§14. Semigroups of operators

14.1. Evolution equations.

The investigations in Chapter 13 are designed particularly for the concretization of elliptic operators, in terms of boundary conditions. But they have a wider applicability. In fact, operators with suitable semiboundedness properties are useful also in the concretization of parabolic or hyperbolic problems, where there is a first- or second-order time derivative in addition to the elliptic operator. We present in the following a basic method for the discussion of such time-dependent problems. More refined methods, building on microlocal techniques (using not only pseudodifferential operators but also Fourier integral operators and wave front sets) have been introduced since the time this method was worked out, but we still think that it can have an interest as a first introduction to time-dependent equations.

A differential operator example of interest in Physics is the Laplace operator Δ, defined on $\mathbb{R}^n$ by

$$\Delta u(x) = \partial_{x_1}^2 u + \cdots + \partial_{x_n}^2 u \,. \tag{14.1}$$

It is used alone e.g. in the equation of a potential field u in an open subset of $\mathbb{R}^3$

$$\Delta u(x) = 0 \;\text{ for }\; x \in \Omega \,, \tag{14.2}$$

and it is used together with a time parameter t e.g. in the *heat equation*

$$\partial_t u(x,t) - \Delta_x u(x,t) = 0 \;\text{ for }\; x \in \Omega \;\text{ og }\; t \geq 0 \,, \tag{14.3}$$

the *Schrödinger equation*

$$\frac{1}{i}\partial_t u(x,t) - \Delta_x u(x,t) = 0 \;\text{ for }\; x \in \Omega \;\text{ and }\; t \in \mathbb{R} \,, \tag{14.4}$$

and the *wave equation*

$$\partial_t^2 u(x,t) - \Delta_x u(x,t) = 0 \;\text{ for }\; x \in \Omega \;\text{ and }\; t \in \mathbb{R} \,. \tag{14.5}$$

The three last equations have the common property that they can formally be considered as equations of the form

$$\partial_t u(t) = Bu(t) \,, \tag{14.6}$$

where $t \mapsto u(t)$ is a function from the time axis into a space of functions of x, where B operates. For (14.3), B acts like Δ, for (14.4) like $i\Delta$. For (14.5) we obtain the form (14.6) by introducing the vector

$$v(t) = \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} \,,$$

which must satisfy

$$\partial_t v(t) = Bv(t) \ , \quad \text{with} \ \ B = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \ . \tag{14.7}$$

The equation (14.6) is called an *evolution equation.* We get a very simple version in the case where u takes its values in $\mathbb{C}$, and B is just multiplication by the constant $\lambda \in \mathbb{C}$

$$\partial_t u(t) = \lambda u(t) \ .$$

It is well-known that the solutions of this equation are

$$u(t) = e^{t\lambda} c \ , \quad c \in \mathbb{C}. \tag{4.7a}$$

We shall show in the following how the abstract functional analysis allows us to define similar solutions $\exp(tB)u_0$, when λ is replaced by an operator B in a Banach space X, under suitable hypotheses.

In preparation for this, we need to consider Banach space valued functions $v(t)$. Let $v : I \to X$, where I is an interval of $\mathbb{R}$ and X is Banach space. The function $v(t)$ is said to be *continuous at* $t_0 \in I$, when $v(t) \to v(t_0)$ in X for $t \to t_0$ in I. In details, this means: For any $\varepsilon > 0$ there is a $\delta > 0$ such that $|t - t_0| < \delta$ implies $\|v(t) - v(t_0)\| < \varepsilon$. When continuity holds for all $t_0 \in I$, v is said to be continuous on I.

The function $v : I \to X$ is said to be *differentiable* at t, if $\lim_{h\to 0} \frac{1}{h}(v(t+h) - v(t))$ exists in X (for t and $t + h \in I$); the limit is denoted $v'(t)$ (or $\partial_t v$ or $\frac{dv}{dt}$). More precisely, one says here that $v(t)$ is *norm differentiable* or *strongly differentiable*, to distinguish this property from the property of being *weakly differentiable.* The latter means that for any continuous linear functional $x^* \in X^*$, the function

$$f_{x^*}(t) = x^*(v(t))$$

is differentiable, in such a way that there exists a $v'(t) \in X$ so that $\partial_t f_{x^*}(t) = x^*(v'(t))$ for all x^*. A norm differentiable function is clearly weakly differentiable, with the same derivative $v'(t)$, but there exist weakly differentiable functions that are not norm differentiable.

The integral of a continuous vector function $v : I \to X$ is defined by the help of middlesums precisely as for real or complex functions (here $\mathbb{R}$ or $\mathbb{C}$ is replaced by X, the modulus is replaced by the norm). One has the usual rules (where the proofs are straightforward generalizations of the proofs for real functions):

$$\int_a^b (\alpha v(t) + \beta w(t))\, dt = \alpha \int_a^b v(t)\, dt + \beta \int_a^b w(t)\, dt \ ,$$

$$\int_a^b v(t)\, dt + \int_b^c v(t)\, dt = \int_a^c v(t)\, dt$$

for arbitrary points a, b and c in I, and

$$\| \int_a^b v(t)\, dt \|_X \leq \int_a^b \|v(t)\|_X \, dt \quad \text{when} \quad a \leq b \, . \tag{14.12}$$

Moreover, $\int_a^t v(s)\, ds$ is differentiable, with

$$\frac{d}{dt} \int_a^t v(s)\, ds = v(t). \tag{14.13a}$$

Like for integrals of complex functions there is not a genuine mean value theorem, but one does have that

$$\frac{1}{h} \int_c^{c+h} v(t)\, dt \to v(c) \quad \text{in} \quad X \quad \text{for} \quad h \to 0 \, . \tag{14.13}$$

When the vector function $v : I \to X$ is differentiable with a continuous derivative $v'(t)$, one has the identity

$$\int_a^b v'(t)\, dt = v(b) - v(a) \tag{14.14}$$

(since the corresponding identity holds for all functions $f_{x^*}(t) = x^*(v(t))$).

We now return to the possible generalizations of (14.7a) to functions valued in a Banach space X. The easiest case is where B is a *bounded* operator on X. Here we can simply put

$$\exp(tB) = \sum_{n \in \mathbb{N}_0} \frac{1}{n!} (tB)^n \tag{14.8}$$

for all $t \in \mathbb{R}$, since the series converges (absolutely) in the operator norm to a bounded operator; this is seen e.g. by noting that

$$\| \sum_{N \leq n < N'} \frac{1}{n!} (tB)^n \| \leq \sum_{N \leq n < N'} \frac{1}{n!} |t|^n \|B\|^n \, ,$$

where the right hand side is a partial sum in the convergent series

$$\exp(|t| \|B\|) = \sum_{n \in \mathbb{N}_0} \frac{1}{n!} (|t| \|B\|)^n.$$

It follows in particular that $\| \exp(tB) \| \leq \exp(|t| \|B\|)$.

The operator family satisfies, for $s, t \in \mathbb{R}$,

$$\exp(sB)\exp(tB) = \sum_{n=0}^{\infty} \frac{(sB)^n}{n!} \sum_{m=0}^{\infty} \frac{(tB)^m}{m!} = \sum_{l=0}^{\infty} \sum_{n+m=l} \frac{s^n t^m}{n!m!} B^l \tag{14.9}$$
$$= \sum_{l=0}^{\infty} \frac{(s+t)^l}{l!} B^l = \exp((s+t)B) \ ,$$

where we have used the corresponding identity for the exponential series; the reorganization is allowed since the norms of the terms form a convergent series. Moreover, we have for $C \ge s \ge t \ge 0$ that

$$\begin{aligned}\| \exp(sB) - \exp(tB)\| &= \| \sum_{n=0}^{\infty} \frac{s^n - t^n}{n!} B^n \| \le \sum_{n=0}^{\infty} \frac{s^n - t^n}{n!} \|B\|^n \\ &= \exp(s\|B\|) - \exp(t\|B\|) \to 0 \ \text{ for } \ s - t \to 0 \ , \\ \| \exp(-sB) - \exp(-tB)\| &= \| \exp(-tB)(\exp(tB) - \exp(sB)) \exp(-sB)\| \\ &\le \exp(|t|\|B\|) \exp(|s|\|B\|) \| \exp(tB) - \exp(sB)\| \\ &\to 0 \ \text{ for } \ (-s) - (-t) \to 0 \ ,\end{aligned}$$

which shows that the operator family is continuous in t with respect to the operator norm.

That the operator function $\exp(tB)$ is differentiable with derivative $B\exp(tB)$ can for example be seen as follows. Integration of the continuous function $\exp(tB)$ and composition with B gives:

$$\begin{aligned}B \int_0^t \exp(sB)\, ds &= B \int_0^t \sum_{n \in \mathbb{N}_0} \frac{1}{n!} (sB)^n \, ds \\ &= \sum_{n \in \mathbb{N}_0} \frac{1}{(n+1)!} t^{n+1} B^{n+1} = \exp(tB) - I,\end{aligned}$$

from which it follows by differentiation of both sides that

$$B\exp(tB) = \frac{d}{dt} \exp(tB). \tag{4.9a}$$

(We could interchange integration and summation in view of the majorized convergence.)

Exponential functions can also be set up for suitable unbounded operators B in X, more on this below. If X is in particular a Hilbert space, one can define the exponential function via spectral theory if B is a selfadjoint or normal operator. This is used in e.g. in the following cases (that we mention without proof to begin with):

1° When B is *selfadjoint* and *upper semibounded*, $\exp(tB)$ is well-defined for $t \geq 0$.

2° When B is *skew-selfadjoint*, i.e., $B^* = -B$, $\exp(tB)$ is well-defined for $t \in \mathbb{R}$.

The case 1° is relevant for the heat equation on $\mathbb{R}^n$, since $-\Delta$ can be given a sense as a selfadjoint unbounded operator $-B \geq 0$ in $L_2(\mathbb{R}^n)$. The case 2° then pertains to the Schrödinger equation, since $i\Delta$ hereby becomes skew-selfadjoint in $L_2(\mathbb{R}^n)$.

For the wave equation in the form (14.7), the skew-selfadjointness can be obtained by use of some other particular Hilbert spaces. If one wants to work with the wave equation in $L_2(\mathbb{R}^n)$, one can instead interpret the solutions of the abstract equation

$$\partial_t^2 u = -Au \tag{14.10}$$

as combinations of the solutions $\cos(tA^{\frac{1}{2}})u_0$ and $A^{-\frac{1}{2}}\sin(tA^{\frac{1}{2}})u_1$ (where $A = -\Delta$ is ≥ 0); also this can be achieved by use of spectral theory.

The spectral theory that is needed here is an extension of the elementary theory to unbounded operators.

We now turn to a more general definition of the exponential function, not requiring normality of the operator B, namely the theory of semigroups of operators. It also covers the cases 1° and 2°.

14.2. Contraction semigroups in Banach spaces.

The following account builds on Appendix 1 in the book of Lax and Phillips [LP67].

A *semigroup* of operators in a Banach space X is a family of operators $G(t) \in \mathbf{B}(X)$, parametrized by $t \in \overline{\mathbb{R}}_+$ and satisfying:

$$\text{(a)} \quad G(0) = I\ , \quad \text{and} \quad G(s+t) = G(s)G(t) \quad \text{for all} \quad s \quad \text{and} \quad t \geq 0\ .$$

A *group* of operators is a family of operators $G(t) \in \mathbf{B}(X)$ parametrized by $t \in \mathbb{R}$ and such that (a) holds for all s and $t \in \mathbb{R}$. Here all the operatorers are invertible, since the second condition implies $G(t)^{-1} = G(-t)$. Note that both $G(t)$ and $G(-t)$ are semigroups for $t \geq 0$.

Note that we have both for semigroups and groups that $G(s)G(t) = G(t)G(s)$.

The semigroups and groups we consider, will furthermore be required to satisfy the following condition:

$$\text{(b)} \quad G(t)x \to x \quad \text{for} \quad t \to 0+\ , \quad \text{for all} \quad x \in X\ .$$

There are other special conditions imposed, that lead to various classes of semigroups, see the comprehensive treatise of Hille and Phillips [HP57]. For the present purposes it will suffice to consider semigroups (and groups) of *contractions*; they are the ones who in addition satisfy

$$\text{(c)} \quad \|G(t)\| \le 1 \ \text{ for all } \ t\,.$$

Lemma 14.1. *When $G(t)$ satisfies* (a)–(c)*, the map $t \mapsto G(t)x$ is for any $x \in X$ a continuous function from $\overline{\mathbb{R}}_+$ to X (resp. from $\mathbb{R}$ to X in case of a group).*

Proof. By (a) and (c) we have for $t_1 \le t_2$ that

$$\|G(t_2)x - G(t_1)x\| = \|G(t_1)(G(t_2 - t_1)x - x)\| \le \|G(t_2 - t_1)x - x\|\,.$$

If we let t_1 and t_2 converge to $t_0 \in [t_1, t_2]$, the expression goes to 0 according to (b). $\square$

The property in Lemma 14.1 is called strong continuity. We can hereafter call the (semi)groups which satisfy (a), (b) and (c) the *strongly continuous contraction (semi)groups.* (For strongly continuous semigroups in general one requires (a) and strong continuity, then (b) follows.)

The *infinitesimal generator B* is now introduced as the operator defined by

$$Bx = \lim_{h\to 0} \frac{1}{h}(G(h) - I)x\,, \tag{14.11}$$

with $D(B)$ consisting of those x for which the limit exists.

The following theorem shows that the vector function $u(t) = G(t)x$ satisfies the differential equation (14.6) when $x \in D(B)$.

Theorem 14.2. *For $x \in D(B)$, the function $G(t)x : \overline{\mathbb{R}}_+ \to X$ is differentiable, and takes its values in $D(B)$:*

$$\lim_{h\to 0} \frac{1}{h}(G(t+h)x - G(t)x) = G(t)Bx = BG(t)x \ \textit{ for all } \ t \ge 0\,. \tag{14.15}$$

Proof. When $h > 0$,

$$\frac{1}{h}(G(t+h)x - G(t)x) = G(t)\frac{G(h) - I}{h}x = \frac{G(h) - I}{h}G(t)x\,.$$

When $x \in D(B)$, the expression in the middle converges to $G(t)Bx$ for $h \to 0+$. This shows that $G(t)D(B) \subset D(B)$, and that (14.15) holds, when

$h \to 0$ is replaced by $h \to 0+$. If $t > 0$, we must also investigate the passage to the limit through negative h; here we use that

$$\frac{1}{h}(G(t+h)x - G(t)x) - G(t)Bx = G(t+h)\frac{G(-h)-I}{-h}x - G(t)Bx$$
$$= G(t+h)\left(\frac{G(-h)-I}{-h}x - Bx\right) + (G(t+h) - G(t))Bx\ .$$

For a given ε we first choose $\delta > 0$ such that $\|\frac{1}{-h}(G(-h) - I)x - Bx\| < \varepsilon$ for $-\delta \leq h < 0$. Then the first term is $< \varepsilon$ because of (c); next we can choose $0 < \delta' \leq \min\{\delta, t\}$ so that the second term is $< \varepsilon$ for $-\delta' \leq h < 0$, in view of Lemma 14.1. $\square$

Corollary 14.3. *For $x \in D(B)$,*

$$G(t)x - x = \int_0^t G(s)Bx\,ds\ . \tag{14.16}$$

This follows from the general property (14.14). We can also show:

Lemma 14.4. *For all $x \in X$, $t > 0$, $\int_0^t G(s)x\,ds$ belongs to $D(B)$ and*

$$G(t)x - x = B\int_0^t G(s)x\,ds\ . \tag{14.17}$$

Proof. It follows from the continuity and the semigroup property (a) that for $h > 0$:

$$\frac{G(h)-I}{h}\int_0^t G(s)x\,dx = \frac{1}{h}\int_0^t (G(s+h) - G(s))x\,ds$$
$$= \frac{1}{h}\int_h^{t+h} G(s)x\,ds - \frac{1}{h}\int_0^t G(s)x\,ds$$
$$= \frac{1}{h}\int_t^{t+h} G(s)x\,ds - \frac{1}{h}\int_0^h G(s)x\,ds,$$

which converges to $G(t)x - x$ for $h \to 0$, by (14.13). $\square$

Lemma 14.5. *B is closed and densely defined.*

Proof. According to Lemma 14.4, $\frac{1}{h}\int_0^h G(s)x\,ds \in D(B)$ for all $x \in X$, $h > 0$, so since this goes to x for $h \to 0$, $D(B)$ is dense in X. Now if $x_n \in D(B)$

with $x_n \to x$ and $Bx_n \to y$, then $G(s)Bx_n \to G(s)y$ *uniformly in* s (by (c)), such that we have for any $h > 0$ that

$$\begin{aligned} \frac{1}{h}(G(h)x - x) &= \lim_{n\to\infty} \frac{1}{h}(G(h)x_n - x_n) \\ &= \lim_{n\to\infty} \frac{1}{h}\int_0^h G(s)Bx_n ds = \frac{1}{h}\int_0^h G(s)y\, ds \ , \end{aligned}$$

by Corollary 14.13. However, $\frac{1}{h}\int_0^h G(s)y\, ds \to y$ for $h \to \infty$, from which we conclude that $x \in D(B)$ with $Bx = y$. □

Lemma 14.6. *A contraction semigroup is uniquely determined from its infinitesimal generator.*

Proof. Assume that $G_1(t)$ and $G_2(t)$ have the same infinitesimal generator B. For $x \in D(B)$, $G_2(t)x \in D(B)$, and we have by a generalization of the Leibniz formula:

$$\frac{d}{ds}G_1(t-s)G_2(s)x = -G_1(t-s)BG_2(s)x + G_1(t-s)G_2(s)Bx = 0 \ .$$

Integration over intervals $[0, t]$ gives

$$G_1(0)G_2(t)x - G_1(t)G_2(0)x = 0 \ ,$$

hence $G_1(t)x = G_2(t)x$ for $x \in D(B)$. Since $D(B)$ is dense in X, and these operators are bounded, we conclude that $G_1(t) = G_2(t)$. □

We can now show an important property of B, namely that the halfplane $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$ lies in the resolvent set. Moreover, the resolvent can be obtained directly from the semigroup, and it satisfies a convenient norm estimate.

Theorem 14.7. *Let $G(t)$ be a strongly continuous contraction semigroup with infinitesimal generator B. Any λ with $\operatorname{Re}\lambda > 0$ belongs to the resolvent set $\rho(B)$, and then*

$$(B - \lambda I)^{-1}x = -\int_0^\infty e^{-\lambda t}G(t)x\, dt \quad \text{for } x \in X \ , \tag{14.18}$$

with

$$\|(B - \lambda I)^{-1}\| \le (\operatorname{Re}\lambda)^{-1} \ . \tag{14.19}$$

Proof. Let $\operatorname{Re}\lambda > 0$. Note first that $e^{-\lambda t}G(t)$ is a strongly continuous contraction semigroup with infinitesimal generator $B - \lambda I$. An application of Corollary 14.3 and Lemma 14.4 to this semigroup gives that

$$e^{-\lambda s}G(s)x - x = (B - \lambda I)\int_0^s e^{-\lambda t}G(t)x\,dt \quad \text{for } x \in X\ , \tag{14.20}$$

$$e^{-\lambda s}G(s)x - x = \int_0^s e^{-\lambda t}G(t)(B - \lambda I)x\,dt \quad \text{for } x \in D(B)\ . \tag{14.21}$$

For any $y \in X$ we have that $\|e^{-\lambda t}G(t)y\| \le e^{-\operatorname{Re}\lambda t}\|y\|$; therefore the limit

$$Ty = \lim_{s\to\infty}\int_0^s e^{-\lambda t}G(t)y\,dt = \int_0^\infty e^{-\lambda t}G(t)y\,dt\ ,$$

exists, and T is clearly a linear operator on X with norm

$$\|T\| \le \int_0^\infty e^{-\operatorname{Re}\lambda t}\,dt = (\operatorname{Re}\lambda)^{-1}.$$

In particular, $e^{-\lambda s}G(s)x \to 0$ for $s \to \infty$. Then (14.20) and (14.21) imply after a passage to the limit:

$$\begin{aligned} -x &= (B - \lambda I)Tx \quad &&\text{for } x \in X \\ -x &= T(B - \lambda I)x \quad &&\text{for } x \in D(B)\ , \end{aligned}$$

which shows that λ is in the resolvent set, with resolvent equal to $-T$. □

We have found some properties of the infinitesimal generator B for a contraction semigroup $G(t)$, and now turn to the question (of interest for applications) of how the operators look that *can* be infinitesimal generators of a contraction semigroup. The question was answered by Hille and Yoshida (around 1945, in non-communicating ends of the world) by different proofs of the following theorem.

Theorem 14.8. (Hille-Yoshida) *When B is a densely defined, closed operator in X with $\mathbb{R}_+$ contained in $\rho(B)$, and*

$$\|(B - \lambda I)^{-1}\| \le \lambda^{-1} \quad \textit{for } \lambda \in \mathbb{R}_+\ , \tag{14.22}$$

then B is the infinitesimal generator of a strongly continuous contraction semigroup.

Proof. The operators

$$B_\lambda = -\lambda^2(B-\lambda I)^{-1} - \lambda I,$$

defined for each $\lambda > 0$, are bounded with norm $\le 2\lambda$, and we can form the operator families

$$G_\lambda(t) = \exp(tB_\lambda) \text{ for } t \in \mathbb{R} ,$$

by (14.9) ff.; they are is continuous in t with respect to the operator norm. Now observe that for $x \in D(B)$,

$$\lambda(B-\lambda I)^{-1}x + x = (B-\lambda I)^{-1}(\lambda x + (B-\lambda I)x) = (B-\lambda I)^{-1}Bx ,$$

so that (14.22) implies that when $x \in D(B)$,

$$-\lambda(B-\lambda I)^{-1}x \to x \text{ for } \lambda \to \infty . \tag{14.23}$$

Since $\| - \lambda(B-\lambda I)^{-1}\| \le 1$ for all $\lambda > 0$ and $D(B)$ is dense in X, (14.23) extends to all $x \in X$. We then get furthermore:

$$B_\lambda x = -\lambda(B-\lambda)^{-1}Bx \to Bx \text{ for } x \in D(B) ,\ \lambda \to \infty , \tag{14.24}$$

and we want to show that $G_\lambda(t)$ converges strongly, for each t, towards a semigroup $G(t)$ with B as generator, for $\lambda \to \infty$.

To do this, note first that an application of the product formula, as in (14.9), gives that $G_\lambda(t) = \exp(-\lambda^2(B-\lambda)^{-1}t)\exp(-\lambda t)$, where

$$\|\exp(-\lambda^2(B-\lambda)^{-1}t)\| \le \sum_{n=0}^{\infty} \frac{\| - \lambda^2(B-\lambda)^{-1}t\|^n}{n!} \le \exp(\lambda t) \text{ by (14.22)} ,$$

so that

$$\|G_\lambda(t)\| \le \exp(-\lambda t)\exp \lambda t = 1 \tag{14.25}$$

for all $t \ge 0$ and $\lambda > 0$. Since all the bounded operators B_λ, B_μ, $G_\lambda(t)$, $G_\mu(s)$ for λ and $\mu > 0$, s and $t \ge 0$, commute, we find that

$$\begin{aligned} G_\lambda(t) - G_\mu(t) &= \int_0^t \frac{d}{ds}[G_\lambda(s)G_\mu(t-s)]\, ds \\ &= \int_0^t G_\lambda(s)G_\mu(t-s)(B_\lambda - B_\mu)\, ds \end{aligned}$$

which implies, by (14.25), that

$$\|G_\lambda(t)x - G_\mu(t)x\| \le t\|B_\lambda x - B_\mu x\| \text{ for } x \in X . \tag{14.26}$$

When $x \in D(B)$, we know that $B_\mu x \to Bx$ for $\mu \to \infty$ according to (14.24); then we see in particular that the sequence $\{G_n(t)x\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Denoting the limit by $G(t)x$, we obtain that

$$\|G_\lambda(t)x - G(t)x\| \le t\|B_\lambda x - Bx\| \text{ for } x \in D(B) . \tag{14.27}$$

This shows that when $x \in D(B)$, then $G_\lambda(t)x \to G(t)x$ for $\lambda \to \infty$, *uniformly for* t *in bounded intervals* $[0, a] \subset \mathbb{R}_+$. Since we also have that $\|G_\lambda(t)x\| \le \|x\|$ for all t and λ (cf. (14.25)), it follows that $\|G(t)x\| \le \|x\|$, so that the operator $x \mapsto G(t)x$ defined for $x \in D(B)$ extends by closure to an operator $G(t) \in \mathbf{B}(X)$ with norm $\|G(t)\| \le 1$. For the extended operator we now also find that

$$G_\lambda(t)x \to G(t)x \text{ for each } x \in X ,$$

uniformly for t in bounded intervals $[0, a]$, for if we let $x_k \in D(B)$, $x_k \to x$, we have when $t \in [0, a]$

$$\begin{aligned}
&\|G_\lambda(t)x - G(t)x\| \\
&\qquad \le \|G_\lambda(t)(x - x_k)\| + \|G_\lambda(t)x_k - G(t)x_k\| + \|G(t)(x - x_k)\| \\
&\qquad \le 2\|x - x_k\| + a\|B_\lambda x_k - Bx_k\| ,
\end{aligned}$$

where the last expression is seen to go to 0 for $\lambda \to \infty$, independently of t, by choosing first x_k close to x and then adapting λ.

The semigroup property (a) carries over to $G(t)$ from the $G_\lambda(t)$'s. That $G(t)$ satisfies (b) is seen from

$$\|G(t)x - x\| \le \|G(t)x - G_\lambda(t)x\| + \|G_\lambda(t)x - x\| ,$$

where one first chooses λ so large that $\|G(t)x - G_\lambda(t)x\| < \varepsilon$ for $t \in [0, 1]$ and next lets $t \to 0$.

The semigroup $G(t)$ now has an infinitesimal generator C, and it remains to show that $C = B$. For $x \in D(B)$ we have:

$$\begin{aligned}
\|G_\lambda(s)B_\lambda x - G(s)Bx\| &\le \|G_\lambda(s)\|\|B_\lambda x - Bx\| + \|(G_\lambda(s) - G(s))Bx\| \\
&\le \|B_\lambda x - Bx\| + \|(G_\lambda(s) - G(s))Bx\| \\
&\to 0 \text{ for } \lambda \to \infty,
\end{aligned}$$

uniformly for s in a bounded interval, by (14.24) and the proved convergence of $G_\lambda(s)y$ at each $y \in X$. This gives by use of Corollary 14.3 and (14.26), that for $x \in D(B)$,

$$\begin{aligned}
\frac{1}{h}(G(h)x - x) &= \lim_{\lambda\to\infty} \frac{1}{h}(G_\lambda(h)x - x) = \lim_{\lambda\to\infty} \frac{1}{h}\int_0^h G_\lambda(s)B_\lambda x\, ds \\
&= \frac{1}{h}\int_0^h G(s)Bx\, dx .
\end{aligned}$$

If we here let $h \to 0$, the last expression will convergere to $G(0)Bx = Bx$, from which we conclude that at $x \in D(C)$ with $Cx = Bx$. Since $B - 1$ and $C - 1$ are bijections of $D(B)$ and $D(C)$, respectively, onto X, B must equal C. $\square$

The operator family $G(t)$ defined from B in this way is also called $\exp(tB)$.

The theory can be extended to semigroups which do not necessarily consist of contractions. For one thing, there is the obvious generalization where we to the operator $B + \mu I$ associate the semigroup

$$\exp(t(B + \mu I)) = \exp(t\mu)\exp(tB) \; ; \tag{14.28}$$

this only gives contractions when $\operatorname{Re}\mu \le 0$. Here $\|\exp(t(B + \mu I))\| \le |\exp(t\mu)| = \exp(t\operatorname{Re}\mu)$. More general strongly continuous semigroups can only be expected to satisfy inequalities of the type $\|G(t)\| \le c_1 \exp tc_2$ with $c_1 \ge 1$, and need a more general theory — see e.g. the books of E. Hille and R. Phillips [HP57] and of N. Dunford and J. Schwartz [DS57].

14.3. Contraction semigroups in Hilbert space.

We now consider the case where X is a Hilbert space H.

Lemma 14.9. *When $G(t)$ is a semigroup in H satisfying* (a), (b) *and* (c), *then its infinitesimal generator B is upper semibounded, with upper bound ≤ 0.*

Proof. For $x \in X$ one has that

$$\begin{aligned}\operatorname{Re} \tfrac{1}{h}(G(h)x - x, x) &= \tfrac{1}{h}\big(\operatorname{Re}(G(h)x, x) - \|x\|^2\big)\\ &\le \tfrac{1}{h}\big(\|G(h)x\|\,\|x\| - \|x\|^2\big) \le 0\end{aligned}$$

according to (c), from which we conclude for $x \in D(B)$ by a passage to the limit:

$$\operatorname{Re}(Bx, x) \le 0 \quad \text{for } x \in D(B) \, . \tag{14.29}$$

This show the lemma. $\square$

In the Hilbert space case we can also consider the family of adjoint operators $G^*(t)$.

Theorem 14.10. *When $G(t)$ is a semigroup in H satisfying* (a), (b) *and* (c), *then $G^*(t)$ is likewise a semigroup in H satisfying* (a), (b) *and* (c); *and when the generator for $G(t)$ is B, then the generator for $G^*(t)$ is precisely B^*.*

Proof. It is seen immediately that $G^*(t)$ satisfies (a) and (c). For (b) we observe that one for x and $y \in H$ has:

$$(G^*(t)x, y) = (x, G(t)y) \to (x, y) \quad \text{for } t \to 0 \, .$$

This implies that

$$\begin{aligned} 0 \le \|G^*(t)x - x\|^2 &= (G^*(t)x, G^*(t)x) + \|x\|^2 - (G^*(t)x, x) - (x, G^*(t)x) \\ &\le \|x\|^2 - (G^*(t)x, x) + \|x\|^2 - (x, G^*(t)x) \\ &\to 0 \ \text{ for } t \to 0 \,, \end{aligned}$$

from which we conclude that $G^*(t)x - x \to 0$ for $t \to 0$.

Now let C be the infinitesimal generator of $G^*(t)$, and let $x \in D(B)$, $y \in D(C)$. Then

$$(Bx, y) = \lim_{h\to 0} \big(\tfrac{1}{h}(G(h)x - x), y\big) = \lim_{h\to 0} \big(x, \tfrac{1}{h}(G^*(h)y - y)\big) = (x, Cy) \;;$$

thus $C \subset B^*$. By Theorem 14.7, $R(C - I) = H$; and $B^* - I$ is injective since $Z(B^* - I) = R(B - I)^\perp = \{0\}$. Then $C \subset B^*$ cannot hold unless $C = B^*$. $\square$

Corollary 14.11. *An operator B in a Hilbert space H is the infinitesimal generator of a strongly continuous contraction semigroup if and only if B is densely defined and closed, satisfies* (14.29)*, and has $\mathbb{R}_+$ contained in its resolvent set.*

Proof. The necessity of the conditions follows from what we have just shown, together with Lemma 14.5 and Theorem 14.7. The sufficiency is seen from the fact that (14.29) implies that $-(B - \lambda I) = -B + \lambda I$ for $\lambda > 0$ has a bounded inverse with norm $\le \lambda^{-1}$, by Theorem 12.9.1°; then the Hille-Yoshida Theorem (Theorem 14.8) can be applied. $\square$

Operators satisfying (14.29) are in part of the literature called *dissipative operators*. There is a variant of the above theorems:

Corollary 14.12. *Let B be a closed, densely defined operator in a Hilbert space H. Then the following properties are equivalent:*

(i) *B is the infinitesimal generator of a strongly continuous contraction semigroup.*
(ii) *B is dissipative (i.e., $m(-B) \ge 0$) and $\mathbb{R}_+ \subset \rho(B)$.*
(iii) *B and B^* are dissipative.*

Proof. The equivalence of (i) and (ii) is shown above. Condition (i) implies (iii) by Theorem 14.10, and (iii) implies (ii) by Theorem 12.9.3°. $\square$

Remark 14.13. It is shown in [P59] that the conditions (i)–(iii) in Corollary 14.12 are equivalent with

(iv) B is maximal dissipative.

Let us here observe that (ii) easily implies (iv), for if B satisfies (ii) and B' is a dissipative extension, then $B' - I$ is injective (by Theorem 12.9.1° applied to $-(B'-I)$); hence, since $B-I$ is already bijective from its domain to H, B' must equal B. For the direction from (iv) to (ii), [P59] carries the problem over to $J = (I+B)(I-B)^{-1}$; it is a contraction with $D(J) = R(I-B)$ and $D(B) = R(I+J)$. Here B is maximal dissipative if and only if J is maximal with respect to being a contraction. Since we take B closed ([P59] considers more general operators), $D(J) = R(I-B)$ is closed (by Theorem 12.9.2°); then there exists a proper contraction extension of J unless $D(J) = H$. We see that maximality of B implies surjectiveness of $I - B$, and similarly of $I - \frac{1}{\lambda}B$ for all $\lambda > 0$, assuring (ii).

So, the infinitesimal generators of contraction semigroups are the maximal dissipative operators, as described by (ii), (iii) or (iv). Note that these operators have

$$\nu(B),\, \nu(B^*),\, \sigma(B),\, \sigma(B^*) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le 0\}. \tag{14.30}$$

We now consider some special cases:

1° $B = -A$, where A is selfadjoint ≥ 0. Here A and A^* are ≥ 0, so that B and B^* are dissipative.

2° $B = -A$, where A is a variational operator with $m(A) \ge 0$ (Section 12.4). Here $m(A^*)$ is likewise ≥ 0, so B and B^* are dissipative. This case is more general than 1°, but less general than the full set of operators satisfying (14.30), for, as we recall from (12.48) (applied to $-B$), we here have $\nu(B)$, $\nu(B^*)$, $\sigma(B)$ and $\sigma(B^*)$ contained in an angular set

$$\widetilde{M} = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le 0,\, |\operatorname{Im}\lambda| \le c(-\operatorname{Re}\lambda + k)\},$$

for some $c \ge 0$, $k \in \mathbb{R}$. The semigroups generated by such operators belong to the so-called *holomorphic semigroups* (where $G(t)x$ extends holomorphically to t in a sector around $\mathbb{R}_+$); they have particularly convenient properties, for example that $G(t)x \in D(B)$ for $t \ne 0$, any $x \in X$. (Besides [HP57], they enter e.g. in Kato [K66], Friedman [F69] and many other works.)

3° $B = iA$ where A is selfadjoint, i.e., $B = -B^*$, B is *skew-selfadjoint*. Here $\nu(B)$ and $\nu(B^*)$ are contained in the imaginary axis, so B and B^* are dissipative.

In the last case we can introduce

$$\begin{aligned} U(t) &= \exp(tB) && \text{for } t \ge 0\,, \\ U(t) &= \exp(-tB^*) = U(t)^* && \text{for } t \le 0\,, \end{aligned}$$

writing $U(t) = \exp(tB)$ also for $t \le 0$. We shall now show that $U(t)$ is a *strongly continuous group of unitary operators.*

That $U(t)x$ is continuous from $t \in \mathbb{R}$ into H follows from the continuity for $t \ge 0$ and $t \le 0$. Next, we can show that $U(t)$ is an isometry for each $t \in \mathbb{R}_+$ or $t \in \mathbb{R}_-$, since we have for $x \in D(B)$:

$$\begin{aligned}\frac{d}{dt}\|U(t)x\|^2 &= \lim_{h\to 0}\frac{1}{h}\big[(U(t+h)x, U(t+h)x) - (U(t)x, U(t)x)\big]\\ &= \lim_{h\to 0}\Big(\frac{U(t+h)x - U(t)x}{h}, U(t)x\Big) + \lim_{h\to 0}\Big(U(t)x, \frac{U(t+h)x - U(t)x}{h}\Big)\\ &\quad + \lim_{h\to 0}\Big(\frac{U(t+h)x - U(t)x}{h}, U(t+h)x - U(t)x\Big)\\ &= (BU(t)x, U(t)x) + (U(t)x, BU(t)x) = 0\ ;\end{aligned}$$

where it was used after the second equality sign that there for any $\varepsilon > 0$ exists h_0 such that $\|U(t+h)x - U(t)x\| < \varepsilon$ for $|h| \le h_0$; then we could let $h \to 0$. Hence $\|U(t)x\|^2$ is constant in t and thus equal to $\|U(0)x\|^2 = \|x\|^2$ for all t; the identity extends by continuity to $x \in H$. In a similar way it is seen that for $x \in D(B)$, $t \in \mathbb{R}_+$ or $t \in \mathbb{R}_-$,

$$\frac{d}{dt}U(t)U(-t)x = U(t)BU(-t)x + U(t)(-B)U(-t)x = 0\ ,$$

so that $U(t)U(-t)x$ is constant for $t \ge 0$ and for $t \le 0$, and hence equal to $U(0)U(0)x = x$. The identity

$$U(t)U(-t)x = x \ \text{ for } t \in \mathbb{R}$$

extends by continuity to all $x \in H$, and shows that

$$U(t)^{-1} = U(-t) \ \text{ for } t \in \mathbb{R}\ ;$$

hence U is unitary. It also implies the group property, since one has e.g. for $s \ge 0$, $0 \ge t \ge -s$:

$$U(s)U(t) = U(s+t)U(-t)U(t) = U(s+t)\ .$$

Let now conversely $U(t)$ be a strongly continuous group of contractions. If $\{U(t)\}_{t\ge 0}$ and $\{U(-t)\}_{t\ge 0}$ have the infinitesimal generators B resp. C, then for $x \in D(B)$,

$$\lim_{h\to 0+}\frac{U(-h) - I}{h}x = \lim_{h\to 0+} U(-h)\frac{I - U(h)}{h}x = Bx\ ,$$

i.e., $-B \subset C$. Similarly, $-C \subset B$, so $B = -C$. For B then holds that $m(-B) = m(B) = 0$, and both $\{\lambda \mid \operatorname{Re}\lambda > 0\}$ and $\{\lambda \mid \operatorname{Re}\lambda < 0\}$ are contained in the resolvent set. This shows that B is skew-selfadjoint, by Corollary 14.12 (or by Theorem 12.10 applied to iB). It is now seen from the preceding analysis that the operators $U(t)$ are unitary.

We have hereby obtained the theorem of M. H. Stone:

Theorem 14.14 (Stone). *An operator B in H is the infinitesimal generator of a strongly continuous group of unitary operators if and only if B is skew-selfadjoint.*

14.4 Applications.

When B is the infinitesimal generator of a strongly continuous contraction semigroup (or group) $G(t)$ in a Banach space or Hilbert space X, the vector function

$$u(t) = G(t)u_0 \tag{14.31}$$

is, according to Theorem 14.2, a solution of the abstract Cauchy problem (initial value problem)

$$\begin{cases} u'(t) = Bu(t)\ , \quad t > 0\ (t \in \mathbb{R})\ , \\ u(0) = u_0\ , \end{cases} \tag{14.32}$$

for any given initial value $u_0 \in D(B)$. (One can furthermore show that $G(t)u_0$ is the only continuously differentiable solution of (14.32).) If $G(t)$ is a holomorphic semigroup, (14.31) solves (14.32) even when $u_0 \in X$.

The semigroup theory can in this way be used to get solutions of the problem (14.32) for various types of operators B.

We have a wealth of examples:

First, we have in Chapter 12 defined the selfadjoint, the semibounded and the variational operators entering in 1°, 2° and 3° above.

Next, we have in Chapter 13 described operators of these types that are given to act in a particular way, namely belonging to the set of closed operators lying between A_0 and A_1 with given properties.

Concrete interpretations to particular differential operators are given in Chapter 4, most prominently for the Laplace operator Δ on bounded sets, but also including Δ on $\mathbb{R}^n$, some variable-coefficient cases, and second-order ordinary differential equations on intervals (considered in some exercises). Here $X = L_2(\mathbb{R}^n)$ or $L_2(\Omega)$ for an open set $\Omega \subset \mathbb{R}^n$.

For example, if $B = -A_\gamma =$ the Dirichlet realization of the Laplace operator (called $-T$ in Theorem 4.27), the problem (14.32) becomes:

$$\begin{aligned} \partial_t u(x,t) &= \Delta_x u(x,t) \text{ for } x \in \Omega,\ t > 0, \\ \gamma_0 u(x,t) &= 0 \text{ for } t > 0, \\ u(x,0) &= u_0(x) \text{ for } x \in \Omega, \end{aligned}$$

and the function $e^{-A_\gamma}u_0$ solves the heat equation with Dirichlet boundary condition.

An interpretation of the general study (from Chapter 13) of closed extensions of a minimal elliptic realization is given in Chapter 9, with details for an accessible example and introductory remarks on the general case. The analysis of lower bounded operators in Chapter 13 is particular suited for application to evolution problems using semigroup theory.

We can moreover get solutions of the Schrödinger equation with an initial condition, as mentioned in the beginning of the present chapter, when $i\Delta$ is concretized as a skew-selfadjoint operator. Also the wave equation can be studied using (14.7); here the matrix is taken to act e.g. in $H_0^1(\Omega) \times L_2(\Omega)$ (as defined in Chapter 4).

The solutions defined in this way are of course somewhat abstract and need further investigation, and one can show much more precisely which spaces the solutions belong to, and discuss their uniqueness and other properties. Further questions can be asked in a framework of functional analysis (as in scattering theory). Parabolic problems generalizing the heat equation have been widely studied, recently also by refined methods using pseudodifferential techniques (as in [G95]). For the constructive analysis of solutions of hyperbolic equations, generalizing the wave equation, the most modern tools come from microlocal analysis (based on refined Fourier analysis). There are many interesting works on this; here we shall just point to the four volumes of L. Hörmander (cf. [H83], [H85]), which form a cornerstone in this development.

Exercises for Chapter 14.

14.1. Show that if S is a densely defined and maximal symmetric operator in a Hilbert space H, then either iS or $-iS$ is the infinitesimal generator of a strongly continuous contraction semigroup.

14.2. Let B be a closed, densely defined operator in a Hilbert space H, and assume that there is a constant $\alpha \geq 0$ so that $m(B)$, $m(-B)$, $m(B^*)$ and $m(-B^*)$ are $\geq -\alpha$.
Let $G(t)$ be the family of operators defined by

$$\begin{aligned} G(t) &= e^{\alpha t} \exp(t(B - \alpha I)) && \text{for } t \geq 0 \\ G(t) &= e^{-\alpha t} \exp(-t(-B - \alpha I)) && \text{for } t \leq 0 \,. \end{aligned}$$

Show that $G(t)$ is a strongly continuous group, satisfying $\|G(t)\| \leq \exp(\alpha|t|)$ for all t.

14.3. Let $G(t)$ be a strongly continuous contraction semigroup on a Banach

space X, and let $f(t)$ be a continuous functions of $t \in \overline{\mathbb{R}}_+$, valued in X. Show that $G(t)f(t)$ is a continuous function of $t \in \overline{\mathbb{R}}_+$.
(*Hint.* To $G(t)f(t) - G(t_0)f(t_0)$ one can add and subtract $G(t)f(t_0)$ and use that $\|G(t)\| \le 1$ for all t.)

14.4. Consider the problem for functions $u(t)$ taking values in a Banach space X:

$$\begin{aligned} u'(t) - Bu(t) &= f(t), \text{ for } t > 0, \\ u(0) &= 0. \end{aligned} \tag{2}$$

It is assumed that B is the infinitesimal generator of a strongly continuous contraction semigroup $G(t)$.

Let $T > 0$. Show that if $f \in C^1([0,T], X)$, then the function

$$u(t) = \int_0^t G(t-s)f(s)\,ds \tag{3}$$

is a solution of (2), with $u \in C^1([0,T], X) \cap C^0([0,T], D(B))$.
(*Hints:* The formula (3) can also be written

$$u(t) = \int_0^t G(s')f(t-s')\,ds',$$

from which it can be deduced that $u \in C^1([0,T], X)$. To verify (2), let $h > 0$ and write

$$\begin{aligned} \frac{u(t+h) - u(t)}{h} &= \frac{1}{h}\int_t^{t+h} G(t+h-s)f(s)\,ds \\ &\quad + \frac{G(h) - I}{h}\int_0^t G(t-s)f(s)\,ds = I_1 + I_2. \end{aligned}$$

Show that $I_1 \to f(t)$ for $h \to 0$. Use the differentiability of u to show that $u(t) \in D(B)$ and $I_2 \to Bu(t)$.)

Give an example of an application to a PDE problem, where B is a realization of an elliptic operator.

(*Comment.* The use of the semigroup in formula (3) is sometimes called the Duhamel principle.)