, Kv)| \le c\|u\|_{0,d+\frac{1}{2}-s}\|v\|_s,$$

for any s, hence with $t = d + \frac{1}{2} - s$,

$$\|Tu\|_{t-d-\frac{1}{2}} \le c\|u\|_{0,t}.$$

Replacing t by $m + t$ with $m \ge 0$, we find moreover

$$\|Tu\|_{m+t-d-\frac{1}{2}} \le c\|u\|_{0,m+t} \le c\|u\|_{m,t},$$

in view of (10.133). This implies (10.140).

3° The extension by continuity is obvious. The validity for general s, as indicated, can be deduced from the preceding statements by interpolation; this is a technique explained e.g. in [LM68] and we refrain from giving details here. □

Corollary 10.30. *When $T = \sum_{0\le j<r} S_j\gamma_j + T'$ is a trace operator of class r and order d, with symbol compactly supported with respect to x' and y', and $s > r - \frac{1}{2}$, $s \ge 0$, $t \in \mathbb{R}$, then T defines a continuous mapping:*

$$T : H^{s,t}(\mathbb{R}^n_+) \to H^{s+t-d-\frac{1}{2}}(\mathbb{R}^{n-1}). \tag{10.143}$$

Proof. For the part T' of class 0 this follows from Theorem 10.29. For the terms $S_j\gamma_j$, it follows by a straightforward generalization of the estimate for γ_0 shown in Remark 9.4: For $u \in \mathscr{S}(\mathbb{R}^n)$, $s > j + \frac{1}{2}$,

$$\begin{aligned}
\|D_n^j u(x', 0)\|^2_{s+t-j-\frac{1}{2}} &= \int_{\mathbb{R}^{-1}} |D_n^j \acute{u}(\xi', 0)|^2 \langle\xi'\rangle^{2s+2t-2j-1}\, d\!\!{}^-\xi' \\
&\le c \int_{\mathbb{R}^{n-1}} \langle\xi'\rangle^{2s+2t-2j-1} \Big(\int_{\mathbb{R}} |\xi_n^j \hat{u}(\xi)|\, d\!\!{}^-\xi_n\Big)^2 d\!\!{}^-\xi' \\
&\le c \int_{\mathbb{R}^{n-1}} \langle\xi'\rangle^{2s+2t-2j-1} \Big(\int_{\mathbb{R}} |\hat{u}(\xi)|^2 \langle\xi\rangle^{2s}\, d\!\!{}^-\xi_n\Big)\Big(\int_{\mathbb{R}} \langle\xi\rangle^{2j-2s}\, d\!\!{}^-\xi_n\Big) d\!\!{}^-\xi' \\
&= c' \int_{\mathbb{R}^n} \langle\xi'\rangle^{2s+2t-2j-1+2j-2s+1} \langle\xi\rangle^{2s} |\hat{u}(\xi)|^2\, d\!\!{}^-\xi = c'\|u\|^2_{s,t};
\end{aligned}$$

here we used (9.19). This implies the boundedness of the mapping γ_j on the dense subset $\mathscr{S}(\overline{\mathbb{R}}^n_+)$, and it extends by continuity to a bounded mapping from $H^{s,t}(\mathbb{R}^n_+)$ to $H^{s+t-j-\frac{1}{2}}(\mathbb{R}^{n-1})$. This is then combined with the fact that S_j has order $d-j$. □

The operators γ_j do not have adjoints within the calculus, hence neither do trace operators of class $r > 0$.

We can also treat s.g.o.s.

Theorem 10.31. *Let $G = \sum_{0\le j<r} K_j\gamma_j + G'$ be a singular Green operator of class r and order d, with symbol compactly supported with respect to x' and y'. When $s > r - \frac{1}{2}$, $s \ge 0$, $t \in \mathbb{R}$, there are estimates for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$:*

$$\|Gu\|_{s-d,t} \le c\|u\|_{s,t},$$

and hence G defines continuous mappings:

$$G : H^{s,t}(\mathbb{R}^n_+) \to H^{s-d,t}(\mathbb{R}^n_+),\ \text{in particular } G : H^s(\mathbb{R}^n_+) \to H^{s-d}(\mathbb{R}^n_+). \tag{10.144}$$

It is also continuous from $H^{0,t}(\mathbb{R}^n_+) \to H^{0,t-d}(\mathbb{R}^n_+)$, all $t \in \mathbb{R}$.

Proof. For the sum over j this is a consequence of the mapping properties shown in Theorem 10.29 and Corollary 10.30. For the part G' of class 0, one can either appeal to the fact that it is a rapidly convergent series of compositions of Poisson and trace operators of class 0 (one should then account for how the norms depend on the enumeration), or one can work out a proof similarly as in Theorem 10.29: For $u, v \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$,

$$\begin{aligned}(G'u,v)_{L_2(\mathbb{R}^n_+)} &\\ &= \int e^{i(x'-y')\cdot\xi'}\tilde g'(x',y',x_n,y_n,\xi')u(y',y_n)\bar v(x',x_n)\,dy'đ\xi'dy_ndx'dx_n\\ &= \int \tilde g'(\widehat{\theta'-\xi'},\widehat{\xi'-\eta'},x_n,y_n,\xi')\acute u(\eta',y_n)\overline{\acute v}(\theta',x_n)\,đ\xi'đ\eta'đ\theta'dx_ndy_n.\end{aligned} \tag{10.145}$$

One finds from the symbol-kernel estimates satisfied by $\tilde g'$ that

$$\|\tilde g'(\widehat{\theta'},\widehat{\eta'},x_n,y_n,\xi')\|_{L_{2,x_n,y_n}(\mathbb{R}^2_{++})} \le M_N\langle\xi'\rangle^d\langle\theta'\rangle^{-2N}\langle\eta'\rangle^{-2N},$$

any N. Applications of the inequality

$$\int_{\mathbb{R}^2_{++}} |g(x_n,y_n)\varphi(x_n)\psi(y_n)|\,dx_ndy_n \le \|g\|_{L_2(\mathbb{R}^2_{++})}\|\varphi\|_{L_2(\mathbb{R}_+)}\|\psi\|_{L_2(\mathbb{R}_+)}$$

give, with norms in $L_2(\mathbb{R}_+)$,

$$|(G'u,v)| \le \\ c\int\langle\theta'-\xi'\rangle^{-2N}\langle\xi'-\eta'\rangle^{-2N}\langle\xi'\rangle^d\|\acute u(\eta',y_n)\|\|\acute v(\theta',x_n)\|\,đ\xi'đ\eta'đ\theta',$$

from which one finds (as in Theorems 7.5 and 10.29)

$$|(G'u,v)| \le c'\|u\|_{0,s}\|v\|_{0,d-s},$$

allowing the conclusion

$$\|G'u\|_{0,s-d} \le c'\|u\|_{0,s},\ \text{any } s \in \mathbb{R}; \tag{10.146}$$

which implies the last statement in the theorem. Let $s \ge 0$. We can write $s-d = m+s'$ with $m \in \mathbb{N}_0$ and $s' < 0$, and apply (10.146) to the operators $D^\alpha G'$ of order $d+|\alpha|$ and class 0. This gives in view of (10.133):

$$\|G'u\|_{s-d,t} = \|G'u\|_{m+s',t} \le \|G'u\|_{m,s'+t} \le c\big(\textstyle\sum_{|\alpha|\le m} \|D^\alpha G'u\|^2_{0,s'+t}\big)^{\frac12}$$
$$\le c'\big(\textstyle\sum_{|\alpha|\le m} \|u\|^2_{0,s'+d+|\alpha|+t}\big)^{\frac12} \le c''\|u\|_{0,s'+d+m+t} = c''\|u\|_{0,s} \le c''\|u\|_{s,t},$$

showing the continuity in (10.144). □

For ψdo's satisfying the transmission condition, we have:

Theorem 10.32. *Let $P = \mathrm{OP}(p(x',y',\xi))$ where p is a ψdo symbol of order $d \in \mathbb{Z}$ satisfying the transmission condition and compactly supported in (x',y'). Then for all $m \in \mathbb{N}_0$, $t \in \mathbb{R}$, there are estimates for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$:*

$$\|P_+u\|_{m-d,t} \le c\|u\|_{m,t},\ \textit{in particular } \|P_+u\|_{m-d} \le c\|u\|_m, \tag{10.147}$$

extending by continuity to $u \in H^{m,t}(\mathbb{R}^n_+)$ resp. $H^m(\mathbb{R}^n_+)$.

Proof. As noted in (10.8), we can write p as the sum of a symbol $\sum_{0\le l\le d} s_l(x',y',\xi')\xi_n^l$ where the s_l are polynomials in ξ', and a symbol p' which satisfies

$$|D^\beta_{x'}D^\alpha_\xi p'(x',\xi)| \le c\langle\xi'\rangle^{d-|\alpha|+1}\langle\xi\rangle^{-1}.$$

In the following we only need the slightly weaker estimates

$$|D^\beta_{x'}D^\alpha_\xi p'(x',\xi)| \le c\langle\xi'\rangle^{d-|\alpha|};$$

they would allow us to include in p' the term that is constant in ξ_n, but this makes no difference for the following. The first part of p reduces to x'-form by a finite and exact version of (7.38); this gives a differential operator of order d, which is known to satisfy (10.147) (by considerations as in Chapter 6). For the other part $P' = \mathrm{OP}(p')$, we can apply the argumentation in the proof of Theorem 7.5 with respect to the tangential variables only; this gives for $w \in \mathscr{S}(\mathbb{R}^n)$, any $r \in \mathbb{R}$:

$$\|P'w\|_{0,r-d} \le c\|w\|_{0,r},$$

which extends to $w \in H^{0,r}(\mathbb{R}^n)$, and hence implies for u in $H^{0,r}(\mathbb{R}^n_+)$ (identifiable with a closed subspace of $H^{0,r}(\mathbb{R}^n)$ by extension by zero),

$$\|P'_+u\|_{0,r-d} \le c\|u\|_{0,r}.$$

If $m \le d$, this implies (cf. (10.133))

$$\|P'_+u\|_{m-d,t} \le \|P'_+u\|_{0,m-d+t} \le c\|u\|_{0,m+t} \le c\|u\|_{m,t},$$

showing the desired estimate for P' and hence also for P, when $m \le d$.

Now let $m > d$ and let $m' = m - d$. Note that $D^\alpha P_+ = (D^\alpha P)_+$, where $D^\alpha P$ is a ψdo of order $d+|\alpha|$ satisfying the transmission condition. Applying the preceding considerations to $(D^\alpha P)_+$ for $|\alpha| \le m'$, we find

$$\|P_+u\|_{m-d,t} = \Big(\sum_{|\alpha|\le m'} \|(D^\alpha P)_+u\|^2_{0,t}\Big)^{\frac12} \le c\Big(\sum_{|\alpha|\le m'} \|u\|^2_{d+|\alpha|,t}\Big)^{\frac12} \le c'\|u\|_{m,t}.$$

□

Remark 10.33. For ψdo's with symbols depending on the normal variable, the result is obtained after an extra reduction. Consider for example a symbol in (x', y', y_n)-form, $p(x', y', y_n, \xi)$. Take a Taylor expansion

$$p(x', y', y_n, \xi) = \sum_{0\le k<K} \tfrac{1}{k!}\partial^k_{y_n} p(x', y', 0, \xi) y_n^k + p_{(K)}(x', y', y_n, \xi) y_n^K.$$

The terms in the sum define operators that map functions in $H^m(\mathbb{R}^n_+)$ with support in a fixed bounded set continuously into $H^{m-d}(\mathbb{R}^n_+)$, by Theorem 10.32. For a given m, we can take $K \ge m$ so that the functions $y_n^K u$ extend by zero to functions in $H^m(\mathbb{R}^n)$; then $\mathrm{OP}(p_{(K)})_+$ maps them continuously into $H^{m-d}(\mathbb{R}^n_+)$.

The result extends to noninteger $s \ge 0$ (in the place of m) by interpolation. There is moreover an extension down to $s > -\frac12$; this hinges on the fact that $H^{s,t}(\mathbb{R}^n_+)$ and $H_0^{s,t}(\overline{\mathbb{R}}^n_+)$ coincide for $-\frac12 < s < \frac12$ (one can show that multiplication by $1_{x_n>0}$ is bounded in $H^{s,t}(\mathbb{R}^n)$ for $|s| < \frac12$). The mapping properties in (10.143) and (10.144) likewise extend to $s > -\frac12$, when $r = 0$.

We define "loc" and "comp" versions of the Sobolev spaces in the usual way:

$$\begin{aligned} H^s_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+) &= \{u \in \mathscr{D}'(\mathbb{R}^n_+) \mid \varphi u \in H^s(\mathbb{R}^n_+) \text{ for any } \varphi \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)\}, \\ H^s_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+) &= \{u \in H^s(\mathbb{R}^n_+) \mid \operatorname{supp} u \text{ is compact in } \overline{\mathbb{R}}^n_+\}; \end{aligned} \tag{10.148}$$

the former is a Fréchet space and the latter an inductive limit of such spaces. Note that by the Sobolev embedding theorem,

$$\bigcap_{s\ge0} H^s_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+) = C^\infty(\overline{\mathbb{R}}^n_+), \quad \bigcap_{s\ge0} H^s_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+) = C^\infty_{(0)}(\overline{\mathbb{R}}^n_+).$$

Then one can obtain as a corollary of the preceding statements:

Theorem 10.34. *When $\mathcal{A}$ is a ψdbo (a Green operator) as in* (10.18) *with entries of order d and class r, it defines continuous mappings from $H^s_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+)^N \times H^{s-\frac12}_{\mathrm{comp}}(\mathbb{R}^{n-1})^M$ to $H^{s-d}_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+)^{N'} \times H^{s-d-\frac12}_{\mathrm{loc}}(\mathbb{R}^{n-1})^{M'}$ for $s > r-\frac12$; it also maps continuously as indicated in* (10.18).

Remark 10.35. A few more observations on adjoints: It was mentioned in (10.40) that the adjoint of a singular Green operator in (x',y')-form $G = \mathrm{OPG}(\tilde g(x',y',x_n,y_n,\xi'))$ of order d and class 0 is a singular Green operator G_1 of the same kind, with symbol-kernel $\tilde g_1 = \overline{\tilde g}(y',x',y_n,x_n,\xi')$. This is seen (when the symbol-kernel is compactly supported in x',y') by changing the order of integration in the first line of (10.145). Here G and G_1 are contained in each other's adjoints as mappings in $\mathscr{S}(\overline{\mathbb{R}}^n_+)$, extending to the duality between $G : H^{0,t}(\mathbb{R}^n_+) \to H^{0,t-d}(\mathbb{R}^n_+)$ and $G_1 : H^{0,d-t}(\mathbb{R}^n_+) \to H^{0,-t}(\mathbb{R}^n_+)$, any $t \in \mathbb{R}$. Relations between symbols in x'-form or y'-form are found by further reductions using Theorem 7.13 1° in the primed variables.

For ψdo's P, we note that when P is of order ≤ 0 and properly supported, P_+ and $(P^*)_+$ are adjoints, since

$$\begin{aligned}(P_+u,v)_{L_2(\mathbb{R}^n_+)} &= (Pe^+u,e^+v)_{L_2(\mathbb{R}^n)} = (e^+u,P^*e^+v)_{L_2(\mathbb{R}^n)}\\ &= (u,(P^*)_+v)_{L_2(\mathbb{R}^n_+)},\end{aligned} \tag{10.149}$$

for $u,v \in C_0^\infty(\mathbb{R}^n_+)$, extending to $L_2(\mathbb{R}^n_+)$ by approximation.

10.6 Elliptic ψdbo's

For the study of invertible elements in our "algebra" of operators, we need to define the concept of *ellipticity*. This really consists of two conditions, namely, invertibility of the principal interior symbol and invertibility of the principal boundary symbol operator.

Definition 10.36. Let $\mathcal{A}$ be a ψdbo (10.18) — a Green operator — of order d and class r, with symbols $p(x,\xi)$, $g(x',\xi,\eta_n)$, $t(x',\xi)$, $k(x',\xi)$ and $s(x',\xi')$.

1° The **principal interior symbol** is the symbol $p^0(x,\xi)$. The **principal boundary symbol operator** is the operator

$$\mathfrak{a}^0(x',\xi',D_n) = \begin{pmatrix} p^0(x',0,\xi',D_n)_+ + g^0(x',\xi',D_n) & k^0(x',\xi',D_n)\\ t^0(x',\xi',D_n) & s^0(x',\xi')\end{pmatrix},$$

going from $\mathscr{S}(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^M$ to $\mathscr{S}(\overline{\mathbb{R}}_+)^{N'} \times \mathbb{C}^{M'}$. The interior symbol, resp. boundary symbol operator, are defined similarly from the full symbols.

2° $\mathcal{A}$ is said to be **elliptic**, when p^0 is bijective for all $|\xi| \ge 1$ and all x, and $\mathfrak{a}^0$ is bijective for all $|\xi'| \ge 1$ and all x'. (In particular, $N = N'$ then.)

When P alone is known to be elliptic, it can be shown that $p^0(x',\xi',D_n)_+$ is a Fredholm operator in $\mathscr{S}^N_+$ (and between suitable Sobolev spaces over

$\overline{\mathbb{R}}_+$, the nullspace and range complement being the same as for $\mathscr{S}_+^N$), with an index depending continuously on (x',ξ'). A necessary condition for the ellipticity of the full system $\mathcal{A}$ is then that $M'-M = \text{index } p^0$.

The following theorem holds:

Theorem 10.37. *When $\mathcal{A}$ is elliptic, the inverse $\mathfrak{b}^0$ of the principal boundary symbol operator $\mathfrak{a}^0$ belongs to the calculus; it is a boundary symbol operator of order $-d$ and class $r' = \max\{r-d, 0\}$.*

Proof (indications). The proof of this theorem in the differential operator case, where the ψdbo is as in (10.19) (with $M = 0$), is not so hard, since one can find the inverse constructively by analysis of the solutions in $\mathscr{S}_+^N$ of the equation $p^0(x',0,\xi',D_n)u = 0$ in terms of the polynomial $\det p^0(x',0,\xi',\xi_n)$ in ξ_n (cf. Example 10.39 below); it is found that the symbol ingredients in the inverse of the principal symbol are rational functions of ξ_n. In the general pseudodifferential case, another technique is needed. Here one can reduce, by composition with suitable invertible operators, to the situation where $P = I$ and the other operators are of order and class 0. Then the boundary symbol operator is similar to an operator of the form $I + g(x',\xi',D_n)$. Now we can use the Laguerre expansions to write $g(D_n)$ as

$$\begin{aligned} g(x',\xi',D_n) &= \sum_{l,m\in\mathbb{N}_0} c_{lm}(x',\xi')k_l(\xi',D_n)t_m(\xi',D_n);\\ k_l &= \mathrm{OPK}_n(\hat{\varphi}_l(\xi_n,\langle\xi'\rangle)), \quad t_m = \mathrm{OPT}_n(\overline{\hat{\varphi}}_m(\xi_n,\langle\xi'\rangle)), \end{aligned}$$

cf. (10.104). When $I + g(x',\xi',D_n)$ is invertible, we can for large enough M replace g (for x' in a compact set, all $|\xi'| \geq 1$) by

$$g_M(x',\xi',D_n) = \sum_{l,m\leq M} c_{lm}(x',\xi')k_l t_m;$$

such that $I + g_M$ is still invertible (since the c_{lm} are rapidly decreasing). This operator is, for each (x',ξ'), invertible on the finite dimensional space V_M spanned by the orthonormal system $(\varphi_l)_{0\leq l\leq M}$, where it acts (with respect to the basis) as the matrix $I + (c_{lm}(x',\xi'))_{l\leq M, m\leq M}$. This is an elliptic ψdo symbol in the standard sense, so the inverse is again a ψdo symbol. One can trace this inverse back to an explicit representation of $(\mathfrak{a}^0)^{-1}$, showing that it belongs to the calculus. (Further details are given e.g. in [G96], or in a more elementary form in the first edition from 1986.) □

This allows a parametrix construction:

Theorem 10.38. *When $\mathcal{A}$ is elliptic, there exists a ψdbo $\mathcal{B}$ (with principal boundary symbol operator $\mathfrak{b}^0$ as in Theorem* 10.37*), which is a* **parametrix** *of $\mathcal{A}$, in the sense that*

$$\mathcal{A}\mathcal{B} - I \text{ and } \mathcal{B}\mathcal{A} - I \text{ are negligible.} \tag{10.150}$$

Proof (indications). A first approximation to $\mathcal{B}$ is the operator $\mathcal{B}'$ with boundary symbol $\mathfrak{b}^0$ and interior symbol q, where $q(x,\xi)$ is a parametrix symbol for p. Then $\mathcal{A}\mathcal{B}'$ equals the identity plus a Green operator with interior symbol ~ 0 and with boundary symbol of order -1. Proceeding as in the proof of Theorem 7.18, one can improve $\mathcal{B}'$ to a parametrix $\mathcal{B}$ (still of order $-d$ and class $r' = \max\{r-d, 0\}$), such that $\mathcal{A}\mathcal{B} - I$ is of order $-\infty$. There is an equivalent left parametrix. □

One-sided ellipticity (surjective or injective) can also be considered (some cases of this are discussed in Section 11.2).

Example 10.39. Let $a^0(x',\xi)$ be the principal symbol at $x_n = 0$ of an elliptic partial differential operator A on $\mathbb{R}^n$ of order d, possibly $N \times N$-matrix-formed. Along with A there is given a trace operator $T = \{T_0, T_1, \dots T_{d-1}\}$, where T_j is $M_j \times N$-matrix-formed with $0 \le M_j \le d-1$ (the cases $M_j = 0$ give void boundary conditions, and are just included for notational convenience), each T_j of the form $\sum_{0\le k\le j} S_{jk}\gamma_k$, with differential operators S_{jk} on $\mathbb{R}^{n-1}$ of order $j-k$ having principal symbols $s^0_{jk}(x',\xi')$. We consider the boundary value problem

$$Au = f \text{ on } \mathbb{R}^n_+, \quad Tu = \varphi \text{ on } \mathbb{R}^{n-1},$$

for given vector functions f and φ, with u sought in a suitable Sobolev space in terms of the spaces where f and φ are given. The problem defined by the principal boundary symbol operator

$$\begin{aligned} a^0(x',\xi',D_n)u(x_n) &= f(x_n) \text{ on } \mathbb{R}_+, \\ t^0_j u \equiv \sum_{k\le j} s^0_{jk}(x',\xi')\gamma_k u &= \varphi_j \text{ at } x_n = 0 \quad (0 \le j \le d-1), \end{aligned} \tag{10.151}$$

is considered for all (x',ξ') with $\xi' \neq 0$, usually called the *model problem.*

The problem is easily reduced to a semihomogeneous problem by use of the inverse symbol $q^0(x',\xi',\xi_n) = (a^0)^{-1}$, defined for $\xi' \neq 0$. In fact, $a^0_+ q^0_+ = I$ on $\mathbb{R}_+$, since a^0 is a differential operator (then $L(a^0, q^0) = 0$, cf. Theorem 10.26). Thus when we set $z(x_n) = u(x_n) - (q^0_+ f)(x_n)$, we get the problem for z,

$$a^0(x',\xi',D_n)z(x_n) = 0 \text{ on } \mathbb{R}_+, \quad t^0_j z = \psi_j \text{ at } x_n = 0 \quad (0 \le j \le d-1), \tag{10.152}$$

where $\psi_j = \varphi_j - t^0_j q^0_+ f$.

For each $(x',\xi') \in \mathbb{R}^{2(n-1)}$, the symbol defines a polynomial in τ,

$$a^0(x',\xi',\tau) = \textstyle\sum_{0\le l\le d} s_l(x',\xi')\tau^l; \tag{10.153}$$

the s_l being polynomials in ξ' of degree $d-l$. By the ellipticity, the coefficient of τ^d, $s_d = s_d(x')$, is a bijective matrix function (check by inserting $(\xi',\xi_n) = (0,1)$). So $\det a^0(x',\xi',\tau)$ has Nd roots in $\mathbb{C}$, counted with multiplicity. When $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, the roots lie in $\mathbb{C} \setminus \mathbb{R}$, for otherwise a root $\tau_0 \in \mathbb{R}$ would give

$\det a^0(x',\xi',\tau_0) = 0$ contradicting the ellipticity. Say $m_+(x',\xi')$ roots lie in $\mathbb{C}_+$ (cf. (A.3)), $m_-(x',\xi') = Nd - m_+(x',\xi')$ roots lie in $\mathbb{C}_-$.

When $n \geq 3$, each $\xi' \neq 0$ can be connected to $-\xi'$ by a curve running in $\mathbb{R}^{n-1} \setminus \{0\}$. The number $m_+(x',\xi')$ must be constant along such a curve (the roots in $\mathbb{C}_+$ can be encircled by a large closed curve in $\mathbb{C}_+$, and the polynomial coefficients depend continuously on ξ', so the collected algebraic multiplicity of these roots must be continuous in ξ', by the theorem of Rouché). Then $m_+(x',\xi') = m_+(x',-\xi') = m_-(x',\xi')$. This must then equal $\frac{1}{2}Nd$, constant in (x',ξ') (in particular, Nd must be even). Symbols with the property $m_+(x',\xi') \equiv \frac{1}{2}Nd$ are called *properly elliptic*. Strongly elliptic symbols are properly elliptic.

It is a standard result in ODE that the solutions of $a^0(x',\xi',D_n)z = 0$ are linear combinations of terms $f_j(x_n)e^{i\tau_j x_n}$ where f_j is a (vector valued) polynomial in x_n and τ_j is a root. For the solutions lying in $\mathscr{S}_+^N$, only the terms where $\tau_j \in \mathbb{C}_+$ are nontrivial. More precisely, the space of null-solutions in $\mathscr{S}_+^N$ is spanned by exponential polynomials $f_j(x_n)e^{i\tau_j x_n}$ with $\tau_j \in \mathbb{C}_+$, and has dimension $m_+(x',\xi')$. (Similarly, the space of null-solutions in $\mathscr{S}(\overline{\mathbb{R}}_-)^N$ is spanned by exponential polynomials $f_j(x_n)e^{i\tau_j x_n}$ with $\tau_j \in \mathbb{C}_-$, and has dimension $m_-(x',\xi')$.) When a^0 is properly elliptic, then it is necessary for ellipticity of $\{a^0, t^0\}$ that the dimension $M = M_0 + \cdots + M_{d-1}$ (the number of boundary conditions) equals $\frac{1}{2}Nd$.

The construction of solutions to the model problem is explained in various ways e.g. in [ADN64], [LM68]. A more global construction goes by use of the so-called Calderón projector, which we take up in Chapter 11.

From here on, the construction of a calculus on manifolds with boundary goes very much like in the theory for ψdo's on manifolds without boundary.

The various types of operators can be defined on a compact manifold X with boundary $\partial X = X'$ by use of local coordinates, as already indicated in Section 10.1 around (10.42). In view of Proposition 10.11, the singular ingredients only need special care in a neighborhood of X'; here one can define the local coordinate systems in terms of a covering of the boundary with open sets with an associated normal coordinate that matches on overlapping sets. One can then deduce the following mapping property from Theorem 10.34:

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} H^s(X)^N \\ \times \\ H^{s-\frac{1}{2}}(X')^M \end{matrix} \to \begin{matrix} H^{s-d}(X)^{N'} \\ \times \\ H^{s-d-\frac{1}{2}}(X')^{M'} \end{matrix}, \quad s > r + \tfrac{1}{2}, \tag{10.154}$$

for matrix-formed operators of order d and class r; there is a similar result for operators between sections of vector bundles.

In the elliptic case there is a parametrix $\mathcal{B}$ (of class $r' = \max\{d-r, 0\}$) going in the opposite direction. One uses the parametrix explained in Theorem 10.38 in coordinate patches intersecting the boundary, and in coordinate patches at a distance from the boundary, one simply uses $\left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$ as symbol of $\mathcal{B}$. The local pieces are put together as in the proof of Theorem 8.6. Then

(10.150) holds with remainder operators $\mathcal{R}_1$ and $\mathcal{R}_2$ negligible of class r' resp. r, hence compact in Sobolev spaces of sufficiently high order. From this, one can show regularity of solutions of problems $\mathcal{A}\{u, \varphi\} = \{f, \psi\}$, and deduce that the elliptic ψdbo $\mathcal{A}$ is a Fredholm operator (in spaces (10.154)). There is a vast literature on the index, beginning with Atiyah and Bott [AB64], see also e.g. [B71], [RS82], [H85], [Gi85], [G96]. The noncommutative residue for ψdbo's was introduced in Fedosov, Golse, Leichtnam and Schrohe [FGLS96]; this and the canonical trace are futher studied e.g. in [G08] and its references.

In the treatment of differential operator problems, an advantage of the present theory in comparison with classical methods of "*a priori* estimates" (as in [ADN64], [LM68]) is that the full solution operator for a given boundary value problem is found in a constructive way.

The theory has been extended to L_p-based Sobolev-type spaces ($1 < p < \infty$) in [G90]. A full presentation of the treatment of elliptic systems is given there, with optimal mapping properties (including operators $P_+ + G$ of negative class). Nonsmooth x-dependence is treated in Abels [A05].

There exist other general theories for boundary value problems. To mention a few: Schulze and coauthors have dealt with ψdo's not satisfying the transmission condition in many works, see e.g. [S98] and its references, which also deal with ψdo's on manifolds with singularities. Melrose and coauthors have treated singular situations by different techniques, see e.g. [M93] and its references.

Exercises for Chapter 10

10.1. Show that if the symbol $p(x, \xi)$ is for each x a rational function of ξ for $|\xi| \geq 1$, homogeneous of degree d, then p satisfies the transmission condition.

10.2. *This and the next exercise illustrate the transmission condition in an elementary way.*

Let $p(\xi_n)$ be an x_n-independent symbol in $S^{-1}_{1,0}(\mathbb{R}, \mathbb{R})$ (with variables denoted (x_n, ξ_n)), defining the operator $p(D_n) = \mathrm{OP}_n(p) = \mathscr{F}^{-1} p \mathscr{F}$. Let $\tilde{p}(x_n) = \mathscr{F}^{-1} p(\xi_n)$.

(a) Show that $p \in L_2(\mathbb{R})$ and $\tilde{p} \in L_2(\mathbb{R})$, and moreover that $D_{\xi_n} p \in L_1(\mathbb{R})$ and hence $x_n \tilde{p} \in C^0_\bullet(\mathbb{R})$, the space of continuous functions going to zero for $|x_n| \to \infty$.

(b) Show that $D^k_{x_n}(x^j_n \tilde{p}) = \mathscr{F}^{-1}(\xi^k_n (-D_{\xi_n})^j p) \in C^0_\bullet(\mathbb{R})$, when $j - k \geq 1$.

(c) Show that $\tilde{p}(x_n)$ is C^∞ for $x_n \neq 0$ and is rapidly decreasing for $x_n \to \pm\infty$.

10.3. With p as in Exercise 10.2, consider the operator $p(D_n)_+$, also called p_+,

$$p_+u = r^+p(D_n)e^+u = r^+\mathscr{F}^{-1}(p(\xi_n)f(\xi_n))\,, \quad f = \mathscr{F}(e^+u)\,,$$

for $u \in \mathscr{S}(\overline{\mathbb{R}}_+)$.

(a) Show that $pf \in L_1(\mathbb{R})$ and hence $p_+u \in C^0_\bullet(\overline{\mathbb{R}}_+)$.

(b) Show, by successive applications of the formula

$$x_np_+u = r^+\mathscr{F}^{-1}[-D_{\xi_n}(p(\xi_n)f(\xi_n))] = -\operatorname{OP}_n(D_{\xi_n}p)_+u - \operatorname{OP}_n(p)_+(x_nu),$$

that $x_np_+u \in C^0_\bullet(\overline{\mathbb{R}}_+)$, $\dots$, $x_n^Np_+u \in C^0_\bullet(\overline{\mathbb{R}}_+)$ for all N, hence $(p_+u)(x_n)$ is $O(x_n^{-N})$ for $x_n \to \infty$, any N.

(c) Show, by use of the formula for differentiation of e^+u,

$$D_{x_n}e^+u = e^+D_{x_n}u - iu(0)\delta,$$

that

$$\begin{aligned} D_{x_n}p_+u &= r^+D_{x_n}p(D_n)e^+u \\ &= r^+p(D_n)e^+D_{x_n}u - i(r^+p(D_n)\delta)\cdot u(0) \\ &= p_+D_{x_n}u - ir^+\tilde{p}(x_n)\cdot u(0)\,. \end{aligned}$$

Use this to see that $D_{x_n}p_+u \in C^0(\overline{\mathbb{R}}_+)$ for general $u \in \mathscr{S}(\overline{\mathbb{R}}_+)$ if and only if $r^+\tilde{p}(x_n) \in C^0(\overline{\mathbb{R}}_+)$.

(d) Show (by repeated applications of (c))

$$D^j_{x_n}p_+u = p_+D^j_{x_n}u - i\sum_{0\le l\le j-1} r^+D^l_{x_n}\tilde{p}(x_n)\cdot D^{j-1-l}_{x_n}u(0)\,,$$

and conclude: *In order for p_+u to be in $C^\infty(\overline{\mathbb{R}}_+)$ for all $u \in \mathscr{S}(\overline{\mathbb{R}}_+)$, it is necessary and sufficient that $r^+\tilde{p} \in C^\infty(\overline{\mathbb{R}}_+)$.*

10.4. Verify the asserted symbol and symbol-kernel estimates for (v)–(viii) of Proposition 10.10.

10.5. Let σ be a complex constant with $\operatorname{Re}\sigma > 0$.

(a) Show that

$$\begin{aligned} &h^+\tfrac{1}{\sigma+i\xi_n} = \tfrac{1}{\sigma+i\xi_n}, \quad h^+\tfrac{\xi_n}{\sigma+i\xi_n} = \tfrac{i\sigma}{\sigma+i\xi_n}, \\ &h^+\tfrac{\xi_n^2}{\sigma+i\xi_n} = \tfrac{(i\sigma)^2}{\sigma+i\xi_n}, \end{aligned}$$

and find a formula valid for all powers ξ_n^k in the numerator, $k \in \mathbb{N}_0$.

(b) Show that

$$\begin{aligned} &h^-\tfrac{1}{\sigma-i\xi_n} = \tfrac{1}{\sigma-i\xi_n}, \quad h^-\tfrac{\xi_n}{\sigma-i\xi_n} = i - \tfrac{i\sigma}{\sigma-i\xi_n}, \\ &h^-\tfrac{\xi_n^2}{\sigma-i\xi_n} = i\xi_n + \sigma - \tfrac{\sigma^2}{\sigma-i\xi_n}, \end{aligned}$$

and find a formula valid for all powers ξ_n^k in the numerator.
(*Hint.* Recall the formula $x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + \dots + y^{k-1})$.)

(c) Find, for $k \in \mathbb{N}_0$,

$$h^+ \tfrac{\xi_n^k}{\sigma - i\xi_n} \text{ and } h^- \tfrac{\xi_n^k}{\sigma + i\xi_n}.$$

10.6. (a) With σ as in Exercise 10.5, let $a(\xi_n) = \sigma^2 + \xi_n^2$ and let $q(\xi_n) = a^{-1} = 1/(\sigma^2 + \xi_n^2)$. Find

$$\begin{aligned} &h^+(q), \quad h^+(\xi_n q), \quad h^+(\xi_n^2 q), \\ &h^-(q), \quad h^-(\xi_n q), \quad h^-(\xi_n^2 q), \\ &\int^+ q(\xi_n)\, đ\xi_n, \quad \int^+ \xi_n q(\xi_n)\, đ\xi_n, \text{ and } \int^+ \xi_n^2 q(\xi_n)\, đ\xi_n. \end{aligned}$$

(b) Let $A = -\Delta + m$ on $\mathbb{R}^n$ for some $m > 0$, and let $Q = A^{-1}$. For Q_+ considered on $\mathbb{R}^n_+$, find the trace operators $T_0 = \gamma_0 Q_+$, $T_1 = \gamma_1 Q_+$ and the Poisson operator $K : v(x') \to r^+ Q(v(x') \otimes \delta(x_n))$.

10.7. Let $P = \mathrm{OP}(p(x, y', \xi))$ be of order $d \le -1$ on $\mathbb{R}^n$, where p satisfies the transmission condition at $x_n = 0$, is compactly supported with respect to x' and y', and is independent of y_n.

(a) Show that $T = \gamma_0 P_+$ is a trace operator of class 0 and order d; find its symbol and symbol-kernel.

(b) Show that $K : v(x') \mapsto r^+ P^*(v(x') \otimes \delta(x_n))$ is a Poisson operator of order $d+1$; find its symbol and symbol-kernel.

(c) Show that T and K are adjoints, e.g., as $T : L_2(\mathbb{R}^n_+) \to H^{-d-\frac{1}{2}}(\mathbb{R}^{n-1})$ and $K : H^{d+\frac{1}{2}}(\mathbb{R}^{n-1}) \to L_2(\mathbb{R}^n_+)$.

(d) Show that when P is a differential operator, then the Poisson operator $K : v(x') \mapsto r^+ P^*(v(x') \otimes \delta(x_n))$ is zero.

(e) Does the conclusion of (a)–(c) extend to operators of order ≥ 0?
(*Hint.* The answer is negative. Try a simple example.)

10.8. Consider a symbol $p(x', \xi) \in S^d_{0,1}(\mathbb{R}^n, \mathbb{R}^n)$, independent of x_n and satisfying the transmission condition.

(a) Show that when the symbol of p is written in the form

$$p = \sum_{1 \le j \le d} s_j(x', \xi')\xi_n^j + p'', \quad p'' = \sum_{k \in \mathbb{Z}} a_k(x', \xi')\hat{\psi}_k(\xi_n, \sigma), \; \sigma = [\xi']$$

(cf. (10.8), (10.106) and Remark 10.16), then the bounded part $p'' = \sum_{k\in\mathbb{Z}} a_k \hat{\psi}_k$ defines a "discrete convolution operator" when functions $v \in \mathscr{S}(\mathbb{R})$ are expanded in the Laguerre orthonormal system $\{\varphi_k(x_n, \sigma)\}_{k\in\mathbb{Z}}$, $v = \sum_{m\in\mathbb{Z}} v_m \varphi_m$:

$$p''(x',\xi',D_n)v = \mathscr{F}^{-1}\sum_{k\in\mathbb{Z}} a_k\hat\psi_k \sum_{m\in\mathbb{Z}} v_m\hat\varphi_m$$
$$= \mathscr{F}^{-1}\sum_{k,m\in\mathbb{Z}} a_k v_m\hat\varphi_{k+m} = \sum_{l,m\in\mathbb{Z}} a_{l-m}v_m\varphi_l.$$

(*Hint.* Observe that $\hat\psi_k\hat\varphi_m = \hat\varphi_{k+m}$.)

(b) Show that the truncation to $\mathbb{R}_+$, $p''_+ = p''(x',\xi',D_n)_+$ is then a *Toeplitz operator* with respect to the Laguerre orthonormal system $\{\varphi_k(x_n,\sigma)\}_{k\in\mathbb{N}_0}$ on $\mathbb{R}_+$, namely, the following infinite matrix with diagonals consisting of identical entries:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots \\ a_{-1} & a_0 & a_1 & \\ a_{-2} & a_{-1} & a_0 & \\ \vdots & & \ddots & \ddots \end{pmatrix};$$

here $a_l \to 0$ rapidly for $l \to \pm\infty$.

10.9. Continuing in the notation of Exercise 10.8, show that also the differential operator part can be viewed as a Toeplitz operator.
(*Hint.* For ∂_{x_n}, this is seen from (10.61):

$$r^+\partial_{x_n}u = \sum_{m\in\mathbb{N}_0}\Big(-\sigma u_m\varphi_m + 2\sigma\sum_{0\le j<m}(-1)^{m-1-j}u_m\varphi_j\Big)$$
$$= \sum_{l,m\in\mathbb{N}_0} c_{l-m}u_m\varphi_l, \text{ with}$$
$$c_j = 2\sigma(-1)^{j-1} \text{ for } j>0,\quad c_0=-\sigma,\quad c_j = 0 \text{ for } j<0.$$

Note that these coefficients are large, not rapidly decreasing for $j\to\infty$.)

10.10. Notation as in Exercise 10.8.

(a) Show that the s.g.o. $g^+(p(x',\xi',D_n))$ (cf. (10.126)) is a *Hankel operator* in the Laguerre system, namely, the following infinite matrix whose second-diagonals consist of identical entries:

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & a_4 & \\ a_3 & a_4 & a_5 & \\ \vdots & & & \ddots \end{pmatrix}.$$

(*Hint.* Note that $\varphi_k = J\varphi_{-k-1}$, cf. (10.56).)

(b) Find the corresponding representation for $g^-(p(D_n))$.

10.11 (Order-reducing operators). Define, for $r\in\mathbb{Z}$,

$$\chi_+^r(\xi) = (\langle\xi'\rangle + i\xi_n)^r,\quad \chi_-^r(\xi) = (\langle\xi'\rangle - i\xi_n)^r,$$

and denote the operators with these symbols

$$\Xi_+^r = \mathrm{OP}((\langle\xi'\rangle + i\xi_n)^r), \quad \Xi_-^r = \mathrm{OP}((\langle\xi'\rangle - i\xi_n)^r);$$

they are pseudodifferential operators in the general sense of Chapter 6, but do not satisfy all requirements for having symbols in $S^r_{1,0}$-spaces (since, for large $|\alpha|$, $D^\alpha\langle\xi'\rangle^r$ is not $O(\langle\xi\rangle^{r-|\alpha|})$, but only $O(\langle\xi'\rangle^{r-|\alpha|})$). However, we can still define $(\Xi^r_\pm)_+$ and $G^\pm(\Xi^r_\pm)$, and leftover operators as in (10.123) (with (10.115) and (10.122)).

(a) Show that $G^\pm(\Xi^r_\pm) = 0$ when $r \geq 0$.

(b) Show that when $r < 0$, $G^+(\Xi^r_-) = 0$ and $G^-(\Xi^r_+) = 0$.

(c) Show that

$$(\Xi^r_-)_+(\Xi^t_-)_+ = (\Xi^{r+t}_-)_+$$

for all $r, t \in \mathbb{Z}$; in particular,

$$(\Xi^r_-)_+(\Xi^{-r}_-)_+ = I \text{ on } \mathscr{S}(\overline{\mathbb{R}}^n_+),$$

for all $r \in \mathbb{Z}$.

(d) Show that $(\Xi^r_-)_+$ maps $H^s(\mathbb{R}^n_+)$ continuously into $H^{s-r}(\mathbb{R}^n_+)$ for all $s \geq 0$, and conclude (by use of (c)) that the mapping properties extend to

$$(\Xi^r_-)_+ : H^s(\mathbb{R}^n_+) \xrightarrow{\sim} H^{s-r}(\mathbb{R}^n_+), \text{ for all } s \in \mathbb{Z},$$

with inverse $(\Xi^{-r}_-)_+$.

(e) Show (by taking adjoints) that the maps $(\Xi^r_+)_+$ extend to maps with the properties

$$(\Xi^r_+)_+ : H^s_0(\mathbb{R}^n_+) \xrightarrow{\sim} H^{s-r}_0(\mathbb{R}^n_+), \text{ for all } s \in \mathbb{Z},$$

with inverse $(\Xi^{-r}_+)_+$.

(f) Extend the results to include the statements

$$(\Xi^r_-)_+ : H^{s,t}(\mathbb{R}^n_+) \xrightarrow{\sim} H^{s-r,t}(\mathbb{R}^n_+),$$
$$(\Xi^r_+)_+ : H^{s,t}_0(\mathbb{R}^n_+) \xrightarrow{\sim} H^{s-r,t}_0(\mathbb{R}^n_+),$$

for all $s \in \mathbb{Z}, t \in \mathbb{R}$.

(*Comment.* The mapping properties extend to $s \in \mathbb{R} \setminus \mathbb{Z}$ by interpolation. There is a more refined choice of symbols:

$$\lambda^r_\pm(\xi) = \big(\langle\xi'\rangle\varphi(c\xi_n/\langle\xi'\rangle) \pm i\xi_n\big)^r,$$

where $\varphi \in \mathscr{S}(\mathbb{R})$ with $\varphi(0) = 1$ and $\operatorname{supp} \mathscr{F}^{-1}\varphi \subset \,]-\infty, 0]$, and c is a small positive constant. They work in the same way and have the advantage that $\Lambda^r_\pm = \mathrm{OP}(\lambda^r_\pm)$ belong to the ψdbo calculus, cf. [G90] for details. Both the $\Xi^r_\pm$ and the $\Lambda^r_\pm$ can be used to define similar operators on manifolds with boundary.)