§13. Families of extensions

13.1 A general parametrization of extensions.

We have seen in Chapter 12 that a closed, densely defined symmetric operator S in general has several selfadjoint extensions: When S is lower bounded with a nontrivial nullspace, there is the Friedrichs extension established in Section 12.5, but there is also the von Neumann or Krein extension defined in Exercise 12.23 when $m(S) > 0$. When S has equal deficiency indices (finite or infinite), cf. (12.27), there is a family of selfadjoint extensions parametrized by the family of linear isometries (unitary operators) of $R(S + iI)^{\perp}$ onto $R(S - iI)^{\perp}$, cf. Exercise 12.19.

The latter family has been used in the study of boundary conditions for ordinary differential operators (where the deficiency indices are finite) in the classical book of Naimark [N68], and more recently e.g. by Everitt and Markus in a number of papers and surveys, see for example [EM99].

For partial differential operators of elliptic type, one is often in a position of having a symmetric and injective operator to depart from; here there is another family of extensions that has proved to be particularly adequate, as initiated by Krein [K47], Vishik [V52] and Birman [B56], and fully developed by Grubb [G68].

We shall give an introduction to this theory below, supplied with some recent results. Since the theory works also in nonsymmetric cases, we shall treat a more general situation than the search for selfadjoint extensions of symmetric operators.

There is given a complex Hilbert space H with scalar product $(u, v)_H$ and norm $\|u\|_H$, usually denoted (u, v) and $\|u\|$, and two densely defined closed operators A_0 and A'_0 satisfying

$$A_0 \subset (A'_0)^*, \quad A'_0 \subset (A_0)^*. \tag{13.1}$$

We denote

$$(A'_0)^* = A_1, \quad (A_0)^* = A'_1. \tag{13.2}$$

Moreover, there is given a closed densely defined operator A_β having a bounded, everywhere defined inverse A_β^{-1} and satisfying

$$A_0 \subset A_\beta \subset A_1;\ \text{then also } A'_0 \subset A_\beta^* \subset A'_1, \tag{13.3}$$

with $(A_\beta^*)^{-1} = (A_\beta^{-1})^*$ (cf. Theorem 12.7). We shall denote by $\mathcal{M}$, resp. $\mathcal{M}'$, the family of linear operators $\widetilde{A}$ (resp. $\widetilde{A}'$) satisfying

$$A_0 \subset \widetilde{A} \subset A_1,\ \text{resp. } A'_0 \subset \widetilde{A}' \subset A'_1; \tag{13.4}$$

note that a closed operator $\widetilde{A}$ is in $\mathcal{M}$ if and only if $\widetilde{A}^* \in \mathcal{M}'$.

Observe in particular that when A_0 is symmetric and A_β is selfadjoint, we can obtain this situation by taking

$$A_0' = A_0, \quad A_\beta^* = A_\beta, \quad A_1' = A_1 = A_0^*; \quad \mathcal{M} = \mathcal{M}'. \tag{13.5}$$

we call this situation *the symmetric case.*

In the applications to partial differential operators A on an open set $\Omega \subset \mathbb{R}^n$, A_0 will be the minimal operator in $L_2(\Omega)$ defined from A, and A_1 will be the maximal operator in $L_2(\Omega)$ defined from A, cf. Chapter 4, and the $\widetilde{A}$ are the realizations of A. In elliptic cases one can often establish the existence of an operator A_β satisfying (13.3). The family of realizations $\widetilde{A}$ are viewed as the general extensions of A_0.

For simplicity in the formulas, we shall use the convention that for all the operators $\widetilde{A}$ in $\mathcal{M}$ (acting like A_1), we can write $\widetilde{A}u$ as Au without extra markings. Similarly, when $\widetilde{A}' \in \mathcal{M}'$, we can write $A'v$ instead of $\widetilde{A}'v$.

The domains $D(\widetilde{A})$ of closed realizations are closed with respect to the graph norm $\|u\|_A = (\|u\|^2 + \|Au\|^2)^{\frac12}$.

Lemma 13.1. *There are decompositions into direct sums:*

$$D(A_1) = D(A_\beta) \dot{+} Z(A_1), \quad D(A_1') = D(A_\beta^*) \dot{+} Z(A_1'); \tag{13.5a}$$

here we write $u \in D(A_1)$ resp. $v \in D(A_1')$ in decomposed form as

$$u = u_\beta + u_\zeta, \quad v = v_{\beta'} + v_{\zeta'}, \tag{13.5b}$$

with the notation for the hereby defined decomposition maps

$$\begin{aligned} \operatorname{pr}_\beta &\colon D(A_1) \to D(A_\beta), \quad \operatorname{pr}_\zeta \colon D(A_1) \to Z(A_1), \\ \operatorname{pr}_{\beta'} &\colon D(A_1') \to D(A_\beta^*), \quad \operatorname{pr}_{\zeta'} \colon D(A_1') \to Z(A_1'). \end{aligned} \tag{13.5c}$$

The maps are projections and satisfy

$$\begin{aligned} \operatorname{pr}_\beta &\colon u \mapsto A_\beta^{-1} Au, \quad \operatorname{pr}_{\beta'} \colon v \mapsto (A_\beta^*)^{-1} A'v, \\ \operatorname{pr}_\zeta &= I - \operatorname{pr}_\beta, \quad \operatorname{pr}_{\zeta'} = I - \operatorname{pr}_{\beta'}, \end{aligned} \tag{13.6}$$

and they are continuous with respect to the graph-norms.

Proof. Clearly, $D(A_\beta) + Z(A_1) \subset D(A_1)$. Define pr_β by (13.6); it sends $D(A_1)$ into $D(A_\beta)$ and is continuous with respect to the graph-norm:

$$\|\operatorname{pr}_\beta u\|^2 + \|A \operatorname{pr}_\beta u\|^2 = \|A_\beta^{-1} Au\|^2 + \|Au\|^2 \le (\|A_\beta^{-1}\|^2 + 1)\|Au\|^2.$$

Then pr_ζ defined in (13.6) is also continuous in the graph norm; and this operator maps $D(A_1)$ into $Z(A_1)$ since

$$A(u - A_\beta^{-1} Au) = Au - Au = 0.$$

It is immediate to check that $\mathrm{pr}_\beta \mathrm{pr}_\beta = \mathrm{pr}_\beta$. Then also $\mathrm{pr}_\zeta \mathrm{pr}_\zeta = (I - \mathrm{pr}_\beta)(I - \mathrm{pr}_\beta) = \mathrm{pr}_\zeta$, and $\mathrm{pr}_\beta \mathrm{pr}_\zeta = \mathrm{pr}_\beta - \mathrm{pr}_\beta \mathrm{pr}_\beta = 0$, so the operators are projections onto complementing subspaces.

There is a similar proof for $D(A_1')$. $\square$

Note that the injections of $D(A_\beta)$ and $Z(A_1)$ into $D(A_1)$ are likewise continuous in the graph norm (they are closed suspaces), so the mapping

$$u \mapsto \{\mathrm{pr}_\beta u, \mathrm{pr}_\zeta u\} \text{ from } D(A_1) \text{ to } D(A_\beta) \times Z(A_1) \tag{13.7}$$

is a homeomorphism with respect to graph norms. The notation is such that

$$\begin{aligned} \mathrm{pr}_\beta u = u_\beta, &\quad \mathrm{pr}_\zeta u = u_\zeta, \\ \mathrm{pr}_{\beta'} v = v_{\beta'}, &\quad \mathrm{pr}_{\zeta'} v = v_{\zeta'}. \end{aligned} \tag{13.8}$$

In the following, the notation U_V is used for the *orthogonal* projection of an element or subspace U of H into a closed subspace V of H. For simplicity in notation, we shall henceforth denote

$$Z(A_1) = Z, \quad Z(A_1') = Z'.$$

Lemma 13.2. *For $u \in D(A_1)$, $v \in D(A_1')$, one has*

$$\begin{aligned} (Au, v) - (u, A'v) &= (Au, v_{\zeta'}) - (u_\zeta, A'v) \\ &= ((Au)_{Z'}, v_{\zeta'}) - (u_\zeta, (A'v)_Z). \end{aligned} \tag{13.9}$$

Proof. Since $Au = Au_\beta$, $A'v = A'v_{\beta'}$, we can write

$$\begin{aligned} (Au, v) - (u, A'v) &= (Au_\beta, v_{\beta'} + v_{\zeta'}) - (u_\beta + u_\zeta, A'v_{\beta'}) \\ &= (Au_\beta, v) - (u, A'v_{\beta'}) + (Au, v_{\zeta'}) - (u_\zeta, A'v). \end{aligned}$$

Here

$$(Au_\beta, v_{\beta'}) - (u_\beta, A'v_{\beta'}) = 0,$$

since u_β and $v_{\beta'}$ belong to the domains of the adjoint operators A_β and A_β^*. In the remaining terms, $(Au, v_{\zeta'}) = ((Au)_{Z'}, v_{\zeta'})$ since $v_{\zeta'} \in Z'$, and similarly $(u_\zeta, Av) = (u_\zeta, (Av)_Z)$. $\square$

We shall say that S_1, S_2 is a pair of adjoint operators, when $S_2 = S_1^*$, $S_1 = S_2^*$; note that they are closed.

The heart of our construction lies in the following result:

Proposition 13.3. *Let $\widetilde{A} \in \mathcal{M}$, $\widetilde{A}^* \in \mathcal{M}'$ be a pair of adjoint operators. Define*

$$\begin{aligned} D(T) &= \operatorname{pr}_\zeta D(\widetilde{A}), \quad V = \overline{D(T)},\\ D(T_1) &= \operatorname{pr}_{\zeta'} D(\widetilde{A}^*), \quad W = \overline{D(T_1)}, \end{aligned} \tag{13.10}$$

closures in H. Then the equations

$$Tu_\zeta = (Au)_W, \text{ for } u \in D(\widetilde{A}), \tag{13.11}$$

define an operator $T\colon V \to W$ with domain $D(T)$; and

$$T_1 v_{\zeta'} = (A'v)_V, \text{ for } v \in D(\widetilde{A}^*), \tag{13.12}$$

define an operator $T_1\colon W \to V$ with domain $D(T_1)$. Moreover, the operators T and T_1 are adjoints of one another.

Proof. Note that V and W are closed subspaces of Z resp. Z'. When $u \in D(\widetilde{A})$ and $v \in D(\widetilde{A}^*)$, then in view of Lemma 13.2,

$$0 = (Au, v) - (u, A'v) = (Au, v_{\zeta'}) - (u_\zeta, A'v),$$

so that, since $u_\zeta \in V$ and $v_{\zeta'} \in W$,

$$((Au)_W, v_{\zeta'}) = (u_\zeta, (A'v)_V), \text{ when } u \in D(\widetilde{A}),\ v \in D(\widetilde{A}^*). \tag{13.13}$$

We shall show that the set $G \subset V \times W$ defined by

$$G = \{\, \{u_\zeta, (Au)_W\} \mid u \in D(\widetilde{A})\} \tag{13.14}$$

is a graph, of an operator T from V to W with domain $D(T)$ defined in (13.10). For this, we have to show that for the pairs in (13.14), $u_\zeta = 0$ implies $(Au)_W = 0$ (recall Lemma 12.3). So let $u_\zeta = 0$ for such a pair. Then by (13.13),

$$((Au)_W, v_{\zeta'}) = 0 \text{ for all } v \in D(\widetilde{A}^*). \tag{13.15}$$

But here $v_{\zeta'}$ runs through a dense subset of W, so it follows that $(Au)_W = 0$. Thus G is the graph of an operator T from V to W; its domain is clearly $D(T)$ defined in (13.10).

The proof that (13.12) defines an operator $T_1\colon W \to V$ with domain $D(T_1)$ defined in (13.10) is similar.

Note that T and T_1 are densely defined, so that their adjoints are well-defined. By (13.13),

$$(Tu_\zeta, v_{\zeta'})_W = (u_\zeta, T_1 v_{\zeta'})_V \text{ for all } u_\zeta \in D(T),\ v_{\zeta'} \in D(T_1), \tag{13.16}$$

so $T \subset T_1^*$. We shall show that this inclusion is an equality.

Let $z \in D(T_1^*)$ (which is contained in $V \subset Z$). Define

$$x = z + A_\beta^{-1} T_1^* z; \tag{13.17}$$

clearly $x \in D(A_1)$ with

$$x_\beta = A_\beta^{-1} T_1^* z,\ x_\zeta = z,\ Ax = T_1^* z. \tag{13.18}$$

We have for all $v \in D(\widetilde{A}^*)$:

$$\begin{aligned}(Ax, v) - (x, A'v) &= (Ax, v_{\zeta'}) - (z, A'v) \ \text{ by Lemma 13.2} \\ &= (T_1^* z, v_{\zeta'}) - (z, (A'v)_V) \ \text{ since } z \in V \\ &= (T_1^* z, v_{\zeta'}) - (z, T_1 v_{\zeta'}) = 0, \ \text{ by definition of } T_1.\end{aligned}$$

Since v is arbitrary in $D(\widetilde{A}^*)$, it follows that $x \in D(\widetilde{A}^{**}) = D(\widetilde{A})$, and then $z = \operatorname{pr}_\zeta x \in D(T)$, as was to be shown.

There is a similar proof that $T_1 = T^*$. $\square$

Proposition 13.3 shows how every pair of adjoint operators $\widetilde{A} \in \mathcal{M}$, $\widetilde{A}^* \in \mathcal{M}'$, gives rise to a pair of adjoint operators $T\colon V \to W$, $T^*\colon W \to V$, with V and W being closed subspaces of Z, resp. Z'. The next proposition shows that all choices of such pairs of adjoints are reached in a one-to-one way, and gives formulas for the domains $D(\widetilde{A})$, $D(\widetilde{A}^*)$ corresponding to T, T^*.

Proposition 13.4. *Let V and W be closed subspaces of Z, resp. Z', and let $T\colon V \to W$, $T^*\colon W \to V$ be a pair of adjoint operators (generally unbounded). Define the operators $\widetilde{A} \subset A_1$, $\widetilde{A}' \subset A_1'$ by*

$$D(\widetilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T),\, (Au)_W = Tu_\zeta\}, \tag{13.19}$$

$$D(\widetilde{A}') = \{v \in D(A_1') \mid v_{\zeta'} \in D(T^*),\, (A'v)_V = T^* v_{\zeta'}\}; \tag{13.20}$$

they are in $\mathcal{M}$ resp. $\mathcal{M}'$. They are adjoints of one another, and the operators derived from $\widetilde{A}$, $\widetilde{A}^*$ by Proposition 13.3 *are exactly $T\colon V \to W$, $T^*\colon W \to V$.*

If $T\colon V \to W$, $T^*\colon W \to V$, are operators derived from a pair of adjoints $\widetilde{A} \in \mathcal{M}$, $\widetilde{A}^* \in \mathcal{M}'$ by Proposition 13.3, *then the formulas* (13.19), (13.20) *give back the domains $D(\widetilde{A})$, $D(\widetilde{A}^*)$.*

Proof. Let $\widetilde{A}$ and $\widetilde{A}'$ be given by (13.19), (13.20); they clearly extend A_0 resp. A_0'. It follows by use of Lemma 13.2 that for $u \in D(\widetilde{A})$, $v \in D(\widetilde{A}')$,

$$(Au, v) - (u, A'v) = ((Au)_W, v_{\zeta'}) - (u_\zeta, (A'v)_V) = (Tu_\zeta, v_{\zeta'}) - (u_\zeta, T^* v_{\zeta'}) = 0,$$

in view of the definitions of $\widetilde{A}$ and $\widetilde{A}'$. So $\widetilde{A}$ and $\widetilde{A}'$ are contained in each other's adjoints. We have to show equality, by symmetry it suffices to show that $\widetilde{A}^* \subset \widetilde{A}'$.

Let $v \in D(\widetilde{A}^*)$. Since A_0 is closed with a bounded inverse, its range is closed, so

$$H = R(A_0) \oplus Z', \tag{13.21}$$

orthogonal direct sum. In particular, $R(A_0) \perp W$. Therefore any element $u = z + A_\beta^{-1}Tz + x$, where $z \in D(T)$ and $x \in D(A_0)$, is in $D(\widetilde{A})$, since $u_\zeta = z$ and $(Au)_W = (Tz + Ax)_W = Tz$. We have for all such u, using Lemma 13.2 again:

$$0 = (Au, v) - (u, A'v) = (Tz + Ax, v_{\zeta'}) - (z, A'v) = (Tz, v_{\zeta'}) - (z, (A'v)_V).$$

Since z is arbitrary in $D(T)$, this shows that $v_{\zeta'} \in D(T^*)$ with $T^*v_{\zeta'} = (A'v)_V$, and hence, by definition, $v \in D(\widetilde{A}')$. So $D(\widetilde{A}^*) \subset D(\widetilde{A}')$, and since $\widetilde{A}' \subset \widetilde{A}^*$, it follows that $\widetilde{A}' = \widetilde{A}^*$. It is now also obvious that T and T^* are determined from $\widetilde{A}$ and $\widetilde{A}^*$ as in Proposition 13.3. This shows the first statement in the proposition.

Finally to see that the pair that gives rise to $T\colon V \to W$ and $T^*\colon W \to V$ is unique, let $\widetilde{A}$, $\widetilde{A}^*$ be the pair defined by (13.19), (13.20), and let $\widetilde{B}$, $\widetilde{B}^*$ be another pair that gives rise to $T\colon V \to W$ and $T^*\colon W \to V$ as in Proposition 13.3. Then according to (13.10)–(13.12) and (13.19)–(13.20),

$$D(\widetilde{A}) \supset D(\widetilde{B}), \quad D(\widetilde{A}^*) \supset D(\widetilde{B}^*).$$

It follows that $\widetilde{A} \supset \widetilde{B}$ and $\widetilde{A}^* \supset \widetilde{B}^*$, but this can only happen when $\widetilde{A} = \widetilde{B}$. $\square$

The two propositions together imply:

Theorem 13.5. *There is a one-to-one correspondence between all pairs of adjoint operators $\widetilde{A} \in \mathcal{M}$, $\widetilde{A}^* \in \mathcal{M}'$, and all pairs of adjoint operators $T\colon V \to W$, $T^*\colon W \to V$, where V and W run through the closed subspaces of Z resp. Z'; the correspondence being given by*

$$D(\widetilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T),\, (Au)_W = Tu_\zeta\}, \tag{13.22}$$

$$D(\widetilde{A}^*) = \{v \in D(A_1') \mid v_{\zeta'} \in D(T^*),\, (A'v)_V = T^*v_{\zeta'}\}. \tag{13.23}$$

In this correspondence,

$$D(T) = \operatorname{pr}_\zeta D(\widetilde{A}), \quad V = \overline{D(T)}, \tag{13.24}$$

$$D(T^*) = \operatorname{pr}_{\zeta'} D(\widetilde{A}^*), \quad W = \overline{D(T^*)}. \tag{13.25}$$

The formulation is completely symmetric in $\widetilde{A}$ and $\widetilde{A}^*$, and in T and T^*. The pair $\widetilde{A}$, $\widetilde{A}^*$ is of course completely determined by the closed operator $\widetilde{A} \in \mathcal{M}$, so we need only mention $\widetilde{A}$; similarly we need only mention $T\colon V \to W$. Before we formulate this in a corollary, we shall show another useful description of $\widetilde{A}$ in terms of T.

Theorem 13.6. *When $\widetilde{A}$ corresponds to $T\colon V \to W$ as above, the mapping*

$$\Psi\colon \{z, f, v\} \mapsto u = z + A_\beta^{-1}(Tz + f) + v \tag{13.26}$$

defines a bijection

$$\Psi\colon D(T) \times (Z' \ominus W) \times D(A_0) \xrightarrow{\sim} D(\widetilde{A}), \tag{13.27}$$

homeomorphic with respect to graph norms on $D(T)$, $D(A_0)$ and $D(\widetilde{A})$ and the H-norm on $Z' \ominus W$ (in a sense a graph norm too).

Proof. Let $u = z + A_\beta^{-1}(Tz+f)+v$, with $\{z, f, v\} \in D(T)\times(Z'\ominus W)\times D(A_0)$. Clearly, u belongs to $D(A_1)$, with $u_\zeta = z$, $u_\beta = A_\beta^{-1}(Tz + f) + v$ according to the decomposition in Lemma 13.1. Moreover, $Au = Tz + f + Av$, with $Tz \in W$, $f \in Z' \ominus W$ and $Av \in R(A_0)$, so since

$$H = W \oplus (Z' \ominus W) \oplus R(A_0), \tag{13.28}$$

(recall (13.21)), $(Au)_W = Tz$. Thus $u \in D(\widetilde{A})$ according to (13.22).

Conversely, when $u \in D(\widetilde{A})$, we can decompose Au according to (13.28), setting $f = (Au)_{Z'\ominus W}$, $v = A_\beta^{-1}[(Au)_{R(A_0)}]$. Then $v \in D(A_0)$. Here $u = u_\zeta + u_\beta$, where $u_\beta = A_\beta^{-1}Au$, and $u_\zeta \in D(T)$, $(Au)_W = Tu_\zeta$ by assumption. Then

$$u = u_\zeta + A_\beta^{-1}Au = u_\zeta + A_\beta^{-1}(Tu_\zeta + f + Av) = u_\zeta + A_\beta^{-1}(Tu_\zeta + f) + v,$$

where $u_\zeta \in D(T)$, $f \in Z' \ominus W$ and $v \in D(A_0)$. This shows that the mapping Ψ is surjective. The injectiveness is easily shown by use of (13.28).

The continuity (with respect to graph-norms) is easily verified, and then the inverse is likewise continuous (in view of the open mapping theorem or by a direct verification). □

As a consequence of Theorems 13.5 and 13.6 we can now formulate:

Theorem 13.7. *There is a one-to-one correspondence between all closed operators $\widetilde{A} \in \mathcal{M}$ and all operators $T\colon V \to W$, where V and W are closed subspaces of Z resp. Z', and T is closed with domain $D(T)$ dense in V; the correspondence is defined by* (13.22), (13.24).

The domain $D(\widetilde{A})$ is also described by (13.26), (13.27).

There are many results on how properties of $\widetilde{A}$ are reflected in properties of T in this correspondence. We begin with the following, that follow straightforwardly from the formulas:

Theorem 13.8. *Let $\widetilde{A} \in \mathcal{M}$ correspond to $T\colon V \to W$ as in Theorem* 13.7. *Then:*

1° $Z(\widetilde{A}) = Z(T)$*. In particular,* $\dim Z(\widetilde{A}) = \dim Z(T)$*, and $\widetilde{A}$ is injective if and only if T is so.*

2° *$R(\widetilde{A})$ has the following decomposition:*

$$R(\widetilde{A}) = R(T) + [(Z' \ominus W) \oplus R(A_0)]. \tag{13.29}$$

where also the first sum is orthogonal. In particular, $R(\widetilde{A})$ is closed if and only if $R(T)$ is closed, and in the affirmative case, $H \ominus R(\widetilde{A}) = W \ominus R(T)$. The ranges of $\widetilde{A}$ and T (in H resp. W) have the same codimension, and $\widetilde{A}$ is surjective if and only if T is so.

3° *$\widetilde{A}$ is Fredholm, if and only if T is so, and then they have the same index.*

Proof. 1°. Let $u \in Z(\widetilde{A})$. Then $u_\zeta = u$, and $Tu_\zeta = (Au)_W = 0$, so $u \in Z(T)$. Conversely, if $u \in Z(T)$, then $u \in D(\widetilde{A})$ (take f and v equal to 0 in (13.26)), so since $u \in D(\widetilde{A}) \cap Z$, $\widetilde{A}u = 0$.

2°. By the proof of Theorem 13.6 and (13.28), the general element of $R(\widetilde{A})$ is orthogonally decomposed as $g = Tz + f + Av$, where $z \in D(T)$, $f \in Z' \ominus W$ and $Av \in R(A_0)$; this gives the decomposition (13.29). Here $Z' \ominus W$ and $R(A_0)$ are closed, so closedness of $R(\widetilde{A})$ holds if and only if it holds for $R(T)$. Moreover, if they are closed, the orthogonal complement of $R(\widetilde{A})$ in H equals the orthogonal complement of $R(T)$ in W.

The remaining statements are straightforward consequences. (Fredholm operators are explained in Section 8.3.) □

Note that $[(Z' \ominus W) \oplus R(A_0)]$ may also be written $H \ominus W$.

In the case where $\widetilde{A}$ and T are injective, we have a simple formula linking their inverses.

Theorem 13.9. *Let $\widetilde{A} \in \mathcal{M}$ correspond to $T\colon V \to W$ as in Theorem* 13.7. *Assume that $\widetilde{A}$ is injective, then so is T. Define $T^{(-1)}$ as the linear extension of*

$$T^{(-1)} f = \begin{cases} T^{-1} f & \text{when } f \in R(T), \\ 0 & \text{when } f \in H \ominus W. \end{cases} \tag{13.30}$$

Then

$$\widetilde{A}^{-1} = A_\beta^{-1} + T^{(-1)}, \text{ defined on } R(\widetilde{A}). \tag{13.31}$$

Proof. Let $f \in R(\widetilde{A})$. Let $u = \widetilde{A}^{-1} f$, $v = A_\beta^{-1} f$. Then $u - v = z$, where $z \in Z$. By the definition of T, z belongs to $D(T)$ and $Tz = (Au)_W = f_W$. Therefore $f_W \in R(T)$, and

$$T^{(-1)} f = T^{-1} f_W = z.$$

Inserting this in $u = v + z$, we find:

$$\widetilde{A}^{-1} f = A_\beta^{-1} f + T^{(-1)} f,$$

which proves the theorem. $\square$

Example 13.10. A simple example of the choice of T is to take $V = Z$, $W = Z'$, $T = 0$ on $D(T) = Z$. This corresponds to an operator in $\mathcal{M}$ that we shall denote A_M (M here indicates that it is maximal in a certain sense). Since $Z' \ominus W = \{0\}$, we see from Theorem 13.6 that $D(A_M) = D(A_0) \dot{+} Z$. The adjoint A_M^* corresponds to T^* equal to the zero operator from Z' to Z, so it is completely analogous to A_M, with $D(A_M^*) = D(A_0') \dot{+} Z'$.

In the symmetric situation where (13.5) holds, we see that A_M is selfadjoint. Moreover, it is the only selfadjoint extension of A_0 containing all of Z in its domain. For, any extension of A_0 having Z in its domain must extend A_M, hence cannot be selfadjoint unless it equals A_M. In the case where $m(A_0) > 0$, A_M is the extension of J. von Neumann, see Exercise 12.23.

Example 13.10a. As an elementary illustration of Theorem 13.7, consider a Sturm-Liouville operator (also studied in Exercises 12.25 and 4.11–14):

Let A be the differential operator $Au = -u'' + qu$ on $I =]\alpha, \beta[$, where $q(t) \geq 0$, continuous on $[\alpha, \beta]$. Let A_0 be the closure of $A|_{C_0^\infty}$ (as an operator in $H = L_2(I)$); it is symmetric. Let $A_1 = A_0^*$. In other words, A_0 is the minimal operator $A_{\min}$, and A_1 is the maximal operator $A_{\max}$ (the L_2 weak definition of A). It is known from Chapter 4 that $D(A_1) = H^2(I)$ and

$$D(A_0) = H_0^2(I) = \{u \in H^2(I) \mid u(\alpha) = u'(\alpha) = u(\beta) = u'(\beta) = 0\}.$$

Together with A we can consider the sesquilinear form

$$a(u, v) = (u', v') + (qu, v), \quad u, v \in H^1(I)$$

(scalar products in $L_2(I)$). Let $V_1 = H^1(I)$, $V_0 = H_0^1(I)$. Clearly, a is V_1-coercive and ≥ 0; moreover, its restriction a_0 to $H_0^1(I)$ is V_0-elliptic in view of the Poincaré inequality (Theorem 4.29). The variational operator A_γ defined from the triple (H, V_0, a_0) acts like A_1 with domain $D(A_\gamma) = H^2(I) \cap H_0^1(I)$; it is invertible with $A_\gamma^{-1} \in \mathbf{B}(H)$, and is the Friedrichs extension of A_0. We take $A_\beta = A_\gamma$.

The nullspace Z of A_1 is spanned by two real null solutions $z_1(t)$ and $z_2(t)$ with $z_1(\alpha) = 0$, $z_1'(\alpha) \neq 0$, resp. $z_2(\beta) = 0$, $z_2'(\beta) \neq 0$ (they are linearly independent since $D(A_\gamma) \cap Z = \{0\}$). Defining

$$\gamma_0 u = \begin{pmatrix} u(\alpha) \\ u(\beta) \end{pmatrix}, \quad \gamma_1 u = \begin{pmatrix} u'(\alpha) \\ -u'(\beta) \end{pmatrix},$$

we have the Lagrange (or Green's) formula

$$(Au,v)-(u,Av)=\gamma_1 u\cdot\overline{\gamma_0 v}-\gamma_0 u\cdot\overline{\gamma_1 v},\ \text{ for } u,v\in H^2(I). \tag{13.31a}$$

A simple calculation shows that the mapping $\gamma_0 : Z \to \mathbb{C}^2$ has the inverse $K : \mathbb{C}^2 \to Z$ defined by

$$K : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1(t) & z_2(t) \end{pmatrix} \begin{pmatrix} 0 & 1/z_1(\beta) \\ 1/z_2(\alpha) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(called a Poisson operator in higher dimensional theories). Let us also define

$$P=\gamma_1 K=\begin{pmatrix} z_2'(\alpha)/z_2(\alpha) & z_1'(\alpha)/z_1(\beta) \\ -z_2'(\beta)/z_2(\alpha) & -z_1'(\beta)/z_1(\beta) \end{pmatrix};$$

note that the off-diagonal entries are nonzero. An application of (13.31a) to $z, w \in Z$,

$$0=(Az,w)-(z,Aw)=\gamma_1 z\cdot\overline{\gamma_0 w}-\gamma_0 z\cdot\overline{\gamma_1 w}=P\gamma_0 z\cdot\overline{\gamma_0 w}-\gamma_0 z\cdot\overline{P\gamma_0 w},$$

shows that $P^*=P$. (P is often called the Dirichlet-to-Neumann operator.)

Consider an operator $\widetilde{A}$ for which $V=W=Z$; we shall investigate the meaning of T. In the defining equation

$$(Au,z)=(Tu_\zeta,z),\ \text{ for } u\in D(\widetilde{A}), z\in Z, \tag{13.31b}$$

we rewrite the left-hand side as

$$(Au,z)=(Au,z)-(u,Az)=\gamma_1 u\cdot\overline{\gamma_0 z}-\gamma_0 u\cdot\overline{\gamma_1 z}=(\gamma_1 u-P\gamma_0 u)\cdot\overline{\gamma_0 z},$$

and the right-hand side (using that $\gamma_0 u_\zeta=\gamma_0 u$) as

$$(Tu_\zeta,z)=(TK\gamma_0 u_\zeta,K\gamma_0 z)=(TK\gamma_0 u,K\gamma_0 z)=K^*TK\gamma_0 u\cdot\overline{\gamma_0 z}.$$

Set $K^*TK=L$ (this carries the operator $T : Z\to Z$ over into a 2×2-matrix L), then (13.31b) is turned into

$$(\gamma_1 u-P\gamma_0 u)\cdot\overline{\gamma_0 z}=L\gamma_0 u\cdot\overline{\gamma_0 z},\ \text{ for } u\in D(\widetilde{A}), z\in Z.$$

Since $\gamma_0 z$ runs through $\mathbb{C}^2$, we conclude that for $u\in D(\widetilde{A})$, $\gamma_1 u-P\gamma_0 u=L\gamma_0 u$, or,

$$\gamma_1 u=(L+P)\gamma_0 u. \tag{13.31c}$$

Conversely, when $u \in H^2(I)$ satisfies this, we see that (13.31b) holds, so $u \in D(\widetilde{A})$. Note that (13.31c) is a general type of Neumann condition.

Some examples: If $T = 0$, corresponding to $L = 0$, (13.31c) is a nonlocal boundary condition (nonlocal in the sense that it mixes information from the two endpoints α and β). The local Neumann-type boundary conditions $u'(\alpha) = b_1 u(\alpha)$, $-u'(\beta) = b_2 u(\beta)$, are obtained by taking $L = \left(\begin{smallmatrix} b_1 & 0 \\ 0 & b_2 \end{smallmatrix}\right) - P$. Properties of $\widetilde{A}$ are reflected in properties of L, as shown in the analysis of consequences of Theorem 13.7 (also in the next sections).

As a case where V and W are nontrivial subspaces of Z, take for example $V = W = \operatorname{span} z_1$. Then T is a multiplication operator, and $\widetilde{A}$ represents a boundary condition of Neumann type at α, Dirichlet type at β ($u(\beta) = 0$).

For PDE in dimension $n > 1$, the nullspaces Z and Z' will in general have infinite dimension, and there are many more choices of $T : V \to W$. Elliptic examples have some of the same flavor as above, but need extensive knowledge of boundary maps and Sobolev spaces, cf. [G68]–[G74], [BGW08].

13.2 The symmetric case.

In this section, we restrict the attention to symmetric A_0. It can be shown in general that when A_0 is a symmetric, closed, densely defined operator in H with a bounded inverse, then there exists a selfadjoint extension A_β (J. W. Calkin [C40], see also the book of F. Riesz and B. Sz.-Nagy [RN53], p. 336). Then we can obtain the situation where (13.5) holds, *the symmetric case*. If A_0 is lower bounded, we can take for A_β the Friedrichs extension of A_0 (cf. Theorem 12.23); for this particular choice, A_β will be called A_γ.

When (13.5) holds, there is just one nullspace Z and one set of decomposition projections pr_β, pr_ζ to deal with. Then we get as an immediate corollary of Theorems 13.5 and 13.7:

Corollary 13.11. *Assume that* (13.5) *holds.*

Let $\widetilde{A}$ correspond to $T\colon V \to W$ as in Theorem 13.7*. Then $\widetilde{A}$ is selfadjoint if and only if: $V = W$ and T is selfadjoint.*

In more detail: If V is a closed subspace of Z and $T\colon V \to V$ is selfadjoint, then the operator $\widetilde{A} \subset A_1$ with domain

$$D(\widetilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T),\, (Au)_V = Tu_\zeta\} \tag{13.32}$$

is a selfadjoint extension of A_0. Conversely, if $\widetilde{A} \in \mathcal{M}$ is selfadjoint, then the operator $T\colon V \to V$, where

$$D(T) = \mathrm{pr}_\zeta D(\widetilde{A}), \quad V = \overline{D(T)}, \quad Tu_\zeta = (Au)_V \text{ for } u \in D(\widetilde{A}), \tag{13.33}$$

is selfadjoint.

It is also easy to see that $\widetilde{A}$ is symmetric if and only if $V \subset W$ and $T: V \to W$ is symmetric as an operator in W, and that $\widetilde{A}$ is maximal symmetric if and only if T is so.

In the following, we consider throughout the situation where $\widetilde{A}$ corresponds to $T: V \to W$ as in Theorem 13.7, and shall not repeat this every time. The following observation will be important:

Lemma 13.12. *When* (13.5) *holds and* $V \subset W$*, then*

$$(Au, v) = (Au_\beta, v_\beta) + (Tu_\zeta, v_\zeta), \tag{13.34}$$

for all $u, v \in D(\widetilde{A})$*.*

Proof. This follows from the calulation

$$\begin{aligned}(Au, v) &= (Au, v_\beta) + (Au, v_\zeta) = (Au_\beta, v_\beta) + (Au, v_\zeta)\\ &= (Au_\beta, v_\beta) + ((Au)_W, v_\zeta) = (Au_\beta, v_\beta) + (Tu_\zeta, v_\zeta),\end{aligned}$$

where we used that $v_\zeta \in V \subset W$ for the third equality, and that $(Au)_W = Tu_\zeta$ for the last equality. $\square$

Now we shall consider lower bounded extensions, in the case where A_β is lower bounded. We can assume that its lower bound is positive. Part of the results can be obtained for general A_β (cf. [G68], [G70]), but the most complete results are obtained when A_β equals the Friedrichs extension A_γ. Recall from Section 12.5 that A_γ is the variational operator defined from (H, H_0, a_γ), where H_0 (called V in Section 12.5) is the completion of $D(A_0)$ in the norm $\|u\|_{H_0} = (A_0u, u)^{\frac{1}{2}}$, and $a_\gamma(u, v)$ is the extension of (A_0u, v) to H_0. In particular, $D(A_0)$ is dense in H_0 (and A_γ is the only selfadjoint positive extension with this denseness property). To simplify the presentation, we go directly to this case, assuming

$$m(A_0) > 0, \quad A_\beta = A_\gamma, \tag{13.35}$$

and recalling from Theorem 12.23 that $m(A_\gamma) = m(A_0)$. The results in the following are from [G70].

Lemma 13.13. *Assume that* (13.5) *and* (13.35) *hold.*

If for some $\lambda \in \mathbb{C}$*,* $c > 0$*,*

$$|(Au, u) - \lambda\|u\|^2| \geq c\|u\|^2 \text{ for all } u \in D(\widetilde{A}), \tag{13.36}$$

then

$$V \subset W, \tag{13.37}$$

$$|(Tz, z) - \lambda\|z\|^2| \geq c\|z\|^2 \text{ for all } z \in D(T). \tag{13.38}$$

Proof. By Theorem 13.6, the general element of $D(\widetilde{A})$ is $u = z + A_\gamma^{-1}(Tz + f) + v$, with $\{z, f, v\} \in D(T) \times (Z \ominus W) \times D(A_0)$; here $u_\zeta = z$, $u_\gamma = A_\gamma^{-1}(Tz + f) + v$. For such u,

$$\begin{aligned}(Au, u) &= (Au, u_\gamma + u_\zeta) = (Au_\gamma, u_\gamma) + (Tz + f + Av, u_\zeta) \\ &= (Au_\gamma, u_\gamma) + (Tz, z) + (f, z),\end{aligned} \tag{13.39}$$

where we used in the last step that $Av \in R(A_0) \perp Z$. Now assume that (13.36) holds, i.e.,

$$\Big|\frac{(Au, u)}{\|u\|^2} - \lambda\Big| \geq c, \text{ all } u \in D(\widetilde{A}) \setminus \{0\},$$

then in view of (13.39),

$$\Big|\frac{(Au_\gamma, u_\gamma) + (Tz, z) + (f, z)}{\|u_\gamma + z\|^2} - \lambda\Big| \geq c, \text{ all } \{z, f, v\} \neq 0. \tag{13.40}$$

We shall now use that $D(A_0)$ is dense in $H_0 = D(a_\gamma)$, where $H_0 \subset H$ algebraically, topologically and densely (cf. (12.35)ff). For any given $z \in D(T)$ and $f \in Z \ominus W$, let v_k be a sequence in $D(A_0)$ converging to $A_\gamma^{-1}(Tz + f)$ in H_0-norm. Then $u_k = A_\gamma^{-1}(Tz + f) - v_k \in D(A_\gamma)$ and satisfies

$$\|u_k\|_{H_0}^2 = (Au_k, u_k) \to 0, \text{ and a fortiori } \|u_k\|_H \to 0, \text{ for } k \to \infty.$$

Inserting the terms defined from the triple $\{z, f, -v_k\}$ in (13.40) and letting $k \to \infty$, we conclude that

$$\Big|\frac{(Tz, z) + (f, z)}{\|z\|^2} - \lambda\Big| \geq c, \text{ when } z \neq 0. \tag{13.41}$$

This can be further improved: Note that if (f, z) is nonzero for some choice of $z \in D(T)$, $f \in Z \ominus W$, a replacement of f by $\frac{a}{(f,z)} f$ in the inequality (13.41) gives

$$\Big|\frac{(Tz, z) + a}{\|z\|^2} - \lambda\Big| \geq c, \text{ any } a \in \mathbb{C},$$

which cannot hold since some choice of a will give 0 in the left hand side. Thus $(f, z) = 0$ for all $z \in D(T)$, $f \in Z \ominus W$, so the closure V of $D(T)$ in Z must satisfy $V \subset W$. This shows (13.37), and then (13.41) simplifies to give (13.38). □

This has an immediate consequence for the numerical ranges:

Theorem 13.14. *Assume that* (13.5) *and* (13.35) *hold, and let* $\widetilde{A}$ *correspond to* $T\colon V \to W$ *as in Theorem* 13.7.

If $\nu(\widetilde{A})$ *is not all of* $\mathbb{C}$*, then* $V \subset W$*, and*

$$\overline{\nu(T)} \subset \overline{\nu(\widetilde{A})}. \tag{13.42}$$

Proof. Let $\nu(\widetilde{A}) \neq \mathbb{C}$; then since $\nu(\widetilde{A})$ is convex (Exercise 12.34 (e)), it is contained in a halfplane, and so is $\overline{\nu(\widetilde{A})}$. By definition, a number $\lambda \in \mathbb{C}$ has a positive distance to $\nu(\widetilde{A})$ if and only if (13.36) holds for some $c > 0$, i.e., $\lambda \in \mathbb{C} \setminus \overline{\nu(\widetilde{A})}$. Such λ's exist, so $V \subset W$, by Lemma 13.13. The lemma shows moreover that $\lambda \in \mathbb{C} \setminus \overline{\nu(\widetilde{A})}$ implies $\lambda \in \mathbb{C} \setminus \overline{\nu(T)}$, so (13.42) holds. □

One can also get information on spectra:

Corollary 13.15. *Assumptions as in Theorem* 13.14.

1° *If* $\widetilde{A}$ *and* $\widetilde{A}^*$ *both have the lower bound* $a \in \mathbb{R}$*, then* $V = W$*, and* T *and* T^* *have the lower bound* a*, their spectra being contained in* $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq a\}$.

2° *Assume moreover that* $\widetilde{A}$ *is variational, with numerical range (and hence spectrum) contained in an angular set*

$$M = \{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| \leq c_1(\operatorname{Re}\lambda + k),\ \operatorname{Re}\lambda \geq a\}; \tag{13.43}$$

(recall that then $\widetilde{A}^*$ *has the same properties). Then* $V = W$*, and* T *and* T^* *are variational, with numerical ranges and spectra contained in* M.

Proof. 1°. The preceding theorem applied to $\widetilde{A}$ gives that $V \subset W$ and $m(T) \geq a$, and when it is applied to $\widetilde{A}^*$ it gives that $W \subset V$ and $m(T^*) \geq a$. The statement on the spectra follows from Theorem 12.9.

2°. Since $\widetilde{A}$ is variational, so is $\widetilde{A}^*$. When $\nu(\widetilde{A}) \subset M$, the associated sesquilinear form $\tilde{a}$ has $\nu(\tilde{a}) \subset M$ (cf. (12.58)ff.), so $\widetilde{A}^*$ likewise has its numerical range (and both $\widetilde{A}$ and $\widetilde{A}^*$ have their spectra) in M; cf. Theorem 12.18. Theorem 13.14 implies immediately that T and T^* have their numerical ranges in M. Now Theorem 12.25 implies that T and T^* are variational, with spectra in M. □

In the converse direction we find:

Theorem 13.16. *Assume that* (13.5) *and* (13.35) *hold, and let* $\widetilde{A}$ *correspond to* $T\colon V \to W$ *as in Theorem* 13.7.

Assume that $V \subset W$.

1° *If* $m(T) > -m(A_\gamma)$, *then*

$$m(\widetilde{A}) \geq \frac{m(A_\gamma)m(T)}{m(A_\gamma)+m(T)}. \tag{13.44}$$

In particular, $m(T) \geq 0$ *(*> 0*) implies* $m(\widetilde{A}) \geq 0$ *(resp.* > 0*).*

2° *If, for some* $\theta \in\,]-\pi/2, \pi/2[\,$, $m(e^{i\theta}T) > -\cos\theta\, m(A_\gamma)$, *then*

$$m(e^{i\theta}\widetilde{A}) \geq \frac{\cos\theta\, m(A_\gamma)m(e^{i\theta}T)}{\cos\theta\, m(A_\gamma)+m(e^{i\theta}T)}. \tag{13.45}$$

Proof. 1°. For $u \in D(\widetilde{A}) \setminus \{0\}$, we have since $V \subset W$ (cf. Lemma 13.12),

$$\frac{\operatorname{Re}(Au,u)}{\|u\|^2} = \frac{\operatorname{Re}((Au_\gamma,u_\gamma)+(Tu_\zeta,u_\zeta))}{\|u_\gamma+u_\zeta\|^2} \geq \frac{m(A_\gamma)\|u_\gamma\|^2+m(T)\|u_\zeta\|^2}{\|u_\gamma+u_\zeta\|^2}. \tag{13.46}$$

When $m(T) \geq 0$, the numerator is ≥ 0 for all u, so we can continue the estimate by

$$\frac{m(A_\gamma)\|u_\gamma\|^2+m(T)\|u_\zeta\|^2}{\|u_\gamma+u_\zeta\|^2} \geq \frac{m(A_\gamma)\|u_\gamma\|^2+m(T)\|u_\zeta\|^2}{(\|u_\gamma\|+\|u_\zeta\|)^2}. \tag{13.47}$$

For the function $f(x,y) = (\alpha x^2+\beta y^2)(x+y)^{-2}$ with $\alpha > 0$, $\beta \geq 0$, considered for $x \geq 0$, $y \geq 0$, $(x,y) \neq (0,0)$, one easily shows the inequality

$$f(x,y) \geq \alpha\beta(\alpha+\beta)^{-1},$$

by finding the minimum of $f(t,1)$ for $t \geq 0$. This implies (13.44) when $m(T) \geq 0$.

When $m(T) < 0$, we cannot get (13.47) in general, since the numerator can be negative. But when $0 > m(T) > -m(A_\gamma)$, we can instead proceed as follows:

If $m(A_\gamma)\|u_\gamma\|^2 + m(T)\|u_\zeta\|^2 \geq 0$, we have that $\operatorname{Re}(Au,u) \geq 0$, confirming (13.44) for such u. It remains to consider the u for which $m(A_\gamma)\|u_\gamma\|^2 + m(T)\|u_\zeta\|^2 < 0$. This inequality implies that $u_\zeta \neq 0$, and that $t = \|u_\gamma\|/\|u_\zeta\|$ satisfies

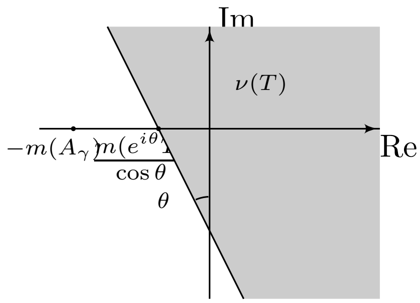
$$t = \|u_\gamma\|/\|u_\zeta\| < (-m(T))^{\frac12} m(A_\gamma)^{-\frac12} < 1.$$

Then since $\|u_\gamma+u_\zeta\|^2 \geq (\|u_\gamma\|-\|u_\zeta\|)^2 > 0$, we find that

$$\frac{m(A_\gamma)\|u_\gamma\|^2+m(T)\|u_\zeta\|^2}{\|u_\gamma+u_\zeta\|^2} \geq \frac{m(A_\gamma)\|u_\gamma\|^2+m(T)\|u_\zeta\|^2}{(\|u_\gamma\|-\|u_\zeta\|)^2} = \frac{m(A_\gamma)t^2+m(T)}{(t-1)^2},$$

since the numerator is negative. The function $g(t) = (\alpha t^2 + \beta)(t-1)^2$, $\alpha > 0$, $0 > \beta > -\alpha$, defined for $0 \le t < (-\beta)^{\frac{1}{2}}\alpha^{-\frac{1}{2}}$, obtains its minimum at $t = -\beta/\alpha$ with value $\alpha\beta(\alpha+\beta)^{-1}$. This confirms (13.44) for such u. The last statement in 1° is an immediate consequence.

2°. The same proof is here applied to the operators multiplied by $e^{i\theta}$, and leads to the result since $m(e^{i\theta}A_\gamma) = \cos\theta\, m(A_\gamma) > 0$. □



The condition $m(e^{i\theta}T) > -\cos\theta m(A_\gamma)$ means geometrically that $\nu(T)$ is contained in a halfplane in $\mathbb{R}^2$ lying to the right of the point $(-m(A_\gamma), 0)$ on the real axis; more precisely it is the halfplane lying to the right of the line through $(m(e^{i\theta}T)/\cos\theta, 0)$ with angle $\theta + \pi/2$ to the real axis. Two inequalities $m(e^{\pm i\theta}T) \ge b$ valid together, with $b > -\cos\theta\, m(A_\gamma)$, $\theta \in]0, \pi/2[$, mean that $\nu(T)$ is contained in

$$M' = \{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| \le \cot\theta\,(\operatorname{Re}\lambda - b/\cos\theta)\} \tag{13.48}$$

with $b/\cos\theta > -m(A_\gamma)$ (a sector to the right of $(-m(A_\gamma), 0)$).

Remark 13.17. This theorem only applies when the lower bound of T resp. $e^{i\theta}T$ is "not too negative". It can be shown by a greater effort that if A_γ^{-1} is a compact operator in H (as it often is in the applications), then one gets lower boundedness of $e^{i\theta}\widetilde{A}$ for any lower bounded $e^{i\theta}T$, see [G74].

Theorem 13.16 also holds when A_γ is replaced by a general A_β with positive lower bound, cf. [G68], [G70]. However, the extended version in [G74], where the condition $m(T) > -m(A_\gamma)$ is removed when A_γ^{-1} is compact, uses the particular properties of A_γ.

We now turn to variational operators.

Let $\widetilde{A}$ be variational, associated with the triple $(H, D(\tilde{a}), \tilde{a})$. (We here denote by $D(\tilde{a})$ the Hilbert space where $\tilde{a}$ is defined and bounded, usually called V in Sections 12.4, 12.6.) As observed in Corollary 13.15, $V = W$, and $\widetilde{A}^*$, T and T^* are likewise variational, with numerical ranges and spectra in the set M (13.43) defined for $\widetilde{A}$.

Theorem 13.18. *Assume that* (13.5) *and* (13.35) *hold, let* $\widetilde{A}$ *be a variational operator associated with a triple* $(H, D(\tilde{a}), \tilde{a})$, *and let* $T: V \to V$ *be the corresponding variational operator as in Corollary* 13.15, *associated with the triple* $(V, D(t), t)$.

Then

$$D(\tilde{a}) = D(a_\gamma)\dot{+}D(t), \text{ with } D(t) = D(\tilde{a}) \cap Z, \tag{13.49}$$

the projections pr_γ *and* pr_ζ *extending continuously to this direct sum, and*

$$\tilde{a}(u, v) = a_\gamma(u_\gamma, v_\gamma) + t(u_\zeta, v_\zeta), \text{ for } u, v \in D(\tilde{a}). \tag{13.50}$$

Proof. We have from Lemma 13.12 that (13.50) holds for $u, v \in D(\widetilde{A})$, and we want to extend this to $D(\tilde{a})$. To see that $D(t) \subset D(\tilde{a})$, let $z \in D(t)$ and let $z_k \in D(T)$, converging to z in $D(t)$ for $k \to \infty$; then it also converges in H. Since $A_\gamma^{-1}Tz_k \in D(A_\gamma)$, and $D(A_0)$ is dense in $D(a_\gamma)$, we can choose $v_k \in D(A_0)$ such that $w_k = v_k + A_\gamma^{-1}Tz_k \to 0$ in $D(a_\gamma)$, hence in H. Now $u_k = w_k + z_k \in D(\widetilde{A})$ (cf. Theorem 13.6), and $u_k \to z$ in H. Moreover, since $\mathrm{pr}_\gamma u_k = w_k$, $\mathrm{pr}_\zeta u_k = z_k$,

$$\tilde{a}(u_k - u_l, u_k - u_l) = a_\gamma(w_k - w_l, w_k - w_l) + t(z_k - z_l, z_k - z_l) \to 0, \text{ for } k, l \to \infty,$$

which implies that u_k has a limit in $D(\tilde{a})$; this must equal z. Hence $z \in D(\tilde{a})$. Since $A_0 \subset \widetilde{A}$, $D(a_\gamma) \subset D(\tilde{a})$, so we conclude that

$$D(a_\gamma) + D(t) \subset D(\tilde{a}). \tag{13.51}$$

We next show that

$$D(a_\gamma) \cap D(t) = \{0\}. \tag{13.52}$$

Here $D(t) \subset V \subset Z$, so it suffices to show that

$$D(a_\gamma) \cap Z = \{0\}. \tag{13.53}$$

But if $z \in D(a_\gamma) \cap Z$, let $u_k \in D(A_0)$ converge to z in $D(a_\gamma)$; then

$$a_\gamma(z, z) = \lim_{k\to\infty} a_\gamma(u_k, z) = \lim_{k\to\infty} (A_0 u_k, z)_H = 0,$$

since $R(A_0) \perp Z$. Then $z = 0$ since $m(a_\gamma) > 0$.

If $m(\widetilde{A}) > 0$, we are almost through, for the real part of the identity (13.50) that we know is valid for $u = v \in D(\widetilde{A})$,

$$\mathrm{Re}\, \tilde{a}(u, u) = a_\gamma(u_\gamma, u_\gamma) + \mathrm{Re}\, t(u_\zeta, u_\zeta) \tag{13.54}$$

shows, since all three terms are squares of norms (on $D(\tilde a)$, $D(a_\gamma)$ resp. $D(t)$), that the maps $\mathrm{pr}_\gamma\colon D(\tilde a)\to D(a_\gamma)$ and $\mathrm{pr}_\zeta\colon D(\tilde a)\to D(t)$ extend from the dense subset $D(\widetilde A)$ to continuous maps from $D(\tilde a)$ to $D(a_\gamma)$ resp. $D(t)$; they remain projections, and must have linearly independent ranges in view of (13.51) and (13.52). Applying them to an arbitrary element of $D(\tilde a)$ we see that $D(\tilde a)\subset D(a_\gamma)+D(t)$. The second statement in (13.49) is obvious from (13.52)ff. The identity (13.50) follows by extension by continuity.

If $m(\widetilde A)\le 0$, we have to do a little more work: Choosing a large constant μ such that $m(\widetilde A+\mu)>0$, we can apply the preceding argument to the set-up with $\widetilde A$, A_γ resp. $T:V\to V$ replaced by $\widetilde A+\mu$, $A_\gamma+\mu$ resp. T_μ : $V_\mu\to V_\mu$ ($V_\mu\subset Z(A_1+\mu)$); here $D(\widetilde A+\mu)=D(\widetilde A)$, $D(A_\gamma+\mu)=D(A_\gamma)$, $D(\tilde a+\mu)=D(\tilde a)$, $D(a_\gamma+\mu)=D(a_\gamma)$. Let t_μ be the sequilinear form with domain $D(t_\mu)\subset V_\mu$ associated with T_μ.

Let $u\in D(\tilde a)$ and decompose it first using $D(\tilde a+\mu)=D(a_\gamma+\mu)\dot+D(t_\mu)$ (by the first part of the proof), and next using $D(A_1)=D(A_\gamma)\dot+Z$ for z_μ:

$$\begin{aligned} u&=v+z_\mu,\quad v\in D(a_\gamma),\ z_\mu\in D(t_\mu)\subset Z(A_1+\mu)\subset D(A_1),\\ &=v+z_1+z_2,\quad z_1\in D(A_\gamma),\ z_2\in Z.\end{aligned}\tag{13.55}$$

Then we have inequalities (with positive constants)

$$\begin{aligned}\|u\|_{D(\tilde a)}&\ge c(\|v\|_{D(a_\gamma)}+\|z_\mu\|_{D(t_\mu)})\\ &\ge c\|v\|_{D(a_\gamma)}+c'(\|z_1\|_{D(A_\gamma)}+\|z_2\|_H)\\ &\ge c''\|v+z_1\|_{D(a_\gamma)}+c'\|z_2\|_H,\end{aligned}$$

since the norm in $D(\widetilde A)$ is stronger than that in $D(a_\gamma)$, and the norm in $D(t_\mu)$ is stronger than the H-norm. Let $u_k\in D(\widetilde A)$, $u_k\to u$ in $D(\tilde a)$; then in the decomposition (13.55), $u_{k,\gamma}=v_k+z_{k,1}\to u_\gamma$ in $D(a_\gamma)$ and $u_{k,\zeta}=z_{k,2}\to u_\zeta$ in H. Now we combine this with (13.54) applied to u_k-u_l; it shows that

$$\operatorname{Re} t(u_{k,\zeta}-u_{l,\zeta},u_{k,\zeta}-u_{l,\zeta})=\operatorname{Re}\tilde a(u_k-u_l,u_k-u_l)-a_\gamma(u_{k,\gamma}-u_{l,\gamma},u_{k,\gamma}-u_{l,\gamma})$$

goes to 0 for $k,l\to\infty$, so $z_{k,2}$ is a Cauchy sequence in $D(t)$.

This shows that $D(\tilde a)\subset D(a_\gamma)\dot+D(t)$ by the decomposition $u=(v+z_1)+z_2$ in (13.55), with continuous maps. So we can conclude that (13.49) holds, and the proof is completed as above. □

The theorem gives a description of all variational operators $\widetilde A\in\mathcal M$. In view of Corollary 13.15, the variational operators $T\colon V\to V$ include all choices $(V,D(t),t)$, where V is closed $\subset Z$, $D(t)\subset V$ algebraically, topologically and densely, and t is $D(t)$-coercive with lower bound $m(t)>-m(a_\gamma)$.

For, such t have

$$\begin{aligned}\nu(t) &\subset \{\lambda\in\mathbb{C} \mid |\operatorname{Im}\lambda| \le c_1(\operatorname{Re}\lambda + k),\ \operatorname{Re}\lambda \ge m(t)\} \\ &\subset \{\lambda\in\mathbb{C} \mid |\operatorname{Im}\lambda| \le c_1'(\operatorname{Re}\lambda - m(t) + \varepsilon)\}\end{aligned} \tag{13.56}$$

for any ε, with a possibly larger c_1' (the angular set M is contained in a right sector with rays emanating from $m(t)-\varepsilon$). Then Corollary 13.15 2° applies to T.

As mentioned in Remark 13.17, the restriction on how negative $m(t)$ is allowed to be, is removed in [G74] when A_γ^{-1} is compact.

Note that the theorem shows a very simple connection between the forms $\tilde a$ and t: The domain of $\tilde a$ is simply the direct sum of $D(a_\gamma)$ and $D(t)$, and the value of $\tilde a$ on $D(a_\gamma)$ is fixed. One can choose any V, closed subspace of Z, and take as $D(t)$ any Hilbert space V_0 that is algebraically, topologically and densely injected in V, and among the possible choices of t there are at least the bounded and V_0-elliptic sesquilinear forms on V_0. So we have:

Corollary 13.19. *Assume* (13.5) *and* (13.35). *The Hilbert spaces U such that there exists a variational $\widetilde A \in \mathcal{M}$ with $D(\tilde a) = U$ are the spaces satisfying*

$$D(a_\gamma) \subset U \subset D(a_\gamma)\dot{+}Z \tag{13.57}$$

with continuous injections.

The largest possible space U here is $D(a_\gamma)\dot{+}Z$. Let t be the zero sesquilinear form on Z; then T is the zero operator on Z, so $\widetilde A$ is a selfadjoint extension of A_0 that has all of Z included in its domain and nullspace, since $Z(\widetilde A) = Z(T)$, cf. Theorem 13.8.1°. This $\widetilde A$ equals the extension A_M considered in Example 13.10. The associated sesquilinear form a_M satisfies:

$$D(a_M) = D(a_\gamma)\dot{+}Z, \quad a_M(u,v) = a_\gamma(u_\gamma, v_\gamma), \tag{13.58}$$

in view of (13.50). We can extend Corollary 13.19 as follows:

Corollary 13.20. *Assume* (13.5) *and* (13.35). *A variational operator $\widetilde A$ belongs to $\mathcal{M}$ if and only if the associated sesquilinear form $\tilde a$ satisfies* (i)–(iii)*:*

(i) $D(a_\gamma) \subset D(\tilde a) \subset D(a_M)$, *with continuous injections;*
(ii) $\tilde a \supset a_\gamma$*;*
(iii) $\tilde a(w,z) = \tilde a(z,w) = 0$, *for* $w \in D(a_\gamma)$, $z \in D(\tilde a) \cap Z$.

Proof. The necessity of (i) and (ii) is immediate from (13.49) and (13.50), and we get (iii) by applying (13.50) with $u=w$, $v=z$, or $u=z$, $v=w$.

For the converse direction, let $\widetilde{A}$ be the variational operator associated with a form $\tilde{a}$ satisfying (i)–(iii). Then $\widetilde{A}$ extends A_0, since for all $w \in D(A_0)$, $v \in D(\tilde{a})$,

$$(A_0 w, v) = (A_0 w, v_\gamma + v_\zeta) = (A_0 w, v_\gamma) = a_\gamma(w, v_\gamma) = \tilde{a}(w, v),$$

where we used $R(A_0) \perp Z$ in the second step and (iii) in the last step. Moreover, the adjoint $\widetilde{A}^*$ has the analogous properties with $\tilde{a}$ replaced by $\tilde{a}^*$, so it also extends A_0. Then $\widetilde{A} \in \mathcal{M}$. □

Restricting the attention to selfadjoint nonnegative operators, we get in particular:

Corollary 13.21. *Assume* (13.5) *and* (13.35). *A selfadjoint nonnegative operator* $\widetilde{A}$ *belongs to* $\mathcal{M}$ *if and only if the associated sesquilinear form* $\tilde{a}$ *satisfies* (i)–(iii)*:*

(i) $D(a_\gamma) \subset D(\tilde{a}) \subset D(a_M)$, *with continuous injections;*
(ii) $\tilde{a} \supset a_\gamma$*;*
(iii) $\tilde{a}(u, u) \geq a_M(u, u)$, *for* $u \in D(\tilde{a})$.

Proof. Here we note that if $\widetilde{A} \in \mathcal{M}$, then the corresponding form t is ≥ 0, so (iii) follows from (13.50).

In the converse direction, we reduce to an application of Corollary 13.20 by observing that (ii)–(iii) imply, for $u \in D(\tilde{a})$, $u = w + z$ according to the decomposition (13.58),

$$\begin{aligned} a_\gamma(w, w) = a_M(u, u) &\leq \tilde{a}(w + z, w + z) \\ &= a_\gamma(w, w) + \tilde{a}(w, z) + \tilde{a}(z, w) + \tilde{a}(z, z), \end{aligned}$$

hence

$$\tilde{a}(w, z) + \tilde{a}(z, w) + \tilde{a}(z, z) \geq 0.$$

Here $u = w + z$ can be replaced by $u = \alpha w + z$, any $\alpha \in \mathbb{C}$, in view of (i). Taking $\alpha = s\,\overline{\tilde{a}(w, z)}$, $s \in \mathbb{R}$, gives

$$2s\,|\tilde{a}(w, z)| + \tilde{a}(z, z) \geq 0 \text{ for all } s \in \mathbb{R},$$

which clearly cannot hold if $\tilde{a}(w, z) \neq 0$. Similarly, $\tilde{a}(z, w)$ must equal 0. Then (iii) of Corollary 13.20 holds, and we can apply that corollary to conclude that $\widetilde{A} \in \mathcal{M}$. □

Some historical remarks: Corollary 13.21 was shown by M. G. Krein in [K47]. Krein used a different method, deducing the result from another equivalent result: The selfadjoint nonnegative extensions of A_0 are precisely the selfadjoint nonnegative operators $\widetilde{A}$ satisfying

$$(A_\gamma + 1)^{-1} \le (\widetilde{A} + 1)^{-1} \le (A_M + 1)^{-1}; \tag{13.59}$$

i.e., $((A_\gamma + 1)^{-1} f, f) \le ((\widetilde{A} + 1)^{-1} f, f) \le ((A_M + 1)^{-1} f, f)$ for all $f \in H$. The two operators A_γ and A_M are extreme in this scale of operators; Krein called A_γ the "hard extension" and A_M the "soft extension". In the sense of (13.59), or in the sense expressed in Corollary 13.21, the operators $\widetilde{A}$ are the selfadjoint operators "lying between" the hard extension A_γ and the soft extension A_M.

Birman [B56] showed a version of the inequality in Theorem 13.16 for selfadjoint operators. The parametrization we have presented in Section 13.1 is related to that of Vishik [V52], except that he sets the $\widetilde{A}$ in relation to operators on the nullspaces going the opposite way of our T's, and in this context mainly considers cases where $\widetilde{A}$ has closed range.

One of the primary interests of the extension theory is its application to the study of boundary value problems for elliptic differential operators; this has been developed in a rather complete way in papers by Grubb [G68]–[G74]. The basic idea relies on the fact that the nullspace Z is in such cases isomorphic to a suitable Sobolev space over the boundary of the set where the differential operator acts; then the operator $T : V \to W$ can be carried over to an operator $L : X \to Y^*$ where X and Y are spaces defined over the boundary, and $\widetilde{A}$ is seen to represent a boundary condition.

Such studies use, among other things, the distribution definition of boundary values established in Lions and Magenes [LM68], and pseudodifferential operators. To illustrate the main idea, we show in Chapter 9 how it is applied in a relatively simple, constant-coefficient situation, that can be understood on the basis of Chapters 1–6. More general variable-coefficient applications are treated in the exercises to Chapter 11, which builds on Chapters 7, 8 and 10.

Extension theories have been developed much further in recent years, often with a focus on applications to ordinary differential equations and other cases with finite dimensional nullspaces. A central work in this development is the book of Gorbachuk and Gorbachuk, [GG91], with references to many other works. Extensions are here characterized in terms of so-called boundary triplets. An important trend in the extension theories has been to replace *operators* by *relations*, as e.g. in Kochubei [K75], Derkach and Malamud [DM91], Malamud and Mogilevski [MM99], and many other works. Many contributions deal with extensions of symmetric operators, fewer with extensions of adjoint couples as in Section 13.1. One of the important issues in such studies is spectral theory: the description of the spectrum and the resolvent of the obtained operators.

We shall show below how the results of Section 13.1 can be extended to cover also resolvent questions. The direct consequences are given in Section 13.3, and the connection with considerations of boundary triplets is explained in Section 13.4. Here we also describe a recent development where boundary triplets are introduced in subspace situations; in the study of closed extensions as in Section 13.1, this replaces the need to discuss relations.

3.3 Consequences for resolvents.

We now return to the general set-up of Section 13.1, and consider the situation where a spectral parameter $\lambda \in \mathbb{C}$ is subtracted from the operators in $\mathcal{M}$. When $\lambda \in \varrho(A_\beta)$, we have a similar situation as in Section 13.1:

$$A_0 - \lambda \subset A_\beta - \lambda \subset A_1 - \lambda, \quad A_0' - \bar\lambda \subset A_\beta^* - \bar\lambda \subset A_1' - \bar\lambda, \tag{13.60}$$

and we use the notation $\mathcal{M}_\lambda$, $\mathcal{M}'_{\bar\lambda}$ for the families of operators $\widetilde{A} - \lambda$ resp. $\widetilde{A}' - \bar\lambda$, and write

$$\begin{gathered} Z(A_1 - \lambda) = Z_\lambda, \quad Z(A_1' - \bar\lambda) = Z'_{\bar\lambda}; \\ \operatorname{pr}^\lambda_\beta = (A_\beta - \lambda)^{-1}(A - \lambda), \quad \operatorname{pr}^{\bar\lambda}_{\beta'} = (A^*_\beta - \bar\lambda)^{-1}(A' - \bar\lambda), \\ \operatorname{pr}^\lambda_\zeta = I - \operatorname{pr}^\lambda_\beta, \quad \operatorname{pr}^{\bar\lambda}_{\zeta'} = I - \operatorname{pr}^{\bar\lambda}_{\beta'}\,. \end{gathered} \tag{13.61}$$

Let us also introduce the notation $\mathrm{i}_{U\to X}$ for the injection $U \hookrightarrow X$.

We have an immediate corollary of the results in Section 13.1:

Theorem 13.22. *Let $\lambda \in \varrho(A_\beta)$. There is a 1–1 correspondence between the closed operators $\widetilde{A} - \lambda$ in $\mathcal{M}_\lambda$ and the closed, densely defined operators $T^\lambda \colon V_\lambda \to W_{\bar\lambda}$, where V_λ and $W_{\bar\lambda}$ are closed subspaces of Z_λ resp. $Z'_{\bar\lambda}$; here*

$$\begin{aligned} &D(T^\lambda) = \operatorname{pr}^\lambda_\zeta D(\widetilde{A}), \quad V_\lambda = \overline{D(T^\lambda)}, \quad W_{\bar\lambda} = \overline{\operatorname{pr}^{\bar\lambda}_{\zeta'} D(\widetilde{A}^*)}, \\ &T^\lambda u^\lambda_\zeta = ((A-\lambda)u)_{W_{\bar\lambda}} \text{ for } u \in D(\widetilde{A}), \\ &D(\widetilde{A}) = \{u \in D(A_1) \mid u^\lambda_\zeta \in D(T^\lambda),\, ((A-\lambda)u)_{W_{\bar\lambda}} = T^\lambda u^\lambda_\zeta\}. \end{aligned} \tag{13.62}$$

In this correspondence,

$$\begin{aligned} Z(\widetilde{A} - \lambda) &= Z(T^\lambda), \\ R(\widetilde{A} - \lambda) &= R(T^\lambda) + (H \ominus W_{\bar\lambda}), \end{aligned} \tag{13.63}$$

orthogonal sum. In particular, if $\lambda \in \varrho(\widetilde{A})$,

$$(\widetilde{A} - \lambda)^{-1} = (A_\beta - \lambda)^{-1} + \mathrm{i}_{V_\lambda \to H} (T^\lambda)^{-1} \operatorname{pr}_{W_{\bar\lambda}}\,. \tag{13.64}$$

Proof. The first part is covered by Theorem 13.7, and the formulas in (13.63) and (13.64) follow from Theorem 3.8 and 3.9. □

In the litterature on extensions, formulas such as (13.64) describing the difference between two resolvents are usually called Krein resolvent formulas.

The next results have has been developed from calculations in [G74, Section 2].

Define, for $\lambda \in \varrho(A_\beta)$, the bounded operators on H:

$$\begin{aligned} E^\lambda &= A_1(A_\beta-\lambda)^{-1} = I + \lambda(A_\beta-\lambda)^{-1}, \\ F^\lambda &= (A_1-\lambda)A_\beta^{-1} = I - \lambda A_\beta^{-1}, \\ E'^{\bar\lambda} &= A_1'(A_\beta^*-\bar\lambda)^{-1} = I + \bar\lambda(A_\beta^*-\bar\lambda)^{-1} = (E^\lambda)^*, \\ F'^{\bar\lambda} &= (A_1'-\bar\lambda)(A_\beta^*)^{-1} = I - \bar\lambda(A_\beta^*)^{-1} = (F^\lambda)^*; \end{aligned} \tag{13.65}$$

Lemma 13.23. *E^λ and F^λ are inverses of one another, and so are $E'^{\bar\lambda}$ and $F'^{\bar\lambda}$. In particular, the operators restrict to homeomorphisms*

$$\begin{aligned} E^\lambda_Z &: Z \xrightarrow{\sim} Z_\lambda, \quad F^\lambda_Z : Z_\lambda \xrightarrow{\sim} Z, \\ E'^{\bar\lambda}_{Z'} &: Z' \xrightarrow{\sim} Z'_{\bar\lambda}, \quad F'^{\bar\lambda}_{Z'} : Z'_{\bar\lambda} \xrightarrow{\sim} Z'. \end{aligned} \tag{13.66}$$

Moreover, for $u \in D(A_1)$, $v \in D(A_1')$,

$$\begin{aligned} \mathrm{pr}^\lambda_\zeta u &= E^\lambda \,\mathrm{pr}_\zeta u, \quad \mathrm{pr}^\lambda_\beta u = \mathrm{pr}_\beta u - \lambda(A_\beta-\lambda)^{-1}\mathrm{pr}_\zeta u, \\ \mathrm{pr}^{\bar\lambda}_{\zeta'} v &= E'^{\bar\lambda}\,\mathrm{pr}_{\zeta'} v, \quad \mathrm{pr}^{\bar\lambda}_{\beta'} v = \mathrm{pr}_{\beta'} v - \bar\lambda(A_\beta^*-\bar\lambda)^{-1}\mathrm{pr}_{\zeta'} v. \end{aligned} \tag{13.67}$$

Proof. To show that E^λ and F^λ are inverses of one another, note that for

$$v \in A_\beta^{-1}H = D(A_\beta) = D(A_\beta-\lambda) = (A_\beta-\lambda)^{-1}H,$$

one has that $v = A_\beta^{-1}A_1 v = (A_\beta-\lambda)^{-1}(A_1-\lambda)v$. Hence

$$\begin{aligned} E^\lambda F^\lambda f &= A_1(A_\beta-\lambda)^{-1}(A_1-\lambda)A_\beta^{-1} f = A_1 A_\beta^{-1} f = f; \\ F^\lambda E^\lambda f &= (A_1-\lambda)A_\beta^{-1}A_1(A_\beta-\lambda)^{-1} f = (A_1-\lambda)(A_\beta-\lambda)^{-1} f = f. \end{aligned}$$

There is the analogous proof for the primed operators.

Now E^λ maps Z into Z_λ, and F^λ maps the other way, since

$$\begin{aligned} A_1 u = 0 &\implies (A_1-\lambda)E^\lambda u = (A_1-\lambda)(u + \lambda(A_\beta-\lambda)^{-1}u) \\ &\qquad = -\lambda u + \lambda u = 0, \\ (A_1-\lambda)v = 0 &\implies A_1 F^\lambda v = A_1(v - \lambda A_\beta^{-1} v) = \lambda v - \lambda v = 0, \end{aligned}$$

so since E^λ and F^λ are bijective in H, E^λ maps Z homeomorphically onto $Z(A_1-\lambda)$, with inverse F^λ. This justifies the first line in (13.66), and the second line is similarly justified.

The relation of the decompositions in (13.61) to the usual decompositions is found by observing that for $u \in D(A_1)$,

$$u = u_\beta + u_\zeta = [u_\beta + (I - E^\lambda)u_\zeta] + E^\lambda u_\zeta,$$

where $E^\lambda u_\zeta \in Z(A_1-\lambda)$ and

$$(I - E^\lambda)u_\zeta = -\lambda(A_\beta-\lambda)^{-1}u_\zeta \in D(A_\beta-\lambda);$$

hence, by the uniqueness in the decomposition $D(A_1-\lambda) = D(A_\beta-\lambda)\dot{+}Z_\lambda$,

$$u_\zeta^\lambda = E^\lambda u_\zeta, \quad u_\beta^\lambda = u_\beta - \lambda(A_\beta-\lambda)^{-1}u_\zeta. \tag{13.68}$$

Similarly for $v \in D(A_1')$,

$$v_{\zeta'}^{\bar\lambda} = E'^{\bar\lambda}v_{\zeta'}, \quad v_{\beta'}^{\bar\lambda} = v_{\beta'} - \bar\lambda(A_\beta^*-\bar\lambda)^{-1}v_{\zeta'}. \quad \square \tag{13.69}$$

Now we can find the relation between T and T^λ:

Theorem 13.24. *Let $T: V \to W$ correspond to $\widetilde{A}$ by Theorem 13.7, let $\lambda \in \varrho(A_\beta)$, and let $\widetilde{A}-\lambda$ correspond to $T^\lambda : V_\lambda \to W_{\bar\lambda}$ by Theorem 13.22.*

For $\lambda \in \varrho(A_\beta)$, define the operator G^λ from Z to Z' by

$$G^\lambda z = -\lambda \operatorname{pr}_{Z'} E^\lambda z, \quad z \in Z. \tag{13.70}$$

Then

$$\begin{gathered} D(T^\lambda) = E^\lambda D(T), \quad V_\lambda = E^\lambda V, \quad W_{\bar\lambda} = E'^{\bar\lambda}W, \\ (T^\lambda E^\lambda v, E'^{\bar\lambda}w) = (Tv,w) + (G^\lambda v, w), \text{ for } v \in D(T),\ w \in W. \end{gathered} \tag{13.71}$$

Proof. The first line in (13.71) follows from (13.62) in view of (13.67). The second line in calculated as follows: For $u \in D(\widetilde{A})$, $w \in W$,

$$\begin{aligned} (Tu_\zeta, w) &= (Au, w) = (Au, F'^{\bar\lambda}E'^{\bar\lambda}w) = (F^\lambda Au, E'^{\bar\lambda}w) \\ &= ((A-\lambda)(A_\beta)^{-1}Au, E'^{\bar\lambda}w) = ((A-\lambda)u_\beta, E'^{\bar\lambda}w) \\ &= ((A-\lambda)u, E'^{\bar\lambda}w) - ((A-\lambda)u_\zeta, E'^{\bar\lambda}w) \\ &= (T^\lambda u_\zeta^\lambda, E'^{\bar\lambda}w) + (\lambda u_\zeta, E'^{\bar\lambda}w) = (T^\lambda E^\lambda u_\zeta, E'^{\bar\lambda}w) + (\lambda E^\lambda u_\zeta, w). \end{aligned}$$

This shows the equation in (13.71) when we set $u_\zeta = v$. $\square$

Denote by E_V^λ the restriction of E^λ to a mapping from V to V_λ, with inverse F_V^λ, and let similarly $E_W'^{\bar\lambda}$ be the restriction of $E'^{\bar\lambda}$ to a mapping from W to $W_{\bar\lambda}$, with inverse $F_W'^{\bar\lambda}$. Then the second line of (13.71) can be written

$$(E_W'^{\bar\lambda})^* T^\lambda E_V^\lambda = T + G_{V,W}^\lambda \text{ on } D(T) \subset V, \tag{13.72}$$

where

$$G_{V,W}^\lambda = \operatorname{pr}_W G^\lambda \operatorname{i}_{V\to Z} . \tag{13.73}$$

(Beware that the adjoint $(E_W'^{\bar\lambda})^*$ is a mapping from $W_{\bar\lambda}$ to W, which is not derived by restriction from the formulas in (13.65).) Equivalently,

$$T^\lambda = (F_W'^{\bar\lambda})^*(T + G_{V,W}^\lambda) F_V^\lambda \text{ on } D(T^\lambda) \subset V_\lambda. \tag{13.74}$$

Then the Krein resolvent formula (13.64) can be made more explicit as follows:

Corollary 13.25. *When $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\beta)$, T^λ is invertible, and*

$$(T^\lambda)^{-1} = E_V^\lambda (T + G_{V,W}^\lambda)^{-1} (E_W'^{\bar\lambda})^*. \tag{13.75}$$

Hence

$$(\widetilde{A} - \lambda)^{-1} = (A_\beta - \lambda)^{-1} + \operatorname{i}_{V_\lambda \to H} E_V^\lambda (T + G_{V,W}^\lambda)^{-1} (E_W'^{\bar\lambda})^* \operatorname{pr}_{W_{\bar\lambda}} . \tag{13.76}$$

Proof. (13.75) follows from (13.74) by inversion, and insertion in (13.64) shows (13.76). $\square$

Note that $G_{V,W}^\lambda$ depends in a simple way on V and W and is independent of T.

13.4 Connections between the extension theory and theories of boundary triplets and M-functions.

We shall now consider the relation of our theory to constructions of extensions in terms of boundary triplets. In the nonsymmetric case, the set-up is the following, according to Brown, Marletta, Naboko and Wood [BMNW08] (with reference to Vainerman [V80], Lyantze and Storoch [LS83], Malamud and Mogilevski [MM02]):

A_0, A_1, A_0' and A_1' are given as in the beginning of Section 13.1, and there is given a pair of Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and two pairs of "boundary operators"

$$\begin{pmatrix} \Gamma_1 \\ \\ \Gamma_0 \end{pmatrix} : D(A_1) \to \begin{matrix} \mathcal{H} \\ \times \\ \mathcal{K} \end{matrix}, \quad \begin{pmatrix} \Gamma_1' \\ \\ \Gamma_0' \end{pmatrix} : D(A_1') \to \begin{matrix} \mathcal{K} \\ \times \\ \mathcal{H} \end{matrix}, \tag{13.77}$$

bounded with respect to the graph norm and surjective, and satifying

$$(Au, v) - (u, A'v) = (\Gamma_1 u, \Gamma_0' v)_{\mathcal{H}} - (\Gamma_0 u, \Gamma_1' v)_{\mathcal{K}}, \text{ all } u \in D(A_1), v \in D(A_1'). \tag{13.78}$$

Moreover it is assumed that

$$D(A_0) = D(A_1) \cap Z(\Gamma_1) \cap Z(\Gamma_0), \quad D(A_0') = D(A_1') \cap Z(\Gamma_1') \cap Z(\Gamma_0'). \tag{13.79}$$

In our setting, these hypotheses are satisfied by the choice

$$\begin{aligned} &\mathcal{H} = Z', \quad \mathcal{K} = Z, \\ &\Gamma_1 = \operatorname{pr}_{Z'} A_1, \quad \Gamma_0 = \operatorname{pr}_{\zeta}, \\ &\Gamma_1' = \operatorname{pr}_Z A_1', \quad \Gamma_0' = \operatorname{pr}_{\zeta'}. \end{aligned} \tag{13.80}$$

The surjectiveness follows since $u \in D(A_1)$ has the form $u = v + z$ with independent v and z, where $z = \operatorname{pr}_\zeta u = \operatorname{pr}_\zeta z$ runs through Z, and v runs through $D(A_\beta)$ so that $Au = Av$ runs through H, hence $(Av)_{Z'}$ runs through Z'. For (13.78), cf. Lemma 13.2. For (13.79), note that $(Au)_{Z'} = 0$ implies $Au \in R(A_0)$, and $\operatorname{pr}_\zeta u = 0$ implies $u \in D(A_\beta)$, so $u = A_\beta^{-1} Au \in D(A_0)$. (The choice (13.80) is relevant for applications to elliptic PDE, cf. [BGW08].)

Following [BMNW08], the boundary triplet is used to define operators $A_B \in \mathcal{M}$ and $A'_{B'} \in \mathcal{M}'$ for any pair of operators $B \in \mathbf{B}(\mathcal{K}, \mathcal{H})$, $B' \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ by

$$D(A_B) = \{u \in D(A_1) \mid \Gamma_1 = B\Gamma_0\}, \quad D(A'_{B'}) = \{v \in D(A_1') \mid \Gamma_1' = B'\Gamma_0'\}. \tag{13.81}$$

We see that for the present choice of boundary triplet (13.80),

$$D(A_B) = \{u \in D(A_1) \mid (Au)_{Z'} = Bu_\zeta\}.$$

Thus the operator $T : V \to W$ that A_B corresponds to by Theorem 13.7 is

$$T = B, \text{ with } V = Z, \quad W = Z'.$$

This covers some particular cases of the operators appearing in Theorem 13.7. For one thing, V and W are the full spaces Z and Z'. Secondly, T is bounded. When Z and Z' are finite dimensional, all operators from Z to Z' will be bounded, but our theory allows $\dim Z = \dim Z' = \infty$, and then we must also allow unbounded T's, to cover general extensions. Therefore we in the following replace B by a closed, densely defined and possibly unbounded operator T from Z to Z', and define A_T, now called $\widetilde{A}$, by

$$D(\widetilde{A}) = \{u \in D(A_1) \mid u_\zeta \in D(T), (Au)_{Z'} = Tu_\zeta\}; \tag{13.82}$$

this is consistent with the notation in Theorem 13.7.

In the discussion of resolvents via boundary triplets, a central object is the so-called M-function, originally introduced by Weyl and Titchmarsh in connection with Sturm-Liouville problems. We define (as an extension of the definition in [BMNW08] to unbounded B's):

Definition 13.26. *For $\lambda \in \varrho(\widetilde{A})$, the M-function $M_{\widetilde{A}}(\lambda)$ is defined as a mapping from $R(\Gamma_1 - T\Gamma_0)$ to $\mathcal{K}$, by*

$$M_{\widetilde{A}}(\lambda)(\Gamma_1 - T\Gamma_0)x = \Gamma_0 x \text{ for all } x \in Z_\lambda \text{ with } \Gamma_0 x \in D(T). \tag{13.83}$$

In terms of (13.80), *$M_{\widetilde{A}}(\lambda)$ is defined as a mapping from $R(\operatorname{pr}_{Z'} A_1 - T \operatorname{pr}_\zeta)$ to Z by*

$$M_{\widetilde{A}}(\lambda)(\operatorname{pr}_{Z'} A_1 - T \operatorname{pr}_\zeta)x = \operatorname{pr}_\zeta x \text{ for all } x \in Z_\lambda \text{ with } \operatorname{pr}_\zeta x \in D(T). \tag{13.84}$$

There is the analogous definition of $M'_{\widetilde{A}'}(\lambda)$ in the adjoint setting.

We shall show that $M_{\widetilde{A}}(\lambda)$ is well-defined. First we observe:

Lemma 13.27. *$R(\Gamma_1 - T\Gamma_0) = R(\operatorname{pr}_{Z'} A_1 - T \operatorname{pr}_\zeta)$ is equal to Z'. In fact, any $f \in Z'$ can be written as*

$$f = (\operatorname{pr}_{Z'} A_1 - T \operatorname{pr}_\zeta)v = \operatorname{pr}_{Z'} A_1 v, \text{ for } v = A_\beta^{-1} f. \tag{13.85}$$

Proof. Let v run through $D(A_\beta)$. Then $\Gamma_0 v = \operatorname{pr}_\zeta v = 0 \in D(T)$, and Av runs through H, so $\Gamma_1 v - T\Gamma_0 v = (Av)_{Z'}$ runs through Z'. In other words, for any $f \in Z'$ we can take $v = A_\beta^{-1} f$. $\square$

Proposition 13.28. *For any $\lambda \in \varrho(\widetilde{A})$, $M_{\widetilde{A}}(\lambda)$ is well-defined as a bounded map from Z' to Z by* (13.83) *or* (13.84), *and ranges in $D(T)$. In fact,*

$$M_{\widetilde{A}}(\lambda) = \operatorname{pr}_\zeta (I - (\widetilde{A} - \lambda)^{-1}(A_1 - \lambda)) A_\beta^{-1} \operatorname{i}_{Z' \to H}. \tag{13.86}$$

Proof. The mapping $\Phi = \Gamma_1 - T\Gamma_0 = \operatorname{pr}_{Z'} A_1 - T \operatorname{pr}_\zeta$ is defined for those $u \in D(A_1)$ for which $\operatorname{pr}_\zeta u \in D(T)$. Let

$$Z_{\lambda,T} = \{z^\lambda \in Z_\lambda \mid \operatorname{pr}_\zeta z^\lambda \in D(T)\},$$

then (13.84) means that $M_{\widetilde{A}}(\lambda)$ should satisfy

$$M_{\widetilde{A}}(\lambda)\Phi z^\lambda = \operatorname{pr}_\zeta z^\lambda \text{ for } z^\lambda \in Z_{\lambda,T}.$$

We first show that Φ maps $Z_{\lambda,T}$ bijectively onto Z', with inverse

$$\Psi = (I - (\widetilde{A} - \lambda)^{-1}(A_1 - \lambda)) A_\beta^{-1} \operatorname{i}_{Z' \to H}. \tag{13.87}$$

For the surjectiveness, let $f \in Z'$. Let $v = A_\beta^{-1} f$, then $f = \Phi v$ according to Lemma 13.27. Next, let $w = (\widetilde{A} - \lambda)^{-1}(A_1 - \lambda)v$, then since $w \in D(\widetilde{A})$, $\operatorname{pr}_\zeta w \in D(T)$ and

$$\Phi w = \operatorname{pr}_{Z'} A_1 w - T \operatorname{pr}_\zeta w = 0,$$

by (13.82). Let $z^\lambda = v - w$, it lies in Z_λ and has $\operatorname{pr}_\zeta z^\lambda = - \operatorname{pr}_\zeta w \in D(T)$, so $z^\lambda \in Z_{\lambda,T}$. It follows that

$$\Phi z^\lambda = \Phi v = f.$$

Thus Φ is indeed surjective from $Z_{\lambda,T}$ to Z'. It is also injective, for if $z^\lambda \in Z_{\lambda,T}$ is such that $\Phi z^\lambda = 0$, then, by (13.82), $z^\lambda \in D(\widetilde{A}) \cap Z_\lambda = \{0\}$ (recall that $\lambda \in \varrho(\widetilde{A})$). Following the steps in the construction of z^λ from f, we see that the inverse of $\Phi : Z_{\lambda,T} \to Z'$ is indeed given by (13.87).

We can now rewrite the defining equation (13.84) as

$$M_{\widetilde{A}}(\lambda) f = \operatorname{pr}_\zeta \Psi f \text{ for all } f \in Z'.$$

This shows that $M_{\widetilde{A}}(\lambda) = \operatorname{pr}_\zeta \Psi$ is the desired operator, and clearly (13.86) holds. Since $\widetilde{A}$ is closed, $M_{\widetilde{A}}(\lambda)$ is closed as a mapping from Z' to Z, hence continuous. $\square$

For a further analysis of $M_{\widetilde{A}}(\lambda)$, assume $\lambda \in \varrho(A_\beta) \cap \varrho(\widetilde{A})$. Then the maps E^λ, F^λ etc. in (13.65) are defined. Let $z^\lambda \in Z_\lambda$, and consider the defining equation

$$M_{\widetilde{A}}(\lambda)((Az^\lambda)_{Z'} - T \operatorname{pr}_\zeta z^\lambda) = \operatorname{pr}_\zeta z^\lambda; \tag{13.88}$$

where $\operatorname{pr}_\zeta z^\lambda$ is required to lie in $D(T)$. By (13.66) and (13.67), there is a unique $z \in Z$ such that $z^\lambda = E^\lambda_Z z$; in fact

$$z^\lambda = E^\lambda_Z z = \operatorname{pr}^\lambda_\zeta z; \quad z = F^\lambda_Z z^\lambda = \operatorname{pr}_\zeta z^\lambda,$$

so the requirement is that $z^\lambda \in E^\lambda_Z D(T)$.

Writing (13.88) in terms of z, and using that $Az^\lambda = \lambda z^\lambda$, we find:

$$\begin{aligned} z = \operatorname{pr}_\zeta z^\lambda &= M_{\widetilde{A}}(\lambda)((Az^\lambda)_{Z'} - Tz) = M_{\widetilde{A}}(\lambda)((\lambda z^\lambda)_{Z'} - Tz) \\ &= M_{\widetilde{A}}(\lambda)((\lambda E^\lambda z)_{Z'} - Tz) = M_{\widetilde{A}}(\lambda)(-G^\lambda - T)z, \end{aligned} \tag{13.89}$$

cf. (13.70). Here we observe that the operator to the right of $M_{\widetilde{A}}(\lambda)$ equals T^λ from Section 13.3 up to homeomorphisms:

$$-G^\lambda - T = -(E'^{\bar\lambda}_{Z'})^* T^\lambda E^\lambda_Z, \tag{13.90}$$

by (13.72); here T^λ is invertible from $E^\lambda_Z D(T)$ onto Z'_λ. We conclude that $M_{\widetilde{A}}(\lambda)$ is the inverse of the operator in (13.90). We have shown:

Theorem 13.29. *When the boundary triplet is chosen as in* (13.80) *and* $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\beta)$, $-M_{\widetilde{A}}(\lambda)$ *equals the inverse of* $T + G^\lambda$, *also equal to the inverse of* T^λ *modulo homeomorphisms:*

$$-M_{\widetilde{A}}(\lambda)^{-1} = T + G^\lambda = (E'^{\bar\lambda}_{Z'})^* T^\lambda E^\lambda_Z. \tag{13.91}$$

In particular, $M_{\widetilde{A}}(\lambda)$ *has range* $D(T)$.

With this insight we have access to the straightforward resolvent formula (13.76), which implies in this case:

Corollary 13.30. *For* $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\beta)$,

$$(\widetilde{A} - \lambda)^{-1} = (A_\beta - \lambda)^{-1} - \mathrm{i}_{Z_\lambda \to H}\, E^\lambda_Z M_{\widetilde{A}}(\lambda)(E'^{\bar\lambda}_{Z'})^* \operatorname{pr}_{Z'_{\bar\lambda}}. \tag{13.92}$$

We also have the direct link between nullspaces and ranges (13.63), when merely $\lambda \in \varrho(A_\beta)$.

Corollary 13.31. *For any* $\lambda \in \varrho(A_\beta)$,

$$\begin{aligned} Z(\widetilde{A} - \lambda) &= E^\lambda_Z Z(T + G^\lambda), \\ R(\widetilde{A} - \lambda) &= (F'^{\bar\lambda}_{Z'})^* R(T + G^\lambda) + R(A_0 - \lambda). \end{aligned} \tag{13.93}$$

One can show that $M_{\widetilde{A}}(\lambda)$ is a holomorphic operator family for $\lambda \in \varrho(\widetilde{A})$. There is a result in [BMNW08] on the relation between poles of $M_{\widetilde{A}}(\lambda)$ and eigenvalues of $\widetilde{A}$; for the points in $\varrho(A_\beta)$, Corollary 13.31 is more informative.

The analysis moreover implies that $M_{\widetilde{A}}(\lambda)$ and $M'_{\widetilde{A}^*}(\bar\lambda)$ are adjoints, at least when $\lambda \in \varrho(A_\beta)$.

Note that T^λ is well-defined for all $\lambda \in \varrho(A_\beta)$, whereas $M_{\widetilde{A}}(\lambda)$ is well-defined for all $\lambda \in \varrho(\widetilde{A})$; the latter fact is useful for other purposes. In this way, the two operator families supply each other.

So far, we have only discussed M-functions for elements of $\mathcal{M}$ with $V = Z$, $W = Z'$ in Theorem 13.7. But, inspired by the result in Theorem 13.29, we can in fact establish useful M-functions for all closed $\widetilde{A}$. They will be homeomorphic to the inverses of the operators T^λ that exist for $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\beta)$, and extend to exist for all $\lambda \in \varrho(\widetilde{A})$.

Theorem 13.32. *Let* $\widetilde{A}$ *be an arbitrary closed densely defined operator between* A_0 *and* A_1, *and let* $T : V \to W$ *be the corresponding operator according to Theorem* 13.7. *For any* $\lambda \in \varrho(\widetilde{A})$ *there is a bounded operator*

$M_{\widetilde{A}}(\lambda) : W \to V$, depending holomorphically on $\lambda \in \varrho(\widetilde{A})$, such that when $\lambda \in \varrho(A_\beta)$, $-M_{\widetilde{A}}(\lambda)$ is the inverse of $T + G^{\lambda}_{V,W}$, and is homeomorphic to T^{λ} (as defined in Section 13.3). It satisfies

$$M_{\widetilde{A}}(\lambda)((Az^{\lambda})_W - T \operatorname{pr}_\zeta z^{\lambda}) = \operatorname{pr}_\zeta z^{\lambda}, \tag{13.94}$$

for all $z^{\lambda} \in Z_\lambda$ such that $\operatorname{pr}_\zeta z^{\lambda} \in D(T)$. Its definition extends to all $\lambda \in \varrho(\widetilde{A})$ by the formula

$$M_{\widetilde{A}}(\lambda) = \operatorname{pr}_\zeta \big(I - (\widetilde{A} - \lambda)^{-1}(A_1 - \lambda)\big) A_\beta^{-1} \operatorname{i}_{W \to H} \tag{13.95}$$

In particular, the resolvent formula

$$(\widetilde{A} - \lambda)^{-1} = (A_\beta - \lambda)^{-1} - \operatorname{i}_{V_\lambda \to H} E^{\lambda}_V M_{\widetilde{A}}(\lambda)(E'^{\bar\lambda}_W)^* \operatorname{pr}_{W_{\bar\lambda}} \tag{13.96}$$

holds when $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\beta)$. For all $\lambda \in \varrho(A_\beta)$,

$$\begin{aligned} Z(\widetilde{A} - \lambda) &= E^{\lambda}_V Z(T + G^{\lambda}_{V,W}), \\ R(\widetilde{A} - \lambda) &= (F'^{\bar\lambda}_W)^* R(T + G^{\lambda}_{V,W}) + H \ominus W_{\bar\lambda}, \end{aligned} \tag{13.97}$$

where $W_{\bar\lambda} = E'^{\bar\lambda} W$.

Proof. Following the lines of proofs of Lemma 13.27 and Proposition 13.28, we define $M_{\widetilde{A}}(\lambda)$ satisfying (13.94) as follows: Let $f \in W$. Let $v = A_\beta^{-1} f$; then $\operatorname{pr}_\zeta v = 0 \in D(T)$, and

$$(Av)_W - T \operatorname{pr}_\zeta v = Av = f. \tag{13.98}$$

Next, let $w = (\widetilde{A} - \lambda)^{-1}(A - \lambda)v$, then $z^{\lambda} = v - w$ lies in Z_λ and satisfies $\operatorname{pr}_\zeta z^{\lambda} \in D(T)$ (since $\operatorname{pr}_\zeta v = 0$ and $\operatorname{pr}_\zeta w \in D(T)$). This z^{λ} satisfies

$$(Az^{\lambda})_W - T \operatorname{pr}_\zeta z^{\lambda} = f, \tag{13.99}$$

in view of (13.98) and the fact that $w \in D(\widetilde{A})$.

Next, observe that for any vector $z^{\lambda} \in Z_\lambda$ with $\operatorname{pr}_\zeta z^{\lambda} \in D(T)$ such that (13.99) holds, $f = 0$ implies $z^{\lambda} = 0$, since such a z^{λ} lies in the two linearly independent spaces $D(\widetilde{A} - \lambda)$ and Z_λ. So there is indeed a mapping Ψ from f to $\operatorname{pr}_\zeta z^{\lambda}$ solving (13.99), for any $f \in W$. Then $M_{\widetilde{A}}(\lambda)$ is the composition $\operatorname{pr}_\zeta \Psi$; it is described by (13.95). The holomorphicity in $\lambda \in \varrho(\widetilde{A})$ is seen from this formula.

The mapping connects with T^{λ} (cf. Corollary 13.23) as follows:

When $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\beta)$, then $z = \operatorname{pr}_\zeta z^\lambda = F_V^\lambda z^\lambda$, and $z^\lambda = E_V^\lambda z$, so the vectors z^λ with $\operatorname{pr}_\zeta z^\lambda \in D(T)$ consitute the space $E_V^\lambda D(T)$. Calculating as in (13.89) we then find that

$$\begin{aligned} z = \operatorname{pr}_\zeta z^\lambda &= M_{\widetilde{A}}(\lambda)((Az^\lambda)_W - Tz) = M_{\widetilde{A}}(\lambda)((\lambda z^\lambda)_W - Tz) \\ &= M_{\widetilde{A}}(\lambda)((\lambda E^\lambda z)_W - Tz) = M_{\widetilde{A}}(\lambda)(-G^\lambda_{V,W} - T)z, \end{aligned}$$

so $M_{\widetilde{A}}(\lambda)$ is the inverse of $-(T + G^\lambda_{V,W}) : D(T) \to W$. The remaining statements follow from Theorem 13.23 and Corollary 13.25. □

Note that when $M_{\widetilde{A}}(\lambda)$ is considered in a neighborhood of a spectral point of $\widetilde{A}$ in $\varrho(A_\beta)$, then we have not only information on the possibility of a pole of $M_{\widetilde{A}}(\lambda)$, but we have an inverse T^λ, from which $Z(\widetilde{A} - \lambda)$ and $R(\widetilde{A} - \lambda)$ can be read off.

An application of these concepts to elliptic PDE is given in [BGW08].

Exercises for Chapter 13.

13.1. Work out the statement and proof of Theorem 13.8 3° in detail.

13.2. Show the assertions in Example 13.10a concerning the case $V = W = \operatorname{span} z_1$.

13.3. Work out the details of Example 13.10a in the case where $\alpha = 0$, $\beta = 1$, $q = 0$.

Consider the realization $\widetilde{A}$ defined by the Neumann condition $u'(0) = b_1 u(0)$, $-u'(1) = b_2 u(1)$. Show that $\widetilde{A}$ is selfadjoint positive if and only if

$$b_1 > -1, \quad b_2 > -1 + (b_1 + 1)^{-1}.$$

13.4. Work out the details of Example 13.10a in the case where $\alpha = 0$, $\beta = 1$, $q = 1$.

For the realization $\widetilde{A}$ defined by the Neumann condition $u'(0) = b_1 u(0)$, $-u'(1) = b_2 u(1)$, find the choices of b_1, b_2 for which $\widetilde{A}$ is selfadjoint positive.

4.13a. *This exercise will be moved to Chapter 4.*
Consider the differential operator $Au = -u'' + \alpha^2 u$ on $\mathbb{R}_+$, where α is a positive constant. (This is a Sturm-Liouville operator with $p = 1$, $q = \alpha^2$ constant > 0.) Verify the following facts:

(a) The maximal operator $A_{\max}$ associated with A in $L_2(\mathbb{R}_+)$ has domain $H^2(\mathbb{R}_+)$.

(b) The minimal operator $A_{\min}$ associated with A in $L_2(\mathbb{R}_+)$ has domain $H_0^2(\mathbb{R}_+)$.

(c) The realization A_γ of A defined by the boundary condition $u(0) = 0$ is variational, associated with the triple (H, V, a), where $H = L_2(\mathbb{R}_+)$, $V = H_0^1(\mathbb{R}_+)$,

$$a(u, v) = (u', v')_{L_2(\mathbb{R}_+)} + \alpha^2 (u, v)_{L_2(\mathbb{R}_+)}.$$

Here $m(A_\gamma) \geq \alpha^2$.

In the next exercises, it is checked how the theory in this chapter looks for a very simple example.

13.5. (Notation from Chapter 4 is used here.) Consider the differential operator $Au = -u'' + \alpha^2 u$ on $\mathbb{R}_+$, where α is a positive constant. With

$A_{\max}$, $A_{\min}$ and A_γ defined as in Exercise 4.13a, denote $A_{\max} = A_1$ and $A_{\min} = A_0$.

(a) Show that the hypotheses of Sections 13.1 and 13.2 are satisfied for these operators in $H = L_2(\mathbb{R}_+)$, as a symmetric case, with A_β equal to the Dirichlet realization A_γ.

(b) Show that Z is one-dimensional, spanned by the vector $e^{-\alpha t}$.

(c) Show that $A_\gamma^{-1} e^{-\alpha t} = \frac{1}{2\alpha} t e^{-\alpha t}$.

(d) Show that the choice $V = W = 0$, $T = 0$ (necessarily), corresponds to the operator $\widetilde{A} = A_\gamma$.

(e) From now on we let $V = W = Z$, and let T be the multiplication by a complex number τ. For the corresponding operator $\widetilde{A}$, describe the elements of $D(\widetilde{A})$ by use of the formula (13.26).

(f) Show that $\widetilde{A}$ is a realization of A determined by a boundary condition $u'(0) = bu(0)$, where

$$\tau = 2\alpha(b + \alpha).$$

Which choice of τ gives the Neumann condition $u'(0) = 0$?

13.6. (Continuation of Exercise 13.5)

(a) Show that the numerical range of T is the point

$$\nu(T) = \{2\alpha(b + \alpha)\}.$$

(b) Show that $m(\widetilde{A}) \geq 0$ if and only if $\operatorname{Re}(b + \alpha) \geq 0$.

(c) Let $\operatorname{Im} b > 0$, and describe the convex hull M of the set

$$[\alpha^2, \infty[\cup \{2\alpha(b + \alpha)\}.$$

Using the result of Exercise 12.34 (e), show that the closure of the numerical range of $\widetilde{A}$ contains M.

(d) Show that if $b = -\alpha$, $\widetilde{A}$ is not bijective.

13.7. (Continuation of Exercise 13.5) The next point is to identify the operators from Sections 13.3 and 13.4 for the present example. Let $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$, and let

$$\sigma = (\alpha^2 - \lambda)^{\frac{1}{2}}, \quad \sigma_1 = (\alpha^2 - \bar{\lambda})^{\frac{1}{2}},$$

where the squareroot is taken to be holomorphic on $\mathbb{C} \setminus \overline{\mathbb{R}}_-$ and positive on $\mathbb{R}_+$. Consider now $\widetilde{A} - \lambda$, where $\widetilde{A}$ is the realization of A defined in Exercise 13.5 by the boundary condition $u'(0) = bu(0)$.

(a) Show that $\sigma_1 = \bar{\sigma}$, and that $\operatorname{Re} \sigma > 0$.

(b) Show that $Z_\lambda = \operatorname{span}(e^{-\sigma t})$, and that $Z'_{\bar{\lambda}} = \operatorname{span}(e^{-\sigma_1 t})$.

(c) Show that when $\operatorname{Im} \lambda \neq 0$,

$$(A_\gamma - \lambda)^{-1} e^{-\sigma_1 t} = (\lambda - \bar{\lambda})^{-1}(e^{-\sigma t} - e^{-\sigma_1 t}).$$

(d) For $\operatorname{Im} \lambda > 0$, find the τ^λ such that the mapping $T^\lambda : Z_\lambda \to Z'_{\bar{\lambda}}$, defined by sending $e^{-\sigma t}$ over into $\tau^\lambda e^{-\sigma_1 t}$, is the operator corresponding to $\widetilde{A} - \lambda$ by Theorem 13.22.

13.8. (Continuation of Exercise 13.5)

(a) Show that the mapping $E_Z^\lambda : Z \to Z_\lambda$ sends $e^{-\alpha t}$ over into $e^{-\sigma t}$. Similarly, $E'^{\bar{\lambda}}_{Z'} : Z \to Z'_{\bar{\lambda}}$ sends $e^{-\alpha t}$ over into $e^{-\sigma_1 t}$.

(b) Show that the adjoint $(E'^{\bar{\lambda}}_{Z'})^* : Z'_{\bar{\lambda}} \to Z$ sends $e^{-\sigma_1 t}$ over into $\alpha(\operatorname{Re} \sigma)^{-1} e^{-\alpha t}$.

(c) Show that $G^\lambda : Z \to Z$ maps $e^{-\alpha t}$ into $g^\lambda e^{-\alpha t}$, where

$$g^\lambda = -2\alpha\lambda(\alpha + \sigma)^{-1} = 2\alpha(\sigma - \alpha).$$

(*Hint.* For the last equality, use that $(\sigma - \alpha)(\sigma + \alpha) = \sigma^2 - \alpha^2 = -\lambda$.)

(d) Conclude that $T + G^\lambda$ is the multiplication by $2\alpha(b + \sigma)$.

(e) Find $M_{\widetilde{A}}(\lambda)$.

(*Comment.* The multiplication by $-\sigma$ can be regarded as the Dirichlet-to-Neumann operator for $A - \lambda$, since it maps $u(0)$ to $u'(0)$ for solutions of $(A - \lambda)u = 0$; in particular, $-\alpha$ is the Dirichlet-to-Neumann operator for A. Point (d) shows that the subtraction of $-\alpha$ is replaced by the subtraction of $-\sigma$ in the passage from T to $T + G^\lambda$.)

13.9. (Continuation of Exercise 13.5)

Show that when $b = ir$ with $r > 0$, then

$$\mathbb{C} \setminus [\alpha^2, \infty[\subset \varrho(\widetilde{A}),$$

and yet the closure of the numerical range of $\widetilde{A}$ contains the set

$$\{x + iy \mid x, y \in \mathbb{R},\ x \geq 2\alpha^2, 0 \leq y \leq 2\alpha r\}.$$

(*Comment.* This shows an interesting case of an operator that has its spectrum contained in $[\alpha^2, \infty[$ but is far from being selfadjoint.)