§11. Pseudodifferential methods for boundary value problems

11.1 The Calderón projector.

As an illustration of the usefulness of the systematic ψdbo calculus, we shall briefly explain the definition and application of the Calderón projector C^+ for an elliptic differential operator $A\colon C^\infty(X,E_1)\to C^\infty(X,E_2)$ of order d, as introduced by Calderón [C63], Seeley [S66], [S69], see also Hörmander [H66], Boutet de Monvel [B66], Grubb [G71] and [G77]. Much of this chapter is written in a compact style; it has been included in order to make the material available in textbook form. We begin with a very simple example.

Example 11.1a. Let $\alpha>0$ and consider the differential operator $Au=-u''+\alpha^2u$ on $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_-$. The solutions in $\mathcal{S}'(\mathbb{R})$ of $-u''+\alpha^2u=0$ on $\mathbb{R}$ are spanned by $e^{\alpha x}$ and $e^{-\alpha x}$; the only solution in $L_2(\mathbb{R})$ is 0. The nonhomogeneous equation $-u''+\alpha^2u=f$ is uniquely solved in $L_2(\mathbb{R})$ by $u(x)=\mathcal{F}^{-1}((\alpha^2+\xi^2)^{-1}\hat f(\xi))\in H^2(\mathbb{R})$.

The initial values or Cauchy data are defined for general $u\in H^2_{\mathrm{loc}}(\mathbb{R})$ as the pair $\varrho u=\begin{pmatrix}u(0)\\u'(0)\end{pmatrix}$. When precision is needed, we use the notation ϱ^+, ϱ^- or $\widetilde\varrho$ for the mapping $u\mapsto\begin{pmatrix}u(0)\\u'(0)\end{pmatrix}$ when $u\in H^2(\mathbb{R}_+)$, $u\in H^2(\mathbb{R}_-)$, resp. $u\in H^2_{\mathrm{loc}}(\mathbb{R})$. Let

$$\begin{aligned}&Z_\pm=\{u\in L_2(\mathbb{R}_\pm)\mid Au=0\text{ on }\mathbb{R}_\pm\};\ \text{clearly,}\\&Z_+=\operatorname{span}\{r^+e^{-\alpha x}\}\subset\mathcal{S}(\overline{\mathbb{R}}_+),\quad Z_-=\operatorname{span}\{r^-e^{\alpha x}\}\subset\mathcal{S}(\overline{\mathbb{R}}_-).\end{aligned}$$

Define

$$\begin{aligned}&N_+=\varrho^+Z_+,\ N_-=\varrho^-Z_-;\ \text{clearly,}\\&N_+=\{\begin{pmatrix}\varphi\\-\alpha\varphi\end{pmatrix}\mid\varphi\in\mathbb{C}\},\quad N_-=\{\begin{pmatrix}\varphi\\\alpha\varphi\end{pmatrix}\mid\varphi\in\mathbb{C}\}.\end{aligned}$$

Here N_+ and N_- are linearly independent, so $\mathbb{C}^2=N_+\dot+N_-$. The Calderón projector C^+ is now simply *the projection of* $\mathbb{C}^2$ *onto* N_+ *along* N_-. A straightforward calculation shows:

$$C^+=\begin{pmatrix}\frac12&-\frac1{2\alpha}\\-\frac\alpha2&\frac12\end{pmatrix}.$$

The map $C^-=I-C^+$ projects $\mathbb{C}^2$ onto N_- along N_+. There are linear maps $K^\pm:\mathbb{C}^2\to Z_\pm$ that act as inverses to $\varrho^\pm:Z_\pm\to N_\pm$ and vanish on $N_\mp$, respectively.

The maps $C^\pm$ and $K^\pm$ are easily found explicitly in this case; the efforts in higher dimensional cases go towards determining them in a constructive way. They serve in the disussion of ellipticity and solvability of boundary value problems.

It should be noted that the general theory would take $\varrho u = \{u(0), Du(0)\}$ with a factor $-i$, this is convenient in Green's formulas in higher-order cases.

We consider a smooth compact n-dimensional manifold X with boundary, provided with two N-dimensional hermitian vector bundles E_1 and E_2. (A reader who is not used to working with vector bundles should just think of $(N\times N)$-matrices of operators.) X can be assumed to be smoothly imbedded in an n-dimensional boundaryless manifold $\widetilde{X}$ such that $X' = \partial X$ is an $(n-1)$-dimensional hypersurface in $\widetilde{X}$ and E_1 and E_2 are restrictions to X of N-dimensional hermitian bundles $\widetilde{E}_1$ and $\widetilde{E}_2$ over $\widetilde{X}$. Denote $X^\circ = X_+$, $\widetilde{X}\setminus X = X_-$, and write $\widetilde{E}_i|_{\overline{X}_\pm} = E_{i,\pm}$, $\widetilde{E}_i|_{X'} = E_i'$. We can assume that near the boundary, the manifold and bundles are described as a product situation, with a chosen normal coordinate x_n. More precisely, there is a neighborhood U of X' in $\widetilde{X}$ where the points are represented as $x = (x', x_n)$, $x' \in X'$ and $x_n \in]-1,1[$, such that $x_n = 0$ on the boundary, $x_n > 0$ in $U_+ = X_+ \cap U$ and $x_n < 0$ in $U_- = X_- \cap U$. Moreover, the bundles $\widetilde{E}_i$ are over U simply the liftings of E_i' from X' to $X'\times\,]-1,1[$. Then $D_n = -i\partial_{x_n}$ has a meaning, over U, on the sections of the bundles $\widetilde{E}_1$, $\widetilde{E}_2$. The coordinate systems on U are taken to be of the form $\kappa_j\colon U_j'\times\,]-1,1[\to V_j'\times\,]-1,1[$, where $\kappa_j(x', x_n) = (\kappa_j'(x'), x_n)$, defined from coordinate systems $\kappa_j'\colon U_j' \to V_j'$ for X'; trivializations act similarly. The volume form dx on $\widetilde{X}$ is chosen such that $dx = dx'dx_n$ on U, for a volume form dx' on X' and the Lebesgue measure dx_n on $\mathbb{R}$. In these coordinates, D_n is formally selfadjoint.

If A extends to an elliptic operator (also denoted A) from $C^\infty(\widetilde{E}_1)$ to $C^\infty(\widetilde{E}_2)$, we let Q denote an parametrix of A on $\widetilde{X}$; then the formulas of Theorem 8.6 hold with $P = A$. The use of Calderón projectors is simplest if $\widetilde{X}$, $\widetilde{E}_1$, $\widetilde{E}_2$ and A can be chosen so that $\widetilde{X}$ is compact and A is *invertible* on $\widetilde{X}$; then Q stands for the inverse, and $\mathcal{R}_1$ and $\mathcal{R}_2$ are zero. *We assume this in the following.* (There can be topological obstructions to a compact choice of $\widetilde{X}$ where A is elliptic; then one can take for $\widetilde{X}$ a neighborhood of X and carry an analysis through modulo negligible operators. Or, if A is elliptic on a compact extension $\widetilde{X}$ but not invertible there, one can get results modulo suitable finite-rank operators. For lack of space we do not treat such cases here.)

As usual, we define $\gamma_0 u = u(x', 0)$ and $\gamma_j u = \gamma_0(D_n^j u)$, and when A is of order d, we also consider the Cauchy data map $\varrho = \{\gamma_0, \dots, \gamma_{d-1}\}$. Each of these maps can be regarded as a map either from sections over $\overline{X}_+$, or from sections over $\overline{X}_-$, or from sections over $\widetilde{X}$, to sections over X'; to distinguish between the three versions, we use notations such as ϱ^+, ϱ^- resp. $\widetilde{\varrho}$ (so what we would originally call ϱ is now ϱ^+). This makes no difference when they act on C^∞ sections, but it is important when various generalizations to function spaces are considered. The operator D_n is the same in all three versions.

When $F = F_0 \oplus \cdots \oplus F_{d-1}$ are vector bundles over X', we denote

$$\begin{aligned}\mathcal{H}^s(F) &= \textstyle\prod_{0\le j<d} H^{s-j-\frac12}(F_j),\\ \widetilde{\mathcal{H}}^s(F) &= \textstyle\prod_{0\le j<d} H^{s+j+\frac12}(F_j) = (\mathcal{H}^{-s}(F))^*\end{aligned} \tag{11.1}$$

(for the last equation we recall the fact that $H^t(F_j)$ and $H^{-t}(F_j)$ are dual spaces). Denoting $\bigoplus_{0\le j<d} E_i' = E_i'^d$, we can then formulate the well-known mapping properties of $\varrho^\pm$ and $\widetilde{\varrho}$ as follows:

$$\begin{aligned}\varrho^\pm &\colon H^s(E_{i,\pm}) \to \mathcal{H}^s(E_i'^d),\\ \widetilde{\varrho} &\colon H^s(\widetilde{E}_i) \to \mathcal{H}^s(E_i'^d), \quad \text{for } s > d-\tfrac12.\end{aligned} \tag{11.2}$$

The "twosided" trace map $\widetilde{\gamma}_0 \colon H^s(\widetilde{E}_i) \to H^{s-\frac12}(E_i')$ has an adjoint $\widetilde{\gamma}_0^* :$ $H^{\frac12-s}(E_i') \to H^{-s}(\widetilde{E}_i)$ for $s > \frac12$. It can also be written

$$\widetilde{\gamma}_0^* v = v(x') \otimes \delta(x_n), \tag{11.3}$$

since

$$\langle v(x') \otimes \delta(x_n), \bar\varphi(x)\rangle = \langle v(x'), \bar\varphi(x',0)\rangle = \langle v, \overline{\widetilde{\gamma}_0 u}\rangle,$$

cf. (10.108′). Note that $\widetilde{\gamma}_0^*$ ranges in distributions on $\widetilde{X}$ supported in X'. Similarly, the two-sided Cauchy data map $\widetilde{\varrho} \colon H^s(\widetilde{E}_i) \to \mathcal{H}^s(E_i'^d)$ has the adjoint

$$\widetilde{\varrho}^* \colon \widetilde{\mathcal{H}}^{-s}(E_i'^d) \to H^{-s}(\widetilde{E}_i) \tag{11.4}$$

for $s > d - \frac12$; here since $\widetilde{\varrho} = \widetilde{\gamma}_0 \begin{pmatrix} 1 & D_n & \dots & D_n^{d-1}\end{pmatrix}^t$,

$$\widetilde{\varrho}^* = \begin{pmatrix} 1 & D_n & \dots & D_n^{d-1}\end{pmatrix} \widetilde{\gamma}_0^*. \tag{11.5}$$

As usual, we use the notation $P_\pm$ for the truncation of a ψdo P on $\widetilde{X}$ to $X_\pm$ (respectively):

$$P_\pm = r^\pm P e^\pm, \text{ when } P \text{ is a } \psi\text{do on } \widetilde{X};$$

here $r^\pm$ means restriction to $X_\pm$ and $e^\pm$ means extension by zero on $X \setminus X_\pm$.

Lemma 11.1. *The boundary maps $\varrho^{\pm}$ and $\tilde\varrho$ in* (11.2) *are surjective for $s=d$. In fact, there exists a bounded operator*

$$\widetilde{\mathcal K}_{(d)} = \begin{pmatrix} \widetilde{\mathcal K}_0 & \cdots & \widetilde{\mathcal K}_{d-1} \end{pmatrix} : \mathcal H^d(E_i'^d) \to H^d(\widetilde E_i) \tag{11.6}$$

such that $\mathcal K^{\pm}_{(d)} = r^{\pm}\widetilde{\mathcal K}_{(d)}$ are Poisson operators ($\mathcal K^{\pm}_{(d)} = \begin{pmatrix} \mathcal K^{\pm}_0 & \cdots & \mathcal K^{\pm}_{d-1}\end{pmatrix}$ with $\mathcal K^{\pm}_j$ of order $-j$), and

$$\varrho^{\pm}\mathcal K^{\pm}_{(d)} = I,\ \tilde\varrho\widetilde{\mathcal K}_{(d)} = I,\ \text{ on } \mathcal H^d(E_i'^d). \tag{11.7}$$

Proof. This follows by localization from Theorem 9.5, which treats the Euclidean case where X_+ is replaced by $\mathbb R^n_+$. Note first that each operator $\mathcal K_j : \mathcal S(\mathbb R^{n-1}) \to \mathcal S(\overline{\mathbb R}^n_+)$ defined in Theorem 9.5 (let us call it $\mathcal K_j^+$ now) is indeed a Poisson operator of order $-j$, with symbol-kernel $\frac{1}{j!}(ix_n)^j\psi(\langle\xi'\rangle x_n)$. The appropriate estimates (as in Definitions 10.15, 10.3) follow from (9.21), which treats $D^k_{x_n}\frac{1}{j!}(ix_n)^j\psi(\langle\xi'\rangle x_n)$ directly, and is easily generalized to expressions with powers of x_n and $D_{\xi'}$ in front. There are similar estimates for this function considered for $x_n \in \mathbb R_-$, so it defines a Poisson operator $\mathcal K_j^-$ of order $-j$ from $\mathbb R^{n-1}$ to $\mathbb R^n_-$. Moreover, $\mathcal K_j^+$ and $\mathcal K_j^-$ can be regarded as $r^+\widetilde{\mathcal K}_j$ resp. $r^-\widetilde{\mathcal K}_j$, where $\widetilde{\mathcal K}_j : \mathcal S(\mathbb R^{n-1}) \to \mathcal S(\mathbb R^n)$ is defined by the formula in (9.19) used for all $x_n \in \mathbb R$. In the estimates worked out after (9.21), one can replace the integration in $x_n \in \mathbb R_+$ by an integration in $x_n \in \mathbb R$, showing that $\widetilde{\mathcal K}_j$ is bounded from $H^{m-j-\frac12}(\mathbb R^{n-1})$ to $H^m(\mathbb R^n)$.

Set $m=d$, then the vectors

$$\widetilde{\mathcal K}_{(d)} = \begin{pmatrix} \widetilde{\mathcal K}_0 & \cdots & \widetilde{\mathcal K}_{d-1} \end{pmatrix},\quad \mathcal K^{\pm}_{(d)} = \begin{pmatrix} \mathcal K^{\pm}_0 & \cdots & \mathcal K^{\pm}_{d-1}\end{pmatrix},$$

are right inverses of $\widetilde\varrho_{(d)}$, $\varrho^{\pm}_{(d)}$ in the Euclidean case.

In the manifold situation, we get the right inverses by using the Euclidean contruction in local coordinates, after applying a partition of unity. When φ is given in $\mathcal H^d(E_i'^d)$, consider a localized piece, say $\underline{\varphi}_l$. It has compact support in $\mathbb R^{n-1}$, say M, so if $\widetilde{\mathcal K}_{(d)}\underline{\varphi}_l$ is multiplied by a function $\underline{\zeta}_l \in C_0^\infty(\mathbb R^n)$ which is 1 on a neighborhood of M, the identity $\widetilde\varrho_{(d)}\underline{\zeta}_l\widetilde{\mathcal K}_{(d)}\underline{\varphi}_l = \underline{\varphi}_l$ still holds. The operators $\underline{\zeta}_l\widetilde{\mathcal K}_{(d)}$ (where $\underline{\zeta}_l$ is taken with support close to M) are carried over to the manifold situation and added together, giving the desired operator. □

Proposition 11.2. *Let A be a differential operator of order d from $\widetilde E_1$ to $\widetilde E_2$, written as*

$$A = \sum_{l=0}^{d} S_l(x', x_n, D')D_n^l \tag{11.8}$$

on U, with differential operators S_l of order $d-l$ acting in X' for each $x_n \in]-1,1[$. The following Green's formula holds for $u \in H^d(E_{1,+})$ and $v \in H^d(E_{2,+})$:

$$(Au,v)_{X_+} - (u,A^*v)_{X_+} = (\mathfrak{A}\varrho^+ u, \varrho^+ v)_{X'}, \tag{11.9}$$

where $\mathfrak{A}$ is a (uniquely determined) matrix

$$\mathfrak{A} = (\mathfrak{A}_{jk})_{j,k=0,\dots,d-1}$$

of differential operators $\mathfrak{A}_{jk}$ (from E_1' to E_2') of orders $d-j-k-1$, with

$$\mathfrak{A}_{jk}(x',D') = iS_{j+k+1}(x',0,D') + \text{lower-order terms} \tag{11.10}$$

(zero if $j+k+1>d$). Here $\mathfrak{A}$ maps $\mathcal{H}^s(E_1'^d)$ continuously into $\widetilde{\mathcal{H}}^{s-d}(E_2'^d)$ for all $s \in \mathbb{R}$, bijectively if S_d is bijective at $x_n = 0$. □

Proof. We show the formula for smooth u and v; then it extends by continuity to H^d spaces. By definition of A^*, $(Au,v)_{X_+} - (u,A^*v)_{X_+} = 0$ if u or v has compact support in X_+, so the only nontrivial contribution comes from cases where u and v are supported in U. For such sections we have the usual formula (by integration by parts)

$$(D_n u, v)_{U_+} - (u, D_n v)_{U_+} = i(\gamma_0^+ u, \gamma_0^+ v)_{X'},$$

and its iterated version (as in (4.48′))

$$(D_n^l u, v)_{U_+} - (u, D_n^l v)_{U_+} = i \sum_{0 \le k \le l-1} (\gamma_{l-1-k}^+ u, \gamma_k^+ v)_{X'}.$$

Then

$$\begin{aligned}(Au,v)_{X_+} - (u,A^*v)_{X_+} &= (\sum_{l\le d} S_l D_n^l u, v)_{X_+} - (u, \sum_{l \le d} D_n^l S_l^* v)_{X_+} \\ &= \sum_{l \le d} [(D_n^l u, S_l^* v)_{X_+} - (u, D_n^l S_l^* v)_{X_+}] \\ &= i \sum_{l\le d} \sum_{k \le l-1} (\gamma_{l-1-k}^+ u, \gamma_k^+ (S_l^* v))_{X'}.\end{aligned} \tag{11.11}$$

Since $D_n^k(S_l^* v) = S_l^* D_n^k v + \sum_{j<k} S_{lj}' D_n^j v$ with differential operators S_{lj}' on X' of order $d-l$, the last expression can be further reorganized to have the form given in the right-hand side of (11.9). The asserted continuity holds since $\mathfrak{A}_{jk}$ is of order $d-j-k-1$.

$\mathfrak{A}$ is uniquely determined since ϱ is *surjective* from the smooth sections over $\overline{X}_+$ to the d-tuples of smooth sections over X'.

Observe that $\mathfrak{A}$ has a skew-triangular character

$$\mathfrak{A} = \mathfrak{A}^0 + \mathfrak{A}' = i \begin{pmatrix} S_1^0 & \dots & S_{d-1}^0 & S_d^0 \\ S_2^0 & \dots & S_d^0 & 0 \\ \vdots & & \vdots & \vdots \\ S_d^0 & \dots & 0 & 0 \end{pmatrix} + \begin{pmatrix} \text{lower} & & \dots & 0 \\ \text{order} & \dots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & & \dots & 0 \end{pmatrix}. \quad (11.12)$$

The terms in the second diagonal of $\mathfrak{A}$ and $\mathfrak{A}^0$ equal $iS_d(x', 0, D') = is_d(x')$; it is a zero-order differential operator and hence a multiplication operator (a vector bundle morphism from E_1' to E_2', trivializing to multiplication by an x'-dependent matrix). The operator $\mathfrak{A}$ (and also $\mathfrak{A}^0$) is then *invertible* if and only if $s_d(x')$ is a *bijective* morphism (the matrix is regular). $\square$

When A is elliptic, $s_d(x')$ is invertible at each point, so $\mathfrak{A}$ is bijective. For more general differential operators, bijectiveness of $s_d(x')$ means that X' is *noncharacteristic* for A.

We see from the example of $\mathfrak{A}$ that it is interesting to allow matrix-formed operators that are *multi-order systems*, with different orders of the various entries, fitting together in a convenient way. When $S = (S_{jk})_{\substack{j=j_0,\dots,j_1 \\ k=k_0,\dots,k_1}}$, we say that

$$S \text{ has multi-order } (t_j, s_k)_{\substack{j=j_0,\dots,j_1 \\ k=k_0,\dots,k_1}}, \text{ when each } S_{jk} \text{ has order } t_j - s_k. \quad (11.13)$$

The principal part then consists of the $(t_j - s_k)$-order parts of the S_{jk}. Ellipticity means that the matrix of principal symbols defined accordingly, is invertible for $|\xi| \geq 1$; it is sometimes called Douglis-Nirenberg ellipticity (also Volevich should be mentioned in this context). The operator $\mathfrak{A}$ in (11.12) is of multi-order $(d-j-1, k)_{j,k=0,\dots,d-1}$, elliptic when $s_d(x')$ is bijective for all x'.

In relation to the given elliptic operator A we define the spaces

$$\begin{aligned} Z_\pm^s &= \{\, z \in H^s(E_{1,\pm}) \mid Az = 0 \text{ on } X_\pm \}, \\ N_\pm^s &= \varrho^\pm Z_\pm^s \subset \mathcal{H}^s(E_1'^d). \end{aligned} \quad (11.14)$$

Although the trace operators $\varrho^\pm$ are defined on $H^s(E_{1,\pm})$ for $s > d - \frac{1}{2}$ only, they can be given a sense on $Z_\pm^s$ (respectively) for all $s \in \mathbb{R}$.

Consider for example ϱ^+. Accounts of the extension to Z_+^s are found in several places: For one thing, there is Seeley's own deduction in [S66] where the Calderón projector was originally worked out with full proofs (many of which are reproduced here); this particular point is shown at the end of the paper, also for L_p Sobolev spaces, $1 < p < \infty$. Another account is by Lions and Magenes in the paper [LM63], where it is shown that when A is elliptic, $C^\infty(\overline{X}_+)$ is dense in $D^s_A = \{ u \in H^s(X_+) \mid Au \in L_2(X_+) \}$ for all $s \le 0$, and this allows an extension of Green's formula to such u; we have seen a particular case of the proof in Theorems 9.8, 9.10. (The paper [LM63] covers also L_p related spaces; more general results in an L_2 framework are found in the book [LM68].) Thirdly, the result follows from Theorems 4.3.1 and 2.5.6 in [H63] for the localized situation in $\mathbb{R}^n_+$, using the spaces $H^{s,t}(\mathbb{R}^n_+)$ (cf. (10.126), (10.128)). The idea is to use the equation $Au = 0$ to conclude that $u \in Z^s_+$ implies $u \in H^{s+k,-k}(\mathbb{R}^n_+)$ for all k ("partial hypoellipticity at the boundary"); then for sufficiently large k, the boundary value $\gamma^+_j u$ makes sense. For completeness, we indicate a proof of the third type.

Theorem 11.3. *The map ϱ^+ extends to a continuous map from Z^s_+ to $\mathcal{H}^s(E_1'^d)$ for all $s \in \mathbb{R}$.*

In fact, when $u \in Z^s_+$, then in local coordinates at the boundary, u is in $H^{s+k,-k}(\mathbb{R}^n_+)$ for any $k \in \mathbb{N}_0$, where $\gamma^+_j \colon H^{s+k,-k}(\mathbb{R}^n_+) \to H^{s-j-\frac12}(\mathbb{R}^{n-1})$ for $s + k - j > \frac12$.

Similarly, ϱ^- extends to a continuous map from Z^s_- to $\mathcal{H}^s(E_1'^d)$ for all $s \in \mathbb{R}$.

Indications of proof. One ingredient in the proof is the fact that for any $r \in \mathbb{Z}$, the operator $(\Xi^r_-)_+ = \mathrm{OP}((\langle \xi' \rangle - i\xi_n)^r)_+$ maps $H^{s,t}(\mathbb{R}^n_+)$ homeomorphically onto $H^{s-r,t}(\mathbb{R}^n_+)$, with inverse $(\Xi^{-r}_-)_+$, *for all $s, t \in \mathbb{R}$*. This can be inferred from [H63] and appears in early works of Vishik and Eskin, in Seeley's proof [S66] and in Eskin's book [E81]. See Exercise 10.8 above, where the reader is guided through a proof.

Consider the localized situation. Let $A = D^d_n + \sum_{0 \le l \le d-1} S_l(x, D') D^l_n$, where the S_l are differential operators with respect to x' of order $d - l$ with bounded smooth coefficients. (An elliptic operator is reduced to this form by dividing out the coefficient of D^d_n.) If $u \in H^{s,t}(\mathbb{R}^n_+)$ solves $Au = 0$, then since $S_l D^l_n u \in H^{s-l,t-d+l}(\mathbb{R}^n_+) \subset H^{s-d+1,t-1}(\mathbb{R}^n_+)$ (recall (10.127)),

$$D^d_n u = -\sum_{0 \le l \le d-1} S_l D^l_n \in H^{s-d+1,t-1}(\mathbb{R}^n_+).$$

Then also $(\Xi^d_-)_+ u = \sum_{0 \le l \le d} c_l \,\mathrm{OP}'(\langle \xi' \rangle^{d-l}) D^l_n u$ lies in $H^{s-d+1,t-1}(\mathbb{R}^n_+)$, and it follows by application of $(\Xi^{-d}_-)_+$ that $u \in H^{s+1,t-1}(\mathbb{R}^n_+)$. This gives the induction step in a proof that when $u \in H^{s,0}(\mathbb{R}^n_+)$ solves $Au = 0$, then $u \in H^{s+k,-k}(\mathbb{R}^n_+)$ for all k. The asserted mapping property of γ^+_j was shown in Corollary 10.25. $\square$

Since $(Au,v)_{\widetilde X}-(u,A^*v)_{\widetilde X}=0$, we have in addition to (11.9):

$$(Au,v)_{X_-}-(u,A^*v)_{X_-}=-(\mathfrak{A}\varrho^-u,\varrho^-v)_{X'}, \tag{11.15}$$

for $u\in H^d(E_{1,-})$ and $v\in H^d(E_{2,-})$.

The central step in our construction is to introduce the operators

$$K=Q\widetilde\varrho^*\mathfrak{A},\quad K^\pm=\mp r^\pm K=\mp r^\pm Q\widetilde\varrho^*\mathfrak{A}. \tag{11.16}$$

Since $\mathfrak{A}\colon \mathcal{H}^s(E_1'^d)\to\widetilde{\mathcal{H}}^{s-d}(E_2'^d)$ for $s\in\mathbb{R}$, $\widetilde\varrho^*\colon\widetilde{\mathcal{H}}^{s-d}(E_2'^d)\to H^{s-d}(\widetilde E_2)$ for $s<\frac12$ (cf. (11.4)), and $Q\colon H^{s-d}(\widetilde E_2)\to H^s(\widetilde E_1)$ for $s\in\mathbb{R}$, we have that

$$\begin{aligned} K\colon \mathcal{H}^s(E_1'^d)&\to H^s(\widetilde E_1),\quad\text{ when } s<\tfrac12;\\ K^\pm\colon \mathcal{H}^s(E_1'^d)&\to H^s(E_{1,\pm}),\quad\text{ when } s<\tfrac12.\end{aligned} \tag{11.17}$$

Observe moreover that the operators $K^\pm$ in fact *map into* $Z^s_\pm$, since

$$AK\varphi=\widetilde\varrho^*\mathfrak{A}\varphi\quad\text{ is supported in } X'. \tag{11.18}$$

The operators $K^\pm$ are *Poisson operators* in view of Theorem 10.20, so by Theorem 10.24, the mapping properties in the second line of (11.17) extend to all $s\in\mathbb{R}$. This is not true for the operator K itself that gives an important singularity at $x_n=0$.

Proposition 11.4. *The Poisson operators $K^\pm$ defined in* (11.17) *satisfy, for any $s\in\mathbb{R}$:*

$$K^+\varrho^+z=z,\ \textit{for } z\in Z^s_+,\quad K^-\varrho^-z=z,\ \textit{for } z\in Z^s_-. \tag{11.19}$$

It follows that for all $s\in\mathbb{R}$:

$$\varrho^+K^+\varphi=\varphi,\ \textit{for } \varphi\in N^s_+,\quad \varrho^-K^-\varphi=\varphi,\ \textit{for } \varphi\in N^s_-. \tag{11.20}$$

Proof. We begin by showing that

$$K^+\varrho^+z=z,\ \text{for } z\in Z^d_+,\quad K^-\varrho^-z=z,\ \text{for } z\in Z^d_-. \tag{11.21}$$

The first statement is seen as follows: Let $z\in Z^d_+$, let $w\in L_2(E_{1,+})$ and let $v=Q^*e^+w$; is lies in $H^d(\widetilde E_2)$. Note that $r^+A^*v=w$. Then by Green's formula, since $Az=0$ on X_+,

$$\begin{aligned} -(z,w)_{X_+}&=(Az,r^+v)_{X_+}-(z,r^+A^*v)_{X_+}=(\mathfrak{A}\varrho^+z,\varrho^+v)_{X'}\\ &=(\mathfrak{A}\varrho^+z,\widetilde\varrho v)_{X'}=\langle\widetilde\varrho^*\mathfrak{A}\varrho^+z,\overline{Q^*e^+w}\rangle_{\widetilde X}=(Q\widetilde\varrho^*\mathfrak{A}\varrho^+z,e^+w)_{\widetilde X}\\ &=-(K^+\varrho^+z,w)_{X_+}.\end{aligned}$$

Since w was arbitrary, it follows that $z = K^+\varrho^+ z$. The second statement in (11.21) is shown similarly, using (11.15).

The identities in (11.19) then also hold for $s \geq d$, by the continuity properties of the maps. Our next task is to show them for $s < d$. We here take recourse to the description in Theorem 11.3 of the elements in the nullspace in local coordinates. Let $z \in Z_+^s$ for some $s < d$. Let $\{\eta_1, \dots, \eta_{J_0}\}$ be a partition of unity subordinate to an atlas of local trivializations ψ_i, U_i, V_i, $i = 1, \dots, I_0$, for E_1, as in Lemma 8.4 2°. Decompose z as $z = \sum_{j \leq J_0} z_j$, $z_j = \eta_j z$. We use the notation $\underline{z}_j$ for the localized version of z_j for a choice of trivialization ψ_i where η_j is supported in U_i (and we likewise indicate localized operators by underlining). By Theorem 11.3, each $\underline{z}_j$ is in $H^{s+k,-k}(\mathbb{R}^n_+)^N$ for $k \in \mathbb{N}_0$, where ϱ^+ is well-defined if $s + k \geq d$, mapping into $\mathcal{H}^s(\mathbb{R}^{n-1}, (\mathbb{C}^N)^d)$. Take k so large that $s + k \geq d$. Since $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ is dense in $H^{s+k,-k}(\mathbb{R}^n_+)$, we can find a sequence $u_{j,l}$ of C^∞ functions supported in U_i such that $\underline{u}_{j,l} \to \underline{z}_j$ in $H^{s+k,-k}(\mathbb{R}^n_+)^N$ for $l \to \infty$. Then $\underline{A}\,\underline{u}_{j,l} \to \underline{A}\,\underline{z}_j$ in $H^{s+k-d,-k}(\mathbb{R}^n_+)^N$. Let $v_{j,l} = Q_+ A u_{j,l}$; then $A v_{j,l} = A u_{j,l}$, so $z_{j,l} = u_{j,l} - v_{j,l}$ has $A z_{j,l} = 0$. We set $z_{(l)} = \sum_{j \leq J_0} z_{j,l}$; it lies in Z_+^∞.

Moreover, we can write each $v_{j,l}$ as

$$v_{j,l} = \sum_{m \leq J_0} v_{j,m,l}, \quad v_{j,m,l} = \eta_m v_{j,l}.$$

Consider a term $v_{j,m,l}$; here when U_i contains the supports of both η_m and η_j, $v_{j,m,l}$ localizes to $\underline{v}_{j,m,l} = \underline{\eta}_m \underline{Q}_+ \underline{A}\,\underline{u}_{j,l}$. It converges to $\underline{v}_{j,m,0} = \underline{\eta}_m \underline{Q}_+ \underline{A}\,\underline{z}_j$ in $H^{s+k,-k}$ for $l \to \infty$, so in this sense,

$$\sum_{j \leq J_0} v_{j,l} = \sum_{j,m \leq J_0} v_{j,m,l} \to Q_+ A z = 0, \text{ for } l \to \infty. \tag{11.22}$$

Now $z_{(l)} \to z$ in the sense that the localized pieces $\underline{z}_{j,l} = \underline{u}_{j,l} - \sum_{m \leq J_0} \underline{v}_{j,m,l}$ have $\underline{u}_{j,l} \to \underline{z}_j$ in $H^{s+k,-k}$, and $\underline{v}_{j,m,l} \to \underline{v}_{j,m,0}$ in $H^{s+k,-k}$, with $\sum_{j,m \leq J_0} v_{j,m,0} = 0$.

Since $z_{(l)} \in Z_+^\infty$, (11.21) applies to show that

$$K^+ \varrho^+ z_{(l)} = z_{(l)}. \tag{11.23}$$

For the localized pieces we have for $l \to \infty$ (after insertion of an extra partition of unity (η_r) to localize the action of K^+):

$$\begin{aligned} \underline{\eta}_r \underline{K}^+ \underline{\varrho}^+ (\underline{\eta}_m \underline{u}_{j,l} - \underline{v}_{j,m,l}) &\to \underline{\eta}_r \underline{K}^+ \underline{\varrho}^+ \underline{\eta}_m \underline{z}_j, \\ \underline{\eta}_m \underline{u}_{j,l} - \underline{v}_{j,m,l} &\to \underline{\eta}_m \underline{z}_j - \underline{v}_{m,j,0}, \end{aligned}$$

in $H^{s+k,-k}$, using that $\underline{\varrho}^+$ maps $H^{s+k,-k}$ to $\mathcal{H}^s$ and $\eta_r \underline{K}^+$ maps $\mathcal{H}^s$ to $H^{s+k,-k}$. The formulas hold in trivializations ψ_i where η_r, η_m, η_j are supported in U_i. Adding the pieces carried back to X and using (11.23) before passing to the limit, we find the desired identity $K^+\varrho^+ z = z$.

There is a similar proof for the minus-case.

Finally, (11.21) follows immediately, since $\varphi \in N_+^s$ means that $\varphi = \varrho^+ z$ for some $z \in Z_+^s$, and

$$\varrho^+ K^+ \varphi = \varrho^+ K^+ \varrho^+ z = \varrho^+ z = \varphi,$$

by the just proved identity (with a similar proof for the minus-case). □

Definition 11.4a. *The Calderón projectors $C^\pm$ associated with A are defined by*

$$C^\pm = \varrho^\pm K^\pm. \tag{11.24}$$

They are ψdo's in $\bigoplus_{0\le j<d} E_1'$, by the composition rule Theorem 10.19 7° and Theorem 10.23, with the continuity property

$$C^\pm \colon \mathcal{H}^s(E_1'^d) \to \mathcal{H}^s(E_1'^d), \text{ all } s \in \mathbb{R}. \tag{11.25}$$

Moreover,

$$C^\pm \text{ maps } \mathcal{H}^s(E_1'^d) \text{ into } N_\pm^s, \tag{11.26}$$

respectively, since $K^\pm$ maps into $Z_\pm^s$, respectively. The projection property will now be shown.

Proposition 11.5. 1° *The ψdo's $C^\pm$ defined in Definition* 11.4a *are projections in $\mathcal{H}^s(E_1'^d)$ for all $s \in \mathbb{R}$,*

$$(C^+)^2 = C^+, \quad (C^-)^2 = C^-.$$

2° *Moreover, C^+ and C^- are complementing projections,*

$$C^+ + C^- = I, \quad C^+C^- = 0 = C^-C^+.$$

Proof. The projection property follows, since

$$(C^+)^2 = \varrho^+ K^+ \varrho^+ K^+ = \varrho^+ K^+ = C^+$$

in view of (11.19); the identity $(C^+)^2 = C^+$ thus holds for smooth sections and extends by continuity to general distribution sections. The identity $(C^-)^2 = C^-$ follows similarly from (11.19). This shows 1°.

To show 2°, let $\varphi \in \mathcal{H}^s(E_1'^d)$, and let $z^\pm = K^\pm\varphi$. For each $\psi \in C^\infty(E_1'^d)$, choose a section $v \in C^\infty(\widetilde{E})$ with $\widetilde{\varrho} v = \psi$. Then we have, using the Green's formulas (11.9), (11.15) "backwards", together with the fact that $z^\pm \in Z^d_\pm$,

$$\begin{aligned}
&(\mathfrak{A}(C^+ + C^-)\varphi, \psi)_{X'} = (\mathfrak{A}\varrho^+ z^+, \varrho^+ r^+ v) + (\mathfrak{A}\varrho^- z^-, \varrho^- r^- v)\\
&= (Az^+, r^+ v)_{X_+} - (z^+, r^+ A^* v)_{X_+} - (Az^-, r^- v)_{X_-} + (z^-, r^- A^* v)_{X_-}\\
&= -(K^+\varphi, r^+ A^* v)_{X_+} + (K^-\varphi, r^- A^* v)_{X_-}\\
&= (Q\widetilde{\varrho}^*\mathfrak{A}\varphi, A^* v)_{\widetilde{X}} = \langle \widetilde{\varrho}^*\mathfrak{A}\varphi, \overline{v}\rangle_{\widetilde{X}} = (\mathfrak{A}\varphi, \psi)_{X'}.
\end{aligned}$$

Since φ and ψ were arbitrary and $\mathfrak{A}$ is invertible, it follows that $C^+ + C^- = I$ holds on $\mathcal{H}^d(E_1'^d)$, and the validity on $\mathcal{H}^s(E_1'^d)$ follows by extension by continuity.

It follows moreover that $C^+C^- = C^+(I - C^+) = C^+ - (C^+)^2 = 0$. $\square$

We have hereby established the essential ingredients in the main theorem:

Theorem 11.6. *Assume that A has the inverse Q on $\widetilde{X}$. Define the spaces $Z^s_\pm$ and $N^s_\pm$ by (11.14). Then the spaces $N^s_\pm$ are complementing closed subspaces of $\mathcal{H}^s(E_1'^d)$;*

$$\mathcal{H}^s(E_1'^d) = N^s_+ \dot{+} N^s_-, \text{ for any } s \in \mathbb{R}. \tag{11.27}$$

When we define

$$K^\pm = \mp r^\pm Q\widetilde{\varrho}^*\mathfrak{A}, \quad C^\pm = \varrho^\pm K^\pm = \mp\varrho^\pm r^\pm Q\widetilde{\varrho}^*\mathfrak{A}, \tag{11.28}$$

the Poisson operators $K^\pm\colon \mathcal{H}^s(E_1'^d) \to H^s(E_{1,\pm})$ have range equal to $Z^s_\pm$; moreover, they act as homeomorphisms

$$K^\pm\colon N^s_\pm \xrightarrow{\sim} Z^s_\pm, \text{ with inverse } \varrho^\pm, \tag{11.29}$$

respectively. This gives us a parametrization of the nullspace $Z^s_\pm$ by its Cauchy data.

The ψdo's $C^\pm$ (the Calderón projectors for A) are the projections of $\mathcal{H}^s(E_1'^d)$ onto $N^s_\pm$ along $N^s_\mp$, respectively. In particular,

$$C^+ + C^- = I, \quad (C^+)^2 = C^+, \quad (C^-)^2 = C^-, \quad C^+C^- = 0. \tag{11.30}$$

Proof. It remains to account for surjectiveness and homeomorphism properties.

The surjectiveness of $K^+\colon \mathcal{H}^s(E_1'^d) \to Z^s_+$ follows from the identity $z = K^+\varrho^+ z$ for $z \in Z^s_+$ shown in Proposition 11.4.

It follows that the $C^{\pm}$ are surjective onto the spaces $N^s_{\pm}$. These are complementing closed subspaces of $\mathcal{H}^s(E_1'^d)$, since they are the range spaces for the complementing projections.

Now the $K^{\pm}$ in (11.29) are injective, since e.g. $K^+\varphi = 0$ for a $\varphi \in N^s_+$ implies $C^+\varphi = 0$, hence $\varphi = 0$ since C^+ acts like the identity on N^s_+. They are also surjective, since $z \in Z^s_+$ can be written as $z = K^+\varphi$ with $\varphi = \varrho^+ z \in N^s_+$, by Proposition 11.4 and the definition of N^s_+. So indeed (11.29) holds with plus; there is a similar proof with minus. $\square$

When Q is merely a parametrix of A, one can still define operators $K^{\pm}$ by formulas as in (11.28) supplied with smoothing terms, setting

$$C^+ = \varrho^+ K^+ = -\varrho^+ r^+ Q \widetilde{\varrho}^* \mathfrak{A} + \mathcal{T} \tag{11.31}$$

and $C^- = I - C^+$ (with a ψdo $\mathcal{T}$ of order $-\infty$); then they have the listed mappping properties only modulo smoothing operators. Such a construction is worked out in [G77] for general multi-order operators A, with applications. Seeley gives in [S 69] an *optimal* construction, where K^+ maps $\mathcal{H}^s$ injectively onto a subspace of Z^s_+ with complement Z_0, and where $C^+ = \varrho^+ K^+$ *is a projection of* $\mathcal{H}^s$ *onto* N^s_+. The book of Booss-Bavnbek and Wojciechowski [BW93] goes through a proof of Theorem 11.6 for first-order operators A.

The principal symbols of $C^{\pm}$ and $K^{\pm}$ are found by carrying out the analogous construction in the one-dimensional model case (as exemplified in Example 11.1a). For fixed x', fixed $\xi' \neq 0$, consider the model operator

$$a^0(x', \xi', D_n) = \mathrm{OP}_n(a^0(x', 0, \xi', \xi_n)),$$

acting on N-vector functions on $\mathbb{R}$. Recall the notation $\mathcal{S}(\overline{\mathbb{R}}_{\pm}) = \mathcal{S}_{\pm}$. The solutions of $a^0 u = 0$ that are bounded on $\mathbb{R}_+$ resp. $\mathbb{R}_-$ form two vector spaces

$$Z_{\pm}(x', \xi') = \{u(x_n) \in \mathcal{S}^N_{\pm} \mid a^0(x', \xi', D_n)u = 0 \text{ on } \mathbb{R}_{\pm}\}. \tag{11.32}$$

We observe that for any s,

$$Z_{\pm}(x', \xi') = \{u(x_n) \in H^s(\mathbb{R}_{\pm})^N \mid a^0(x', \xi', D_n)u = 0 \text{ on } \mathbb{R}_{\pm}\}, \tag{11.33}$$

in view of the description of solutions as exponential polynomials recalled in Example 10.33, so there is no need to work with a scale of nullspaces as in the n-dimensional case. Here the $Z_{\pm}(x', \xi')$ have finite dimension, namely $m_{\pm}(x', \xi')$, respectively. They are parametrized by the Cauchy data (in view of standard existence- and uniqueness theorems for constant-coefficient ODE), so if we define

$$N_+(x', \xi') = \varrho^+ Z_+(x', \xi'), \quad N_-(x', \xi') = \varrho^- Z_-(x', \xi'), \tag{11.34}$$

we have immediately that $N_\pm(x',\xi')$ have dimension $m_\pm(x',\xi')$. Moreover, $N_+(x',\xi')$ and $N_-(x',\xi')$ must be complementing subspaces of $\mathbb{C}^{Nd}$, for their intersection is zero (a linear combination of the occurring exponential polynomials cannot vanish both for $x_n\to\infty$ and for $x_n\to-\infty$ without being 0), and the sum of their dimensions is Nd.

So, just from the knowledge of the solution structure of ODE, we have the existence of homemorphisms $K^\pm(x',\xi')\colon N_\pm(x',\xi')\to Z_\pm(x',\xi')$ inverse to $\varrho^\pm$, and complementing projections $C^\pm(x',\xi')$ of $\mathbb{C}^{Nd}$ onto $N_\pm(x',\xi')$, respectively. The $K_\pm(x',\xi')$ are extended linearly to map $N_\mp(x',\xi')$ to 0, respectively.

The Calderón construction gives us useful formulas for these operators. There is a matrix $\mathfrak{a}^0(x',\xi')$ such that

$$(a^0(D_n)u,v)_{\mathbb{R}_+}-(u,a^0(D_n)^*v)_{\mathbb{R}_+}=(\mathfrak{a}^0\varrho^+u,\varrho^+v)_{\mathbb{C}^{Nd}}, \tag{11.35}$$

for $u,v\in H^d(\mathbb{R}_+)^N$, it is the principal symbol of $\mathfrak{A}$ from Proposition 11.2. (It is a $d\times d$-matrix of $N\times N$-matrices.) The inverse of $a^0(x',\xi',D_n)$ is defined from the principal symbol of Q, $q^0=(a^0)^{-1}$, by

$$a^0(x',\xi',D_n)^{-1}=q^0(x',\xi',D_n)=\mathrm{OP}_n(q^0(x',0,\xi)).$$

We can consider the adjoint of $\widetilde\varrho\colon H^s(\mathbb{R})^N\to\mathbb{C}^{Nd}$ $(s>\frac12)$, writing

$$\widetilde\varrho^*v=\begin{pmatrix}1 & D_n & \dots & D_n^{d-1}\end{pmatrix}\widetilde\gamma_0^*v;\quad \widetilde\gamma_0^*v=v\delta(x_n)\text{ for }v\in\mathbb{C}^{Nd}. \tag{11.36}$$

Then when we define

$$\begin{aligned}K(x',\xi')&=q^0(x',\xi',D_n)\widetilde\varrho^*\mathfrak{a}^0(x',\xi'),\\ K^\pm(x',\xi')&=\mp r^\pm K(x',\xi')=\mp r^\pm q^0(x',\xi',D_n)\widetilde\varrho^*\mathfrak{a}^0(x',\xi'),\end{aligned} \tag{11.37}$$

the $K^\pm(x',\xi')$ are verified to be the desired operators (that we already have defined), with $C^\pm(x',\xi')=\varrho^\pm K^\pm(x',\xi')$, by simple variants of the proofs of Propositions 11.4–5 and Theorem 11.6.

The reader is encouraged to check the details. In these calculations, doing the $\circ_n$ compositions starting with the originally given principal symbols, we arrive at the *principal symbols* of the operators that were first defined in the full calculus. Thus the operators $K^\pm(x',\xi')$ and $C^\pm(x',\xi')$ are the principal boundary symbol operators for the operators $K^\pm$, $C^\pm$ in the n-dimensional situation, usually denoted $k^{\pm,0}(x',\xi',D_n)$ resp. $c^{\pm,0}(x',\xi')$.

For later reference, we sum up the results in a proposition.

Proposition 11.7. *Let $\xi' \neq 0$. The spaces $Z_\pm(x',\xi')$ and $N_\pm(x',\xi')$ defined in* (11.32)–(11.34) *have dimension $m_\pm(x',\xi')$ (cf. Example* 10.33*).*

The principal boundary symbol operators for $K^\pm$ and $C^\pm$ in Theorem 11.6 *are determined as*

$$k^{\pm,0}(x',\xi',D_n) = K^\pm(x',\xi'), \quad c^{\pm,0}(x',\xi') = \varrho^\pm k^{\pm,0}(x',\xi',D_n)$$

(through formulas given in (11.35)–(11.37)*); here*

$$k^{\pm,0}(x',\xi',D_n)\colon N_\pm(x',\xi') \xrightarrow{\sim} Z_\pm(x',\xi'),$$

and $c^{\pm,0}(x',\xi')$ project $\mathbb{C}^{Nd}$ onto its complementing subspaces $N_\pm(x',\xi')$, respectively.

11.2 Application to boundary value problems.

The Calderón projectors are very useful in a treatment of boundary value problems for A. We now return to the notation ϱ for ϱ^+. Let there be given a problem:

$$Au = f \text{ on } X, \quad S\varrho u = \varphi \text{ on } X', \tag{11.38}$$

where S is a system of ψdo's S_{jk} of order $j-k$ ($j,k = 0,\dots,d-1$) going from E_1' to bundles F_j of dimension ≥ 0 over X'; $M = \sum_{0\leq j<d} \dim F_j$. (The zero-dimensional bundles could be omitted; they are just included for notational convenience.)

In the following, we consider $\{A, S\varrho\}$ as a mapping from $H^s(E_1)$ to $H^{s-d}(E_2)\times\mathcal{H}^s(F)$, for some $s > d-\frac{1}{2}$, and discuss right/left inverses that are continuous in the opposite direction; here S is considered as a mapping from $\mathcal{H}^s(E_1'^d)$ to $\mathcal{H}^s(F)$ and the $C^\pm$ act in $\mathcal{H}^s(E_1'^d)$. We assume as above that A is invertible on $\widetilde{X}$, with inverse Q.

Theorem 11.8.

1° *If SC^+ has a right inverse S_1, then $\begin{pmatrix} A \\ S\varrho \end{pmatrix}$ has the right inverse*

$$(\, R_S \quad K_S \,) = (\, Q_+ - K^+S_1S\varrho Q_+ \quad K^+S_1 \,)\,. \tag{11.39}$$

Conversely, if $\begin{pmatrix} A \\ S\varrho \end{pmatrix}$ has a right inverse $(\, R_S \quad K_S \,)$, then SC^+ has the right inverse

$$S_1 = \varrho K_S. \tag{11.40}$$

2° *If $\begin{pmatrix} S \\ C^- \end{pmatrix}$ has a left inverse $(\, S_1 \quad S_2 \,)$, then $\begin{pmatrix} A \\ S\varrho \end{pmatrix}$ has the left inverse* (11.39)*.*

Conversely, if $\begin{pmatrix} A \\ S\varrho \end{pmatrix}$ has a left inverse $(\, R_S \quad K_S \,)$, then $\begin{pmatrix} S \\ C^- \end{pmatrix}$ has the left inverse

$$(\, S_1 \quad S_2 \,) = (\, \varrho K_S \quad I - \varrho K_S S \,)\,. \tag{11.41}$$

Proof. We first observe some auxiliary formulas:

$$AQ_+ = I, \quad Q_+A = I - K^+\varrho, \quad K^+C^- = 0. \tag{11.42}$$

The first formula holds since $AQ = I$ on $\widetilde{X}$ and A is local. Next, we note that Green's formula (11.9) can be written in distributional form (compare with (10.78)):

$$e^+r^+A\tilde{u} = Ae^+r^+\tilde{u} + \widetilde{\varrho}^*(\mathfrak{A}\varrho u) \text{ for } \tilde{u} \in H^{s+d}(\widetilde{E}_1),\ u = r^+\tilde{u}, \tag{11.43}$$

for $s > -\frac{1}{2}$. The second formula follows from this by composition with r^+Q, using (11.28) and the fact that $QA = I$ on $\widetilde{X}$; it holds on $H^{s+d}(E_1)$, $s > -\frac{1}{2}$. The third formula holds since $K^+C^+ = K^+\varrho K^+ = K^+$, cf. Proposition 11.4.

For statement 1°, let S_1 be a right inverse of SC^+. Then, by the above rules,

$$\begin{aligned} A(Q_+ - K^+S_1S\varrho Q_+) &= I, \\ S\varrho(Q_+ - K^+S_1S\varrho Q_+) &= S\varrho Q_+ - SC^+S_1S\varrho Q_+ = 0, \\ AK^+S_1 &= 0, \\ S\varrho K^+S_1 &= SC^+S_1 = I. \end{aligned}$$

Conversely, when $(\,R_S \quad K_S\,)$ is a right inverse of $\begin{pmatrix} A \\ S\varrho \end{pmatrix}$, then $AK_S = 0$, $S\varrho K_S = I$, so K_S maps into Z_+^s, whereby $C^-\varrho K_S = 0$ and consequently $SC^+\varrho K_S = S\varrho K_S - SC^-\varrho K_S = I$. Thus ϱK_S is a right inverse of SC^+. This proves 1°.

For 2°, we check the composition of (11.39) to the left with $\begin{pmatrix} A \\ S\varrho \end{pmatrix}$ as follows, using (11.42) and the fact that $C^-C^+ = 0$:

$$\begin{aligned} (\,Q_+ - K^+S_1S\varrho Q_+ \quad K^+S_1\,)\begin{pmatrix} A \\ S\varrho \end{pmatrix} &= (I - K^+S_1S\varrho)Q_+A + K^+S_1S\varrho \\ &= (I - K^+S_1S\varrho)(I - K^+\varrho) + K^+S_1S\varrho = I - K^+(I - S_1SC^+)\varrho \\ &= I - K^+(I - (I - S_2C^-)C^+)\varrho = I - K^+C^-\varrho = I. \end{aligned} \tag{11.44}$$

Conversely, define $(\,S_1 \quad S_2\,)$ by (11.41) and check its left composition with $\begin{pmatrix} S \\ C^- \end{pmatrix}$:

$$(\,\varrho K_S \quad I - \varrho K_S S\,)\begin{pmatrix} S \\ C^- \end{pmatrix} = \varrho K_S S + C^- - \varrho K_S SC^- = \varrho K_S SC^+ + I - C^+. \tag{11.45}$$

When $w = K^+C^+\varphi$ for some $\varphi \in C^\infty(E_1'^d)$, then $Aw = 0$, $\varrho w = C^+C^+\varphi = C^+\varphi$ and $S\varrho w = SC^+\varphi$, so since $(\,R_S \quad K_S\,)$ is a left inverse of $\begin{pmatrix} A \\ S\varrho \end{pmatrix}$,

$$w = K_SS\varrho w = K_SSC^+\varphi.$$

It follows that $\varrho K_S SC^+\varphi = \varrho w = C^+\varphi$ for $\varphi \in C^\infty(E_1'^d)$. Then the expression in (11.45) equals I. This ends the proof of 2°. $\square$

The statements have generalizations where the word "inverse" is replaced by "parametrix", also when Q is merely a parametrix of A (here one can keep track of the smoothing terms as in [G77]).

The result holds in particular on the principal boundary symbol level, when we discuss solutions in $\mathcal{S}_+$. We can extend the terminology from Chapter 7 of surjectively elliptic, resp. injectively elliptic operators and symbols, to boundary value problems, calling the systems with surjectiveness, resp. injectiveness, of the model operator (for all x', all $|\xi'| = 1$) *surjectively elliptic*, resp. *injectively elliptic*.

Theorem 11.9. *Consider the model operator (principal boundary symbol operator)*

$$\begin{pmatrix} a^0(x',0,\xi',D_{x_n}) \\ \\ s^0(x',\xi')\varrho \end{pmatrix} : \mathcal{S}_+^N \to \begin{matrix} \mathcal{S}_+^N \\ \times \\ \mathbb{C}^M \end{matrix},$$

for $\{A, S\varrho\}$ *in Theorem* 11.8. *For all* x', *all* $|\xi'| \geq 1$, *the statements of Theorem* 11.8 *hold for the principal boundary symbol operators:* $\begin{pmatrix} a^0(x',0,\xi',D_{x_n}) \\ s^0(x',\xi')\varrho \end{pmatrix}$ *has a right inverse if and only if* $\begin{pmatrix} s^0(x',\xi') \\ c^{-,0}(x',\xi') \end{pmatrix}$ *does so, and* $\begin{pmatrix} a^0(x',0,\xi',D_{x_n}) \\ s^0(x',\xi')\varrho \end{pmatrix}$ *has a left inverse if and only if* $s^0(x',\xi')c^{+,0}(x',\xi')$ *does so (with the corresponding versions of* (11.39)–(11.41)*).*

In particular,

$$\begin{aligned} &\begin{pmatrix} A \\ S\varrho \end{pmatrix} \text{ is injectively elliptic } \iff \begin{pmatrix} S \\ C^- \end{pmatrix} \text{ is injectively elliptic;} \\ &\begin{pmatrix} A \\ S\varrho \end{pmatrix} \text{ is surjectively elliptic } \iff SC^+ \text{ is surjectively elliptic.} \end{aligned} \tag{11.46}$$

Here $\begin{pmatrix} s^0(x',\xi') \\ c^{-,0}(x',\xi') \end{pmatrix}$ is an $(M+Nd)\times Nd$-matrix, whereas $s^0(x',\xi')c^{+,0}(x',\xi')$ is an $M \times Nd$-matrix.

In view of Proposition 11.7, the injectively resp. surjectively elliptic problems can also be characterized by *injectiveness resp. surjectiveness of* $s^0(x',\xi')$ *from* $N_+(x',\xi')$ *to* $\mathbb{C}^M$ *for all* x', $|\xi'| = 1$. In particular, this requires $M \geq m_+(x',\xi')$ resp. $M \leq m_+(x',\xi')$. Thus for two-sided elliptic problems, M must equal $m_+(x',\xi')$ (which must be constant in (x',ξ') then). In the properly elliptic case (in all cases when $n \geq 3$), one has $m_+(x',\xi') = m_-(x',\xi') = \frac{1}{2}Nd$, so two-sided ellipticity is possible only when $M = \frac{1}{2}Nd$.

We observe that injective ellipticity holds if and only if

$$v \in \mathbb{C}^{Nd},\ s^0(x',\xi')v = 0,\ c^{-,0}(x',\xi')v = 0 \implies v = 0; \tag{11.47}$$

i.e., the nullspaces of s^0 and $c^{-,0}$ are *linearly independent*.

Example 11.10. The systems $\begin{pmatrix} A \\ \varrho \end{pmatrix}$ and $\begin{pmatrix} A \\ C^+\varrho \end{pmatrix}$ are injectively elliptic; they both have the left inverse $(Q_+ \quad K^+)$. In fact, by (11.42),

$$Q_+A + K^+\varrho = I; \quad Q_+A + K^+C^+\varrho = I.$$

This left inverse is also found from (11.39), when we use that $\begin{pmatrix} I \\ C^- \end{pmatrix}$ and $\begin{pmatrix} C^+ \\ C^- \end{pmatrix}$ both have the left inverse $(C^+ \quad C^-)$. The case $S = C^+$ for first-order systems is closely related to the so-called Atiyah-Patodi-Singer problem (see e.g. [G99] which gives references to many other studies of this problem). More on this case in Exercises 11.19 and 11.20.

(11.42) also shows that Q_+ is a right inverse of A *without boundary condition*; i.e., in the case $F = 0$. This is also confirmed by the formulas in the theorem.

The boundary conditions in this example are too strong, resp. too weak, to have unique solvability for all data.

The cases in Example 11.10 are somewhat atypical. It is more interesting to consider two-sided elliptic cases, and there are numerous studies of such cases in the literature. As a fundamental example, we consider the Dirichlet problem for strongly elliptic operators; some other cases are treated in the exercises.

11.3 The solution operator for the Dirichlet problem.

In this section we consider a strongly elliptic differential operator A of order $d = 2m$ on $\widetilde{X}$. It satisfies a Gårding inequality (cf. Theorem 7.16) on $\widetilde{X}$ as well as on X (on $C_0^\infty(X^\circ)$), and it defines the Dirichlet realization A_γ of A on X with domain $D(A_\gamma) = D(A_{\max}) \cap H_0^m(X)$ (cf. Theorem 7.17). We shall show that in fact $D(A_\gamma) = H^{2m}(X) \cap H_0^m(X)$, and we shall describe the solution operator for the nonhomogeneous Dirichlet problem under a hypothesis of invertibility.

We assume that A is invertible on $\widetilde{X}$, and that

$$A_\gamma \colon D(A_\gamma) \to L_2(X) \text{ is bijective.} \tag{11.48}$$

This can always be obtained by adding a large enough constant to A such that

$$\operatorname{Re}(Au, u) \geq c_0 \|u\|_m^2 \text{ for } u \in C^\infty(X), \tag{11.48a}$$

with $c_0 > 0$; then A on $\widetilde{X}$ and its Dirichlet realizations on X_+ and X_- have positive lower bound.

Remark 11.11a. Using the compactness of these inverses as operators in $L_2(\widetilde{X})$ resp. $L_2(X_\pm)$ (which holds since the injections of H^m-spaces into L_2-spaces are compact, cf. Section 8.2), and the fact that the spectra are contained in proper subsectors of $\mathbb{C}$, one can show that the resolvent sets are complements of a countable set of eigenvalues with finite multiplicities and no accumulation points — in short, the spectrum is discrete. Then in fact it suffices to add a small constant to a strongly elliptic operator, to get invertible Dirichlet realizations on $X_\pm$. Without the assumption of invertibility, the results in the following hold modulo negligible operators.

The inverse of A_γ will as usual be denoted R_γ.

The operator R_γ solves the semihomogeneous Dirichlet problem

$$Au = f \text{ on } X, \quad \gamma u = 0 \text{ at } X', \tag{11.49}$$

where $\gamma = \{\gamma_0, \dots, \gamma_{m-1}\}$, f given in $L_2(X)$. The other semihomogeneous Dirichlet problem

$$Az = 0 \text{ on } X, \quad \gamma z = \varphi \text{ at } X', \tag{11.50}$$

can be solved as follows, for $\varphi \in \prod_{0\le j<m} H^{2m-j-\frac12}(X')$: Let $v = \mathcal{K}_{(m)}\varphi$ according to Lemma 11.1; it lies in $H^{2m}(X)$ and has $\gamma v = \varphi$. Setting $u = z - v$, we can then reduce problem (11.50) to the problem

$$Au = -Av \text{ on } X, \quad \gamma u = 0 \text{ at } X', \tag{11.51}$$

which has the unique solution $u = -R_\gamma A\mathcal{K}_{(m)}\varphi$ in $D(A_\gamma)$. Then $z = u + v = (\mathcal{K}_{(m)} - R_\gamma A\mathcal{K}_{(m)})\varphi$ solves (11.50). We conclude that the fully nonhomogeneous Dirichlet problem

$$Au = f \text{ on } X, \quad \gamma u = \varphi \text{ at } X', \tag{11.52}$$

has the solution operator

$$\begin{gathered} (\,R_\gamma \quad K_\gamma\,) : \begin{matrix} L_2(X) \\ \times \\ \prod_{0\le j<m} H^{2m-j-\frac12}(X') \end{matrix} \to D(A_{\max}) \cap H^m(X); \\ R_\gamma = A_\gamma^{-1}, \quad K_\gamma = \mathcal{K}_{(m)} - R_\gamma A\mathcal{K}_{(m)}. \end{gathered} \tag{11.53}$$

These considerations also work for the boundary symbol operator, showing that

$$\begin{pmatrix} a^0(x',\xi',D_n) \\ \\ \gamma \end{pmatrix} : \begin{matrix} \mathcal{S}_+ \\ \times \\ \mathbb{C}^m \end{matrix} \to \mathcal{S}_+$$

has an inverse $(\, r^0_\gamma(x',\xi',D_n) \quad k^0_\gamma(x',\xi',D_n)\,)$, and it belongs to the calculus, by Theorem 10.31. Using this, we have according to Theorem 10.32 a *parametrix*, continuous for $s\ge 0$,

$$(\, R'_\gamma \quad K'_\gamma\,) : \begin{matrix} H^s(X) \\ \times \\ \prod_{0\le j<m} H^{s+2m-j-\frac12}(X') \end{matrix} \to H^{s+2m}(X). \tag{11.54}$$

It satisfies

$$\begin{aligned} \begin{pmatrix} A \\ \gamma \end{pmatrix} (\, R'_\gamma \quad K'_\gamma\,) &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \mathcal{R}_1, \\ (\, R'_\gamma \quad K'_\gamma\,) \begin{pmatrix} A \\ \gamma \end{pmatrix} &= I + \mathcal{R}_2, \end{aligned} \tag{11.54a}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are negligible Green operators; $\mathcal{R}_1$ of class 0 and $\mathcal{R}_2$ of class m (since γ is of class m).

The parametrix can be used to show *regularity of solutions*, as follows: Let $u = R_\gamma f$ for some $f\in L_2(X)$. Since $u\in H^m(X)$, $\mathcal{R}_2$ applies to it, and we can write, in view of (11.54a),

$$\begin{aligned} u &= \big[(\, R'_\gamma \quad K'_\gamma\,) \begin{pmatrix} A \\ \gamma \end{pmatrix} - \mathcal{R}_2\big] u \\ &= (\, R'_\gamma \quad K'_\gamma\,) \begin{pmatrix} f \\ 0 \end{pmatrix} - \mathcal{R}_2 u = R'_\gamma f - \mathcal{R}_2 u. \end{aligned} \tag{11.54b}$$

This lies in $H^{2m}(X)$ since $R'_\gamma f$ does so and $\mathcal{R}_2$ maps into $C^\infty(X)$. It follows that

$$D(A_\gamma) = H^{2m}(X)\cap H^m_0(X). \tag{11.55}$$

Going back to the definition of K_γ, we see moreover that it maps $\prod_{0\le j<m} H^{2m-j-\frac12}(X')$ into $H^{2m}(X)$, so the exact solution operator has a continuity property as in (11.54) for $s=0$.

It is not hard to pursue this technique still further to show that the full mapping property as in (11.54) holds for $(\, R_\gamma \quad K_\gamma\,)$ (Exercise 11.3).

We have then obtained:

Theorem 11.11. *When A is strongly ellipticity and satifies* (11.48)*, the Dirichlet problem*

$$Au = f \text{ on } X, \quad \gamma u = \varphi \text{ at } X', \tag{11.49a}$$

has a solution operator

$$\begin{pmatrix} A \\ \gamma \end{pmatrix}^{-1} = (\, R_\gamma \quad K_\gamma\,) \tag{11.49b}$$

sending $H^s(X) \times \prod_{0\le j<m} H^{s+2m-j-\frac12}(X')$ *into* $H^{s+2m}(X)$ *for all* $s \ge 0$.

Hereby we have finally obtained the general statement on the regularity of solutions of the Dirichlet problem, mentioned several times earlier in this book. The above considerations works also for strongly elliptic *systems* A.

If we do not assume (11.48), Theorem 11.11 assures that $D(A_\gamma + k) = H^{2m}(X) \cap H_0^m(X)$ for a suitable constant k, and it follows that $D(A_\gamma) = H^{2m}(X) \cap H_0^m(X)$.

One can moreover show that $(\, R_\gamma \quad K_\gamma\,)$ itself belongs to the ψdbo calculus. On one hand, this is a consequence of a general principle, that we cannot find space to include the proof of here: When an elliptic element has an inverse operator, then the inverse belongs to the calculus (this is part of a property called *spectral invariance*). But it can also be shown for the Dirichlet problem in a more elementary way, passing via the Dirichlet-to-Neumann operator and using that we already have the ψdo properties of the Calderón projector; this will be done below.

The Dirichlet-to-Neumann operator (called the Steklov-Poincaré operator in some texts) enters both in modern and older literature on PDE's, e.g. in the applications of the abstract theory of Chapter 13 that were developed in [G68]–[G74]; see also [BGW08] and its references.

Definition 11.12. *Let A be given as above, strongly elliptic and satisfying* (11.48)*, with* $(\, R_\gamma \quad K_\gamma\,)$ *solving the Dirichlet problem, and denote*

$$\nu = \{\gamma_m, \dots, \gamma_{2m-1}\}, \tag{11.55a}$$

the Neumann trace operator. Then the Dirichlet-to-Neumann operator is defined by

$$P_{\gamma,\nu} = \nu K_\gamma. \tag{11.56}$$

In other words, $P_{\gamma,\nu}$ maps the Dirichlet data into the Neumann data for solutions of $Au = 0$ on X. In view of (11.54) for K_γ, it maps

$$P_{\gamma,\nu} \colon \prod_{0\le j<m} H^{s+2m-j-\frac12}(X') \to \prod_{0\le j<m} H^{s+m-j-\frac12}(X'), \tag{11.57}$$

for $s \ge 0$. We shall show that it is a pseudodifferential operator over X', and is elliptic.

Note that the Dirichlet-to-Neumann operator is $m \times m$-matrix formed, whereas the Calderón projector for A is $2m \times 2m$-matrix formed. They are in fact closely related.

Let us write C^+ in blocks according to the splitting of $\{0, 1, \dots, 2m-1\}$ into $\{0, 1, \dots, m-1\}$ and $\{m, \dots, 2m-1\}$:

$$C^+ = \begin{pmatrix} C^+_{00} & C^+_{01} \\ C^+_{10} & C^+_{11} \end{pmatrix}, \tag{11.60}$$

then

$$C^+_{10}\psi_0 + C^+_{11}\psi_1 = P_{\gamma,\nu}(C^+_{00}\psi_0 + C^+_{01}\psi_1) \tag{11.61}$$

holds for all

$$\{\psi_0, \psi_1\} \in \prod_{0\le j<m} H^{2m-j-\frac12}(X') \times \prod_{0\le j<m} H^{m-j-\frac12}(X').$$

Lemma 11.13. *Each of the blocks* C^+_{ij} *in* (11.60) *is elliptic.*

The proof is developed in Exercises 11.4–7 (originally shown in [G71]).

Now, note that in particular,

$$C^+_{10}\psi_0 = P_{\gamma,\nu}C^+_{00}\psi_0, \tag{11.62}$$

for $\psi_0 \in \prod_{0\le j<m} H^{2m-j-\frac12}(X')$. Let S be a parametrix of C^+_{00}, so that $C^+_{00}S = I - \mathcal{R}$ with a negligible ψdo $\mathcal{R}$. Then

$$C^+_{10}S = P_{\gamma,\nu}C^+_{00}S = P_{\gamma,\nu}(I - \mathcal{R}),$$

from which it follows that

$$P_{\gamma,\nu} = C^+_{10}S + P_{\gamma,\nu}\mathcal{R}. \tag{11.63}$$

Here $\mathcal{R}$ maps distributions to C^∞ functions, and $P_{\gamma,\nu}$ has the mapping properties (11.57), so it follows that $P_{\gamma,\nu}\mathcal{R}$ maps $\mathcal{D}'(X')$ continuously into $C^\infty(X')$, hence is a negligible ψdo. Thus $P_{\gamma,\nu}$ is a ψdo. Finally, since both C^+_{10} and S are elliptic, it follows from (11.63) that $P_{\gamma,\nu}$ is elliptic. We have shown:

Theorem 11.14. $P_{\gamma,\nu}$ *is an elliptic* ψ*do.*

Behind Lemma 11.13 lies a consideration of both the Dirichlet problem on X_+ and that on X_-. It always works for the model problems (which is what is used in Exercises 11.4–7), but let us now look at the full operators. Assuming that (11.48) holds also for the Dirichlet realization on X_-, we can introduce the notation $K^\pm_\gamma$ for the operators solving

$$Au = 0 \text{ in } X_+,\ \gamma^+ u = \varphi, \quad \text{resp.} \quad Au = 0 \text{ in } X_-,\ \gamma^- u = \varphi,$$

(so $K^+_\gamma = K_\gamma$), and denote

$$\nu^\pm K^\pm_\gamma = P^\pm_{\gamma,\nu}; \tag{11.58}$$

here

$$\nu^\pm = \{\gamma^\pm_m, \dots, \gamma^\pm_{2m-1}\}, \tag{11.59}$$

and $P^+_{\gamma,\nu} = P_{\gamma,\nu}$.

Then the above argumentation for $P^+_{\gamma,\nu}$ works also for $P^-_{\gamma,\nu}$, showing that it is a ψdo. The calculations in Exercises 11.4–7 now imply:

Theorem 11.15. *When* (11.48) *holds also for the Dirichlet realization on* X_- *(so that* $P^-_{\gamma,\nu}$ *is well-defined), then* $P^-_{\gamma,\nu}$ *and* $P^+_{\gamma,\nu} - P^-_{\gamma,\nu}$ *are elliptic* ψ*do's.*

In this case, one can moreover show that $P^+_{\gamma,\nu} - P^-_{\gamma,\nu}$ is invertible. Then C^+ can be described by an explicit formula from $P^+_{\gamma,\nu}$ and $P^-_{\gamma,\nu}$, and vice versa (details are worked out in Exercise 11.22):

$$\begin{pmatrix} C^+_{00} & C^+_{01} \\ C^+_{10} & C^+_{11} \end{pmatrix} = \begin{pmatrix} -(P^+_{\gamma,\nu} - P^-_{\gamma,\nu})^{-1} P^-_{\gamma,\nu} & (P^+_{\gamma,\nu} - P^-_{\gamma,\nu})^{-1} \\ -P^+_{\gamma,\nu}(P^+_{\gamma,\nu} - P^-_{\gamma,\nu})^{-1} P^-_{\gamma,\nu} & P^+_{\gamma,\nu}(P^+_{\gamma,\nu} - P^-_{\gamma,\nu})^{-1} \end{pmatrix}, \tag{11.64a}$$

$$P^+_{\gamma,\nu} = C^+_{11}(C^+_{01})^{-1}. \tag{11.64b}$$

Returning to K_γ, we note that when $\varphi \in \prod_{0\le j<m} H^{2m-j-\frac12}(X')$, then, with K^+ as in Theorem 11.6,

$$K^+ \begin{pmatrix} \varphi \\ P_{\gamma,\nu}\varphi \end{pmatrix}$$

solves the Dirichlet problem (11.50), so

$$K_\gamma = K^+ \begin{pmatrix} I \\ P_{\gamma,\nu} \end{pmatrix}. \tag{11.65}$$

This shows that K_γ is a Poisson operator. In particular, the mapping property mentioned in Theorem 11.11 extends to all s:

$$K_\gamma \colon \prod_{0\le j<m} H^{s-j-\frac12}(X') \to H^s(X) \text{ for all } s \in \mathbb{R}. \tag{11.65a}$$

As for R_γ, it can be expressed in terms of $Q = A^{-1}$ and K_γ by

$$R_\gamma = Q_+ - K_\gamma \gamma Q_+, \tag{11.66}$$

exactly as in the calculations leading to Theorem 9.18, so we conclude that R_γ likewise belongs to the ψdbo calculus.

Theorem 11.16. *When* A *is strongly elliptic, invertible on* $\widetilde{X}$*, and satisfies* (11.48)*, then the solution operator* $(R_\gamma \quad K_\gamma)$ *of the Dirichlet problem satisfies:*

K_γ *is a Poisson operator (mapping as in* (11.56a)*), and* R_γ *is the sum of a truncated* ψ*do* Q_+ *and a singular Green operator* $-K_\gamma\gamma Q_+$*.*

One can moreover show that the continuity of R_γ from $H^s(X)$ to $H^{s+2m}(X)$ for $s \ge 0$ extends down to $s > -m - \frac12$, see e.g. the analysis in [G90].

Exercises for Chapter 11.

11.1a. Let A be as in Section 11.1, with $E_1 = E_2 = E$. Define $A_{\max}$ as the operator in $L_2(E) = H^0(E)$ acting like A and with domain

$$D(A_{\max}) = \{u \in L_2(E) \mid Au \in L_2(E)\}.$$

Show that ϱ can be defined on $D(A_{\max})$, mapping it continuously into $\mathcal{H}^0(E'^d)$ (here $D(A_{\max})$ is provided with the graph-norm).
(*Hint.* Begin by showing that when $u \in D(A_{\max})$, Q_+Au is in $H^d(E)$, and $z = u - Q_+Au$ is in Z_+^0.)

11.2a. Let A and $S\varrho$ be as in the beginning of Section 11.2, with $E_1 = E_2 = E$. Define the realization A_S of A as the operator in $L_2(E)$ with domain (using Exercise 11.1a)

$$D(A_S) = \{u \in D(A_{\max}) \mid S\varrho u = 0\}.$$

Show that A_S is closed, densely defined.

11.3a. Continuation of Exercise 11.2a. Assume moreover that

$$\begin{pmatrix} A \\ S\varrho \end{pmatrix} \text{ has the inverse } \begin{pmatrix} R_S & K_S \end{pmatrix},$$

with all the properties in Theorem 11.8; in particular it belongs to the ψdbo calculus.

(a) Show that $D(A_S) \subset H^d(E)$.

(b) Show that when $u \in D(A_S)$ with $Au \in H^s(E)$ for some $s \geq 0$, then $u \in H^{s+d}(E)$.

(*Comment.* It is possible to show similar regularity results when $\{A, S\varrho\}$ is just assumed to be elliptic, e.g. by constructing a parametrix that acts as an inverse on suitable subspaces with finite codimension, using Fredholm theory. A systematic treatment can be found in [G90], including general Green operators and L_p-based spaces.)

11.1. Let A be a second-order scalar differential operator on $\mathbb{R}^n$ with principal symbol independent of x_n:

$$a^0(x', \xi', \xi_n) = s_2(x')\xi_n^2 + s_1(x', \xi')\xi_n + s_0(x', \xi').$$

(a) Assume that A is properly elliptic, such that for all (x', ξ') with $\xi' \neq 0$, the two roots $\tau^{\pm}(x', \xi')$ of the second-order polynomial $a^0(x', \xi', \tau)$ in τ lie in $\mathbb{C}_\pm$, respectively. Denote $\sigma^+ = -i\tau^+$, $\sigma^- = i\tau^-$; show that σ^+ and σ^- have positive real part and are homogeneous in ξ' of degree 1.

(b) Show that the principal symbol of a parametrix Q of A is (for $|\xi'| \geq 1$)

$$\begin{aligned} q^0(x', \xi', \xi_n) &= \frac{1}{s_2(\xi_n - \tau^+)(\xi_n - \tau^-)} \\ &= \frac{1}{s_2(\sigma^+ + \sigma^-)}\Big(\frac{1}{\sigma^+ + i\xi_n} + \frac{1}{\sigma^- - i\xi_n}\Big). \end{aligned}$$

(c) Find (at each (x', ξ'), $|\xi'| \geq 1$) the Poisson operator $r^+ q^0(x', \xi', D_n)\widetilde{\varrho}^*$ and the ψdo symbol $\varrho^+ q^0(x', \xi', D_n)\widetilde{\varrho}^*$, expressed in terms of $\sigma^\pm$.
(*Hint.* One can apply the rules in Theorem 10.20 and Lemma 10.13, see also Exercise 10.3.)

(d) Find the principal symbol of the Calderón projector C^+ for A.

11.2. Continuation of Exercise 11.1. Consider a boundary condition

$$b_1(x')D_1 u + \cdots + b_n(x')D_n u = 0 \text{ for } x_n = 0,$$

where the b_j are complex C^∞ functions, with $b_n(x') \neq 0$ for all x'. Give examples of choices of coefficients b_j where the condition defines an elliptic problem for A, and where it does not so.

11.3. In the setting of Section 11.3, show how (11.54b) can be used to conclude that $(\, R_\gamma \quad K_\gamma \,)$ has a mapping property as in (11.54) for all $s \geq 0$.

11.4. Let A be as in Section 11.3, and define ν by (11.55a). Show that the Neumann problem

$$Au = f \text{ in } X, \quad \nu u = \varphi \text{ on } X',$$

is elliptic.
(*Hint.* Consider the boundary symbol operator at an (x', ξ') with $\xi' \neq 0$. For dimensional reasons, it suffices to show that

$$\begin{pmatrix} a^0(x',\xi',D_n) \\ \nu \end{pmatrix} z = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies z = 0,$$

when $z \in \mathcal{S}_+$. Show that $u = D_n^m z$ solves the Dirichlet problem with zero data, hence is 0. Use Theorem 4.19 to conclude that $z = 0$.)

11.5. For the boundary symbol operator $a^0(x',\xi',D_n)$ for A considered in Section 11.3 (at an (x',ξ') with $\xi'\neq 0$), let k^+_γ and k^-_γ be the solution operators for the Dirichlet problems on $\mathbb{R}_+$ and $\mathbb{R}_-$:

$$k^\pm_\gamma\colon \mathbb{C}^m \to \mathcal{S}_\pm \quad \text{solves} \quad \begin{pmatrix} a^0(x',\xi',D_n) \\ \gamma \end{pmatrix} z = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \text{ on } \mathbb{R}_\pm.$$

With $\nu^\pm$ defined in (11.59), let

$$p^\pm = \nu^\pm k^\pm_\gamma,$$

respectively. Show that $p^\pm$ are invertible matrices.
(*Hint.* One can use the result of Exercise 11.4.)

11.6. Continuation of Exercise 11.5. Show that

$$p^+ - p^-$$

is an invertible matrix. (It is important for this calculation to recall that $\gamma_j^+ u = \gamma_0^+ D_n^j u$, $\gamma_j^- u = \gamma_0^- D_n^j u$, where D_n^j has the same meaning in the two expressions, namely $(-i\partial_{x_n})^j$ — with the same direction of x_n.)
(*Hint.* Let $\varphi\in\mathbb{C}^m$ be such that $(p^+ - p^-)\varphi = 0$, and let $u^\pm \in Z_\pm$ with $\gamma^\pm u^\pm = \varphi$. Since $p^+\varphi = p^-\varphi$, $\nu^+u^+ = \nu^-u^-$. Then $u = e^+u^+ + e^-u^-$ is a solution in $H^{2m}(\mathbb{R})$ of $a^0(D_n)u = 0$.)

11.7. Continuation of Exercises 11.5ff. Show that the Calderón projector for $a^0(x',\xi',D_n)$ (equal to the principal symbol of the Calderón projector for A) satisfies

$$c^{+,0} = \begin{pmatrix} -(p^+ - p^-)^{-1}p^- & (p^+ - p^-)^{-1} \\ -p^+(p^+ - p^-)^{-1}p^- & p^+(p^+ - p^-)^{-1} \end{pmatrix}.$$

In particular, the four blocks are invertible.
(More information can be found in [G71, Appendix].)

11.8. Let $A = I - \Delta$ on $\mathbb{R}^n_+$. With the notation of Exercise 11.5, show that $p^+ = -\langle\xi'\rangle$ and $p^- = \langle\xi'\rangle$, and that

$$c^{+,0} = \begin{pmatrix} \frac12 & -\frac12\langle\xi'\rangle^{-1} \\ -\frac12\langle\xi'\rangle & \frac12 \end{pmatrix}.$$

Moreover, C^+ is the ψdo on $\mathbb{R}^{n-1}$ with this symbol.

11.9. Show that the calculations in Exercises 11.4–7 are valid also for strongly elliptic systems (acting on vector valued functions), cf. (7.50).

11.10. Let A be as in Section 11.3.

(a) Show that for any $s \in \mathbb{R}$, N_+^s (cf. (11.14)) identifies with the graph of $P_{\gamma,\nu}$ going from $\prod_{0\le j<m} H^{s-j-\frac12}(X')$ to $\prod_{0\le j<m} H^{s-m-j-\frac12}(X')$. In other words,

$$\begin{pmatrix} I \\ P_{\gamma,\nu} \end{pmatrix} \text{ maps } \prod_{0\le j<m} H^{s-j-\frac12}(X') \text{ onto } N_s^+$$

with the inverse $\mathrm{pr}_1 = (\,I \quad 0\,)$, for any $s \in \mathbb{R}$.

(b) Show that K_γ defines a homeomorphism

$$K_\gamma \colon \prod_{0\le j<m} H^{s-j-\frac12}(X') \overset{\sim}{\to} Z_+^s$$

with inverse

$$\gamma \colon Z_+^s \to \prod_{0\le j<m} H^{s-j-\frac12}(X'),$$

for all $s \in \mathbb{R}$.

(*Hint.* Use (11.65), (11.29) and the mapping properties of the operators in the ψdbo calculus.)

11.11. Continuation of Exercise 11.10.

(a) Show that $Z_+^\infty = \bigcap_{s\in\mathbb{R}} Z_+^s$ is dense in Z_+^t for all $t \in \mathbb{R}$, and that $N_+^\infty = \bigcap_{s\in\mathbb{R}} N_+^s$ is dense in N_+^t for all $t \in \mathbb{R}$.
(*Hint.* One can combine the results of Exercise 11.10 with the fact that $C^\infty(X')$ is dense in $H^s(X')$ for any s.)

(b) Show that $D(A_{\max})$ is the direct sum of $D(A_\gamma)$ and Z_+^0.

(c) Show that $H^{2m}(X)$ is the direct sum of $D(A_\gamma)$ and Z_+^{2m}.

(d) Show that $H^{2m}(X)$ is dense in $D(A_{\max})$ (with respect to the graph-norm).
(*Hint.* One can use that (a) implies the denseness of Z_+^{2m} in Z_+^0.)

11.12. Let A be as in Section 11.3, now with $m = 1$ but acting on N-vectors $u = (u_1, \dots, u_N)$. In order to use ideas from Chapter 9, where X has a different meaning, we now denote $\widetilde{X} = \Xi$, $X_\pm = \Omega_\pm$, $X' = \Gamma = \partial\Omega_\pm$, so $X = \overline{\Omega}_+$, $U = \Gamma\times\,]-1,1[$. We also denote $\Omega_+ = \Omega$. Let $A_0 = A_{\min}$, $A_1 = A_{\max}$; they are operators in $L_2(\Omega)^N$.

Show that the results of Exercise 11.10 imply that

$$K_\gamma\colon H^{-\frac12}(\Gamma)^N \overset{\sim}{\to} Z(A_1)$$

with inverse

$$\gamma\colon Z(A_1) \overset{\sim}{\to} H^{-\frac12}(\Gamma)^N;$$

the latter operator can more specifically be called γ_Z.

11.13. Continuation of Exercise 11.12. Write A on $U = \Gamma \times \,]-1,1[$ as

$$A = S_2 D_n^2 + S_1 D_n + S_0,$$

as in Proposition 11.2, where S_2, S_1 and S_0 are x_n-dependent $N\times N$-matrices of functions, first-order differential operators, resp. second-order differential operators on Γ.

(a) Verify that there is a Green's formula for $u, v \in H^{2m}(\Omega)^N$:

$$(Au,v)_\Omega - (u, A'v)_\Omega = (\mathfrak{A}\varrho u, \varrho v)_\Gamma = (iS_2\gamma_1 u, \gamma_0 v)_\Gamma - (\gamma_0 u, iS_2^*\gamma_1 v - \mathfrak{A}_{00}^*\gamma_0 v)_\Gamma,$$

where

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{A}_{00} & iS_2 \\ iS_2 & 0 \end{pmatrix},$$

with $\mathfrak{A}_{00}$ equal to iS_1 plus a matrix of functions (S_2 and S_1 taken at $x_n = 0$); A' denotes the adjoint of A as a differential operator on Ξ. Show that Green's formula extends to $u \in H^{2m}(\Omega)^N$, $v \in D(A_1')$, with Sobolev space dualities in the right-hand side.

(*Hint.* Use that ϱ is well-defined on the nullspace Z_+^0 for A'.)

(b) Define the modified Neumann trace operators

$$\beta u = iS_2\gamma_1 u, \quad \beta' v = iS_2^*\gamma_1 v - \mathfrak{A}_{00}^*\gamma_0 v,$$

which allow writing Green's formula as

$$(Au,v)_\Omega - (u,A'v)_\Omega = (\beta u, \gamma_0 v)_\Gamma - (\gamma_0 u, \beta' v)_\Gamma;$$

and define the modified Dirichlet-to-Neumann operators

$$P_{\gamma,\beta} = iS_2\gamma_1 P_{\gamma_0,\gamma_1}, \quad P'_{\gamma,\beta'} = iS_2^*\gamma_1 P'_{\gamma_0,\gamma_1} - \mathfrak{A}_{00}^*,$$

where P_{γ_0,γ_1} and P'_{γ_0,γ_1} stand for $P_{\gamma,\nu}$ for A resp. A' (with $m = 1$). Define moreover the trace operators

$$\mu u = \beta u - P_{\gamma,\beta}\gamma_0 u, \quad \mu' v = \beta' v - P'_{\gamma,\beta'}\gamma_0 v.$$

Show that

$$\mu = \beta A_\gamma^{-1} Au, \text{ continuous from } D(A_1) \text{ to } H^{\frac{1}{2}}(\Gamma)^N,$$

with a similar result for μ', and that

$$(Au, v)_\Omega - (u, A'v)_\Omega = \langle \mu u, \overline{\gamma_0 v} \rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}} - \langle \gamma_0 u, \overline{\mu' v} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}$$

holds for all $u \in D(A_1)$, $v \in D(A_1')$.
(*Hint.* Proceed as in Proposition 9.27.)

(c) Carry the construction in Section 9.4, up to and including Theorem 9.29, over to the present situation, showing a 1–1 correspondence between closed realizations $\widetilde{A}$ of A and closed densely defined operators $L\colon X \to Y^*$, where X and Y are closed subspaces of $H^{-\frac{1}{2}}(\Gamma)^N$.

11.14. Continuation of Exercise 11.12ff. Show that in the 1–1 correspondence established in Exercise 11.13, the Dirichlet realization A_γ corresponds to the case $X = Y = \{0\}$, $L = 0$.

11.15. Continuation of Exercise 11.12ff. Show that in the 1–1 correspondence established in Exercise 11.13, the realization A_M with domain $D(A_M) = D(A_0) \dot{+} Z(A_1)$ corresponds to the case $X = Y = H^{-\frac{1}{2}}(\Gamma)^N$, $L = 0$.

11.16. Continuation of Exercise 11.12ff. Let A_ν be the realization defined by the Neumann condition $\gamma_1 u = 0$,

$$D(A_\nu) = \{u \in D(A_{\max}) \mid \gamma_1 u = 0\},$$

cf. Exercises 11.1a–11.2a.

(a) Show that $D(A_\nu) \subset H^2(\Omega)^N$.
(*Hint.* Show first that when $u \in D(A_\nu)$, $z = u - Q_+ Au$ has $\nu z \in H^{\frac{1}{2}}(\Gamma)^N$. Then use the ellipticity of P_{γ_0,γ_1} to conclude that $\varrho z \in H^{\frac{3}{2}}(\Gamma)^N \times H^{\frac{1}{2}}(\Gamma)^N$.)

(b) Show that in the 1–1 correspondence established in Exercise 11.13, A_ν corresponds to the case $X = Y = H^{-\frac{1}{2}}(\Gamma)^N$, $L = -P_{\gamma,\beta}$ with domain $D(L) = H^{\frac{3}{2}}(\Gamma)^N$.

11.17. Continuation of Exercise 11.12ff. Consider the realization $\widetilde{A}$ defined by a boundary condition

$$\gamma_1 u + B\gamma_0 u = 0,$$

where B is a first-order ψdo on N-vectors of functions on Γ such that $B + P_{\gamma_0,\gamma_1}$ is elliptic of order 1; here

$$D(\widetilde{A}) = \{u \in D(A_{\max}) \mid \gamma_1 u + B\gamma_0 u = 0\}.$$

Show that in the 1–1 correspondence established in Exercise 11.13, $\widetilde{A}$ corresponds to the case $X = Y = H^{-\frac{1}{2}}(\Gamma)^N$, $L = -iS_2(B + P_{\gamma_0,\gamma_1})$ with domain $D(L) = H^{\frac{3}{2}}(\Gamma)^N$; in particular, $D(\widetilde{A}) \subset H^2(\Omega)^N$.

11.18. Continuation of Exercise 11.12ff. For simplicity of the calculations in the following, assume that $S_2 = I$. Let J be an integer in $]1, N[$, and consider the realization $\widetilde{A}$ defined by a boundary condition

$$\gamma_0(u_1, \dots, u_J) = 0, \quad \gamma_1(u_{J+1}, \dots, u_N) + B\gamma_0(u_{J+1}, \dots, u_N) = 0,$$

applied to column vectors $u = (u_1, \dots, u_N)$; here B is an $(N-J) \times (N-J)$-matrix of ψdo's on Γ of order 1. Assume that the first-order ψdo

$$\mathcal{L} = -iB - i\,(\,0_{N-J,J} \quad I_{N-J}\,)\,P_{\gamma_0,\gamma_1}\begin{pmatrix} 0_{J,N-J} \\ I_{N-J} \end{pmatrix}$$

is elliptic. Here I_J stands for the $J \times J$-identity-matrix, and $0_{J,J'}$ stands for the $J \times J'$-zero-matrix.

Show that in the 1–1 correspondence established in Exercise 11.12, $\widetilde{A}$ corresponds to the case $X = Y = H^{-\frac{1}{2}}(\Gamma)^{N-J}$, with L acting like $\mathcal{L}$ with domain $D(L) = H^{\frac{3}{2}}(\Gamma)^{N-J}$. In particular, $D(\widetilde{A}) \subset H^2(\Omega)^N$.

(*Comment.* Similar boundary conditions, where the splitting into the first J and last $N-J$ components is replaced by more general projections, are considered e.g. in Avramidi and Esposito [AE99], [G03] (and earlier). For higher order studies, see e.g. Fujiwara and Shimakura [FS70], [G70]–[G74]. Also positivity issues are treated in these papers, see in particular [G74]; then β and β' in Exercise 11.12 are replaced by more symmetric choices of modified Neumann operators, letting each of them carry half of $\mathfrak{A}_{00}$. See also [BGW08].)

11.19. Consider the boundary value problem

$$Au = f, \quad C^+\varrho u = 0$$

(cf. Example 11.10), with f given in $H^0(E_2)$. Show that it has the unique solution

$$u = Q_+ f - K^+ C^+ \varrho Q_+ f,$$

lying in $H^d(E_1)$.
(Note however, that the nonhomogeneous problem

$$Au = f, \quad C^+\varrho u = \varphi,$$

does *not* have a solution for every $f \in H^0(E_2)$, $\varphi \in \mathcal{H}^d(E_1'^d)$. Why?)

11.20. Let A be a first-order differential operator on $\widetilde{X}$ such that $A = I_N\partial_{x_n} + P$ near X', where P is an $N\times N$-matrix formed first-order selfadjoint elliptic differential operator on X'. Let $n \geq 3$.

(a) Show that the principal symbol $p^0(x',\xi')$ of P (considered at an (x',ξ') with $|\xi'| \geq 1$) has $N/2$ positive eigenvalues $\lambda_1^+, \dots, \lambda_{N/2}^+$ and $N/2$ negative eigenvalues $\lambda_1^-, \dots, \lambda_{N/2}^-$ (repeated according to multiplicities); in particular, N is even.
(*Hint.* Note that $p^0(x',-\xi') = -p^0(x',\xi')$.)

(b) Show that the roots of $\det(i\tau I_N + p^0(x',\xi'))$ are: $\tau_j^+ = i\lambda_j^+$ lying in $\mathbb{C}_+$, $j = 1, \dots, N/2$, and $\tau_j^- = i\lambda_j^-$ lying in $\mathbb{C}_-$, $j = 1, \dots, N/2$.

(c) Assume from now on that all eigenvalues $\lambda_j^\pm$ are simple. Show that the functions in $Z_+(x',\xi')$ are linear combinations of functions $e^{-\lambda_j^+(x',\xi')x_n}v_j(x',\xi')$, where v_j is an eigenvector for the eigenvalue λ_j^+.

(d) Show that $N^+(x',\xi')$ equals the space spanned by the eigenvectors for eigenvalues λ_j^+, $j = 1, \dots, N/2$, also called the positive eigenspace for $p^0(x',\xi')$.

(d) Show that $c^{+,0}(x',\xi')$ is the *orthogonal* projection onto the positive eigenspace for $p^0(x',\xi')$.

11.21. Consider the biharmonic operator $A = \Delta^2$ on a smooth bounded set $\Omega \subset \mathbb{R}^n$.

(a) Find the principal symbol of the Dirichlet-to-Neumann operator (for $|\xi'| \geq 1$).
(*Hint.* Show first that in the model problem on $\mathbb{R}_+$, the null solutions are linear combinations of $e^{-\sigma x_n}$ and $x_n e^{-\sigma x_n}$, $\sigma = |\xi'|$.)

(b) Find the principal symbol of the Calderón projector (for $|\xi'| \geq 1$).
(*Hint.* One can use the results of Exercises 11.6 and 11.7.)

11.22. Assumptions of Theorem 11.15.

(a) Show that $P_{\gamma,\nu}^+ - P_{\gamma,\nu}^-$ is injective.
(*Hint.* One can use a generalization of the method of Exercise 11.6)

(b) Show that $P_{\gamma,\nu}^+ - P_{\gamma,\nu}^-$ is surjective.

(*Hint.* Apply C^+ and C^-, written in blocks as in (11.60), to functions $\begin{pmatrix} 0 \\ \psi \end{pmatrix}$. Conclude that

$$P^+_{\gamma,\nu} C^+_{01}\psi = C^+_{11}\psi, \quad P^-_{\gamma,\nu} C^-_{01}\psi = C^-_{11}\psi,$$

and use the fact that $C^+ + C^- = I$ to see that $(P^+_{\gamma,\nu} - P^-_{\gamma,\nu})C^+_{01}\psi = \psi$.

(c) Show that $P^+_{\gamma,\nu} - P^-_{\gamma,\nu}$ has the inverse C^+_{01}.

(d) Show (11.64a) and (11.64b).