§9. Boundary value problems in a constant-coefficient case

9.1. Boundary maps for the half-space.

We have considered various realizations in Section 4.4 of the Laplacian and similar operators, defined by homogeneous boundary conditions, and shown how their invertibility properties lead to statements about the solvability of homogeneous boundary value problems. It is of great interest to study also nonhomogeneous boundary value problems

$$\begin{aligned} Au &= f \text{ in } \Omega, \\ Tu &= g \text{ on } \partial\Omega, \end{aligned} \tag{9.1}$$

with $g \neq 0$. Here Ω is an open subset of $\mathbb{R}^n$, A is an elliptic partial differential operator, and Tu stands for a set of combinations of boundary values and (possibly higher) normal derivatives (T is generally called a *trace operator*). There are several possible techniques for treating such problems. In specific cases where A is the Laplacian there are integral formulas (depending on the shape of the boundary), but when A has variable coefficients the methods are more qualitative.

When the boundary is smooth (cf. Section 2.5), one succesful method can be described as follows:

At a fixed point x_0 of the boundary, consider the problem with constant coefficients having the value they have at that point, and identify the tangent plane with $\mathbb{R}^{n-1}$ and the interior normal direction with the x_n-axis. The discussion of the resulting constant-coefficient problem is essential for the possibility to obtain solutions of (9.1). Drop the lower-order terms in the equation and boundary condition, and perform a Fourier transformation $\mathcal{F}_{x' \to \xi'}$ in the tangential variables (in $\mathbb{R}^{n-1}$), then we have for each ξ' a one-dimensional problem on $\mathbb{R}_+$. This is what is called the "model problem" at the point x_0.

The original problem (9.1) is called *elliptic* precisely when *the model problem is uniquely solvable in $L_2(\mathbb{R}_+)$ for each $\xi' \neq 0$, at all points x_0 of the boundary.*

It is possible to construct a good approximation of the solution in the elliptic case by piecing together the solutions of the model problems in some sense. A very efficient way of doing this is by setting up a pseudodifferential calculus of boundary value problems, somewhat like in Chapter 7, but necessarily somewhat more complicated, since we have operators going back and forth between the boundary and the interior of Ω.

We shall not go into the systematic treatment in this text. However, we can give the "flavor" of how the Sobolev spaces and solution operators enter into the picture, by studying in detail a simple special case, namely that of $1-\Delta$ on $\mathbb{R}^n_+$.

In the same way as the properties of $I-\Delta$ on $\mathbb{R}^n$ gave a motivational introduction to the properties of general elliptic operators on open sets (or manifolds), the properties of the Dirichlet and Neumann problems for $1-\Delta$ on $\mathbb{R}^n_+$ serve as a motivational introduction to properties of general elliptic boundary value problems. We shall treat this example in full detail. Here we also study the interpretation of the families of extensions of symmetric (and more general) operators described in Chapter 11. At the end we give an overview of what can be shown in general.

The Fourier transform was an important tool for making the treatment of $I-\Delta$ on $\mathbb{R}^n$ easy. When it is considered on $\mathbb{R}^n_+$, we shall use the partial Fourier transform (with respect to the $n-1$ first variables) to simplify things, so we start by setting up how that works.

We shall denote the *restriction* operator from $\mathbb{R}^n$ to $\mathbb{R}^n_+$ by r^+, and the restriction operator from $\mathbb{R}^n$ to $\mathbb{R}^n_-$ by r^-, where

$$\mathbb{R}^n_\pm = \{\, x = (x', x_n) \in \mathbb{R}^n \mid x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1},\ x_n \gtrless 0 \,\}.$$

Here we generally use the notation

$$r_\Omega u = u|_\Omega, \text{ and in particular } r^\pm = r_{\mathbb{R}^n_\pm}, \tag{9.2}$$

defined for $u \in \mathcal{D}'(\mathbb{R}^n)$. Moreover we need the *extension by* 0 operators e_Ω, in particular $e^+ = e_{\mathbb{R}^n_+}$ and $e^- = e_{\mathbb{R}^n_-}$, defined for functions f on Ω by

$$e_\Omega f = \begin{cases} f & \text{on } \Omega, \\ 0 & \text{on } \mathbb{R}^n \setminus \Omega; \end{cases} \qquad e^\pm = e_{\overline{\mathbb{R}}^n_+}. \tag{9.3}$$

The spaces of smooth functions on $\overline{\mathbb{R}}^n_+$ that we shall use will be the following:

$$\begin{aligned} C^\infty_{(0)}(\overline{\mathbb{R}}^n_+) &= r^+ C^\infty_0(\mathbb{R}^n) \\ \mathcal{S}(\overline{\mathbb{R}}^n_+) &= r^+ \mathcal{S}(\mathbb{R}^n), \end{aligned} \tag{9.4}$$

where r^+ is defined in (9.2). Clearly, $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+) \subset \mathcal{S}(\overline{\mathbb{R}}^n_+) \subset H^m(\mathbb{R}^n_+)$ for any m. The space $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ was defined in Chapter 4 as the space of functions on $\overline{\mathbb{R}}^n_+$ that are continuous and have continuous derivatives of all orders on $\overline{\mathbb{R}}^n_+$, and have compact support in $\overline{\mathbb{R}}^n_+$; that definition is consistent with (9.4). (In fact, every such function can be extended to a C^∞ function on $\mathbb{R}^n$; this is achieved by an old principle of Borel showing how to construct a C^∞ function with a given set of Taylor coefficients. A proof of this and related extension theorems can be found in Seeley [S 1964].)

We note that even though r^+ is the restriction to the *open* set $\mathbb{R}^n_+$ (which makes sense for arbitrary distributions in $\mathcal{D}'(\mathbb{R}^n)$), the functions in $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ and $\mathcal{S}(\overline{\mathbb{R}}^n_+)$ extend smoothly to the closed set $\overline{\mathbb{R}}^n_+$. When we consider (measurable) *functions*, it makes no difference whether we speak of the restriction to $\mathbb{R}^n_+$ or to $\overline{\mathbb{R}}^n_+$, but for *distributions* it would, since there exist nonzero distributions supported in the set $\{\, x \in \mathbb{R}^n \mid x_n = 0 \,\}$.

Let us define the partial Fourier transform (in the x' variable) for functions $u \in \mathcal{S}(\overline{\mathbb{R}}^n_+)$, by

$$\mathcal{F}_{x' \to \xi'} u = \acute{u}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} u(x', x_n)\, dx'. \tag{9.5}$$

The partial Fourier transform (9.5) can also be given a sense for arbitrary $u \in H^m(\mathbb{R}^n_+)$, and for suitable ditributions, but we shall do our calculations on rapidly decreasing functions whenever possible, and extend the statements by continuity.

Just as the standard norm $\|u\|_m$ on $H^m(\mathbb{R}^n)$ could be replaced by the norm $\|u\|_{m,\wedge}$ (6.16) involving $\langle\xi\rangle^m$, which is particularly suited for considerations using the Fourier transform, we can replace the standard norm $\|u\|_m$ on $H^m(\mathbb{R}^n_+)$ by an expression involving powers of $\langle\xi'\rangle$. Define for $u \in \mathcal{S}(\overline{\mathbb{R}}^n_+)$:

$$\begin{aligned} \|u\|^2_{m,\prime} &= (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} \int_0^\infty \sum_{j=0}^m \langle\xi'\rangle^{2(m-j)} |D^j_{x_n} \acute{u}(\xi', x_n)|^2\, dx_n d\xi' \\ &= (2\pi)^{1-n} \sum_{j=0}^m \|\langle\xi'\rangle^{2(m-j)} D^j_{x_n} \acute{u}(\xi', x_n)\|^2_{L_2(\mathbb{R}^n_+)}, \end{aligned} \tag{9.6}$$

with the associated scalar product

$$(u, v)_{m,\prime} = (2\pi)^{1-n} \sum_{j=0}^m (\langle\xi'\rangle^{(m-j)} D^j_{x_n} \acute{u}(\xi', x_n), \langle\xi'\rangle^{(m-j)} D^j_{x_n} \acute{v}(\xi', x_n))^2_{L_2(\mathbb{R}^n_+)}. \tag{9.7}$$

Lemma 9.1. *For $m \in \mathbb{N}_0$, there are inequalities*

$$\|u\|^2_m \le \|u\|^2_{m,\prime} \le C'_m \|u\|^2_m, \tag{9.8}$$

valid for all $u \in \mathcal{S}(\overline{\mathbb{R}}^n_+)$, and hence $\|u\|_{m,\prime}$ (with its associated scalar product) extends by continuity to a Hilbert space norm on $H^m(\mathbb{R}^n_+)$ equivalent with the standard norm. Here $C'_0 = 1$.

Proof. We have as in (5.2), for $k \in \mathbb{N}$:

$$\sum_{\beta \in \mathbb{N}_0^{n-1}, |\beta| \le k} (\xi')^{2\beta} \le \langle \xi' \rangle^{2k} = \sum_{|\beta| \le k} C_{k,\beta} (\xi')^{2\beta} \le C'_k \sum_{|\beta| \le k} (\xi')^{2\beta}, \tag{9.9}$$

with positive integers $C_{k,\beta}$ $(= \frac{k!}{\beta!(k-|\beta|)!})$ and $C'_k = \max_{\beta \in \mathbb{N}_0^{n-1}, |\beta| \le k} C_{k,\beta}$. (The prime indicates that we get another constant than in (5.2) since the dimension has been replaced by $n-1$.)

We then calculate as follows, denoting $D' = (D_1, \dots, D_{n-1})$, and using the Parseval-Plancherel theorem with respect to the x'-variables:

$$\begin{aligned}
\|u\|_m^2 &= \sum_{|\alpha| \le m} \|D^\alpha u\|_0^2 = \sum_{j=0}^m \sum_{|\beta| \le m-j} \|(D')^\beta D_{x_n}^j u\|_0^2 \\
&= \int_{\mathbb{R}^{n-1}} \sum_{j=0}^m \sum_{|\beta| \le m-j} (\xi')^{2\beta} \int_0^\infty |D_{x_n}^j \acute{u}(\xi', x_n)|^2 \, dx_n đ\xi' \\
&\le \int_{\mathbb{R}^{n-1}} \int_0^\infty \sum_{j=0}^m \langle \xi' \rangle^{2(m-j)} |D_{x_n}^j \acute{u}(\xi', x_n)|^2 \, dx_n đ\xi' \\
&\equiv \|u\|_{m,'}^2 \le C'_m \|u\|_m^2,
\end{aligned} \tag{9.10}$$

using (9.9) in the inequalities.

This shows (9.8), and the rest follows since the subset $C_{(0)}^\infty(\overline{\mathbb{R}}_+^n)$ of $\mathcal{S}(\overline{\mathbb{R}}_+^n)$, hence $\mathcal{S}(\overline{\mathbb{R}}_+^n)$ itself, is dense in $H^m(\mathbb{R}_+^n)$ (Theorem 4.10). $\square$

The extended norm is likewise denoted $\|u\|_{m,'}$. Note that in view of the formulas (9.9), we can also write:

$$\|u\|_{m,'}^2 = \sum_{j=0}^m \sum_{|\beta| \le m-j} C_{m-j,\beta} \|(D')^\beta D_n^j u\|_0^2, \tag{9.11}$$

where the norm is defined without reference to the partial Fourier transform.

The following is a sharper version of the trace theorem shown in Theorem 4.24 (and Exercise 4.21).

Theorem 9.2. *Let m be an integer > 0. For $0 \le j \le m-1$, the mapping*

$$\gamma_j \colon u(x', x_n) \mapsto D_{x_n}^j u(x', 0) \tag{9.12}$$

from $C_{(0)}^\infty(\overline{\mathbb{R}}_+^n)$ to $C_0^\infty(\mathbb{R}^{n-1})$ extends by continuity to a continuous linear mapping (also called γ_j) of $H^m(\mathbb{R}_+^n)$ into $H^{m-j-\frac{1}{2}}(\mathbb{R}^{n-1})$.

Proof. As in Theorem 4.24, we shall use an inequality like (4.45), but now in a slightly different and more precise way. For $v(t) \in C_0^\infty(\mathbb{R})$ one has:

$$
\begin{aligned}
|v(0)|^2 &= -\int_0^\infty \frac{d}{dt}[v(t)\bar{v}(t)]\,dt = -\int_0^\infty [v'(t)\bar{v}(t) + v(t)\overline{v'}(t)]\,dt \\
&\le 2\|v\|_{L_2(\mathbb{R}_+)}\|\frac{dv}{dt}\|_{L_2(\mathbb{R}_+)} \\
&\le a^2\|v\|^2_{L_2(\mathbb{R}_+)} + a^{-2}\|\frac{dv}{dt}\|^2_{L_2(\mathbb{R}_+)},
\end{aligned}
\tag{9.13}
$$

valid for any $a > 0$. For general functions $u \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ we apply the partial Fourier transform in x' (cf. (9.5)) and apply (9.13) with respect to x_n, setting $a = \langle\xi'\rangle^{\frac12}$ for each ξ' and using the norm (6.16) for $H^{m-j-\frac12}(\mathbb{R}^{n-1})$:

$$
\begin{aligned}
\|\gamma_j u\|^2_{H^{m-j-\frac12}(\mathbb{R}^{n-1})} &= \int_{\mathbb{R}^{n-1}} \langle\xi'\rangle^{2m-2j-1}|D^j_{x_n}\acute{u}(\xi',0)|^2 đ\xi' \\
&\le \int_{\mathbb{R}^{n-1}} \langle\xi'\rangle^{2m-2j-1}\int_0^\infty \big(a^2|D^j_{x_n}\acute{u}(\xi',x_n)|^2 \\
&\qquad + a^{-2}|D^{j+1}_{x_n}\acute{u}(\xi',x_n)|^2\big)\,dx_n đ\xi' \\
&= \int \big[\langle\xi'\rangle^{2(m-j)}|D^j_{x_n}\acute{u}(\xi',x_n)|^2 \\
&\qquad + \langle\xi'\rangle^{2(m-j-1)}|D^{j+1}_{x_n}\acute{u}(\xi',x_n)|^2\big]\,dx_n đ\xi' \\
&\le \|u\|^2_{m,\prime} \le C'_m\|u\|^2_m,
\end{aligned}
\tag{9.14}
$$

cf. (9.8). This shows the continuity of the mapping, for the dense subset $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ of $H^m(\mathbb{R}^n_+)$, so the mapping γ_j can be extended to all of $H^m(\mathbb{R}^n_+)$ by closure. $\square$

The range space in Theorem 9.2 is optimal, for one can show that the mappings γ_j are surjective. In fact this holds for the whole *system* of trace operators γ_j with $j = 0, \dots, m-1$. Let us for each $m > 0$ define the *Cauchy trace operator* $\rho_{(m)}$ associated with the order m by

$$
\rho_{(m)} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{m-1} \end{pmatrix} : H^m(\mathbb{R}^n_+) \to \prod_{j=0}^{m-1} H^{m-j-\frac12}(\mathbb{R}^{n-1});
\tag{9.15}
$$

for $u \in H^m(\mathbb{R}^n_+)$ we call $\rho_{(m)}u$ the *Cauchy data* of u. (The indexation with (m) may be omitted if it is understood from the context.) The space

$\prod_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\mathbb{R}^{n-1})$ is provided with the product norm $\|\varphi\|_{\prod H^{m-j-\frac{1}{2}}} = \big(\|\varphi_0\|^2_{m-\frac{1}{2}} + \cdots + \|\varphi_{m-1}\|^2_{\frac{1}{2}}\big)^{\frac{1}{2}}$. Then one can show that the mapping $\varrho_{(m)}$ in (9.15) is surjective.

We first give a proof of the result for γ_0, that shows the basic idea and moreover gives some insight into boundary value problems for $I - \Delta$. For $\varphi \in \mathcal{S}'(\mathbb{R}^{n-1})$ we denote its Fourier transform by $\hat{\varphi}$ (the usual notation, now applied with respect to the variable x').

Theorem 9.3. *Define the Poisson operator K_γ from $\mathcal{S}(\mathbb{R}^{n-1})$ to $\mathcal{S}(\overline{\mathbb{R}}^n_+)$ by*

$$K_\gamma \colon \varphi(x') \mapsto \mathcal{F}^{-1}_{\xi'\to x'}(e^{-\langle\xi'\rangle x_n}\hat{\varphi}(\xi')). \tag{9.16}$$

It satisfies:

1° $\gamma_0 K_\gamma \varphi = \varphi$ *for* $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})$.

2° $(I - \Delta)K_\gamma\varphi = 0$ *for* $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})$.

3° K_γ *extends to a continuous mapping (likewise denoted K_γ) from $H^{m-\frac{1}{2}}(\mathbb{R}^{n-1})$ to $H^m(\mathbb{R}^n_+)$ for any $m \in \mathbb{N}_0$; the identity in 1° extends to $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$, and the identity in 2° extends to $\varphi \in H^{-\frac{1}{2}}(\mathbb{R}^{n-1})$.*

Proof. It is easily checked that $e^{-\langle\xi'\rangle x_n}$ belong to $\mathcal{S}(\overline{\mathbb{R}}^n_+)$ (one needs to check derivatives of $e^{-\langle\xi'\rangle x_n}$, where (5.7) is useful); then also $e^{-\langle\xi'\rangle x_n}\hat{\varphi}(\xi')$ is in $\mathcal{S}(\overline{\mathbb{R}}^n_+)$, and so is its inverse partial Fourier transform; hence K_γ maps $\mathcal{S}(\mathbb{R}^{n-1})$ into $\mathcal{S}(\overline{\mathbb{R}}^n_+)$. Now

$$\gamma_0 K_\gamma \varphi = \big[\mathcal{F}^{-1}_{\xi'\to x'}(e^{-\langle\xi'\rangle x_n}\hat{\varphi}(\xi'))\big]_{x_n=0} = \mathcal{F}^{-1}_{\xi'\to x'}(\hat{\varphi}(\xi')) = \varphi$$

shows 1°. By partial Fourier transformation,

$$\begin{aligned}\mathcal{F}_{x'\to\xi'}((I-\Delta)K_\gamma\varphi) &= (1 + |\xi'|^2 - \partial^2_{x_n})(e^{-\langle\xi'\rangle x_n}\hat{\varphi}(\xi')) \\ &= (\langle\xi'\rangle^2 - \langle\xi'\rangle^2)e^{-\langle\xi'\rangle x_n}\hat{\varphi}(\xi') = 0;\end{aligned}$$

this implies 2°.

For 3°, consider first the case $m = 0$. Here we have:

$$\begin{aligned}\|K_\gamma\varphi\|_0^2 &= \int_{\mathbb{R}^{n-1}}\int_0^\infty |K_\gamma\varphi(x)|^2\,dx'dx_n \\ &= \int_{\mathbb{R}^{n-1}}\int_0^\infty e^{-2\langle\xi'\rangle x_n}|\hat{\varphi}(\xi')|^2\,đ\xi'dx_n \\ &= \int_{\mathbb{R}^{n-1}} (2\langle\xi'\rangle)^{-1}|\hat{\varphi}(\xi')|^2\,đ\xi'dx_n = \tfrac{1}{2}\|\varphi\|^2_{-\frac{1}{2},\wedge},\end{aligned}$$

showing that K_γ is not only continuous from $H^{-\frac12}(\mathbb{R}^{n-1})$ to $L_2(\mathbb{R}^n_+)$, but even proportional to an isometry (into the space).

In the cases $m > 0$, we must work a little more, but get simple formulas using the new norms:

$$
\begin{aligned}
\|K_\gamma\varphi\|_{m,\prime}^2 &= \int_{\mathbb{R}^{n-1}}\int_0^\infty \sum_{j=0}^m \langle\xi'\rangle^{2(m-j)}|D_{x_n}^j e^{-\langle\xi'\rangle x_n}\hat\varphi(\xi')|^2\, đ\xi' dx_n\\
&= \int_{\mathbb{R}^{n-1}}\int_0^\infty \sum_{j=0}^m \langle\xi'\rangle^{2m} e^{-2\langle\xi'\rangle x_n}|\hat\varphi(\xi')|^2\, đ\xi' dx_n \qquad (9.17)\\
&= \tfrac{m+1}{2}\int_{\mathbb{R}^{n-1}}\langle\xi'\rangle^{2m-1}|\hat\varphi(\xi')|^2\, đ\xi' dx_n = \tfrac{m+1}{2}\|\varphi\|^2_{m-\frac12,\wedge};
\end{aligned}
$$

again the mapping is proportional to an isometry.

Since γ_0 is well-defined as a continuous operator from $H^1(\mathbb{R}^n_+)$ to $H^{\frac12}(\mathbb{R}^{n-1})$ and K_γ is continuous in the opposite direction, the identity in 1° extends from the dense subset $\mathcal{S}(\mathbb{R}^{n-1})$ to $H^{\frac12}(\mathbb{R}^{n-1})$.

For 2°, let $\varphi\in H^{-\frac12}(\mathbb{R}^{n-1})$ and let φ_k be a sequence in $\mathcal{S}(\mathbb{R}^{n-1})$ converging to φ in $H^{-\frac12}(\mathbb{R}^{n-1})$. Then $K_\gamma\varphi_k$ converges to $v = K_\gamma\varphi$ in $L_2(\mathbb{R}^n_+)$. Here $(I-\Delta)K_\gamma\varphi_k = 0$ for all k, so in fact $K_\gamma\varphi_k$ converges to v in the graph norm for the maximal realization $A_{\max}$ defined from $I-\Delta$. Then $v\in D(A_{\max})$ with $A_{\max}v = 0$. So $(I-\Delta)v = 0$ in the distribution sense, in other words, $(I-\Delta)K_\gamma\varphi = 0$. This shows that 2° extends to $H^{-\frac12}(\mathbb{R}^{n-1})$. $\square$

Remark 9.4. The mapping K_γ can be extended still further down to "negative Sobolev spaces". One can define $H^s(\mathbb{R}^n_+)$ for all orders (including noninteger and negative values, and consistently with the definitions for $s\in\mathbb{N}_0$) by

$$
\begin{aligned}
H^s(\mathbb{R}^n_+) &= \{u\in\mathcal{D}'(\mathbb{R}^n_+) \mid u = U|_{\mathbb{R}^n_+} \text{ for some } U\in H^s(\mathbb{R}^n)\},\\
\|u\|_{s,\wedge} &= \inf_{\text{such } U}\|U\|_{s,\wedge};
\end{aligned} \qquad (9.18)
$$

and there is for any $s\in\mathbb{R}$ a continuous linear extension mapping $p_s : H^s(\mathbb{R}^n_+)\to H^s(\mathbb{R}^n)$ such that r^+p_s is the identity on $H^s(\mathbb{R}^n_+)$. Then one can show that K_γ extends to map $H^{s-\frac12}(\mathbb{R}^{n-1})$ continuously into $H^s(\mathbb{R}^n_+)$ for any $s\in\mathbb{R}$.

On the other hand, the mapping γ_0 *does not* extend to negative Sobolev spaces. More precisely, one can show that γ_0 makes sense on $H^s(\mathbb{R}^n_+)$ if and only if $s > \frac12$. To show the sufficiency, we use the inequality $|v(0)| \le \int_{\mathbb{R}}|\hat v(t)|\, đt$, and the calculation, valid for $s > \frac12$,

$$
\int\langle\xi\rangle^{-2s}\, đ\xi_n = \langle\xi'\rangle^{-2s+1}\int(1+|\xi_n/\langle\xi'\rangle|^2)^{-s}\, đ(\xi_n/\langle\xi'\rangle) = c_s\langle\xi'\rangle^{-2s+1}.
$$

Then we have for $u \in \mathcal{S}(\mathbb{R}^n)$:

$$
\begin{aligned}
\|u(x',0)\|^2_{s-\frac12,\wedge} &= \int_{\mathbb{R}^{-1}} |\acute{u}(\xi',0)|^2 \langle \xi' \rangle^{2s-1}\, d\!\!\bar{}\,\xi' \\
&\le c_1 \int_{\mathbb{R}^{n-1}} \langle \xi' \rangle^{2s-1} \big(\int_{\mathbb{R}} |\hat{u}(\xi)|\, d\!\!\bar{}\,\xi_n \big)^2\, d\!\!\bar{}\,\xi' \\
&\le c_2 \int_{\mathbb{R}^{n-1}} \langle \xi' \rangle^{2s-1} \big(\int_{\mathbb{R}} |\hat{u}(\xi)|^2 \langle \xi \rangle^{2s}\, d\!\!\bar{}\,\xi_n \big) \big(\int_{\mathbb{R}} \langle \xi \rangle^{-2s}\, d\!\!\bar{}\,\xi_n \big)\, d\!\!\bar{}\,\xi' \\
&= c_3 \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2\, d\!\!\bar{}\,\xi = c_3 \|u\|^2_{s,\wedge}.
\end{aligned}
$$

By density, this defines a bounded mapping $\tilde{\gamma}_0$ from $H^s(\mathbb{R}^n)$ to $H^{s-\frac12}(\mathbb{R}^{n-1})$, and it follows that γ_0 extends to a bounded mapping from $H^s(\mathbb{R}^n_+)$ to $H^{s-\frac12}(\mathbb{R}^{n-1})$. The necessity of $s > \frac12$ is seen in case $n = 1$ as follows: Assume that $\gamma_0 : u \mapsto u(0)$ is bounded from $H^s(\mathbb{R})$ to $\mathbb{C}$ when $u \in \mathcal{S}(\mathbb{R})$. Write

$$
u(0) = \int_{\mathbb{R}} \hat{u}(t)\, d\!\!\bar{}\,t = \int_{\mathbb{R}} \langle t \rangle^{-s} \langle t \rangle^{s} \hat{u}(t)\, d\!\!\bar{}\,t = \langle \langle t \rangle^{-s}, g(t) \rangle,
$$

where $g(t) = \frac{1}{2\pi} \langle t \rangle^s \hat{u}(t)$ likewise runs through $\mathcal{S}(\mathbb{R})$. The boundedness implies that

$$
|u(0)| = |\langle \langle t \rangle^{-s}, g(t) \rangle| \le c\|u\|_{s,\wedge} = c'\|g\|_{L_2(\mathbb{R})},
$$

which can only hold when $\langle t \rangle^{-s} \in L_2$, i.e., $s > \frac12$. The general case is treated in [LM 1968, Th. I 4.3].

Nevertheless, γ_0 does have a sense, for any $s \in \mathbb{R}$, on the subset of distributions $u \in H^s(\mathbb{R}^n_+)$ for which $(I - \Delta)u = 0$. We shall show this below for $s = 0$; the general result is found in Lions and Magenes [LM 1968].

To show the surjectiveness of the Cauchy data map $\varrho_{(m)}$, one can construct a right inverse as a linear combination of operators defined as in (9.16) with $x_n^j e^{-\langle \xi' \rangle x_n}$, $j = 0, \dots, m-1$, inserted. Here is another choice that is easy to check (and has the advantage that it could be used for $\mathbb{R}^n_-$ too):

Theorem 9.5. *Let $\psi \in \mathcal{S}(\mathbb{R})$ with $\psi(t) = 1$ on a neighborhood of 0. Define the Poisson operator $\mathcal{K}_{(m)}$ from $\varphi = \{\varphi_0, \dots, \varphi_{m-1}\} \in \mathcal{S}(\mathbb{R}^{n-1})^m$ to $\mathcal{K}_{(m)}\varphi \in \mathcal{S}(\overline{\mathbb{R}}^n_+)$ by:*

$$
\begin{aligned}
&\mathcal{K}_{(m)}\varphi = \mathcal{K}_0\varphi_0 + \dots + \mathcal{K}_{m-1}\varphi_{m-1}, \ \text{where} \\
&\mathcal{K}_j\varphi_j = \mathcal{F}^{-1}_{\xi' \to x'}\big(\tfrac{1}{j!} i^j x_n^j \psi(\langle \xi' \rangle x_n) \hat{\varphi}_j(\xi')\big), \ \text{for } j = 0, \dots, m-1.
\end{aligned}
\tag{9.19}
$$

It satisfies
1° $\varrho_{(m)}\mathcal{K}_{(m)}\varphi = \varphi$ for $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})^m$.

2° *$\mathcal{K}_{(m)}$ extends to a continuous mapping (likewise denoted $\mathcal{K}_{(m)}$) from $\Pi_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\mathbb{R}^{n-1})$ to $H^m(\mathbb{R}^n_+)$, and the identity in 1° extends to $\varphi \in \Pi_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\mathbb{R}^{n-1})$.*

Proof. Since $\psi(0) = 1$ and $D^j_{x_n}\psi(0) = 0$ for $j > 0$, and $[D^k_{x_n}(\frac{1}{j!}i^j x_n^j)]_{x_n=0} = \delta_{kj}$ (the Kronecker delta),

$$\gamma_k \mathcal{K}_j \varphi_j = (D^k_{x_n} \tfrac{1}{j!} i^j x_n^j)|_{x_n=0}\varphi_j = \delta_{kj}\varphi_j; \tag{9.20}$$

showing 1°.

For the estimates in 2°, we observe the formulas that arise from replacing x_n by $t = \langle \xi' \rangle x_n$ in the integrals:

$$\int_0^\infty |D^k_{x_n}(x_n^j \psi(\langle \xi' \rangle x_n)|^2\, dx_n \\ = \langle \xi' \rangle^{2k-2j-1} \int_0^\infty |D^k_t(t^j\psi(t))|^2\, dt \equiv \langle \xi' \rangle^{2k-2j-1} c_{kj}. \tag{9.21}$$

Then

$$\begin{aligned} &\|\mathcal{K}_j\varphi_j\|^2_{m,\prime} \\ &= \int_{\mathbb{R}^{n-1}} \int_0^\infty \sum_{k\le m} \langle \xi' \rangle^{2m-2k} |D^k_{x_n}[\tfrac{(ix_n)^j}{j!}\psi(\langle \xi' \rangle x_n)]\hat{\varphi}_j(\xi')|^2\, đ\xi' dx_n \\ &= \int_{\mathbb{R}^{n-1}} \sum_{k\le m} \langle \xi' \rangle^{2m-2k} \langle \xi' \rangle^{2k-2j-1} c_{kj} \tfrac{1}{j!} |\hat{\varphi}_j(\xi')|^2\, đ\xi' \\ &= c_j \|\varphi_j\|^2_{m-j-\frac{1}{2},\wedge}, \end{aligned}$$

which shows the desired boundedness estimate. Now the identity in 1° extends by continuity. $\square$

We know from Theorem 4.25 that the space $H^1_0(\mathbb{R}^n_+)$, defined as the closure of $C_0^\infty(\mathbb{R}^n_+)$ in $H^1(\mathbb{R}^n_+)$, is exactly the space of elements u of $H^1(\mathbb{R}^n_+)$ for which $\gamma_0 u = 0$. This fact extends to higher Sobolev spaces. As usual, $H^m_0(\mathbb{R}^n_+)$ is defined as the closure of $C_0^\infty(\mathbb{R}^n_+)$ in $H^m(\mathbb{R}^n_+)$.

Theorem 9.6. *For all $m \in \mathbb{N}$,*

$$H^m_0(\mathbb{R}^n_+) = \{u \in H^m(\mathbb{R}^n_+) \mid \gamma_0 u = \cdots = \gamma_{m-1} u = 0\}. \tag{9.22}$$

This is proved by a variant of the proof of Theorem 4.25, or by the method described after it, and will not be written in detail here. We observe another interesting fact concerning $H^m_0(\mathbb{R}^n_+)$:

Theorem 9.7. *For each $m \in \mathbb{N}_0$, $H_0^m(\mathbb{R}_+^n)$ identifies, by extension by zero on $\mathbb{R}_-^n$, with the subspace of $H^m(\mathbb{R}^n)$ consisting of the functions supported in $\overline{\mathbb{R}}_+^n$.*

Proof. When $u \in H_0^m(\mathbb{R}_+^n)$, it is the limit of a sequence of function $u_k \in C_0^\infty(\mathbb{R}_+^n)$ in the m-norm. Extending the u_k by 0 (to e^+u_k), we see that $e^+u \in H^m(\mathbb{R}^n)$ and is supported in $\overline{\mathbb{R}}_+^n$. All such functions are reached, for if $v \in H^m(\mathbb{R}^n)$ has support in $\overline{\mathbb{R}}_+^n$, we can approximate it in m-norm by functions in $C_0^\infty(\mathbb{R}_+^n)$ by 1) truncation, 2) translation into $\mathbb{R}_+^n$ and 3) mollification. □

For any $s \in \mathbb{R}$, one can define the closed subspace of $H^s(\mathbb{R}^n)$:

$$H_0^s(\overline{\mathbb{R}}_+^n) = \{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\mathbb{R}}_+^n\}; \tag{9.22a}$$

it identifies with $H_0^m(\mathbb{R}_+^n)$ when $s = m \in \mathbb{N}_0$. For negative s (more precisely, when $s \le -\frac{1}{2}$), this is *not* a space of distributions on $\mathbb{R}_+^n$. It serves as a dual space: For any s one can show that $H^s(\mathbb{R}_+^n)$ and $H_0^{-s}(\overline{\mathbb{R}}_+^n)$ are dual spaces of one another, with a duality extending the scalar product in $L_2(\mathbb{R}_+^n)$, similarly to Theorem 6.13. (The duality is shown e.g. in [H 1963, Sect. 2.5].)

9.2. The Dirichlet problem for $I - \Delta$ on the half-space.

We now consider the elliptic operator $A = I - \Delta$ on $\mathbb{R}_+^n$. First we will show that γ_0 can be extended to $D(A_{\max})$. An important ingredient is the following denseness result (adapted from [LM 1968]):

Theorem 9.8. *The space $C_{(0)}^\infty(\overline{\mathbb{R}}_+^n)$ is dense in $D(A_{\max})$.*

Proof. This follows if we show that when ℓ is a continuous antilinear (conjugate linear) functional on $D(A_{\max})$ which vanishes on $C_{(0)}^\infty(\overline{\mathbb{R}}_+^n)$, then $\ell = 0$. So let ℓ be such a functional; it can be written

$$\ell(u) = (f, u)_{L_2(\mathbb{R}_+^n)} + (g, Au)_{L_2(\mathbb{R}_+^n)} \tag{9.23}$$

for some $f, g \in L_2(\mathbb{R}_+^n)$. We know that $\ell(\varphi) = 0$ for $\varphi \in C_{(0)}^\infty(\overline{\mathbb{R}}_+^n)$. Any such φ is the restriction to $\mathbb{R}_+^n$ of a function $\Phi \in C_0^\infty(\mathbb{R}^n)$, and in terms of such functions we have

$$\ell(r^+\Phi) = (e^+f, \Phi)_{L_2(\mathbb{R}^n)} + (e^+g, (I-\Delta)\Phi)_{L_2(\mathbb{R}^n)} = 0, \text{ all } \Phi \in C_0^\infty(\mathbb{R}^n). \tag{9.24}$$

Now use that $I - \Delta$ on $\mathbb{R}^n$ has the formal adjoint $I - \Delta$, so the equations (9.24) imply

$$\langle e^+f + (I-\Delta)e^+g, \overline{\Phi}\rangle = 0, \text{ all } \Phi \in C_0^\infty(\mathbb{R}^n),$$

i.e.,

$$e^+f + (I-\Delta)e^+g = 0, \text{ or } (I-\Delta)e^+g = -e^+f, \tag{9.25}$$

as distributions on $\mathbb{R}^n$. Here we know that e^+g and e^+f are in $L_2(\mathbb{R}^n)$, and Theorem 6.10 then gives that $e^+g \in H^2(\mathbb{R}^n)$. Since it has support in $\overline{\mathbb{R}}^n_+$, it identifies with a function in $H^2_0(\mathbb{R}^n_+)$ by Theorem 9.7, i.e., $g \in H^2_0(\mathbb{R}^n_+)$. Then by Theorem 6.20, g is in $D(A_{\min})$! And (9.25) implies that $Ag = -f$. But then, for any $u \in D(A_{\max})$,

$$\ell(u) = (f,u)_{L_2(\mathbb{R}^n_+)} + (g,Au)_{L_2(\mathbb{R}^n_+)} = -(Ag,u)_{L_2(\mathbb{R}^n_+)} + (g,Au)_{L_2(\mathbb{R}^n_+)} = 0,$$

since $A_{\max}$ and $A_{\min}$ are adjoints. $\square$

Lemma 9.9. *For u and v in $H^2(\mathbb{R}^n_+)$ one has Green's formula*

$$(Au,v)_{L_2(\mathbb{R}^n_+)} - (u,Av)_{L_2(\mathbb{R}^n_+)} = (i\gamma_1 u, \gamma_0 v)_{L_2(\mathbb{R}^{n-1})} - (\gamma_0 u, i\gamma_1 v)_{L_2(\mathbb{R}^{n-1})}. \tag{9.26}$$

Proof. For smooth, compactly supported functions, the formula follows directly from (1.30), when we note that $\gamma_1 u = \frac{1}{i}\frac{\partial u}{\partial n}$. (It is easily verified directly by integration by parts.) Then it extends by continuity to $H^2(\mathbb{R}^n_+)$, using Theorem 9.2. $\square$

Denoting $\{\gamma_0 u, \gamma_1 u\} = \varrho u$ (column vector), we can also write Green's formula as

$$(Au,v) - (u,Av) = (\mathcal{A}\varrho u, \varrho v), \quad \varrho u = \begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{9.27}$$

where the scalar product in the right-hand side is in $L_2(\mathbb{R}^{n-1}) \times L_2(\mathbb{R}^{n-1})$.

Now we can show:

Theorem 9.10. *Let $A = I - \Delta$ on $\mathbb{R}^n_+$. The Cauchy trace operator $\varrho = \{\gamma_0, \gamma_1\}$, defined on $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, extends by continuity to a continuous mapping from $D(A_{\max})$ to $H^{-\frac{1}{2}}(\mathbb{R}^{n-1}) \times H^{-\frac{3}{2}}(\mathbb{R}^{n-1})$. Here Green's formula (9.26) extends to the formula*

$$(Au,v)_{L_2(\mathbb{R}^n_+)} - (u,Av)_{L_2(\mathbb{R}^n_+)} = \langle i\gamma_1 u, \overline{\gamma_0 v}\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}} - \langle \gamma_0 u, \overline{i\gamma_1 v}\rangle_{H^{-\frac{3}{2}},H^{\frac{3}{2}}}, \tag{9.29}$$

for $u \in D(A_{\max})$, $v \in H^2(\mathbb{R}^n_+)$.

Moreover, $\gamma_0 K_\gamma = I$ on $H^{-\frac{1}{2}}(\mathbb{R}^{n-1})$.

Proof. Let $u \in D(A_{\max})$. In the following, we write $H^s(\mathbb{R}^{n-1})$ as H^s. We want to define $\varrho u = \{\gamma_0 u, \gamma_1 u\}$ as a continuous antilinear functional on $H^{\frac{1}{2}} \times$

$H^{\frac{3}{2}}$, depending continuously (and of course linearly) on $u \in D(A_{\max})$. For a given $\varphi = \{\varphi_0, \varphi_1\} \in H^{\frac{1}{2}} \times H^{\frac{3}{2}}$, let

$$w_\varphi = \mathcal{K}_{(2)}(\mathcal{A}^*)^{-1}\varphi = i\mathcal{K}_0\varphi_1 + i\mathcal{K}_1\varphi_0$$

(using Theorem 9.5), and set

$$\begin{aligned} &\ell_u(\varphi) = (Au, w_\varphi) - (u, Aw_\varphi), \text{ noting that} \\ &|\ell_u(\varphi)| \le C\|u\|_{D(A_{\max})}\|w_\varphi\|_{H^2(\mathbb{R}^n_+)} \le C'\|u\|_{D(A_{\max})}\|\varphi\|_{H^{\frac{1}{2}}\times H^{\frac{3}{2}}}. \end{aligned} \tag{9.30}$$

So, ℓ_u is a continuous antilinear functional on $\varphi \in H^{\frac{1}{2}} \times H^{\frac{3}{2}}$, hence defines an element $\psi = \{\psi_0, \psi_1\} \in H^{-\frac{1}{2}} \times H^{-\frac{3}{2}}$ such that

$$l_u(\varphi) = \langle \psi_0, \overline{\varphi_0}\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} + \langle \psi_1, \overline{\varphi_1}\rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}}.$$

Moreover, it depends continuously on $u \in D(A_{\max})$, in view of the estimates in (9.30). If u is in $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, the defining formula in (9.30) can be rewritten using Green's formula (9.27), which shows that

$$l_u(\varphi) = (Au, w_\varphi) - (u, Aw_\varphi) = (\varrho u, \mathcal{A}^*\varrho w_\varphi) = (\begin{pmatrix}\gamma_0 u \\ \gamma_1 u\end{pmatrix}, \begin{pmatrix}\varphi_0 \\ \varphi_1\end{pmatrix})$$

then. Since φ_0 and φ_1 run through full Sobolev spaces, it follows that $\psi_0 = \gamma_0 u$, $\psi_1 = \gamma_1 u$, when $u \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, so the functional ℓ_u is consistent with $\{\gamma_0 u, \gamma_1 u\}$ then. Since $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ is dense in $D(A_{\max})$, we have found the unique continuous extension.

(9.29) is now obtained in general by extending (9.26) by continuity from $u \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, $v \in H^2(\mathbb{R}^n_+)$.

For the last statement, let $\eta \in H^{-\frac{1}{2}}(\mathbb{R}^{n-1})$. Note that we have already shown in Theorem 9.3 that K_γ maps $\eta \in H^{-\frac{1}{2}}$ into a function in $L_2(\mathbb{R}^n_+)$ satisfying $(I - \Delta)K_\gamma\eta = 0$. But then $K_\gamma\eta \in D(A_{\max})$, so γ_0 can be applied in the new sense. The formula $\gamma_0 K_\gamma \eta = \eta$ is then obtained by extension by continuity from the case $\eta \in \mathcal{S}(\mathbb{R}^{n-1})$. $\square$

The defintion goes via a choice w_φ of the "lifting" of φ, but the final result shows that the definition is independent of this choice.

Consider now the Dirichlet problem

$$\begin{aligned} Au &= f \text{ in } \mathbb{R}^n_+, \\ \gamma_0 u &= \varphi \text{ on } \mathbb{R}^{n-1}, \end{aligned} \tag{9.31}$$

and its two semi-homogeneous versions

$$\begin{aligned} Au &= f \text{ in } \mathbb{R}^n_+, \\ \gamma_0 u &= 0 \text{ on } \mathbb{R}^{n-1}, \end{aligned} \tag{9.32}$$

$$\begin{aligned} Az &= 0 \text{ in } \mathbb{R}^n_+, \\ \gamma_0 z &= \varphi \text{ on } \mathbb{R}^{n-1}. \end{aligned} \tag{9.33}$$

For (9.32), we find by application of the variational theory in Section 10.4 to the triple $(L_2(\mathbb{R}^n_+), H^1_0(\mathbb{R}^n_+), a(u,v))$ with $a(u,v) = (u,v)_1$, the variational operator A_γ, which is selfadjoint with lower bound 1. It is seen as in Section 4.4 (cf. Exercise 4.22), that A_γ is a realization of A with domain $D(A_\gamma) = D(A_{\max}) \cap H^1_0(\mathbb{R}^n_+)$; the Dirichlet realization. Thus problem (9.32) has for $f \in L_2(\mathbb{R}^n_+)$ the unique solution $u = A_\gamma^{-1} f$ in $D(A_{\max}) \cap H^1_0(\mathbb{R}^n_+)$.

For (9.33), we have just shown that when $m \in \mathbb{N}_0$, it has a solution in $H^m(\mathbb{R}^n_+)$ for any $\varphi \in H^{m-\frac12}(\mathbb{R}^{n-1})$, namely $K_\gamma \varphi$, by Theorem 9.3 and the last statement in Theorem 9.10. We have not yet investigated the uniqueness of the latter solution; it will be obtained below.

We can improve the information on (9.32) by use of K_γ and the structure of $I - \Delta$: To solve (9.32) for a given $f \in L_2(\mathbb{R}^n_+)$, introduce

$$v = r^+ Q e^+ f, \quad Q = \operatorname{Op}(\langle \xi \rangle^{-2}). \tag{9.34}$$

Since Q maps $L_2(\mathbb{R}^n)$ homeomorphically onto $H^2(\mathbb{R}^n)$ (Section 6.2), $r^+ Q e^+$ is continuous from $L_2(\mathbb{R}^n_+)$ to $H^2(\mathbb{R}^n_+)$. Application of the differential operator $A = I - \Delta$ to v gives that

$$Av = Ar^+ Q e^+ f = r^+ (I - \Delta) Q e^+ f = f,$$

so when u solves (9.32), the function $z = u - v$ must solve (9.33) with $\varphi = -\gamma_0 r^+ Q e^+ f$; here φ lies in $H^{\frac32}(\mathbb{R}^{n-1})$. The problem for z has a solution $z = K_\gamma \varphi = -K_\gamma \gamma_0 r^+ Q e^+ f \in H^2(\mathbb{R}^n_+)$. Writing $u = v + z$, we finally obtain a solution of (9.32) of the form

$$u = r^+ Q e^+ f - K_\gamma \gamma_0 r^+ Q e^+ f. \tag{9.35}$$

It lies in $H^2(\mathbb{R}^n_+) \subset D(A_{\max})$, and in $H^1_0(\mathbb{R}^n_+)$ since $\gamma_0 u = 0$, so it must equal the unique solution already found. This improves the regularity! We have shown:

Theorem 9.11. *The solution operator A_γ^{-1} of* (9.32) *equals*

$$A_\gamma^{-1} = r^+Qe^+ - K_\gamma\gamma_0 r^+Qe^+, \text{ also denoted } R_\gamma; \tag{9.36}$$

it maps $L_2(\mathbb{R}^n_+)$ continuously into $H^2(\mathbb{R}^n_+)$ and hence

$$D(A_\gamma) = H^2(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+). \tag{9.37}$$

For the special case we are considering, this gives the optimal regularity statement for the problem (9.32) with $f \in L_2(\mathbb{R}^n_+)$. And not only that; it also gives a formula (9.36) which is more constructive than the existence-and-uniqueness statement we had from the variational theory.

Remark 9.12. With further efforts it can be shown that r^+Qe^+ maps $H^k(\mathbb{R}^n_+)$ into $H^{k+2}(\mathbb{R}^n_+)$ also for $k \geq 1$; this is not clear from its form, since e^+ does not map $H^k(\mathbb{R}^n_+)$ into $H^k(\mathbb{R}^n)$ for $k \geq 1$, but hinges on the so-called transmission property of Q. Then one can obtain that u defined by (9.35) is in $H^{k+2}(\mathbb{R}^n_+)$, when $f \in H^k(\mathbb{R}^n_+)$: higher elliptic regularity.

As a corollary, we obtain a uniqueness statement for (9.33) in a regular case:

Corollary 9.13. *For any $\varphi \in H^{\frac{3}{2}}(\mathbb{R}^{n-1})$, $K_\gamma\varphi$ is the unique solution in $H^2(\mathbb{R}^n_+)$ of* (9.33).

Proof. Let $v = K_\gamma\varphi$, then $v \in H^2(\mathbb{R}^n_+)$ with $Av = 0$ and $\gamma_0 v = \varphi$. So a function z solves (9.33) if and only if $w = z - v$ solves

$$Aw = 0, \quad \gamma_0 w = 0.$$

Here if $z \in H^2(\mathbb{R}^n_+)$, also $w \in H^2(\mathbb{R}^n_+)$, so by the uniqueness of solutions of (9.32) in $H^2(\mathbb{R}^n_+)$, w must equal 0. $\square$

The argument can be extended to cover $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$, $z \in H^1(\mathbb{R}^n_+)$, but since we are aiming for $z \in L_2(\mathbb{R}^n_+)$ we go on towards that case (and include the H^1-case afterwards).

Define, for $k \in \mathbb{N}_0$,

$$Z^k(A) = \{z \in H^k(\mathbb{R}^n_+) \mid Az = 0\}, \text{ closed subspace of } H^k(\mathbb{R}^n_+); \tag{9.38}$$

note that $Z^0(A) = Z(A_{\max})$.

Corollary 9.13 and Theorem 9.3 together imply:

Corollary 9.14. *The mappings*

$$\gamma_0\colon Z^2(A) \to H^{\frac{3}{2}}(\mathbb{R}^{n-1}) \text{ and } K_\gamma\colon H^{\frac{3}{2}}(\mathbb{R}^{n-1}) \to Z^2(A)$$

are inverses of one another.

Proof. The mappings are well-defined according to Theorem 9.3, which also shows the identity $\gamma_0 K_\gamma = I$ on $H^{\frac{3}{2}}(\mathbb{R}^{n-1})$; it implies surjectiveness of γ_0 and injectiveness of K_γ. Corollary 9.13 implies that when $z \in Z^2(A)$, $K_\gamma \gamma_0 z = z$, so $K_\gamma\colon H^{\frac{3}{2}}(\mathbb{R}^{n-1}) \to Z^2(A)$ is surjective. □

Now we can show:

Proposition 9.15. *$Z^2(A)$ is dense in $Z^0(A)$.*

Proof. Let $z \in Z^0(A)$. By Theorem 9.8 there exists a sequence $u_k \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ such that $u_k \to z$ in $L_2(\mathbb{R}^n_+)$, $Au_k \to 0$. Let $v_k = A_\gamma^{-1} A u_k$, then $v_k \to 0$ in $H^2(\mathbb{R}^n_+)$ by Theorem 9.11, so also $z_k = u_k - v_k \to z$ in $D(A_{\max})$, with $z_k \in H^2(\mathbb{R}^n_+)$. Here $Au_k = Av_k$ by definition, so indeed $z_k \in Z^2(A)$. □

Then we finally obtain:

Theorem 9.16. *The mappings*

$$\gamma_0\colon Z^0(A) \to H^{-\frac{1}{2}}(\mathbb{R}^{n-1}) \text{ and } K_\gamma\colon H^{-\frac{1}{2}}(\mathbb{R}^{n-1}) \to Z^0(A)$$

are inverses of one another.

Proof. We already have the identity $\gamma_0 K_\gamma \varphi = \varphi$ for $\varphi \in H^{-\frac{1}{2}}(\mathbb{R}^{n-1})$ from Theorem 9.10, and the other identity $K_\gamma \gamma_0 z = z$ for $z \in Z^0(A)$ now follows from the identity valid on $Z^2(A)$ by extension by continuity, using Proposition 9.15. □

Corollary 9.17. *Let $m \in \mathbb{N}_0$. The mappings*

$$\gamma_0\colon Z^m(A) \to H^{m-\frac{1}{2}}(\mathbb{R}^{n-1}) \text{ and } K_\gamma\colon H^{m-\frac{1}{2}}(\mathbb{R}^{n-1}) \to Z^m(A) \tag{9.39}$$

are inverses of one another.

Proof. That the mappings are well-defined follows from Theorem 9.3 and its extension in Theorem 9.10. Then the statements $\gamma_0 K_\gamma \varphi = \varphi$ and $K_\gamma \gamma_0 z = z$ for the relevant spaces follow by restriction from Theorem 9.16. □

Theorem 9.11 and Corollary 9.17 give highly satisfactory results on the solvability of the semi-homogeneous problems (9.32) and (9.33). We collect some consequences for the nonhomogeneous problem (9.31):

Theorem 9.18. *The problem* (9.31) *is uniquely solvable in* $D(A_{\max})$ *for* $f \in L_2(\mathbb{R}^n_+)$, $\varphi \in H^{\frac{3}{2}}(\mathbb{R}^{n-1})$*; the solution belongs to* $H^2(\mathbb{R}^n_+)$ *and is defined by the formula*

$$u = r^+Qe^+f - K_\gamma\gamma_0 r^+Qe^+f + K_\gamma\varphi = R_\gamma f + K_\gamma\varphi; \tag{9.40}$$

cf. (9.36)*. In particular, the mapping*

$$\begin{pmatrix} A \\ \\ \gamma_0 \end{pmatrix} : H^2(\mathbb{R}^n_+) \to \begin{matrix} L_2(\mathbb{R}^n_+) \\ \times \\ H^{\frac{3}{2}}(\mathbb{R}^{n-1}) \end{matrix} \tag{9.41}$$

has the inverse

$$(R_\gamma \quad K_\gamma) : \begin{matrix} L_2(\mathbb{R}^n_+) \\ \times \\ H^{\frac{3}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to H^2(\mathbb{R}^n_+). \tag{9.42}$$

The above gives a complete clarification of the basic solvability properties for the Dirichlet problem (9.31) in the very special case of the simple constant-coefficient operator $I - \Delta$ on $\mathbb{R}^n_+$.

The deduction is of general interest. In fact, one can to some extent treat Dirichlet problems for elliptic operators A (with a certain positivity, so-called strong ellipticity) on bounded smooth domains Ω according to the same scheme:

First, by use of localizations as in Sections 2.4 and 4.2, one can establish mapping properties of the Cauchy trace operator analogous to (9.15). Moreover, one can establish a lifting operator $\mathcal{K}$ (a Poisson operator) with properties similar to those in Theorem 9.5. For A one can define a variational realization A_γ representing the Dirichlet problem and (possibly after addition of a constant) invertible. The denseness of $C^\infty(\overline{\Omega})$ in the maximal domain, the generalized trace operators and the extended Green' formula can be obtained much as in Theorems 9.7 and 9.9.

It is somewhat harder to extend Theorem 9.3 which defines the Poisson operator K_γ mapping into the nullspace, to the general situation, though; one can easily get a first approximation, but from then on there is as much work in it as in the general treatment of (9.31). The strategy in Lions and Magenes [LM 1968] is to begin with the regularity theory for the fully nonhomogeneous boundary value problem, next to pass to an adjoint situation in negative Sobolev spaces, and finally use interpolation theory. (Their method works for far more general boundary conditions too.) Methods establishing K_γ more directly can be inferred from the theory of Boutet de Monvel [BM 1971] (for which a detailed presentation can be found in [G 1996]), and the approach of Seeley [S 1966] (also taken up in [G 1996, 1.3.5], and in lecture notes for the course Math. 4MA given some years ago at Copenhagen University).

Note also that in the case of $I-\Delta$ we could use the straightforward inverse $Q=\operatorname{Op}(\langle\xi\rangle^{-2})$ on $\mathbb{R}^n$; in general cases it will be replaced by more complicated expressions.

In the calculus of pseudodifferential boundary problems initiated by Boutet de Monvel, operators like r^+Qe^+ are called *truncated pseudodifferential operators*, operators like K_γ and $\mathcal{K}_{(m)}$ are called *Poisson operators* (we have already used this name), and operators like $K_\gamma\gamma_0$ and $K_\gamma\gamma_0 r^+Qe^+$, acting on functions on $\mathbb{R}^n_+$ but not of the form r^+Pe^+ with a ps.d.o. P, are called *singular Green operators*.

9.3. The Neumann problem for $I-\Delta$ on the half-space.

For the special operator $I-\Delta$ it is now also easy to discuss the Neumann problem on $\mathbb{R}^n_+$. To stick with standard notation, we let the entering trace operator ν be defined in terms of the normal derivative ∂_{x_n} *without a factor* $-i$, i.e.,

$$\nu u=\gamma_0\partial_{x_n}u\;\big(=i\gamma_1 u\big). \tag{9.43}$$

First we have, similarly to Theorem 9.3:

Theorem 9.19. *Define the Poisson operator K_ν from $\mathcal{S}(\mathbb{R}^{n-1})$ to $\mathcal{S}(\overline{\mathbb{R}}^n_+)$ by*

$$K_\nu\colon \varphi(x')\mapsto \mathcal{F}^{-1}_{\xi'\to x'}(-\langle\xi'\rangle^{-1}e^{-\langle\xi'\rangle x_n}\hat\varphi(\xi')). \tag{9.44}$$

It satisfies:

1° $\nu K_\nu\varphi=\varphi$ *for* $\varphi\in\mathcal{S}(\mathbb{R}^{n-1})$.

2° $(I-\Delta)K_\nu\varphi=0$ *for* $\varphi\in\mathcal{S}(\mathbb{R}^{n-1})$.

3° K_ν *extends to a continuous mapping (likewise denoted K_ν) from $H^{m-\frac32}(\mathbb{R}^{n-1})$ to $H^m(\mathbb{R}^n_+)$ for any $m\in\mathbb{N}_0$; the identity in 1° extends to $\varphi\in H^{\frac12}(\mathbb{R}^{n-1})$, and the identity in 2° extends to $\varphi\in H^{-\frac32}(\mathbb{R}^{n-1})$.*

Proof. 1° is seen from

$$\nu K_\nu\varphi=\big[\mathcal{F}^{-1}_{\xi'\to x'}(-\partial_{x_n}\langle\xi'\rangle^{-1}e^{-\langle\xi'\rangle x_n}\hat\varphi(\xi'))\big]_{x_n=0}=\mathcal{F}^{-1}_{\xi'\to x'}(\hat\varphi(\xi'))=\varphi.$$

2° holds since

$$\begin{aligned}\mathcal{F}_{x'\to\xi'}((I-\Delta)K_\nu\varphi)&=(\langle\xi'\rangle^2-\partial^2_{x_n})(-\langle\xi'\rangle^{-1}e^{-\langle\xi'\rangle x_n}\hat\varphi(\xi'))\\&=(-\langle\xi'\rangle+\langle\xi'\rangle)e^{-\langle\xi'\rangle x_n}\hat\varphi(\xi')=0.\end{aligned}$$

For the continuity statements in 3°, we calculate as in (9.17), with the modification that the extra factor $-\langle\xi'\rangle^{-1}$ changes the norm on φ to be the norm in $H^{m-\frac32}(\mathbb{R}^{n-1})$.

Since ν is well-defined as a continuous operator from $H^2(\mathbb{R}^n_+)$ to $H^{\frac12}(\mathbb{R}^{n-1})$ by Theorem 9.2, and K_ν is continuous in the opposite direction, the identity in 1° extends by continuity to $H^{\frac12}(\mathbb{R}^{n-1})$. The proof of the extension of 2° goes in the same way as in Theorem 9.3. $\square$

Now consider the Neumann problem

$$\begin{aligned} Au &= f \text{ in } \mathbb{R}^n_+, \\ \nu u &= \varphi \text{ on } \mathbb{R}^{n-1}, \end{aligned} \tag{9.45}$$

and its two semi-homogeneous versions

$$\begin{aligned} Au &= f \text{ in } \mathbb{R}^n_+, \\ \nu u &= 0 \text{ on } \mathbb{R}^{n-1}, \end{aligned} \tag{9.46}$$

$$\begin{aligned} Az &= 0 \text{ in } \mathbb{R}^n_+, \\ \nu z &= \varphi \text{ on } \mathbb{R}^{n-1}. \end{aligned} \tag{9.47}$$

For (9.46) we find by application of the variational theory in Section 10.4 to the triple $(L_2(\mathbb{R}^n_+), H^1(\mathbb{R}^n_+), a(u,v))$ with $a(u,v) = (u,v)_1$, the variational operator A_ν, which is selfadjoint with lower bound 1. It is seen as in Section 4.4 (cf. Exercise 4.22), that A_ν is a realization of A with domain

$$D(A_\nu) = \{u \in D(A_{\max}) \cap H^1(\mathbb{R}^n_+) \mid (Au, v) = a(u,v) \text{ for all } v \in H^1(\mathbb{R}^n_+)\}; \tag{9.48}$$

the so-called Neumann realization. The smooth elements of $D(A_\nu)$ satisfy the Neumann condition $\nu u = 0$, but the interpretation of the conditions in (9.48) in the general case was not fully clarified. However, in this generalized sense, problem (9.46) has for $f \in L_2(\mathbb{R}^n_+)$ the unique solution $u = A_\nu^{-1} f$ in $D(A_\nu)$.

For the problem (9.47), Theorem 9.19 shows that when $m \geq 2$, it has the solution $K_\nu \varphi \in H^m(\mathbb{R}^n_+)$ for any $\varphi \in H^{m-\frac32}(\mathbb{R}^{n-1})$. To include the value $m = 0$, we use the statement from Theorem 9.10 that $\nu = i\gamma_1$ extends to a continuous operator from $D(A_{\max})$ to $H^{-\frac32}$. The identity $\varphi = \nu K_\nu \varphi$, valid for $\varphi \in H^{\frac12}$, then extends by continuity to $H^{-\frac32}$. We also know from Theorem 9.19 that $(1-\Delta)K_\nu\varphi = 0$ for $\varphi \in H^{-\frac32}$. Thus $K_\nu \varphi$ is indeed a solution of (9.47) when $\varphi \in H^{-\frac32}$.

Now the information on (9.46) can be improved by our knowledge of K_ν. Define again v by (9.34) and subtract it from u in (9.46), then calculations as after (9.34) with γ_0, K_γ replaced by ν, K_ν lead to the formula

$$u = r^+ Q e^+ f - K_\nu \nu\, r^+ Q e^+ f, \tag{9.49}$$

giving a solution in $H^2(\mathbb{R}^n_+)$ of (9.46). It belongs to the domain of A_ν, since the "halfways Green's formula"

$$(Au, v) - a(u, v) = (\nu u, \gamma_0 v), \tag{9.50}$$

known from (1.30) for smooth functions, extends by continuity (using Theorem 9.2) to $u \in H^2(\mathbb{R}^n_+)$ and $v \in H^1(\mathbb{R}^n_+)$, where νu and $\gamma_0 v$ are in $H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \subset L_2(\mathbb{R}^{n-1})$.

Then u defined by (9.49) must be equal to $A_\nu^{-1} f$, and we conclude, similarly to Theorem 9.11:

Theorem 9.20. *The solution operator A_ν^{-1} of (9.46) equals*

$$A_\nu^{-1} = r^+ Q e^+ - K_\nu \nu\, r^+ Q e^+, \text{ also denoted } R_\nu; \tag{9.51}$$

it maps $L_2(\mathbb{R}^n_+)$ continuously into $H^2(\mathbb{R}^n_+)$, and hence

$$D(A_\nu) = \{u \in H^2(\mathbb{R}^n_+) \mid \nu u = 0\}. \tag{9.52}$$

Having thus obtained optimal regularity for the Neumann problem (9.46), we can also clear up (9.47) completely. The proofs of Corollaries 9.12 and 9.13 generalize immediately to show:

Corollary 9.21. *For any $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$, $K_\nu \varphi$ is the unique solution in $H^2(\mathbb{R}^n_+)$ of (9.47). The mappings*

$$\nu\colon Z^2(A) \to H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \text{ and } K_\nu\colon H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \to Z^2(A)$$

are inverses of one another.

Then we get, using the denseness of $Z^2(A)$ in $Z^0(A)$ shown in Proposition 9.15:

Theorem 9.22. *Let $m \in \mathbb{N}_0$. The mappings*

$$\nu\colon Z^m(A) \to H^{m-\frac{3}{2}}(\mathbb{R}^{n-1}) \text{ and } K_\nu\colon H^{m-\frac{3}{2}}(\mathbb{R}^{n-1}) \to Z^m(A) \tag{9.53}$$

are inverses of one another.

This is shown just as in the proofs of Theorem 9.16 and Corollary 9.16.

We collect some facts from Theorems 9.19 and 9.21 in a theorem on the fully nonhomogeneous problem:

Theorem 9.23. *The problem* (9.45) *is uniquely solvable in* $D(A_{\max})$ *for* $f \in L_2(\mathbb{R}^n_+)$, $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$*; the solution belongs to* $H^2(\mathbb{R}^n_+)$ *and is defined by the formula*

$$u = r^+Qe^+f - K_\nu \nu\, r^+Qe^+f + K_\nu\varphi = R_\nu f + K_\nu\varphi; \tag{9.54}$$

cf. (9.51)*. In particular, the mapping*

$$\begin{pmatrix} A \\ \\ \nu \end{pmatrix} : H^2(\mathbb{R}^n_+) \to \begin{matrix} L_2(\mathbb{R}^n_+) \\ \times \\ H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} \tag{9.55}$$

has the inverse

$$\begin{pmatrix} R_\nu & K_\nu \end{pmatrix} : \begin{matrix} L_2(\mathbb{R}^n_+) \\ \times \\ H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to H^2(\mathbb{R}^n_+). \tag{9.56}$$

9.4. Other realizations of $I - \Delta$.

Besides the Dirichlet and the Neumann boundary conditions, one calls the following boundary conditions *normal*:

$$\nu u + B\gamma_0 u = \varphi, \tag{9.57}$$

when B is a differential operator on $\mathbb{R}^{n-1}$ of order 1, the point being that the normal derivative νu appears with coefficient 1. To keep things simple, we shall here only discuss constant-coefficient operators B. On the other hand, we shall allow the operators B to be pseudodifferential — however just with x'-independent symbols. We take them closely related to the example $\langle \xi' \rangle$, to avoid relying on the general definition of orders of ps.d.o.s at this moment.

The problem

$$\begin{aligned} Au &= f \text{ in } \mathbb{R}^n_+, \\ \nu u + B\gamma_0 u &= \varphi \text{ on } \mathbb{R}^{n-1}, \end{aligned} \tag{9.58}$$

will for some B behave like the Neumann problem (having $H^2(\mathbb{R}^n_+)$-regularity of solutions), for other B not, in particular when complex coefficients or pseudodifferential terms are allowed.

Still more general problems can be considered, for example with a Dirichlet boundary condition on part of the boundary and a Neumann-type condition on the rest of the boundary; here further complications arise.

An interesting question is, how all this fits into the abstract analysis established in Chapter 11, when it can be applied with $A_0 = A_{\min}$ and $A_1 = A_{\max}$: How do the concrete boundary conditions fit into the picture? Can all the operators $\widetilde{A} \in \mathcal{M}$ be interpreted as representing boundary conditions? We shall deal with this question in the following.

With $A = I - \Delta$ on $\mathbb{R}^n_+$, consider the operators in $H = L_2(\mathbb{R}^n_+)$:

$$A_0 = A_{\min}, \quad A_1 = A_{\max}, \quad A_\gamma = \text{ the Dirichlet realization;} \tag{9.59}$$

then we have a set-up as in Section 11.2 with

$$D(A_\gamma) = H^2(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+), \quad Z(A_1) = Z(A_{\max}) = Z^0(A), \tag{9.60}$$

and the positive selfadjoint operator A_γ has lower bound 1. We shall use that $Z(A_{\max})$ is isomorphic with $H^{-\frac12}(\mathbb{R}^{n-1})$, as established in Theorem 9.16. In view of (9.17) we have in fact an isometry:

$$2^{\frac12} K_\gamma \colon H^{-\frac12}(\mathbb{R}^{n-1}) \to Z(A_{\max}) \text{ is a surjective isometry.} \tag{9.61}$$

The inverse of K_γ acts like γ_0 and will be denoted γ_Z (so of course $K_\gamma = \gamma_Z^{-1}$).

In the following, we generally omit the indication "$(\mathbb{R}^{n-1})$" from the boundary Sobolev spaces. Recall from Section 6.3 for the spaces $H^{-\frac12}$ and $H^{\frac12}$ that although they are Hilbert spaces provided with well-defined norms, and are of course self-dual, we put greater emphasis on their identification as dual spaces of one another with respect to the extension $\langle \varphi, \overline{\psi} \rangle_{H^{-\frac12}, H^{\frac12}}$ of the scalar product in $H^0 = L_2(\mathbb{R}^{n-1})$ (consistent with the distribution duality):

$$H^{\frac12} \subset H^0 \subset H^{-\frac12}, \quad H^{-\frac12} \simeq (H^{\frac12})^*, \quad H^{\frac12} \simeq (H^{-\frac12})^*. \tag{9.62}$$

We take the same point of view for subspaces of them:

Let X be a closed subspace of $H^{-\frac12}$. Then we make very little use of the identification of the dual space X^* (the space of antilinear continuous functionals) with X, but regard X^* as a separate object. Since X is not dense in $H^{-\frac12}$ when different from $H^{-\frac12}$, there is not a natural inclusion between X^* and $H^{\frac12}$ (except what comes from identifying them with their duals). But there is a surjective mapping:

Definition 9.24. *For $\psi \in H^{\frac12}$ we define the element $\psi|_X$ of X^* as the functional acting like ψ on X:*

$$\langle \psi|_X, \overline{\varphi} \rangle_{X^*, X} = \langle \psi, \overline{\varphi} \rangle_{H^{\frac12}, H^{-\frac12}} \text{ for } \varphi \in X. \tag{9.63}$$

If $\psi \in H^{\frac12}$ and $\eta \in X^*$, the identity $\psi|_X = \eta$ may also be expressed as:

$$\psi = \eta \text{ on } X. \tag{9.64}$$

The functional $\psi|_X$ is continuous on X since the norm on this space is inherited from $H^{-\frac{1}{2}}$. All elements of X^* are obtained in this way. (For example, if η is in X^*, one can extend it to an element $\tilde{\eta}$ of $H^{\frac{1}{2}}$ by defining it to be zero on $H^{-\frac{1}{2}} \ominus X$. Other choices of complement of X in $H^{-\frac{1}{2}}$ will give other extensions.) The perhaps slightly abusive formulation (9.64) is standard for functions.

The following operator will play an important role:

Definition 9.25. *Define the Dirichlet-to-Neumann operator P by:*

$$P\varphi = \nu\, K_\gamma \varphi; \tag{9.65}$$

it is continuous from $H^{m-\frac{1}{2}}$ to $H^{m-\frac{3}{2}}$ for all $m \in \mathbb{N}_0$.

For $m \geq 2$ and $m = 0$, the continuity follows from the continuity of K_γ from $H^{m-\frac{1}{2}}$ to $Z^m(A)$ by Theorem 9.3, and the continuity of $\nu = i\gamma_1$ from $H^m(\mathbb{R}^n_+)$ to $H^{m-\frac{3}{2}}$ for $m \geq 2$ in Theorem 9.2 and from $D(A_{\max})$ to $H^{-\frac{3}{2}}$ for $m = 0$ by Theorem 9.10. In fact, we can say more about P: From the exact form of K_γ in (9.16) (for smooth functions) we see that P is the pseudodifferential operator

$$P = \mathrm{Op}(-\langle \xi' \rangle), \tag{9.66}$$

and it is clearly an isometry of H^s onto H^{s-1} for all $s \in \mathbb{R}$. Moreover, P is formally selfadjoint and $-P$ is positive with lower bound 1 in $L_2(\mathbb{R}^{n-1})$, since the symbol $-\langle \xi' \rangle$ is real and ≤ -1; cf. Theorem 6.3.

Next, we define a somewhat strange trace operator:

Definition 9.26. *The trace operator M is defined on $D(A_{\max})$ by*

$$Mu = \nu u - P\gamma_0 u. \tag{9.67}$$

At first sight, it ranges in $H^{-\frac{3}{2}}$, since ν and $P\gamma_0$ do so. But this information can be improved:

Proposition 9.27. *The trace operator M satisfies*

$$Mu = \nu A_\gamma^{-1} Au, \text{ for } u \in D(A_{\max}), \tag{9.68}$$

hence maps $D(A_{\max})$ continuously into $H^{\frac{1}{2}}$. The following Green's formula holds **for all** *$u, v \in D(A_{\max})$:*

$$(Au, v) - (u, Av) = \langle Mu, \overline{\gamma_0 v} \rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}} - \langle \gamma_0 u, \overline{Mv} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}. \tag{9.69}$$

Furthermore

$$Mz = 0 \text{ for } z \in Z(A_{\max}), \tag{9.70}$$

and

$$(Au, z) = \langle Mu, \overline{\gamma_0 z}\rangle_{H^{\frac12}, H^{-\frac12}} \text{ for } u \in D(A_{\max}), z \in Z(A_{\max}). \tag{9.71}$$

Proof. When $u \in D(A_{\max})$, we decompose it in

$$u = u_\gamma + u_\zeta, \quad u_\gamma = A_\gamma^{-1} Au \in D(A_\gamma),$$

where $u_\zeta \in Z(A_{\max})$ (as in Lemma 11.1). Then since $\gamma_0 u_\gamma = 0$,

$$\gamma_0 u = \gamma_0 u_\zeta, \text{ so } P\gamma_0 u = P\gamma_0 u_\zeta = \nu u_\zeta \tag{9.72}$$

by definition of P, and hence

$$\nu u - P\gamma_0 u = \nu u - \nu u_\zeta = \nu u_\gamma = \nu A_\gamma^{-1} Au.$$

This shows (9.68), and the continuity follows from the continuity properties of the three factors.

Consider Green's formula (9.26) for functions $u, v \in \mathcal{S}(\overline{\mathbb{R}}^n_+)$ and subtract from it the identity $(P\gamma_0 u, \gamma_0 v) - (\gamma_0 u, P\gamma_0 v) = 0$, then we get

$$(Au, v) - (u, Av) = (\nu u - P\gamma_0 u, \gamma_0 v) - (\gamma_0 u, \nu v - P\gamma_0 v),$$

for functions in $\mathcal{S}(\overline{\mathbb{R}}^n_+)$; this shows (9.69) for smooth functions. It extends by continuity to $u, v \in D(A_{\max})$ in view of the continuity of M shown above and the continuity of γ_0 shown in Theorem 9.10.

The equation (9.70) is clear from the fact that $\nu z = P\gamma_0 z$ when $z \in Z(A_{\max})$, and (9.71) follows from (9.69) with $v = z$ since Az and Mz vanish. $\square$

Recall from Section 11.2 that any closed realization $\widetilde{A}$ corresponds (in a unique way) to a closed, densely defined operator $T\colon V \to W$, where V and W are closed subspaces of $Z(A_{\max})$. We carry this over to a situation with spaces over the boundary as follows:

Definition 9.28. *Let V and W be closed subspaces of $Z(A_{\max})$, and let $T\colon V \to W$ be closed, densely defined. The corresponding set-up over the boundary is then defined by setting*

$$X = \gamma_0 V,\ Y = \gamma_0 W,\ \text{closed subspaces of } H^{-\frac12}, \tag{9.73}$$

and defining $L\colon X \to Y^*$ *by*

$$\begin{aligned} D(L) &= \gamma_0 D(T), \\ \langle L\gamma_0 v, \overline{\gamma_0 w}\rangle_{Y^*,Y} &= (Tv, w), \text{ for all } v \in D(T), w \in W. \end{aligned} \tag{9.74}$$

Since γ_0 equals the invertible operator γ_Z in all the formulas in this definition, T is determined from L, and when $T\colon V \to W$ runs through all choices of closed subspaces V, W of $Z(A_{\max})$ and closed densely defined operators from V to W, then $L\colon X \to Y^*$ runs through all choices of closed subsaces X, Y of $H^{-\frac12}$ and closed densely defined operators from X to Y^*. This is obvious if we do provide X and Y with the norm in $H^{-\frac12}$, identify them with their duals, and use that $2^{-\frac12}\gamma_Z$ is an isometry. (So the norm comes in useful here, but in general we focus on properties that are expressed without reference to a choice of norm.)

Note that T and L have similar properties: They are simultaneously injective, or surjective. The nullspace of L is $Z(L) = \gamma_Z Z(T)$. The adjoint of T corresponds to the adjoint of L defined as an operator from Y to X^*:

$$\begin{aligned} (v, T^*w) = (Tv, w) = \langle L\gamma_0 w, \overline{\gamma_0 w}\rangle_{Y^*,Y} = \langle \gamma_0 w, \overline{L^*\gamma_0 w}\rangle_{X,X^*}, \\ v \in D(T), w \in D(T^*). \end{aligned}$$

(It is understood that $\langle \varphi, \overline{\psi}\rangle_{X,X^*} = \overline{\langle \psi, \overline{\varphi}\rangle}_{X^*,X}$.) When $V = W$, i.e., $X = Y$, T is selfadjoint if and only if L is so. Also lower boundedness is preserved in the correspondence: When $V \subset W$ then $X \subset Y$, and

$$\operatorname{Re}(Tv, v) \ge c\|v\|^2 \text{ for } v \in D(T) \tag{9.75}$$

holds if and only if

$$\operatorname{Re}\langle L\varphi, \overline{\varphi}\rangle_{Y^*,Y} \ge c'\|\varphi\|_Y^2 \text{ for } \varphi \in D(L), \tag{9.76}$$

for some c' whose *value* depends on the choice of norm in Y but whose *sign* (positive, negative or zero) is the same as that of c, independently of the choice of norm.

We can now interpret the general realizations of A by boundary conditions.

Theorem 9.29. *Consider a closed realization $\widetilde{A}$ of A, corresponding to $T\colon V \to W$ as in Theorem* 11.7 *(with $A_1 = A_1' = A_{\max}$), and let $L\colon X \to Y^*$ be the corresponding operator introduced in Definition* 9.28*. Then $D(\widetilde{A})$ consists of the functions $u \in D(A_{\max})$ for which*

$$\gamma_0 u \in D(L), \quad Mu|_Y = L\gamma_0 u. \tag{9.77}$$

In this correspondence, $D(L) = \gamma_0 D(\widetilde{A})$.

Proof. We have from Theorem 11.5 that the elements of $D(\widetilde{A})$ are characterized by the two conditions:

$$u_\zeta \in D(T), \quad (Au, w) = (Tu_\zeta, w) \text{ for all } w \in W. \tag{9.78}$$

We just have to translate this to boundary conditions. In view of the definition of L, we have for any $u \in D(A_{\max})$, since $\gamma_0 u_\gamma = 0$, that

$$u_\zeta \in D(T) \iff \gamma_0 u_\zeta \in D(L) \iff \gamma_0 u \in D(L), \tag{9.79}$$

showing that the first conditions in (9.77) and (9.78) are equivalent. When $w \in W \subset Z(A_{\max})$, we have in view of (9.71):

$$(Au, w) = \langle Mu, \overline{\gamma_0 w} \rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}},$$

whereas

$$(Tu_\zeta, w) = \langle L\gamma_0 u, \overline{\gamma_0 w} \rangle_{Y^*, Y},$$

in view of (9.72) and (9.74). Then the second conditions in (9.77) and (9.78) are equivalent, in view of Definition 9.24.

The last assertion follows from (9.79), since $D(T) = \operatorname{pr}_\zeta D(\widetilde{A})$; cf. Theorem 11.5. $\square$

Let us consider some examples.

Example 9.30. For A_γ itself, $V = W = \{0\}$, and T is trivial (zero) then. So $X = Y = \{0\}$ and $L = 0$. The boundary condition is:

$$\gamma_0 u = 0, \tag{9.80}$$

as we know very well.

Example 9.31. Consider A_M from Example 11.10, the realization with domain

$$D(A_M) = D(A_{\min}) \dot{+} Z(A_{\max}) = H_0^2(\mathbb{R}_+^n) \dot{+} Z^0(A). \tag{9.81}$$

Here $V = W = Z(A_{\max})$, and $T = 0$ on $D(T) = V$. So $X = Y = H^{-\frac{1}{2}}$ with $L = 0$ on $D(L) = H^{-\frac{1}{2}}$. The boundary condition is

$$Mu = 0, \text{ i.e., } \nu u - P\gamma_0 u = 0. \tag{9.82}$$

This is a normal boundary condition (cf. (9.57)ff.), with $B = -P$. The realization, it defines, has *no regularity:* The domain is not in $H^m(\mathbb{R}_+^n)$ for any $m > 0$, since it contains $Z^0(A)$. (By Corollary 9.17, the $Z^m(A)$ are strictly different for different m, since the $H^{m-\frac{3}{2}}$ are so.)

Example 9.32. The Neumann realization is defined by the boundary condition

$$\nu u = 0. \tag{9.83}$$

We know from Section 9.3 that the domain of the hereby defined realization A_ν equals $\{u \in H^2(\mathbb{R}^n_+) \mid \nu u = 0\}$.

Since $\varrho = \{\gamma_0, \gamma_1\}$ is surjective from $H^2(\mathbb{R}^n_+)$ to $H^{\frac{3}{2}} \times H^{\frac{1}{2}}$, $\gamma_0 u$ runs through $H^{\frac{3}{2}}$ when u runs through the domain, so $D(L) = H^{\frac{3}{2}}$, and $X = H^{-\frac{1}{2}}$ since $H^{\frac{3}{2}}$ is dense in $H^{-\frac{1}{2}}$. Since A_ν is selfadjoint, also $Y = H^{-\frac{1}{2}}$. Then $D(A_\nu)$ must be characterized by a condition of the form

$$Mu - L\gamma_0 u = 0, \text{ for } \gamma_0 u \in H^{\frac{3}{2}},$$

so since $Mu = \nu u - P\gamma_0 u$,

$$\nu u - P\gamma_0 u - L\gamma_0 u = 0, \text{ for } \gamma_0 u \in H^{\frac{3}{2}},$$

when $u \in D(A_\nu)$. Since $\nu u = 0$ there, $L + P = 0$ on $H^{\frac{3}{2}}$. This shows that

$$L \text{ acts like } -P, \quad D(L) = H^{\frac{3}{2}}, \tag{9.84}$$

in the case of A_ν.

Let us now consider more general realizations defined by normal boundary conditions

$$\nu u + B\gamma_0 u = 0. \tag{9.85}$$

Here we let B be a pseudodifferential operator with symbol

$$b(\xi') = ib_1\xi_1 + \cdots + ib_{n-1}\xi_{n-1} + c\langle\xi'\rangle, \text{ some } c \in \mathbb{R}; \tag{9.86}$$

i.e., it is the sum of a differential operator and a multiple of P:

$$B = \operatorname{Op}(b) = B_1 - cP, \quad B_1 = b_1\partial_{x_1} + \cdots + b_{n-1}\partial_{x_{n-1}}. \tag{9.87}$$

(9.85) defines the realization $\widetilde{A}$ with domain

$$D(\widetilde{A}) = \{u \in D(A_{\max}) \mid \nu u + B\gamma_0 u = 0\}. \tag{9.88}$$

Since the trace operator $\nu + B\gamma_0$ is continuous from $D(A_{\max})$ to $H^{-\frac{3}{2}}$, the domain is closed.

Clearly,

$$D(\widetilde{A}) \supset \{u \in H^2(\mathbb{R}^n_+) \mid \nu u + B\gamma_0 u = 0\} = D(\widetilde{A}) \cap H^2(\mathbb{R}^n_+), \tag{9.88}$$

but we do not know à priori whether this is an equality; in fact, we shall look for criteria for whether it is so.

Since $\varrho = \{\gamma_0, \gamma_1\}$ is surjective from $H^2(\mathbb{R}^n_+)$ to $H^{\frac{3}{2}} \times H^{\frac{1}{2}}$, $\gamma_0 u$ runs through $H^{\frac{3}{2}}$ when u runs through $D(\widetilde{A}) \cap H^2(\mathbb{R}^n_+)$, so $D(L) \supset H^{\frac{3}{2}}$; since the latter space is dense in $H^{-\frac{1}{2}}$, $X = H^{-\frac{1}{2}}$.

Then in view of Corollary 9.17 and the decomposition of $D(A_{\max})$ into $D(A_\gamma)$ and $Z(A_{\max})$,

$$D(L) = H^{\frac{3}{2}} \iff D(\widetilde{A}) = D(\widetilde{A}) \cap H^2(\mathbb{R}^n_+). \tag{9.89}$$

Concerning the adjoint $\widetilde{A}^*$, we have from the general Green's formula (9.29) that when $v \in D(\widetilde{A}^*)$, $u \in D(\widetilde{A}) \cap H^2(\mathbb{R}^n_+)$,

$$\begin{aligned} 0 = (Av, u) - (v, Au) &= \langle \nu v, \overline{\gamma_0 u} \rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}} - \langle \gamma_0 v, \overline{\nu u} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \\ &= \langle \nu v, \overline{\gamma_0 u} \rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}} - \langle \gamma_0 v, -\overline{B\gamma_0 u} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \\ &= \langle \nu v + B^*\gamma_0 v, \overline{\gamma_0 u} \rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}}, \end{aligned}$$

which shows that

$$v \in D(\widetilde{A}^*) \implies \nu v + B^*\gamma_0 v = 0,$$

since $\gamma_0 u$ runs freely in $H^{\frac{3}{2}}$. It was used here that $B\colon H^s \to H^{s-1}$ has the adjoint $B^*\colon H^{-s+1} \to H^{-s}$, any s, where B^* acts as the ps.d.o. with symbol $\overline{b}(\xi')$. On the other hand, if $v \in H^2(\mathbb{R}^n_+)$ with $\nu v + B^*\gamma_0 v = 0$, then an application of (9.29) with $u \in D(\widetilde{A})$ shows that

$$\begin{aligned} (Au, v) - (u, Av) &= \langle \nu u, \overline{\gamma_0 v} \rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}} - \langle \gamma_0 u, \overline{\nu v} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \\ &= \langle \nu u, \overline{\gamma_0 v} \rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}} - \langle \gamma_0 u, -\overline{B^*\gamma_0 v} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \\ &= \langle \nu u + B\gamma_0 u, \overline{\gamma_0 v} \rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}} = 0, \end{aligned}$$

which implies that $v \in D(\widetilde{A}^*)$. So we conclude that

$$\begin{aligned} \{v \in H^2(\mathbb{R}^n_+) \mid \nu v + B^*\gamma_0 v = 0\} &\subset D(\widetilde{A}^*) \\ &\subset \{v \in D(A_{\max}) \mid \nu v + B^*\gamma_0 v = 0\}. \end{aligned} \tag{9.90}$$

This gives us the information that $D(L^*)$ contains $H^{\frac{3}{2}}$ in its domain, so $Y = H^{-\frac{1}{2}}$ where it is dense.

It follows that L is an operator from $D(L) \subset H^{-\frac{1}{2}}$ to $Y^* = H^{\frac{1}{2}}$. We can now use that $L\gamma_0 u = Mu$ on $D(\widetilde{A})$ to find how L acts, when (9.85) holds:

$$0 = L\gamma_0 u - Mu = L\gamma_0 u - \nu u + P\gamma_0 u = L\gamma_0 u + B\gamma_0 u + P\gamma_0 u,$$

so

$$L = -B - P \text{ on } D(L). \tag{9.91}$$

Similarly,

$$L^* = -B^* - P \text{ on } D(L^*). \tag{9.92}$$

A further precision of L can be made when L is elliptic. Note that

$$\begin{aligned} &L \text{ acts like } -B_1 + cP - P = \operatorname{Op}(l(\xi')), \\ &l(\xi') = -ib_1\xi_1 - \cdots - ib_{n-1}\xi_{n-1} + (1-c)\langle\xi'\rangle. \end{aligned} \tag{9.93}$$

The symbol $\langle\xi'\rangle$ has a series expansion

$$\langle\xi'\rangle = (1 + |\xi'|^2)^{\frac{1}{2}} = |\xi'|(1 + |\xi'|^{-2})^{\frac{1}{2}} = |\xi'| + |\xi'| \sum_{k\geq 1} \tbinom{\frac{1}{2}}{k} |\xi'|^{-2},$$

converging for $|\xi'| > 1$. (Here $\binom{\frac{1}{2}}{k} = \frac{1}{2}(\frac{1}{2} - 1)\cdots(\frac{1}{2} - k + 1)/k!$.) Borrowing a bit of notation from Section 7.2, we have that the *principal part* of $l(\xi')$ is

$$l_1(\xi') = -ib_1\xi_1 - \cdots - ib_{n-1}\xi_{n-1} + (1-c)|\xi'|, \tag{9.94}$$

and l is *elliptic* precisely when $l_1(\xi') \neq 0$ for $\xi' \neq 0$. In this case the requirement for Theorem 6.18 with $m = 1$ is satisfied.

Thus in the elliptic case we can conclude from Theorem 6.18 that $\operatorname{Op}(l)\varphi \in H^{\frac{1}{2}}$ implies $\varphi \in H^{\frac{3}{2}}$. Since L acts like $\operatorname{Op}(l)$ and ranges in $H^{\frac{1}{2}}$, we conclude that $D(L) = H^{\frac{3}{2}}$. This shows:

Theorem 9.33. *Let $\widetilde{A}$ be the realization defined by the boundary condition* (9.85)*. If $l(\xi')$ in* (9.93) *is elliptic, then*

$$D(\widetilde{A}) = \{u \in H^2(\mathbb{R}^n_+) \mid \nu u + B\gamma_0 u = 0\}. \tag{9.94}$$

In this case also

$$D(\widetilde{A}^*) = \{u \in H^2(\mathbb{R}^n_+) \mid \nu u + B^*\gamma_0 u = 0\}. \tag{9.95}$$

The last statement follows since the adjoint symbol $\bar{l}(\xi')$ is then likewise elliptic.

Ellipticity clearly holds if

$$c \neq 1,\ b_1, \ldots, b_{n-1} \in \mathbb{R}, \tag{9.96}$$

in particular when $c = 0$ and the b_j are real. So we have as a corollary:

Corollary 9.34. *When B is a differential operator with real coefficients, the boundary condition* (9.85) *defines a realization with domain in $H^2(\mathbb{R}^n_+)$.*

Non-elliptic examples are found when $c = 1$ — this is so in Example 9.30 — or if some of the coefficients b_j are nonreal, e.g. if one of them equals $\pm i(1-c)$, so that there are points $\xi' \neq 0$ with $l_1(\xi') = 0$. One can also show that ellipticity of $l(\xi')$ is *necessary* for having $D(L) \subset H^{\frac{3}{2}}$, i.e., $D(\widetilde{A}) \subset H^2(\mathbb{R}^n_+)$.

We can also discuss lower boundedness, in view of the results in Section 11.2 and the equivalence of (9.75) and (9.76). The most immediate results are:

Theorem 9.35. *Let $\widetilde{A}$ be the realization determined by the boundary condition* (9.85)*.*

If $\widetilde{A}$ is lower bounded, so is L, with a similar sign of the lower bound.

If L has positive or zero lower bound, so has $\widetilde{A}$.

Proof. We use the equivalence of (9.75) and (9.76). Note that we are in a case where we know on beforehand that $V = W = Z(A_1)$. The first statement follows from Theorem 11.14. The second statement follows from Theorem 11.16. □

Again this applies easily to the differential case (where $c = 0$ in (9.86)):

Corollary 9.36. *When B is a differential operator with real coefficients, the realization defined by the boundary condition* (9.85) *is variational with positive lower bound.*

Proof. In this case, $\operatorname{Re} l(\xi') = \langle \xi' \rangle \geq 1$ and $D(L) = H^{\frac{3}{2}}$, so L has lower bound 1, as an operator in $L_2 = H^0$. The same holds for L^*. Moreover,

$$\begin{aligned} |\operatorname{Im} l(\xi')| &= |b_1\xi_1 + \cdots + b_{n-1}\xi_{n-1}| \leq C|\xi'|, \text{ so} \\ |\operatorname{Re} l(\xi')| &= \langle \xi' \rangle \geq C^{-1} |\operatorname{Im} l(\xi')|, \end{aligned} \tag{9.97}$$

which imply similar inequalities for L and L^* (as in Theorems 10.13, 6.4 and Exercises 10.35–36). It follows in view of (9.74) that T and T^* have their numerical ranges in an angular set

$$M = \{ \lambda \in \mathbb{C} \mid |\operatorname{Im} \lambda| \leq C' \operatorname{Re} \lambda,\ \operatorname{Re} \lambda \geq c' \}$$

with C' and $c' > 0$. Now Theorem 11.16 applies to show that $\widetilde{A}$ and $\widetilde{A}^*$ have their numerical ranges in a similar set, and the variational property follows from Theorem 10.25. □

We observe in general that if L has lower bound > 0, then it is elliptic, and both $\widetilde{A}$ and $\widetilde{A}^*$ are lower bounded. A closer look at $l_1(\xi')$ will show that then necessarily $c < 1$; moreover, $\widetilde{A}$ and $\widetilde{A}^*$ are variational.

Conditions for the coerciveness inequality

$$\operatorname{Re}(Au, u) \geq c_1 \|u\|_1^2 - c_0 \|u\|_0^2 \text{ for } u \in D(\widetilde{A}), \tag{9.98}$$

can be fully analyzed, and have been done so in the litterature (more on this in [G 1971] and subsequent works).

Example 9.37. Assume that $n \geq 3$. Consider $\widetilde{A}$ defined by the boundary condition (9.85) with

$$B = \chi(x_1)\partial_{x_1} - P, \tag{9.99}$$

where χ is as defined in Section 2.1 (here we allow a variable coefficient in the differential operator). By (9.91)ff.,

$$L = -\chi(x_1)\partial_{x_1},$$

in this case. It is not elliptic on $\mathbb{R}^{n-1}$. Moreover, L has a large nullspace, containing the functions that are constant in x_1 for $|x_1| \leq 2$. The nullspace is clearly infinite dimensional, and it is not contained in $H^1(\mathbb{R}^{n-1})$ (let alone $H^{\frac{3}{2}}$), since it contains for example all products of L_2-functions of $(x_2, \dots, x_{n-1})$ with L_2-functions of x_1 that are constant on $[-2, 2]$. We have from Theorem 11.18 that the nullspace of $\widetilde{A}$ equals that of T, and it equals $K_\gamma Z(L)$, cf. Definition 9.27 ff.

So this is an example of a realization with low regularity and a large infinite dimensional nullspace.

9.5. Variable-coefficient cases, higher orders, systems.

The case of $I - \Delta$ on the halfspace is just a very simple example, having the advantage that precise results can be easily obtained. Variable coefficient cases require suitable approximations, for example by introduction of pseudodifferential operator techniques building on those in Chapter 7. We end by listing some of the ingredients of the general theory.

Consider a differential operator of a general order $m > 0$ with C^∞ coefficients on an open set $\Omega \subset \mathbb{R}^n$ with smooth boundary:

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha; \tag{9.100}$$

its principal symbol is $a_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ (cf. (6.5), (6.42)). A is elliptic when $a_m(x, \xi) \neq 0$ for $\xi \neq 0$ (and all x); and A is said to be *strongly elliptic* when

$$\operatorname{Re} a_m(x, \xi) > 0 \text{ for } \xi \neq 0, \text{ all } x. \tag{9.101}$$

It is not hard to see that strongly elliptic operators are necessarily of *even order*, $m = 2k$.

Odd-order elliptic operators can occur when we consider *systems* — matrix-formed operators — where the $a_\alpha(x)$ are matrix functions (this is considered for ps.d.o.s in Section 7.4). Ellipticity here means that the principal symbol $a_m(x,\xi)$ is an invertible matrix for all $\xi \neq 0$, all x. Then it must be a square matrix, and the ellipticity means that the determinant of $a_m(x,\xi)$ is nonzero for $\xi \neq 0$.

Examples of first-order systems that are of current interest are Dirac operators, used in Physics and Geometry.

The strongly elliptic systems are those for which

$$a_m(x,\xi) + a_m(x,\xi)^* \text{ is positive definite when } \xi \neq 0. \tag{9.102}$$

Again the order must be even, $m = 2k$.

In the strongly elliptic case where Ω is bounded and (9.102) holds for all $x \in \overline{\Omega}$, there is a variational realization A_γ of the Dirichlet problem, representing the boundary condition $\gamma u = 0$, where

$$\gamma u = \{\gamma_0 u, \gamma_1 u, \dots, \gamma_{k-1} u\}. \tag{9.103}$$

The domain is $D(A_\gamma) = D(A_{\max}) \cap H_0^k(\Omega)$. One can then show that in fact

$$D(A_\gamma) = H^{2k}(\Omega) \cap H_0^k(\Omega), \tag{9.104}$$

and that A_γ is a Fredholm operator (cf. Section 8.3) from its domain to $L_2(\Omega)$. Moreover, one can consider the fully nonhomogeneous problem

$$\begin{aligned} Au &= f \text{ in } \Omega, \\ \gamma u &= \varphi \text{ on } \partial\Omega, \end{aligned} \tag{9.105}$$

showing that it defines a Fredholm operator

$$\begin{pmatrix} A \\ \\ \gamma \end{pmatrix} : H^{2k}(\Omega) \to \begin{matrix} L_2(\Omega) \\ \times \\ \Pi_{j=0}^{k-1} H^{2k-j-\frac{1}{2}}(\partial\Omega) \end{matrix} . \tag{9.106}$$

In the case of bijectiveness, the semi-homogeneous problem

$$\begin{aligned} Au &= 0 \text{ in } \Omega, \\ \gamma u &= \varphi \text{ on } \partial\Omega, \end{aligned} \tag{9.107}$$

has a solution operator K_γ which is bijective:

$$K_\gamma\colon \Pi_{j=0}^{k-1} H^{s-j-\frac12}(\partial\Omega) \xrightarrow{\sim} \{u \in H^s(\Omega) \mid Au = 0\}, \tag{9.108}$$

for all $s \in \mathbb{R}$, with inverse γ (defined in a generalized sense for low s). When the system in (9.106) is merely Fredholm, (9.108) holds modulo finite dimensional subspaces.

The classical method to show the regularity (9.104) of the solutions to (9.105) with $\varphi = 0$ relies on approximating the derivatives with difference quotients where the inequalities resulting from the ellipticity hypotheses can be used. This is described e.g. in Agmon [A 1965] (including some other variational cases), and in Lions and Magenes [LM 1968] for general normal boundary problems with nonhomogeneous boundary condition; the latter use the adjoint problem to obtain the Fredholm property (the surjectiveness in the best case). The difference quotient method allows keeping track of how much smoothness of the coefficients $a_\alpha(x)$ is needed for a specific result, it is described for the second-order case in Evans [E 1998].

The treatment of general boundary conditions can be described as follows: Straighten out the boundary in the neighborhood of a boundary point. Consider the constant-coefficient case obtained by "freezing" the coefficients in the principal part at a point on the straight boundary, letting the normal variable play the role of x_n and the variable in the tangent plane play the role of x'. Solve this problem by, for example, Fourier analysis. This gives a first approximation of a solution operator when the solution operators for the various boundary points are pieced together. Use some further tricks to get a more precise solution.

If one is not worried about minimal smoothness assumptions on the coefficients $a_\alpha(x)$, one can formulate the strategy in a pseudodifferential language. Here we need other operators than just ps.d.o.s, namely trace operators T going from functions on Ω to functions on $\partial\Omega$, Poisson operators K going from functions on $\partial\Omega$ to functions on Ω, and composed operators KT, called singular Green operators. A quite general framework for the systematic use of such operators was given in Boutet de Monvel [BM 1971] (and further developed in [G 1996]). The use of this framework makes the solvability discussion more operational; one does not just obtain existence of regular solutions, but one constructs in a concrete way the operator that maps the data into the solution (obtaining general versions of (9.36), (9.51)).

Instead of open sets $\Omega \subset \mathbb{R}^n$, one can consider manifolds with boundary, using local coordinates in a similar way as described in Chapter 8. Note however that even when Ω is a subset of euclidean space, its boundary $\partial\Omega$ is a genuine manifold.

When one studies other boundary conditions for A than the Dirichlet condition, a certain comparison with the Dirichlet problem is crucial for deciding whether the problem has regularity of solutions. The explanations in Section 9.4 give a hint of how this can be understood (recall that there were purely differential boundary conditions with complex coefficients that had a non-elliptic L). Operators analogous to the Dirichlet-to-Neumann operator P can be defined in general as pseudodifferential operators on the manifold $\partial\Omega$.

When A is not strongly elliptic, one may not have a Dirichlet problem to start out with, and to compare other problems with. In this case one can still define appropriate boundary value problems with regularity, though. The criterion for regularity and Fredholmness can be formulated as follows: It is that the constant-coefficient problem obtained by freezing the coefficients in the principal part (of the differential operator as well as the boundary condition) at a boundary point and Fourier transforming in the x'-variable, is uniquely solvable for all $\xi' \neq 0$; problems satisfying this are called *elliptic*. (Earlier, the word "coercive" was also used, but it is now customary to reserve that word for problems defining a variational operator. The conditions for ellipticity are sometimes called the "Shapiro-Lopatinskii conditions", referring to early contributors to the regularity question.) A classical reference for the general systems case is the paper of Agmon, Douglis and Nirenberg [ADN 1964]. For a constructive approach, a fundamental ingredient is the Calderón projector (a certain matrix-formed ps.d.o. on the boundary), as worked out in detail in Seeley [S 1966] and Hörmander [H 1967]. The way it applies to general boundary conditions is described e.g. in [G 1999], Section 7, in a compact form.

Exercises for Chapter 9.

9.1. A similar analysis to that in Section 9.2 can be set up for the Dirichlet problem for $-\Delta$ on the circle, continuing the notation of Exercise 4.5. The Sobolev spaces over the boundary can here be defined in terms of the coefficient series (c_n) of Fourier expansions (H^s being provided with the norm $(\sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |c_n|^2)^{\frac{1}{2}}$), as mentioned in Remark 8.3.

9.2. [Here should be hints for showing the properties mentioned in Remark 9.12.]

9.3. Consider $l(\xi')$ defined in (9.93). Show that if $\operatorname{Re} l(\xi') > 0$ for all ξ', then $c < 1$. Show that then L is variational.

9.4. Let $B = c_0 + ic_1\partial_{x_1} + \dots + ic_{n-1}\partial_{x_{n-1}}$, with real numbers $c_0, \dots, c_{n-1}$. Show that if $c_1^2 + \dots + c_{n-1}^2 < 1$, then the realization $\widetilde{A}$ defined by (9.88) with this B is selfadjoint and has $D(\widetilde{A}) \subset H^2(\mathbb{R}^n_+)$.