§8. Pseudodifferential operators on manifolds, index of elliptic operators

8.1 Coordinate changes.

In order to define ψdo's on manifolds, we shall investigate how they behave under coordinate changes. Let Ω and $\underline{\Omega}$ be open subsets of $\mathbb{R}^n$ together with a diffeomorphism κ of Ω onto $\underline{\Omega}$. When P is a ψdo defined on Ω, we define $\underline{P}$ on $\underline{\Omega}$ by

$$\underline{P}u = P(u \circ \kappa) \circ \kappa^{-1}, \text{ when } u \in C_0^\infty(\underline{\Omega}), \tag{8.1}$$

the definition extends to larger spaces as explained earlier.

Theorem 8.1.

1° *Let $P = \operatorname{Op}(q(x,y,\xi))$ be a ψdo with $q(x,y,\xi) \in S^m_{1,0}(\Omega \times \Omega, \mathbb{R}^n)$ and let K be a compact subset of Ω. With κ' denoting the Jacobian matrix $(\partial\kappa_i/\partial x_j)$, let $M(x,y)$ be the matrix defined by (8.6) below; it satisfies*

$$\underline{x} - \underline{y} = M(x,y)(x-y), \tag{8.2}$$

and is invertible on $U_\varepsilon = \{(x,y) \mid x, y \in K, |x-y| < \varepsilon\}$ for a sufficiently small $\varepsilon > 0$, with M, M^{-1} and their derivatives bounded there. In particular, $M(x,x) = \kappa'(x)$.

If $q(x,y,\xi)$ vanishes for $(x,y) \notin U_\varepsilon$, then $\underline{P}$ is a ψdo with a symbol $\underline{q}(\underline{x},\underline{y},\underline{\xi}) \in S^m_{1,0}(\underline{\Omega} \times \underline{\Omega}, \mathbb{R}^n)$ (vanishing for $(\underline{x},\underline{y})$ outside the image of U_ε; it satisfies, with $\underline{x} = \kappa(x)$, $\underline{y} = \kappa(y)$,

$$\underline{q}(\underline{x},\underline{y},\underline{\xi}) = q(x,y,{}^tM(x,y)\underline{\xi}) \cdot |\det {}^tM(x,y)| \ |\det \kappa'(y)^{-1}|. \tag{8.3}$$

In particular, $\underline{q}(\underline{x},\underline{x},\underline{\xi}) = q(x,x,{}^t\kappa'(x)\underline{\xi})$. If q is polyhomogeneous, then so is $\underline{q}$.

If $q(x,y,\xi)$ vanishes for $(x,y) \notin K \times K$, let $\varrho(x,y) = \chi(|x-y|/r)$, then $\underline{P}$ is, for sufficiently small r, the sum of an operator with symbol (8.3) *multiplied by $\chi(|x-y|/r)$ and a smoothing operator.*

For general $q(x,y,\xi)$, let $(\varphi_j)_{j\in\mathbb{N}_0}$ be a locally finite partition of unity on Ω and write $q(x,y,\xi) = \sum_{j,k} \varphi_j(x) q(x,y,\xi) \varphi_k(y)$, so that $P = \sum_{j,k} \varphi_j P \varphi_k$; then $\underline{P} = \sum_{j,k} \underline{\varphi}_j \underline{P} \underline{\varphi}_k$ with $\underline{\varphi}(\kappa(x)) = \varphi(x)$. The terms where $\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_k \neq \emptyset$ *are treated as above and the others define a negligible operator that transforms to a negligible operator.*

2° *When $P = \operatorname{Op}(p(x,\xi))$, $\underline{P}$ is the sum of a ψdo in $\underline{x}$-form $\operatorname{Op}(p_\kappa(\underline{x},\underline{\xi}))$ and a negligible operator, where the symbol p_κ has the asymptotic expansion (Hörmander's formula)*

$$\begin{aligned} p_\kappa(\kappa(x),\underline{\xi}) &= e^{-i\kappa(x)\cdot\underline{\xi}} P(e^{i\kappa(x)\cdot\underline{\xi}}) \\ &\sim \textstyle\sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} D^\alpha_{\underline{\xi}} p(x, {}^t\kappa'(x)\underline{\xi}) \partial_y^\alpha e^{i\mu_x(y)\cdot\underline{\xi}}|_{y=x} \end{aligned} \tag{8.4}$$

in $S^m_{1,0}(\Omega,\mathbb{R}^n)$*; here* $\mu_x(y)=\kappa(y)-\kappa(x)-\kappa'(x)(y-x)$*, and each factor* $\varphi_\alpha(x,\underline{\xi})=\partial_y^\alpha e^{i\mu_x(y)\cdot\underline{\xi}}|_{y=x}$ *is in* $S^{|\alpha|/2}(\Omega,\mathbb{R}^n)$ *and is a polynomial in* $\underline{\xi}$ *of degree* $\le|\alpha|/2$*. In particular,* $\varphi_0=1$ *and* $\varphi_\alpha=0$ *for* $|\alpha|=1$*, and in the polyhomogeneous case,*

$$p^0_\kappa(\underline{x},\underline{\xi})=p^0(x,{}^t\kappa'(x)\underline{\xi}), \tag{8.5}$$

on the set where it is homogeneous.

Proof. By Taylor's formula applied to each κ_i, (8.2) holds with

$$M(x,y)=\int_0^1\kappa'(x+t(y-x))dt, \tag{8.6}$$

a smooth function of x and y where it is defined; the domain includes a neighborhood of the diagonal in $\Omega\times\Omega$. (Note that an M satisfying (8.2) is uniquely determined only if $n=1$.) In particular, $M(x,x)=\kappa'(x)$, hence is invertible. Since $M(x,y)=M(x,x)[I+M(x,x)^{-1}(M(x,y)-M(x,x))]$, it is seen by a Neumann series construction that $M(x,y)^{-1}$ exists and is bounded (with bounded derivatives) for $(x,y)\in U_\varepsilon$, for a sufficiently small $\varepsilon>0$.

If $q(x,y,\xi)$ vanishes for $(x,y)\notin U_\varepsilon$, we have for $u\in C_0^\infty(\underline{\Omega})$, setting $\xi={}^tM(x,y)\underline{\xi}$:

$$\begin{aligned}(\underline{P}u)(\kappa(x))&=\int e^{i(x-y)\cdot\xi}q(x,y,\xi)u(\kappa(y))\,dy đ\xi\\&=\int e^{i(x-y)\cdot\,{}^tM\underline{\xi}}\,q(x,y,\xi)u(\underline{y})|\det\kappa'(y)^{-1}|\;|\det{}^tM(x,y)|\,d\underline{y}đ\underline{\xi}\\&=\int e^{i(\underline{x}-\underline{y})\cdot\underline{\xi}}\,\underline{q}(\underline{x},\underline{y},\underline{\xi})u(\underline{y})\,d\underline{y}đ\underline{\xi},\end{aligned}$$

with $\underline{q}$ defined by (8.3). Clearly, $\underline{q}$ is a symbol in $S^d_{1,0}$ as asserted. When $x=y$, $\det\kappa'(y)^{-1}$ and $\det{}^tM(x,y)$ cancel out. The formula (8.3) shows moreover that polyhomogeneity is preserved.

If $q(x,y,\xi)$ vanishes for $(x,y)\notin K\times K$, we write

$$P=\mathcal{R}+P_1,$$

where $\mathcal{R}=\mathrm{Op}((1-\chi(|x-y|/r))q(x,y,\xi))$ is negligible by Lemma 7.4, and P_1 is as above; we can e.g. take $r=\varepsilon/2$. Since $\mathcal{R}$ is an integral operator with C^∞-kernel, so is the transformed operator $\underline{\mathcal{R}}$.

For the general q, one uses that the summation of the terms with $\operatorname{supp}\varphi_j\cap\operatorname{supp}\varphi_k\ne\emptyset$ is finite locally in (x,y).

If we now consider an operator given in x-form, one can find the x-form p_κ of the symbol of $\underline{P}$ by an application of Theorem 7.7 1°. The formula (8.5) follows easily from this.

As for (8.4), the first formula is the usual characterization, as in (7.26). The second formula is worked out in [H 1985, Th. 18.1.17], to which we refer for details. It was first proved in [H 1965]. $\square$

8.2 Operators on manifolds.

The definition of an n-dimensional C^∞ manifold X is explained e.g. in [H 1963, Sect. 1.8] and [H 1985 I, pp. 143–144]. Consider just compact manifolds, then a finite atlas $\{\kappa_j \colon U_j \to V_j \mid j = 1, \dots, J\}$ suffices to describe the structure. Here the U_j are open subsets of X such that $X = \bigcup_{j=1}^J U_j$, the V_j are open subsets of $\mathbb{R}^n$, and each κ_j is a homeomorphism of U_j onto V_j, such that when U_j and U_k are overlapping,

$$\kappa_j \kappa_k^{-1} : \kappa_k(U_j \cap U_k) \to \kappa_j(U_j \cap U_k) \tag{8.7}$$

is a diffeomorphism. For other systems $\{\tilde\kappa_j \colon \tilde U_j \to \tilde V_j \mid j = 1, \dots, \tilde J\}$, there should be similar relations with the κ_j's on overlaps. We define that a function u on X is C^∞, C^m or $L_{p,\mathrm{loc}}$, when the function $y \mapsto u(\kappa_j^{-1}(y))$ is so on V_j, for each j. Since X is compact, the $L_{p,\mathrm{loc}}$-functions are in fact in $L_p(X)$, which can be provided with a Banach space norm $(\sum_{j=1}^J \|(\varphi_j u) \circ \kappa_j^{-1}\|^p_{L_p(V_j)})^{\frac{1}{p}}$, defined with the help of a partition of unity as in Lemma 8.4 below.

We can also define the space of distributions $\mathcal{D}'(X)$ by use of local coordinates, when we recall the rule for coordinate changes in Definition 3.19, which, in the application to test functions, carries a functional determinant factor J along in order to make the rule consistent with that for continuous functions. Namely, when $\kappa \colon x \mapsto \underline{x}$ is a diffeomorphism from V to $\underline{V}$ in $\mathbb{R}^n$, $\underline{u} = u \circ \kappa^{-1}$ satisfies

$$\langle \underline{u}, \underline{\varphi} \rangle_{\underline{V}} = \langle u, J\varphi \rangle_V, \tag{8.8}$$

with $J(x) = |\det \kappa'(x)|$, $\underline{\varphi} = \varphi \circ \kappa^{-1}$. Distributions u on X are then defined such that $u \circ \kappa_j^{-1}$ for each j is a distribution on V_j, whose application to test functions follows the rule (8.8) under diffeomorphisms (applied e.g. to $\kappa = \kappa_j \kappa_k^{-1}$ going from $\kappa_k(U_j \cap U_k)$ to $\kappa_j(U_j \cap U_k)$). Full details are given in [H 1963].

A preferred scalar product and norm in $L_2(X)$ can be defined when one has chosen a smooth measure dx (compatible with Lebesgue measure in local coordinates), e.g. by providing the manifold with a Riemannian structure.

A more elegant presentation of distribution spaces over X is given in [H 1971] (and in [H 1985 I]), where the introduction of densities of order $\frac{1}{2}$

(essentially carrying $J^{\frac{1}{2}}$ along with the measure) makes the situation for distributions and test functions more symmetric under coordinate changes. We shall make do with the oldfashioned explanations given above.

Sobolev spaces $H^s(X)$ can be defined by use of local coordinates and a partition of unity as in Lemma 8.4 1°: $u \in H^s(X)$ when, for each j, $(\varphi_j u) \circ \kappa_j^{-1} \in H^s(\mathbb{R}^n)$ (here an extension by zero in $\mathbb{R}^n \setminus V_j$ is understood), and a Hilbert space norm on $H^s(X)$ can be defined by

$$\|u\|_s = \Big(\sum_{j=1}^{J} \|(\varphi_j u) \circ \kappa_j^{-1}\|_{H^s}^2\Big)^{\frac{1}{2}}. \tag{8.9}$$

This formula depends on many choices and is in no way "canonical", so $H^s(X)$ could be viewed as a "hilbertable" space rather than a Hilbert space (with an expression heard in a lecture by Seeley).

It is not hard to see that $C^\infty(X)$ is dense in $H^s(X)$ for all s. Indeed, when $u \in H^s(X)$, one can approximate each piece $\varphi_j u \circ \kappa_j^{-1}$ in H^s-norm by C_0^∞ functions $(v_{jk})_{k\in\mathbb{N}_0}$ on V_j; then $u_k(x) = \sum_j v_{jk}(\kappa_j(x))$ is an approximating sequence for u.

If a choice of dx on X has been made, one can, after fixing the norms on $H^s(X)$ for $s \geq 0$, choose the norms in the $H^{-s}(X)$ so that $H^{-s}(X)$ identifies with the dual space of $H^s(X)$ in such a way that the duality is consistent with the L_2-duality.

Theorem 8.2. (Rellich's Theorem).

The injection of $H^s(X)$ into $H^{s'}(X)$ is compact when $s > s'$.

This can be proved by 1) a reduction to compactly supported distributions in each coordinate patch by a partition of unity, 2) an imbedding of a compact subset of a coordinate patch into $\mathbb{T}^n$ (the n-dimensional torus), 3) a proof of the property for $\mathbb{T}^n$ by use of Fourier series expansions.

We shall carry this program out below. To begin with, let us explain the Sobolev spaces over the torus.

A basic result in the theory of Fourier series is that the system $\{e^{ik\cdot x} \mid k \in \mathbb{Z}^n\}$ is an orthonormal basis of $L_2(\mathbb{T}^n, đx)$; here $đx = (2\pi)^{-n}dx$, and $\mathbb{T}^n$ is identified with $Q = [-\pi, \pi]^n$ glued together at the edges (in other words, the functions on $\mathbb{T}^n$ identify with functions on $\mathbb{R}^n$ that are periodic with period 2π in each coordinate $x_1, \dots, x_n$). Then the mapping $f \mapsto (c_k(f))_{k\in\mathbb{Z}^n}$, $c_k(f) = (f, e^{ik\cdot x})$, defines an isometry of $L_2(\mathbb{T}^n, đx)$ onto $\ell_2(\mathbb{Z}^n)$.

There is an easy way to define distributions on the torus. We have the explicit bilinear form $\int_{\mathbb{T}^n} f(x)g(x)\, đx = \int_Q f(x)g(x)\, đx$. The test functions are the C^∞ functions on $\mathbb{T}^n$ (C^∞ functions on $\mathbb{R}^n$ with period 2π

in each coordinate $x_1, \dots, x_n$), and we can identify $\mathcal{D}'(\mathbb{T}^n)$ with the dual space, such that an L_2-function f identifies with the distribution acting like $\varphi \mapsto \int_{\mathbb{T}^n} f(x)\varphi(x)\,đx$.

For a function $f \in C^m(\mathbb{T}^n)$ having the Fourier series $\sum_{k\in\mathbb{Z}^n} c_k e^{ik\cdot x}$, an integration by parts shows that

$$D^\alpha f = \sum_{k\in\mathbb{Z}^n} k^\alpha c_k e^{ik\cdot x}.$$

For $u \in C^m(\mathbb{T}^n)$, the m'th Sobolev norm therefore satisfies, in view of (5.2),

$$\|u\|_m = \Big(\sum_{|\alpha|\le m} \|D^\alpha u\|_{L_2}^2\Big)^{\frac12} \begin{cases} \le \|\big(\langle k\rangle^m c_k(u)\big)_{k\in\mathbb{Z}^n}\|_{\ell_2}, \\ \ge c\|\big(\langle k\rangle^m c_k(u)\big)_{k\in\mathbb{Z}^n}\|_{\ell_2}. \end{cases}$$

Since $C^\infty(\mathbb{T}^n)$ and hence $C^m(\mathbb{T}^n)$ is dense in $H^m(\mathbb{T}^n)$, we conclude that the m'th Sobolev space ($m \in \mathbb{N}_0$) satisfies

$$\begin{aligned} H^m(\mathbb{T}^n) &= \{\, u \in L_2(\mathbb{T}^n) \mid (k^\alpha c_k(u))_{k\in\mathbb{Z}^n} \in \ell_2 \text{ for } |\alpha| \le m \,\} \\ &= \{\, u \in L_2(\mathbb{T}^n) \mid (\langle k\rangle^m c_k(u))_{k\in\mathbb{Z}^n} \in \ell_2 \,\}. \end{aligned}$$

Denote by $\ell_2^s(\mathbb{Z}^n)$ (or ℓ_2^s) the space of sequences $\underline{a} = (a_k)_{k\in\mathbb{Z}^n}$ for which $\sum_{k\in\mathbb{Z}^n} |\langle k\rangle^s a_k|^2 < \infty$. It is a Hilbert space with scalar product and norm

$$(\underline{a}, \underline{b})_{\ell_2^s} = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2s} a_k \bar{b}_k, \quad \|\underline{a}\|_{\ell_2^s} = (\sum_{k\in\mathbb{Z}^n} |\langle k\rangle^s a_k|^2)^{\frac12};$$

this follows immediately from the fact that multiplication $M_{\langle k\rangle^s}\colon (a_k) \mapsto (\langle k\rangle^s a_k)$ maps ℓ_2^s isometrically onto the well-known Hilbert space ℓ_2 ($= \ell_2^0$). Then the above calculations show that $H^m(\mathbb{T}^n)$ may be equivalently be provided with the scalar product and norm

$$\begin{aligned} (u, v)_{m,\wedge} &= \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2m} c_k(u)\overline{c_k(v)} = \big((c_k(u))_{k\in\mathbb{Z}^n}, (c_k(v))_{k\in\mathbb{Z}^n}\big)_{\ell_2^m}, \\ \|u\|_{m,\wedge} &= (u, u)_{m,\wedge}^{\frac12} = \|(c_k(u))_{k\in\mathbb{Z}^n}\|_{\ell_2^m}. \end{aligned}$$

For $s \in \mathbb{R}_+$, this generalizes immediately to define, as subspaces of $L_2(\mathbb{T}^n)$,

$$H^s(\mathbb{T}^n) = \{u \mid (c_k(u))_{k\in\mathbb{Z}^n} \in \ell_2^s(\mathbb{Z}^n)\}, \tag{8.10}$$

provided with the scalar product and norm

$$\begin{aligned} (u, v)_{s,\wedge} &= \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2s} c_k(u)\overline{c_k(v)} = \big((c_k(u)), (c_k(v))\big)_{\ell_2^s}, \\ \|u\|_{s,\wedge} &= (u, u)_{s,\wedge}^{\frac12} = \|(c_k(u))\|_{\ell_2^s}. \end{aligned} \tag{8.11}$$

Also for noninteger s, the definition of $H^s(\mathbb{T}^n)$ is consistent with the general definition on compact manifolds given further above; this can be shown e.g. by use of interpolation theory (cf. Lions and Magenes [LM 1968]; when $s \in\,]0,2[$, the space $H^s(X)$ is the domain of $A^{s/2}$, whenever A is a selfadjoint positive operator in $L_2(X)$ with domain $H^2(X)$). Details will not be given here.

We can moreover make a generalization to arbitrary $s \in \mathbb{R}$.

Observe that when f_k is a sequence in $\ell_2^s(\mathbb{Z}^n)$ for some $s \in \mathbb{R}$, then the series $\sum_{k\in\mathbb{Z}^n} f_k e^{ik\cdot x}$ converges in $\mathcal{D}'$ to a distribution f: Take a test function $\varphi \in C^\infty(\mathbb{T}^n)$; its Fourier series $\sum_{k\in\mathbb{Z}^n} a_k e^{ik\cdot x}$ has the coefficient sequence $(a_k)_{k\in\mathbb{Z}^n}$ lying in ℓ_2^r for any r. Let $f^N = \sum_{|k|\le N} f_k e^{ik\cdot x}$; then

$$\begin{aligned}\langle f^N, \varphi\rangle = \sum_{|k|\le N} f_k \bar{a}_k, \text{ where } &\sum_{|k|\le N} |f_k \bar{a}_k| = \sum_{|k|\le N} |\langle k\rangle^s f_k||\langle k\rangle^{-s} a_k| \\ &\le \big(\sum_{|k|\le N} |\langle k\rangle^s f_k|^2\big)^{\frac12}\big(\sum_{|k|\le N} |\langle k\rangle^{-s} a_k|^2\big)^{\frac12} \le \|(f_k)\|_{\ell_2^s}\|(a_k)\|_{\ell_2^{-s}}.\end{aligned}$$

Thus $\langle f^N, \varphi\rangle$ converges for each φ when $N \to \infty$, and it follows from the limit theorem (Theorem 3.9) that f^N converges to a distribution f. In particular, $\langle f, e^{-ik\cdot x}\rangle = \lim_{N\to\infty}\langle f^N, e^{-ik\cdot x}\rangle = f_k$ for each k.

So there is a subset of the distributions $f \in \mathcal{D}'(\mathbb{T}^n)$ that can be written as $\sum_{k\in\mathbb{Z}^n} f_k e^{ik\cdot x}$, with $f_k = c_k(f) = \langle f, e^{-ik\cdot x}\rangle$ and $(f_k) \in \ell_2^s$, and we define $H^s(\mathbb{T}^n)$ to consist of these; in other words it is defined by (8.10)–(8.11).

It can now be remarked that $H^s(\mathbb{T}^n)$ and $H^{-s}(\mathbb{T}^n)$ identify with each other's dual spaces with a duality extending the L_2 scalar product. The proof is similar to that of Theorem 6.13: First of all, ℓ_2 identifies with its own dual space by the Riesz representation theorem. By use of the isometry $M_{\langle k\rangle^s}$ this extends to an identification of ℓ_2^{-s} and ℓ_2^s with each other's dual spaces, and this carries over to the duality between H^{-s} and H^s when we carry $(a_k)_{k\in\mathbb{Z}^n}$ over to $\sum_{k\in\mathbb{Z}^n} a_k e^{ik\cdot x}$.

Another observation is that since $\mathbb{T}^n$ is compact, any distribution u has a finite order M. Now when φ is as above we have for $|\alpha| \le M$,

$$\begin{aligned}\sup|D^\alpha\varphi(x)| = \sup\big|\sum_{k\in\mathbb{Z}^n} k^\alpha a_k e^{ik\cdot x}\big| &\le \sum_{k\in\mathbb{Z}^n} |\langle k\rangle^{|\alpha|} a_k| \\ &\le \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{M+b}|a_k|\langle k\rangle^{-b} \le \|(a_k)\|_{\ell_2^{M+b}}\|(\langle k\rangle^{-b})\|_{\ell_2^0},\end{aligned}$$

where $\|(\langle k\rangle^{-b})\|_{\ell_2^0} < \infty$ for $b > \frac{n}{2}$ (cf. (18.13)ff. below). Thus $\|\varphi\|_{C^M} \le c_b\|(a_k)\|_{\ell_2^{M+b}}$ for $b > \frac{n}{2}$. Hence

$$|\langle u, \varphi\rangle| \le C_M \sup\{|D^\alpha\varphi(x)| \mid |\alpha| \le M,\, x \in \mathbb{T}^n\} \le C'\|\varphi\|_{M+b,\wedge},$$

for all φ, so u defines a continuous functional on $H^{M+b}(\mathbb{T}^n)$. It follows that $u \in H^{-M-b}(\mathbb{T}^n)$. So

$$\mathcal{D}'(\mathbb{T}) = \bigcup_{s\in\mathbb{R}} H^s(\mathbb{T}^n) = \bigcup_{s\in\mathbb{Z}} H^s(\mathbb{T}^n).$$

Let us define Λ_s for $s \in \mathbb{R}$ as the operator

$$\Lambda_s \colon \sum_{k\in\mathbb{Z}^n} c_k e^{ik\cdot x} \mapsto \sum_{k\in\mathbb{Z}^n} \langle k\rangle^s c_k e^{ik\cdot x}; \tag{8.12}$$

(corresponding to the multiplication operator $M_{\langle k\rangle^s}$ on the coefficient sequence); then clearly

$$\Lambda_s \text{ maps } H^t(\mathbb{T}^n) \text{ isometrically onto } H^{t-s}(\mathbb{T}^n),$$

when the norms $\|u\|_{r,\wedge}$ are used, and Λ_{-s} is the inverse of Λ_s for any s.

Theorem 8.3.

1° *When $s > 0$, Λ_{-s} defines a bounded selfadjoint operator in $L_2(\mathbb{T}^n)$, which is a compact operator in $L_2(\mathbb{T}^n)$.*

The injection of $H^s(\mathbb{T}^n)$ into $L_2(\mathbb{T}^n)$ is compact.

2° *When $s > s'$, the injection of $H^s(\mathbb{T}^n)$ into $H^{s'}(\mathbb{T}^n)$ is compact.*

Proof. Let $s > 0$. Since $\|u\|_{s,\wedge} \geq \|u\|_{0.\wedge}$, Λ_{-s} defines a bounded operator T in $L_2(\mathbb{T}^n)$, and it is clearly symmetric, hence selfadjoint. Moreover, the orthonormal basis $(e^{ik\cdot x})_{k\in\mathbb{Z}^n}$ is a complete system of eigenvectors of T, with eigenvalues $\langle k\rangle^{-s}$. Then T is a compact operator in $L_2(\mathbb{T}^n)$, since $\langle k\rangle^{-s} \to 0$ for $|k| \to \infty$.

It follows that the injection of $H^s(\mathbb{T}^n)$ into $H^0(\mathbb{T}^n)$ is compact. For, when u_l is a bounded sequence in $H^s(\mathbb{T}^n)$, we can write $u_l = \Lambda_{-s} f_l = T f_l$, where (f_l) is bounded in $L_2(\mathbb{T}^n)$, so the compactness of T in $L_2(\mathbb{T}^n)$ implies that u_l has a convergent subsequence in $L_2(\mathbb{T}^n)$. This shows 1°.

For more general s and s', one carries the injection $J : H^s(\mathbb{T}^n) \hookrightarrow H^{s'}(\mathbb{T}^n)$ over into the injection $J' : H^{s-s'}(\mathbb{T}^n) \hookrightarrow H^0(\mathbb{T}^n)$ by use of the isometries $\Lambda_{-s'}$ and $\Lambda_{s'}$, setting $J = \Lambda_{-s'} J' \Lambda_{s'}$; it is compact in $H^{s'}(\mathbb{T}^n)$, since J' is so in $H^0(\mathbb{T}^n)$. $\square$

Proof of Theorem 8.2. Let u_l be a bounded sequence in $H^s(X)$, then for each j (cf. (8.9)), $(\varphi_j u) \circ \kappa_j^{-1}$ is bounded in $H^s(\mathbb{R}^n)$. The support is in a fixed compact subset of V_j, and we can assume (after a scaling and translation if necessary) that this is a compact subset of Q°, so that the sequence identifies with a bounded sequence in $H^s(\mathbb{T}^n)$. Then Theorem 8.3 gives that there is a subsequence that converges in $H^{s'}(\mathbb{T}^n)$. Taking subsequences in this way successively for $j = 1, \dots, J$, we arrive at a numbering such that the corresponding subsequence of u_l converges in $H^{s'}(X)$. $\square$

The statement on the compactness of the operator defined from Λ_{-s} for $s > 0$ can be made more precise by reference to *Schatten classes.* A compact selfadjoint operator $T \geq 0$ is said to belong to the Schatten class $\mathcal{C}_p$ (for some $p > 0$), when the eigenvalue sequence $\{\lambda_j\}_{j\in\mathbb{N}_0}$ satisfies $\sum_{j\in\mathbb{N}_0} \lambda_j^p < \infty$. In particular, the operators in $\mathcal{C}_1$ are the trace-class operators, those in $\mathcal{C}_2$ are the Hilbert-Schmidt operators. For nonselfadjoint T, the Schatten class is defined according to the behavior of the eigenvalues of $(T^*T)^{\frac{1}{2}}$.

We here observe that Λ_{-s}, or the imbedding $H^s(\mathbb{T}^n) \hookrightarrow H^0(\mathbb{T}^n)$, is in the Schatten classes $\mathcal{C}_p$ with $p > n/s$, since

$$\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{-sp} < \infty \text{ for } s > n/p \tag{8.13}$$

(which is seen by comparison with $\int_{\mathbb{R}^n} \langle x\rangle^{-sp}\,dx$). In particular, the injection is trace-class for $s > n$, and it is Hilbert-Schmidt for $s > n/2$.

An operator $P\colon C^\infty(X) \to C^\infty(X)$ is said to be a pseudodifferential operator of order d, when $P_j\colon C_0^\infty(V_j) \to C^\infty(V_j)$, defined by

$$P_j v = P(v \circ \kappa_j) \circ \kappa_j^{-1}, v \in C_0^\infty(V_j) \tag{8.14}$$

is a ψdo of order d on V_j for each j.

To see that the definition makes good sense and is independent of the choice of a particular atlas, we appeal to Theorem 8.1, which shows that the property of being a ψdo on an open set is preserved under diffeomorphisms. Here the pieces P_j generally have $S^d_{1,0}$ symbols, but if they are polyhomogeneous with respect to one atlas, they are so in all atlases, and we say that P is polyhomogeneous. In the following we restrict the attention to the polyhomogeneous case.

The symbols of the localized pieces P_j of course depend on the choice of atlas. However, there is a remarkable fact, namely that the principal symbol has an invariant meaning.

To explain this, we need the concept of vector bundles over a manifold. A *trivial vector bundle* over X with fiber dimension N is simply the manifold $X \times \mathbb{C}^N$; the points are denoted for example $\{x, v\}$ ($x \in X$ and $v \in \mathbb{C}^N$), and for each $x \in X$, the subset $\{x\} \times \mathbb{C}^N$ is called the *fiber* (or fibre) over x. X is then called the base space. The *sections* of $X \times \mathbb{C}^N$ are the vector-valued functions $f\colon X \to \mathbb{C}^N$.

A general C^∞ vector bundle E over X is a C^∞ manifold provided with a projection $\pi\colon E \to X$ such that $\pi^{-1}(\{x\})$ is an N-dimensional complex vector space (the fiber over x). Moreover, one requires that E is covered by open sets of the form $\pi^{-1}(U)$ associated with mappings $\psi\colon \pi^{-1}(U) \to V \times \mathbb{C}^N$, such that the restriction of ψ to U is a coordinate mapping $\kappa\colon U \to V$ (where V is identified with $V \times \{0\}$), and at each $x \in U$, ψ maps $\pi^{-1}(\{x\})$ *linearly* onto $\mathbb{C}^N$, the fiber over $\kappa(x)$. Such a mapping ψ (or rather, a triple $\{\psi, U, V\}$) is called a local trivialization, and the associated mapping $\kappa\colon U \to V$ is called the base space mapping. When ψ_1 and ψ_2 are local trivializations with $U_1 \cap U_2 \neq \emptyset$, the mapping $g_{12} = \psi_1 \circ \psi_2^{-1}$ from $\kappa_2(U_1 \cap U_2) \times \mathbb{C}^N$ to $\kappa_1(U_1 \cap U_2) \times \mathbb{C}^N$ is called a transition function (skiftefunktion); it is a smooth family of regular $N \times N$-matrices parametrized by $y \in \kappa_2(U_1 \cap U_2)$. The (continuous, say) sections of E are the continuous functions $f\colon X \to E$ such that $f(x) \in \pi^{-1}(\{x\})$ (i.e., $f(x)$ lies in the fiber over x). In each local trivialization, they carry over to continuous functions from V to $\mathbb{C}^N$. They are said to be C^k ($k \leq \infty$) when they carry over to C^k functions in the local trivializations. — Each section can be viewed as a subset of E (a "graph").

The zero section sends x into the origin of the fiber over x; one often identifies the zero section with X itself.

One can of course also define real vector bundles, where the fibres are real N-dimensional vector spaces.

We use these concepts in two ways: For one thing, we can let our ψdo's be matrix-valued, and then we can allow the matrix structure to "twist" when one moves around on X, by letting the ψdo's act on sections of vector bundles. Here it is natural to take complex vector bundles. One can provide such vector bundles with a hermitian structure — this means that one chooses a scalar product in each fiber, varying smoothly along X; then L_2 scalar products can be defined for the sections (not just for functions), and all that was said about function spaces (and distribution spaces) above extends to sections of vector bundles.

The other use we make of vector bundles is crucial even for scalar pseudodifferential operators: There is a special real vector bundle with fiber dimension n associated with X called the *cotangent bundle* $T^*(X)$. It can be described as follows: It has an atlas consisting of open sets $\pi^{-1}(U_j)$, $j = 1, \dots, J$, and local trivializations $\psi_i\colon \pi^{-1}(U_j) \to V_j \times \mathbb{R}^n$, such that the associated base space mappings $\kappa_j\colon U_j \to V_j$ are connected with the transition functions in the following way:

When $x \in U_i \cap U_j$, the linear map $\psi_j \circ \psi_i^{-1}$ from $\{\kappa_i(x)\} \times \mathbb{R}^n$ to $\{\kappa_j(x)\} \times \mathbb{R}^n$ equals the inverse transpose of the Jacobian of $\kappa_j \circ \kappa_i^{-1}$ at $\kappa_i(x)$.

(A full discussion can be found in textbooks on differential geometry. One can also find a detailed description in [H 1985 I] pages 146–148. The cotangent bundle is the dual bundle of the *tangent bundle* $T(X)$ where the transition functions are the Jacobians of the $\kappa_j \circ \kappa_i^{-1}$.)

This is a way of describing the cotangent bundle that fits directly with our purpose, which is to define the principal symbol of a ψdo as a function on $T^*(X)\setminus 0$ (the cotangent bundle with the zero section removed). Indeed, formula (8.5) shows that when $p^0(x,\xi)$ is given in some coordinate system, one gets the same value after a change to new coordinates $\underline{x}=\kappa(x)$ if one maps $\{x,\xi\}$ to $\{\underline{x},\underline{\xi}\}=\{\kappa(x),\,{}^t\kappa'(x)^{-1}\xi\}$. (We assume here that p^0 is taken homogeneous outside $\xi=0$.) So it is independent of the choice of local trivializations.

When P is a polyhomogeneous ψdo of order d on X, there is defined a principal symbol in each local coordinate system, and this allows us to define a principal symbol that is a function on the nonzero cotangent bundle $T^*(X)\setminus 0$ (since the value can be found in a consistent way from the value in any local coordinate system). We call it $p^0(x,\xi)$ again, where $x\in X$ and ξ indicates a point in the fiber over x.

One can even let p^0 take its values in a vector bundle E over X (so that p^0 is matrix-formed in local trivializations).

Now it is meaningful to speak of *elliptic* ψdo's on X. For ψdo's acting on sections of bundles — sending $C^\infty(E)$ into $C^\infty(E')$ where E and E' are vector bundles of fiber dimension N resp. N' — we can even speak of injectively elliptic resp. surjectively elliptic ψdo's (meaning that they are so in local trivializations).

There are some considerations on cut-off functions that are useful when dealing with ψdo's on manifolds.

Lemma 8.4. *Let X be a compact C^∞ manifold.*

1° *To every finite open cover $\{U_1, \dots, U_J\}$ of X there exists an* **associated** *partition of unity $\{\varphi_1, \dots, \varphi_J\}$, that is, a family of nonnegative functions $\varphi_j \in C_0^\infty(U_j)$ such that $\sum_{j \le J} \varphi_j = 1$.*

2° *There exists a finite family of local coordinates $\kappa_i \colon U_i \to V_i$, $i = 1, \dots, I_1$ for which there is a* **subordinate** *partition of unity $\{\varrho_1, \dots, \varrho_{J_0}\}$ (nonnegative smooth functions having the sum 1), such that any four of the functions $\varrho_j, \varrho_k, \varrho_l, \varrho_m$ have their support in some U_i. (This extends to families of trivializations when a vector bundle is considered.)*

Proof. The statement in 1° is a simple generalization of the well-known statement for compact sets in $\mathbb{R}^n$. (We can choose compact subsets K_j, K_j' of the U_j such that $K_j' \subset K_j^\circ$, $X = \bigcup K_j'$, and we can find smooth functions ψ_j that are 1 on K_j' and have support in K_j°; then take $\varphi_j = \psi_j / \sum_k \psi_k$.)

For 2°, a proof goes as follows: We can assume that X is provided with a Riemannian metric. Consider a system of coordinates $\{\kappa_i \colon U_i \to V_i\}_{i=1}^{I_0}$; we assume that the patches V_i are disjoint, and note that they need not be connected sets. By the compactness of X, there is a number $\delta > 0$ such that any subset of X with geodesic diameter $\le \delta$ is contained in one of the sets U_i. Now cover X by a finite system of open balls B_j, $j = 1, \dots, J_0$, of radius $\le \delta/8$. We claim that this system has the following 4-*cluster property:* Any four sets B_{j_1}, B_{j_2}, B_{j_3}, B_{j_4} can be grouped in clusters that are mutually disjoint and where each cluster lies in one of the sets U_i.

The 4-cluster property is seen as follows: Let $j_1, j_2, j_3, j_4 \le J_0$ be given. First adjoin to B_{j_1} those of the B_{j_k}, $k = 2, 3, 4$, that it intersects with; next, adjoin to this union those of the remaining sets that it intersects with, and finally do it once more; this gives the first cluster. If any sets are left, repeat the procedure with these (at most three). Now the procedure is repeated with the remaining sets, and so on; this ends after at most four steps. The clusters are clearly mutually disjoint, and by construction, each cluster has diameter $\le \delta$, hence lies in a set U_i. (One could similarly obtain covers with an N-cluster property, taking balls of radius $\le \delta/2N$.)

To the original coordinate mappings we now adjoin the following new ones: Assume that B_{j_1}, B_{j_2}, B_{j_3}, B_{j_4} gave rise to the disjoint clusters $U', U'', \dots$, where $U' \subset U_{i'}$, $U'' \subset U_{i''}, \dots$. Then use $\kappa_{i'}$ on U', $\kappa_{i''}$ on U'', $\dots$ (if necessary followed by linear transformations $\Phi'', \dots$ to separate the images) to define the mapping $\kappa \colon U' \cup U'' \cup \cdots \to \kappa_{i'}(U') \cup \Phi''\kappa_{i''}(U'') \cup \cdots$. This gives a new coordinate mapping, for which $B_{j_1} \cup B_{j_2} \cup B_{j_3} \cup B_{j_4}$ equals the initial set $U' \cup U'' \cup \cdots$. In this way, finitely many new coordinate mappings, say $\{\kappa_i \colon U_i \to V_i\}_{i=I_0+1}^{I_1}$, are adjoined to the original ones, and we have established a mapping $(j_1, j_2, j_3, j_4) \mapsto i = i(j_1, j_2, j_3, j_4)$ for which

$$B_{j_1} \cup B_{j_2} \cup B_{j_3} \cup B_{j_4} \subset U_{i(j_1, j_2, j_3, j_4)}.$$

Let $\{\varrho_j\}_{j=1}^{J_0}$ be a partition of unity associated with the cover $\{B_j\}_{j=1}^{J_0}$ (here we use 1°), then it has the desired property with respect to the system $\{\kappa_i : U_i \to V_i\}_{i=1}^{I_1}$. $\square$

The refined partition of unity in 2° is convenient when we consider compositions of operators. (It was used in this way in [S 1969] but a proof was not included there.) If P and Q are ψdo's on X, write

$$R = PQ = \sum_{j,k,l,m} \varrho_j P \varrho_k \varrho_l Q \varrho_m, \tag{8.15}$$

then each term $\varrho_j P \varrho_k \varrho_l Q \varrho_m$ has support in a set U_i that carries over to $V_i \subset \mathbb{R}^n$ by κ_i, so that we can use the composition rules for ψdo's on $\mathbb{R}^n$. It follows that R is again a ψdo on X; in the local coordinates it has symbols calculated by the usual rules. In particular, the principal symbol of the composition is found from the principal symbols, carried back to $T^*(X) \setminus 0$ and added up:

$$r^0(x,\xi) = \sum_{j,k,l,m} \varphi_j(x)\varphi_k(x)p^0(x,\xi)\varphi_l(x)\varphi_m(x)q^0(x,\xi) = p^0(x,\xi)q^0(x,\xi). \tag{8.16}$$

One shows that the adjoint P^* of a ψdo P (with respect to the chosen scalar product in $L_2(X)$) is again a ψdo, by using a partition of unity as in 2° in the consideration of the identities

$$\int_X (Pf)\overline{g}\, dx = \int_X f\overline{(P^*g)}\, dx. \tag{8.17}$$

We can also use the partitions of unity to show continuity in Sobolev spaces over X:

Theorem 8.5. *Let P be a pseudodifferential operator on X of order d. Then P is continuous from $H^s(X)$ to $H^{s-d}(X)$ for all $s \in \mathbb{R}$.*

Proof. Let ϱ_k be a partition of unity as in Lemma 8.4 2°. Then $P = \sum_{j,k \le J_0} \varrho_j P \varrho_k$. For each j,k there is an $i \le I_1$ such that ϱ_j and ϱ_k are supported in U_i. Then when $\varrho_j P \varrho_k$ is carried over to V_i, we find a ψdo with symbol (in (x,y)-form) vanishing for (x,y) outside a compact subset of $V_i \times V_i$; it identifies (by extension by 0) with a symbol in $S^d_{1,0}(\mathbb{R}^{2n}, \mathbb{R}^n)$, and hence defines a continuous operator from $H^s(\mathbb{R}^n)$ to $H^{s-d}(\mathbb{R}^n)$ for any $s \in \mathbb{R}$. It follows that $\varrho_j P \varrho_k$ is continuous from $H^s(X)$ to $H^{s-d}(X)$. Adding the pieces, we find the statement in the theorem. $\square$

For the construction of a parametrix of a given elliptic operator we use Lemma 8.4 1° combined with some extra cut-off functions:

Theorem 8.6. *Let P be an elliptic ψdo of order d on X. Then there is a ψdo Q on X, elliptic of order $-d$, such that*

$$\begin{aligned} &\text{(i)} \quad PQ = I + \mathcal{R}_1, \\ &\text{(ii)} \quad QP = I + \mathcal{R}_2, \end{aligned} \tag{8.18}$$

with $\mathcal{R}_1$ and $\mathcal{R}_2$ of order $-\infty$.

Proof. Along with the φ_j in 1° we can find ψ_j and $\theta_j \in C_0^\infty(U_j)$ such that $\psi_j(x) = 1$ on a neighborhood of $\operatorname{supp}\varphi_j$, $\theta_j(x) = 1$ on a neighborhood of $\operatorname{supp}\psi_j$. Since P_j (recall (8.14)) is elliptic on V_j, it has a parametrix Q'_j there. Denoting the functions $\varphi_j, \psi_j, \theta_j$ carried over to V_j (i.e. composed with κ_j^{-1}) by $\underline{\varphi}_j, \underline{\psi}_j, \underline{\theta}_j$, we observe that

$$C'_j = \underline{\theta}_j P_j \underline{\theta}_j \underline{\psi}_j Q'_j \underline{\varphi}_j - \underline{\varphi}_j \sim 0.$$

For,

$$\begin{aligned} C'_j &= (\underline{\theta}_j P_j \underline{\theta}_j \underline{\psi}_j Q'_j - I)\underline{\varphi}_j = (\underline{\theta}_j P_j \underline{\psi}_j Q'_j - I)\underline{\varphi}_j \\ &= (\underline{\theta}_j P_j Q'_j - I)\underline{\varphi}_j + \underline{\theta}_j P_j (\underline{\psi}_j - 1) Q'_j \underline{\varphi}_j \sim (\underline{\theta}_j - 1)\underline{\varphi}_j = 0; \end{aligned}$$

it is used here that $1 - \psi_j$ and φ_j have disjoint supports, and that $\theta_j = 1$ on $\operatorname{supp}\varphi_j$. Moreover, C'_j carries over to X as a negligible operator C_j with support properties corresponding to the fact that C'_j has kernel support in $\operatorname{supp}\theta_j \times \operatorname{supp}\varphi_j$. Now let $Qu = \sum_j (\underline{\psi}_j Q'_j((\varphi_j u) \circ \kappa_j^{-1})) \circ \kappa_j$. Then

$$\begin{aligned} (PQ - I)u &= \sum_j (P[\underline{\psi}_j Q'_j((\varphi_j u) \circ \kappa_j^{-1}) \circ \kappa_j] - \varphi_j u) \\ &= \sum_j (\theta_j P \theta_j [\underline{\psi}_j Q'_j((\varphi_j u) \circ \kappa_j^{-1}) \circ \kappa_j] - \varphi_j u) + \mathcal{R}u, \end{aligned}$$

with $\mathcal{R}$ negligible, since $1 - \theta_j$ and ψ_j have disjoint supports. The j'th term in the last sum equals $C_j u$, and $\sum_j C_j$ is negligible. This shows (8.18) (i). There is a similar construction of a left parametrix, satisfying (8.18) (ii), and then each of them is a two-sided parametrix, by the usual argument.

In each coordinate system, the principal symbol of Q'_j is $\underline{p}^0(\underline{x}, \underline{\xi})^{-1}$ (for those $\underline{\xi}$ where it is homogeneous). Then the principal symbol of Q, the sum of the $\underline{\psi}_j Q'_j \underline{\varphi}_j$ carried over to X, is equal to

$$\sum_j \psi_j(x) p^0(x, \xi)^{-1} \varphi_j(x) = (p^0)^{-1}(x, \xi),$$

since $\psi_j \varphi_j = \varphi_j$ and $\sum_j \varphi_j = 1$. In particular, Q is elliptic of order $-d$. $\square$

There are also one-sided versions, that we state here in full generality:

Theorem 8.7. *Let E and E' be complex vector bundles over X of fiber dimensions N resp. N', and let P be a pseudodifferential operator of order d from the sections of E to the sections of E'.*

$1°$ *If $N' \le N$ and P is surjectively elliptic of order d, then there exists a right parametrix Q, which is a ψdo from the sections of E' to E, injectively elliptic of order $-d$, such that* (8.18) (i) *holds.*

$2°$ *If $N' \ge N$ and P is injectively elliptic of order d, then there exists a left parametrix Q, which is a ψdo from the sections of E' to E, surjectively elliptic of order $-d$, such that* (8.18) (ii) *holds.*

$3°$ *If $N = N'$ and either $1°$ or $2°$ holds for some Q, then P is elliptic of order d, and both* (i) *and* (ii) *in* (7.10) *hold with that Q.*

In the case of trivial bundles over X, the proof can be left to the reader (to deduce it from Theorem 7.10 by the method in Theorem 8.6). In the situation of general bundles one should replace the coordinate changes κ_j by the local trivializations ψ_j, with an appropriate notation for the trivialized sections. (Here the proof can be left to readers that are familiar with working with vector bundles.) There is again a corollary on regularity of solutions of elliptic problems as in Corollary 7.12. For existence and uniqueness questions we can get much better results than in Chapter 7, as will be seen in the following.

8.3 Fredholm theory.

Let V_1 and V_2 be vector spaces. A Fredholm operator from V_1 to V_2 is a linear operator T from V_1 to V_2 such that $\ker T$ and $\operatorname{coker} T$ have finite dimension; here $\ker T$ is the nullspace (also denoted $Z(T)$) and $\operatorname{coker} T$ is the quotient space $H_2/R(T)$, where $R(T)$ is the range of T.

We shall briefly recall some facts concerning Fredholm operators between Hilbert spaces. Full proofs can be found in various books, e.g. [H 1985, Sect. 19.1] or the lecture notes "Linear Functional Analysis" by Hörmander (Lund 1989), [H 1989]. Fredholm operators can also be studied in Banach spaces (and even more general spaces) but we shall just need them in Hilbert spaces, where some proofs are simpler to explain.

Let H_1 and H_2 be Hilbert spaces; we denote as usual the space of bounded linear operators from H_1 to H_2 by $\mathbf{B}(H_1, H_2)$. First of all one can observe that when $T \in \mathbf{B}(H_1, H_2)$ is a Fredholm operator, then $R(T)$ is closed in H_2. To see this, note that $T\colon H_1 \ominus Z(T) \to H_2$ is again bounded and is a Fredholm operator. If $\dim \operatorname{coker} T = n$, we can choose a linear mapping $S\colon \mathbb{C}^n \to H_2$ that maps $\mathbb{C}^n$ onto a complement of $R(T)$ in H_2; then $T_1\colon \{x, y\} \to Tx + Sy$ from $(H_1 \ominus Z(T)) \oplus \mathbb{C}^n$ to H_2 is bijective. T_1 is continuous, hence so is its inverse (by the closed graph theorem). But then $R(T) = T_1((H_1 \ominus Z(T)) \oplus \{0\})$ is closed.

The property of having closed range is often included in the definition of Fredholm operators, but we see that it holds automatically here.

When T is a Fredholm operator, its index is defined by

$$\operatorname{index} T = \dim \ker T - \dim \operatorname{coker} T. \tag{8.19}$$

Fredholm operators have the following properties:

Theorem 8.8.
1° MULTIPLICATIVE PROPERTY OF THE INDEX. *When $T_1 \in \mathbf{B}(H_1, H_2)$ and $T_2 \in \mathbf{B}(H_2, H_3)$ are Fredholm operators, then $T_2T_1 \in \mathbf{B}(H_1, H_3)$ is also Fredholm, and*

$$\operatorname{index} T_2T_1 = \operatorname{index} T_2 + \operatorname{index} T_1. \tag{8.20}$$

2° INVARIANCE OF FREDHOLM PROPERTY AND INDEX UNDER SMALL PERTURBATIONS. *Let $T \in \mathbf{B}(H_1, H_2)$ be a Fredholm operator. There is a constant $c > 0$ such that for all operators $S \in \mathbf{B}(H_1, H_2)$ with norm $< c$, $T + S$ is Fredholm and*

$$\operatorname{index}(T + S) = \operatorname{index} T. \tag{8.21}$$

3° INVARIANCE OF FREDHOLM PROPERTY AND INDEX UNDER COMPACT PERTURBATIONS. *Let $T \in \mathbf{B}(H_1, H_2)$ be a Fredholm operator. Then for any compact operator $S \in \mathbf{B}(H_1, H_2)$, $T + S$ is Fredholm and* (8.21) *holds.*

Hints of proof.
Property 1° is just linear algebra; it is shown e.g. in [H 1989, Th. 1.3.2].
For 2°, note that it is obvious when T is bijective, for then

$$T + S = T(I + T^{-1}S), \text{ where } (I + T^{-1}S)^{-1} = \sum_{k\in\mathbb{N}_0} (-T^{-1}S)^k, \tag{8.22}$$

converging in norm when $\|S\| < \|T^{-1}\|^{-1}$. The general case is reduced to this situation by removing the kernel and cokernel in a suitable way, see details e.g. in [H 1989, proof of Th. 2.5.3 b)].

For 3°, recall that a compact operator maps any bounded sequence into a sequence that has a convergent subsequence. Take first T invertible, and let us show why $Z(T + S)$ is finite dimensional:

$Z(T + S)$ is a linear space, and for $x \in Z(T + S)$,

$$Tx = -Sx.$$

Let x_j be a bounded sequence in $Z(T + S)$. By the compactness of S, Sx_j has a convergent subsequence Sx_{j_k}, but then Tx_{j_k} is also convergent, and so is x_{j_k} since T^{-1} is bounded. We have shown that any bounded sequence in $Z(T + S)$ has a convergent subsequence. Then the Hilbert space $Z(T + S)$

must be finite dimensional, for an infinite dimensional Hilbert space has an infinite orthonormal sequence (with no convergent subsequences).

We recall the general property $H_2 = \overline{R(T+S)} \oplus Z(T^*+S^*)$. Since T^* is invertible and S^* is compact, $Z(T^*+S^*)$ has finite dimension. So to see that $R(T+S)$ has finite codimension, we just have to show that it is closed.

Consider $(T+S)\colon \tilde{H}_1 \to H_2$ where $\tilde{H}_1 = H_1 \ominus Z(S+T)$. We claim that for all $x \in \tilde{H}_1$,

$$\|x\| \le c\|(T+S)x\| \text{ for some } c > 0. \tag{8.23}$$

For if not, then there exist sequences $x_j \in \tilde{H}_1$ and c_j such that $\|x_j\| = 1$ and $c_j \to \infty$, with

$$1 = \|x_j\| \ge c_j\|(T+S)x_j\|, \text{ for all } j. \tag{8.24}$$

But then $\|(T+S)x_j\| \le 1/c_j \to 0$. Since S is compact and $\|x_j\| = 1$, there is a subsequence x_{j_k} with $Sx_{j_k} \to v$ for some $v \in H_2$. Then $Tx_{j_k} \to -v$, so $x_{j_k} = T^{-1}Tx_{j_k} \to w = -T^{-1}v$; note that $w \in \tilde{H}_1$ and has norm 1, since this holds for the x_{j_k}. But $(T+S)w = \lim_k(Tx_{j_k} + Sx_{j_k}) = 0$, contradicting the fact that $\tilde{H}_1 \perp Z(T+S)$. It follows from (8.23) that $R(T+S)$ is closed.

This shows 3° when T is invertible. For the general case, see e.g. [H 1989, Th. 2.5.10]. $\square$

One sometimes attaches the Fredholm property to unbounded operators too. This can be justified when T is a closed operator, as follows:

Let T be a closed operator from $D(T) \subset H_1$ to H_2. Then T is bounded as an operator from the Hilbert space $D(T)$ provided with the graph norm $\|u\|_{\text{graph}} = (\|u\|_{H_1}^2 + \|Tu\|_{H_2}^2)^{\frac{1}{2}}$, to H_2. It is said to be Fredholm when $\dim\ker T$ and $\dim\operatorname{coker} T$ are finite, and the index is defined as usual by (8.19). $R(T)$ is again found to be closed in H_2, and Theorem 8.8 holds when the graph norm is used on $D(T)$.

We shall now show that elliptic pseudodifferential operators on compact manifolds are Fredholm operators.

Theorem 8.9. *Let X be a compact n-dimensional C^∞ manifold, and let P be an elliptic pseudodifferential operator of order d on X. Then $P\colon H^s(X) \to H^{s-d}(X)$ is a Fredholm operator for any $s \in \mathbb{R}$.*

Moreover, the kernel is the same finite dimensional subspace V of $C^\infty(X)$ for all $s \in \mathbb{R}$, and there is a finite dimensional subspace W of $C^\infty(X)$ such that the range for all $s \in \mathbb{R}$ consists of the $f \in H^{s-d}(X)$ satisfying $(f, w) = 0$ for $w \in W$. These statements hold also for $s = \infty$, $H^\infty(X) = C^\infty(X)$, so P is a Fredholm operator in $C^\infty(X)$.

Thus $P\colon H^s(X) \to H^{s-d}(X)$ has an index

$$\operatorname{index} P = \dim\ker P - \dim\operatorname{coker} P, \tag{8.25}$$

that is independent of $s \le \infty$.

When Q is a parametrix of P,

$$\operatorname{index} Q = -\operatorname{index} P. \tag{8.26}$$

Proof. We use Theorem 8.6. According to this theorem, P has a parametrix Q, continuous in the opposite direction and satisfying (8.18). Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are bounded operators from $H^s(X)$ to $H^{s+1}(X)$, they are compact operators in $H^s(X)$ by Theorem 8.2, for all s. Then by Theorem 8.8 3°, $PQ = I + \mathcal{R}_1$ and $QP = I + \mathcal{R}_2$ are Fredholm operators in $H^s(X)$ with index 0.

We can write (as in [SR 1991]) $C^\infty(X) = H^\infty(X)$ (since $\bigcap_{s\in\mathbb{R}} H^s(X) = C^\infty(X)$ by the Sobolev imbedding theorem). Denote by $Z^s(P)$ the nullspace of $P\colon H^s(X) \to H^{s-d}(X)$, and similarly by $Z^s(QP)$ the nullspace of $QP\colon H^s(X) \to H^s(X)$. Clearly, $Z^\infty(P) \subset Z^s(P)$, $Z^\infty(QP) \subset Z^s(QP)$, for $s \in \mathbb{R}$. On the other hand, for $u \in H^s(X)$, any s,

$$Pu = 0 \implies QPu = 0 \implies u = -\mathcal{R}_2 u \in C^\infty(X),$$

by (8.18) (ii), so $Z^s(P) \subset Z^s(QP) \subset C^\infty(X)$. Thus $Z^s(P) = Z^\infty(P)$ and $Z^s(QP) = Z^\infty(QP)$ for all s; we denote $Z^\infty(P) = V$. Moreover, since $Z^s(QP)$ has finite dimension, so has $Z^s(P) = V$.

For the consideration of cokernels, we similarly define $R^s(P)$ and $R^s(PQ)$ to be the ranges of $P\colon H^{s+d}(X) \to H^s(X)$ and $PQ\colon H^s(X) \to H^s(X)$. Since $PQ = I + \mathcal{R}_1$ is Fredholm in $H^s(X)$, $R^s(PQ)$ has finite codimension in $H^s(X)$ and then so has $R^s(P) \supset R^s(PQ)$. Thus $P\colon H^{s+d}(X) \to H^s(X)$ is Fredholm, for any s; in particular, $R^s(P)$ is closed in $H^s(X)$. The adjoint $P^*\colon \mathcal{D}'(X) \to \mathcal{D}'(X)$ is defined such that its restrictions to Sobolev spaces satify:

$P^*\colon H^{-s}(X) \to H^{-s-d}(X)$ is the adjoint of $P\colon H^{s+d}(X) \to H^s(X)$, when $H^{-s}(X)$ is identified with the dual space of $H^s(X)$ with respect to a duality (u, v) consistent with the chosen scalar product in $L_2(X)$.

Moreover, since $R^s(P)$ is closed,

$$R^s(P) = \{f \in H^s(X) \mid (f, v) = 0 \text{ for } v \in Z^{-s}(P^*)\}.$$

Since P^* is elliptic, $Z^{-s}(P^*) = Z^\infty(P^*) = W$, a finite dimensional subspace of $C^\infty(X)$ independent of $s \in \mathbb{R}$. This shows that

$$R^s(P) = \{f \in H^s(X) \mid (f, v) = 0 \text{ for } v \in W\},$$

for $s \in \mathbb{R}$, and we finally show it for $s = \infty$. Here the inclusion $\subset$ is obvious, since $R^\infty(P) \subset R^s(P)$ for any s; on the other hand, when $f \in H^\infty(X)$ with

$(f, w) = 0$ for $w \in W$, then there is a $u \in H^s(X)$ (with an arbitrarily chosen $s < \infty$) such that $f = Pu$, and then $u = QPu - \mathcal{R}_2 u \in H^\infty(X)$, so that $f \in R^\infty(P)$.

The last statement follows by another application of Theorem 8.8: Since $\mathcal{R}_1$ is compact in $L_2(X)$, $I + \mathcal{R}_1$ has index 0 there by 3°, and hence $\operatorname{index} P + \operatorname{index} Q = 0$ by 1°. This extends to the operators in the other spaces since the indices are independent of s. $\square$

It may be observed that in the proof we only used that the $\mathcal{R}_i$ are of order -1, not that they are of order $-\infty$. Therefore the last statement is also valid when Q is just a rough parametrix for which the $\mathcal{R}_i$ are of order -1, and we had not really needed the fine adaptation of lower order symbols in the proof of this theorem.

Similarly, if P is replaced by another pseudodifferential operator P_1 of order d with *the same principal symbol*, then

$$\operatorname{index} P_1 = -\operatorname{index} Q = \operatorname{index} P. \tag{8.27}$$

This shows:

Corollary 8.10. *The index of P depends only on the principal symbol of P.*

For Riemannian manifolds X, the index of the elliptic operator defined from the Riemannian metric, and for related elliptic operators defined in suitable vector bundles over X, have been studied intensively. There is a famous theorem of Atiyah and Singer (from the 1960's) showing how the analytical index (the one we have defined above) is related to the so-called topological index defined in terms of algebraic topology associated with the manifold.