Chapter 9
Boundary value problems in a constant-coefficient case

9.1 Boundary maps for the half-space

This chapter can be read in direct succession to Chapters 1–6 and does not build on Chapters 7 and 8.

We have in Section 4.4 considered various realizations of the Laplacian and similar operators, defined by homogeneous boundary conditions, and shown how their invertibility properties lead to statements about the solvability of homogeneous boundary value problems. It is of great interest to study also nonhomogeneous boundary value problems

$$\begin{aligned} Au &= f \text{ in } \Omega, \\ Tu &= g \text{ on } \partial\Omega, \end{aligned} \tag{9.1}$$

with $g \neq 0$. Here Ω is an open subset of $\mathbb{R}^n$, A is an elliptic partial differential operator and Tu stands for a set of combinations of boundary values and (possibly higher) normal derivatives (T is generally called a *trace operator*). There are several possible techniques for treating such problems. In specific cases where A is the Laplacian, there are integral formulas (depending on the shape of the boundary), but when A has variable coefficients the methods are more qualitative.

When the boundary is smooth (cf. Appendix C), one succesful method can be described as follows:

At a fixed point x_0 of the boundary, consider the problem with constant coefficients having the value they have at that point, and identify the tangent plane with $\mathbb{R}^{n-1}$ and the interior normal direction with the x_n-axis. The discussion of the resulting constant-coefficient problem is essential for the possibility to obtain solutions of (9.1). Drop the lower-order terms in the equation and boundary condition, and perform a Fourier transformation $\mathscr{F}_{x'\to\xi'}$ in the tangential variables (in $\mathbb{R}^{n-1}$), then we have for each ξ' a one-dimensional problem on $\mathbb{R}_+$. This is what is called the “model problem” at the point x_0.

The original problem (9.1) is called *elliptic* precisely when *the model problem is uniquely solvable in $L_2(\mathbb{R}_+)$ for each $\xi' \neq 0$, at all points x_0 of the boundary.* The conditions for ellipticity are sometimes called the "Shapiro-Lopatinskiĭ conditions".

It is possible to construct a good approximation of the solution in the elliptic case by piecing together the solutions of the model problems in some sense. A very efficient way of doing this is by setting up a pseudodifferential calculus of boundary value problems, based on the theory in Chapter 7 but necessarily a good deal more complicated, since we have operators going back and forth between the boundary and the interior of Ω. An introduction to such a theory is given in Chapter 10. In the present chapter, we give "the flavor" of how the Sobolev spaces and solution operators enter into the picture, by studying in detail a simple special case, namely, that of $1 - \Delta$ on $\mathbb{R}^n_+$.

In the same way as the properties of $I - \Delta$ on $\mathbb{R}^n$ gave a motivational introduction to the properties of general elliptic operators on open sets (or manifolds), the properties of the Dirichlet and Neumann problems for $1 - \Delta$ on $\mathbb{R}^n_+$ serve as a motivational introduction to properties of general elliptic boundary value problems. We shall treat this example in full detail. Here we also study the interpretation of the families of extensions of symmetric (and more general) operators described in Chapter 13. At the end we give some comments on variable-coefficient cases.

The Fourier transform was an important tool for making the treatment of $I - \Delta$ on $\mathbb{R}^n$ easy, cf. Example 5.20. When the operator is considered on $\mathbb{R}^n_+$, we shall use the partial Fourier transform (with respect to the $n-1$ first variables) to simplify things, so we start by setting up how that works.

We shall denote the *restriction* operator from $\mathbb{R}^n$ to $\mathbb{R}^n_+$ by r^+, and the restriction operator from $\mathbb{R}^n$ to $\mathbb{R}^n_-$ by r^-, where

$$\mathbb{R}^n_\pm = \{\, x = (x', x_n) \in \mathbb{R}^n \mid x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1},\, x_n \gtrless 0 \,\}.$$

Here we generally use the notation

$$r_\Omega u = u|_\Omega, \text{ and in particular } r^\pm = r_{\mathbb{R}^n_\pm}, \tag{9.2}$$

defined for $u \in \mathscr{D}'(\mathbb{R}^n)$. Moreover we need the *extension by* 0 operators e_Ω, in particular $e^+ = e_{\mathbb{R}^n_+}$ and $e^- = e_{\mathbb{R}^n_-}$, defined for functions f on Ω by

$$e_\Omega f = \begin{cases} f & \text{on } \Omega, \\ 0 & \text{on } \mathbb{R}^n \setminus \Omega; \end{cases} \qquad e^\pm = e_{\overline{\mathbb{R}}^n_+}. \tag{9.3}$$

The spaces of smooth functions on $\overline{\mathbb{R}}^n_+$ that we shall use will be the following:

$$C^\infty_{(0)}(\overline{\mathbb{R}}^n_+) = r^+ C_0^\infty(\mathbb{R}^n), \quad \mathscr{S}(\overline{\mathbb{R}}^n_+) = r^+ \mathscr{S}(\mathbb{R}^n), \tag{9.4}$$

where r^+ is defined in (9.2). Clearly, $C_{(0)}^\infty(\overline{\mathbb{R}}^n_+) \subset \mathscr{S}(\overline{\mathbb{R}}^n_+) \subset H^m(\mathbb{R}^n_+)$ for any m. The space $C_{(0)}^\infty(\overline{\mathbb{R}}^n_+)$ was defined in Chapter 4 as the space of functions on $\overline{\mathbb{R}}^n_+$ that are continuous and have continuous derivatives of all orders on $\overline{\mathbb{R}}^n_+$, and have compact support in $\overline{\mathbb{R}}^n_+$; this is consistent with (9.4), cf. (C.7).

We note that even though r^+ is the restriction to the *open* set $\mathbb{R}^n_+$ (which makes sense for arbitrary distributions in $\mathscr{D}'(\mathbb{R}^n)$), the functions in $C_{(0)}^\infty(\overline{\mathbb{R}}^n_+)$ and $\mathscr{S}(\overline{\mathbb{R}}^n_+)$ are used as C^∞-functions on the closed set $\overline{\mathbb{R}}^n_+$. When we consider (measurable) *functions*, it makes no difference whether we speak of the restriction to $\mathbb{R}^n_+$ or to $\overline{\mathbb{R}}^n_+$, but for *distributions* it would, since there exist nonzero distributions supported in the set $\{ x \in \mathbb{R}^n \mid x_n = 0 \}$.

Let us define the partial Fourier transform (in the x'-variable) for functions $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$, by

$$\mathscr{F}_{x' \to \xi'} u = \acute{u}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix' \cdot \xi'} u(x', x_n)\, dx'. \tag{9.5}$$

The partial Fourier transform (9.5) can also be given a sense for arbitrary $u \in H^m(\mathbb{R}^n_+)$, and for suitable distributions, but we shall perform our calculations on rapidly decreasing functions whenever possible, and extend the statements by continuity.

Just as the standard norm $\|u\|_m$ on $H^m(\mathbb{R}^n)$ could be replaced by the norm $\|u\|_{m,\wedge}$ (6.16) involving $\langle \xi \rangle^m$, which is particularly well suited for considerations using the Fourier transform, we can replace the standard norm $\|u\|_m$ on $H^m(\mathbb{R}^n_+)$ by an expression involving powers of $\langle \xi' \rangle$. Define for $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$:

$$\begin{aligned} \|u\|^2_{m,\prime} &= (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} \int_0^\infty \sum_{j=0}^m \langle \xi' \rangle^{2(m-j)} |D^j_{x_n} \acute{u}(\xi', x_n)|^2 \, dx_n d\xi' \\ &= (2\pi)^{1-n} \sum_{j=0}^m \| \langle \xi' \rangle^{(m-j)} D^j_{x_n} \acute{u}(\xi', x_n) \|^2_{L_2(\mathbb{R}^n_+)}, \end{aligned} \tag{9.6}$$

with the associated scalar product

$$(u, v)_{m,\prime} = (2\pi)^{1-n} \sum_{j=0}^m (\langle \xi' \rangle^{(m-j)} D^j_{x_n} \acute{u}(\xi', x_n), \langle \xi' \rangle^{(m-j)} D^j_{x_n} \acute{v}(\xi', x_n))_{L_2}. \tag{9.7}$$

Lemma 9.1. *For $m \in \mathbb{N}_0$, there are inequalities*

$$\|u\|^2_m \le \|u\|^2_{m,\prime} \le C'_m \|u\|^2_m, \tag{9.8}$$

valid for all $u \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$, and hence $\|u\|_{m,\prime}$ (with its associated scalar product) extends by continuity to a Hilbert space norm on $H^m(\mathbb{R}^n_+)$ equivalent with the standard norm. Here $C'_0 = 1$.

Proof. We have as in (5.2), for $k \in \mathbb{N}$:

$$\sum_{\beta \in \mathbb{N}_0^{n-1}, |\beta| \le k} (\xi')^{2\beta} \le \langle \xi' \rangle^{2k} = \sum_{|\beta| \le k} C_{k,\beta} (\xi')^{2\beta} \le C'_k \sum_{|\beta| \le k} (\xi')^{2\beta}, \tag{9.9}$$

with positive integers $C_{k,\beta}$ $(= \frac{k!}{\beta!(k-|\beta|)!})$ and $C'_k = \max_{\beta \in \mathbb{N}_0^{n-1}, |\beta| \le k} C_{k,\beta}$. (The prime indicates that we get another constant than in (5.2) since the dimension has been replaced by $n-1$.)

We then calculate as follows, denoting $D' = (D_1, \dots, D_{n-1})$, and using the Parseval-Plancherel theorem with respect to the x'-variables:

$$\begin{aligned} \|u\|_m^2 &= \sum_{|\alpha| \le m} \|D^\alpha u\|_0^2 = \sum_{j=0}^m \sum_{|\beta| \le m-j} \|(D')^\beta D_{x_n}^j u\|_0^2 \\ &= \int_{\mathbb{R}^{n-1}} \sum_{j=0}^m \sum_{|\beta| \le m-j} (\xi')^{2\beta} \int_0^\infty |D_{x_n}^j \acute{u}(\xi', x_n)|^2 \, dx_n đ\xi' \\ &\le \int_{\mathbb{R}^{n-1}} \int_0^\infty \sum_{j=0}^m \langle \xi' \rangle^{2(m-j)} |D_{x_n}^j \acute{u}(\xi', x_n)|^2 \, dx_n đ\xi' \\ &\equiv \|u\|_{m,\prime}^2 \le C'_m \|u\|_m^2, \end{aligned} \tag{9.10}$$

using (9.9) in the inequalities.

This shows (9.8), and the rest follows since the subset $C_{(0)}^\infty(\overline{\mathbb{R}}_+^n)$ of $\mathscr{S}(\overline{\mathbb{R}}_+^n)$, hence $\mathscr{S}(\overline{\mathbb{R}}_+^n)$ itself, is dense in $H^m(\mathbb{R}_+^n)$ (Theorem 4.10). $\square$

The extended norm is likewise denoted $\|u\|_{m,\prime}$. Note that in view of the formulas (9.9), we can also write

$$\|u\|_{m,\prime}^2 = \sum_{j=0}^m \sum_{|\beta| \le m-j} C_{m-j,\beta} \|(D')^\beta D_n^j u\|_0^2, \tag{9.11}$$

where the norm is defined without reference to the partial Fourier transform.

The following is a sharper version of the trace theorem shown in Theorem 4.24 (and Exercise 4.22).

Theorem 9.2. *Let m be an integer > 0. For $0 \le j \le m-1$, the trace mapping*

$$\gamma_j : u(x', x_n) \mapsto D_{x_n}^j u(x', 0) \tag{9.12}$$

from $C_{(0)}^\infty(\overline{\mathbb{R}}_+^n)$ to $C_0^\infty(\mathbb{R}^{n-1})$ extends by continuity to a continuous linear mapping (also called γ_j) of $H^m(\mathbb{R}_+^n)$ into $H^{m-j-\frac{1}{2}}(\mathbb{R}^{n-1})$.

Proof. As in Theorem 4.24, we shall use an inequality like (4.47), but now in a slightly different and more precise way. For $v(t) \in C_0^\infty(\mathbb{R})$ one has

$$\begin{aligned}|v(0)|^2 &= -\int_0^\infty \frac{d}{dt}[v(t)\overline{v}(t)]\,dt = -\int_0^\infty [v'(t)\overline{v}(t) + v(t)\overline{v'}(t)]\,dt \\ &\le 2\|v\|_{L_2(\mathbb{R}_+)}\|\frac{dv}{dt}\|_{L_2(\mathbb{R}_+)} \\ &\le a^2\|v\|^2_{L_2(\mathbb{R}_+)} + a^{-2}\|\frac{dv}{dt}\|^2_{L_2(\mathbb{R}_+)},\end{aligned} \tag{9.13}$$

valid for any $a > 0$. For general functions $u \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ we apply the partial Fourier transform in x' (cf. (9.5)) and apply (9.13) with respect to x_n, setting $a = \langle\xi'\rangle^{\frac12}$ for each ξ' and using the norm (6.16) for $H^{m-j-\frac12}(\mathbb{R}^{n-1})$:

$$\begin{aligned}\|\gamma_j u\|^2_{H^{m-j-\frac12}(\mathbb{R}^{n-1})} &= \int_{\mathbb{R}^{n-1}} \langle\xi'\rangle^{2m-2j-1}|D^j_{x_n}\acute{u}(\xi',0)|^2 d\!\!\,\bar{}\,\xi' \\ &\le \int_{\mathbb{R}^{n-1}} \langle\xi'\rangle^{2m-2j-1}\int_0^\infty \big(a^2|D^j_{x_n}\acute{u}(\xi',x_n)|^2 \\ &\qquad + a^{-2}|D^{j+1}_{x_n}\acute{u}(\xi',x_n)|^2\big)\,dx_n d\!\!\,\bar{}\,\xi' \\ &= \int \big[\langle\xi'\rangle^{2(m-j)}|D^j_{x_n}\acute{u}(\xi',x_n)|^2 \\ &\qquad + \langle\xi'\rangle^{2(m-j-1)}|D^{j+1}_{x_n}\acute{u}(\xi',x_n)|^2\big]\,dx_n d\!\!\,\bar{}\,\xi' \\ &\le \|u\|^2_{m,\prime} \le C'_m\|u\|^2_m,\end{aligned} \tag{9.14}$$

cf. (9.8). This shows the continuity of the mapping, for the dense subset $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ of $H^m(\mathbb{R}^n_+)$, so the mapping γ_j can be extended to all of $H^m(\mathbb{R}^n_+)$ by closure. □

The range space in Theorem 9.2 is *optimal*, for one can show that the mappings γ_j are surjective. In fact this holds for the whole *system* of trace operators γ_j with $j = 0, \dots, m-1$. Let us for each $m > 0$ define the *Cauchy trace operator* $\rho_{(m)}$ associated with the order m by

$$\rho_{(m)} = \begin{pmatrix}\gamma_0\\ \gamma_1\\ \vdots\\ \gamma_{m-1}\end{pmatrix} : H^m(\mathbb{R}^n_+) \to \prod_{j=0}^{m-1} H^{m-j-\frac12}(\mathbb{R}^{n-1}); \tag{9.15}$$

for $u \in H^m(\mathbb{R}^n_+)$ we call $\rho_{(m)}u$ the *Cauchy data* of u. (The indexation with (m) may be omitted if it is understood from the context.) The space $\prod_{j=0}^{m-1} H^{m-j-\frac12}(\mathbb{R}^{n-1})$ is provided with the product norm $\|\varphi\|_{\prod H^{m-j-\frac12}} = \big(\|\varphi_0\|^2_{m-\frac12} + \dots + \|\varphi_{m-1}\|^2_{\frac12}\big)^{\frac12}$. Then one can show that the mapping $\varrho_{(m)}$ in (9.15) is surjective.

We first give a proof of the result for γ_0, that shows the basic idea and moreover gives some insight into boundary value problems for $I - \Delta$. For

$\varphi \in \mathscr{S}'(\mathbb{R}^{n-1})$ we denote its Fourier transform by $\hat{\varphi}$ (the usual notation, now applied with respect to the variable x').

Theorem 9.3. *Define the Poisson operator K_γ from $\mathscr{S}(\mathbb{R}^{n-1})$ to $\mathscr{S}(\overline{\mathbb{R}}^n_+)$ by*

$$K_\gamma : \varphi(x') \mapsto \mathscr{F}^{-1}_{\xi' \to x'}(e^{-\langle \xi' \rangle x_n} \hat{\varphi}(\xi')). \tag{9.16}$$

It satisfies

1° $\gamma_0 K_\gamma \varphi = \varphi$ *for* $\varphi \in \mathscr{S}(\mathbb{R}^{n-1})$.

2° $(I - \Delta) K_\gamma \varphi = 0$ *for* $\varphi \in \mathscr{S}(\mathbb{R}^{n-1})$.

3° K_γ *extends to a continuous mapping (likewise denoted* K_γ*) from* $H^{m-\frac{1}{2}}(\mathbb{R}^{n-1})$ *to* $H^m(\mathbb{R}^n_+)$ *for any* $m \in \mathbb{N}_0$*; the identity in* 1° *extends to* $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$*, and the identity in* 2° *extends to* $\varphi \in H^{-\frac{1}{2}}(\mathbb{R}^{n-1})$.

Proof. It is easily checked that $e^{-\langle \xi' \rangle x_n}$ belong to $\mathscr{S}(\overline{\mathbb{R}}^n_+)$ (one needs to check derivatives of $e^{-\langle \xi' \rangle x_n}$, where (5.7) is useful); then also $e^{-\langle \xi' \rangle x_n} \hat{\varphi}(\xi')$ is in $\mathscr{S}(\overline{\mathbb{R}}^n_+)$, and so is its inverse partial Fourier transform; hence K_γ maps $\mathscr{S}(\mathbb{R}^{n-1})$ into $\mathscr{S}(\overline{\mathbb{R}}^n_+)$. Now

$$\gamma_0 K_\gamma \varphi = \big[\mathscr{F}^{-1}_{\xi' \to x'}(e^{-\langle \xi' \rangle x_n} \hat{\varphi}(\xi'))\big]_{x_n=0} = \mathscr{F}^{-1}_{\xi' \to x'}(\hat{\varphi}(\xi')) = \varphi$$

shows 1°. By partial Fourier transformation,

$$\begin{aligned} \mathscr{F}_{x' \to \xi'}((I - \Delta) K_\gamma \varphi) &= (1 + |\xi'|^2 - \partial^2_{x_n})(e^{-\langle \xi' \rangle x_n} \hat{\varphi}(\xi')) \\ &= (\langle \xi' \rangle^2 - \langle \xi' \rangle^2) e^{-\langle \xi' \rangle x_n} \hat{\varphi}(\xi') = 0; \end{aligned}$$

this implies 2°.

For 3°, consider first the case $m = 0$. Here we have

$$\begin{aligned} \|K_\gamma \varphi\|_0^2 &= \int_{\mathbb{R}^{n-1}} \int_0^\infty |K_\gamma \varphi(x)|^2 \, dx_n đ\xi' \\ &= \int_{\mathbb{R}^{n-1}} \int_0^\infty e^{-2\langle \xi' \rangle x_n} |\hat{\varphi}(\xi')|^2 \, dx_n đ\xi' \\ &= \int_{\mathbb{R}^{n-1}} (2\langle \xi' \rangle)^{-1} |\hat{\varphi}(\xi')|^2 \, đ\xi' = \tfrac{1}{2} \|\varphi\|^2_{-\frac{1}{2},\wedge}, \end{aligned}$$

showing that K_γ is not only continuous from $H^{-\frac{1}{2}}(\mathbb{R}^{n-1})$ to $L_2(\mathbb{R}^n_+)$, but even proportional to an isometry (into the space).

In the cases $m > 0$, we must work a little more, but get simple formulas using the new norms:

$$\|K_\gamma \varphi\|^2_{m,'} = \int_{\mathbb{R}^{n-1}} \int_0^\infty \sum_{j=0}^m \langle \xi' \rangle^{2(m-j)} |D^j_{x_n} e^{-\langle \xi' \rangle x_n} \hat{\varphi}(\xi')|^2 \, dx_n đ\xi'$$

$$\begin{aligned}
&= \int_{\mathbb{R}^{n-1}} \int_0^\infty \sum_{j=0}^{m} \langle \xi' \rangle^{2m} e^{-2\langle \xi' \rangle x_n} |\hat\varphi(\xi')|^2 \, dx_n d\!\!{}^-\xi' \\
&= \tfrac{m+1}{2} \int_{\mathbb{R}^{n-1}} \langle \xi' \rangle^{2m-1} |\hat\varphi(\xi')|^2 \, d\!\!{}^-\xi' = \tfrac{m+1}{2} \|\varphi\|^2_{m-\frac12,\wedge};
\end{aligned} \tag{9.17}$$

again the mapping is proportional to an isometry.

Since γ_0 is well-defined as a continuous operator from $H^1(\mathbb{R}^n_+)$ to $H^{\frac12}(\mathbb{R}^{n-1})$ and K_γ is continuous in the opposite direction, the identity in 1° extends from the dense subset $\mathscr{S}(\mathbb{R}^{n-1})$ to $H^{\frac12}(\mathbb{R}^{n-1})$.

For 2°, let $\varphi \in H^{-\frac12}(\mathbb{R}^{n-1})$ and let φ_k be a sequence in $\mathscr{S}(\mathbb{R}^{n-1})$ converging to φ in $H^{-\frac12}(\mathbb{R}^{n-1})$. Then $K_\gamma\varphi_k$ converges to $v = K_\gamma\varphi$ in $L_2(\mathbb{R}^n_+)$. Here $(I-\Delta)K_\gamma\varphi_k = 0$ for all k, so in fact $K_\gamma\varphi_k$ converges to v in the graph norm for the maximal realization $A_{\max}$ defined from $I-\Delta$. Then $v \in D(A_{\max})$ with $A_{\max}v = 0$. So $(I-\Delta)v = 0$ in the distribution sense, in other words, $(I-\Delta)K_\gamma\varphi = 0$. This shows that 2° extends to $H^{-\frac12}(\mathbb{R}^{n-1})$. □

Remark 9.4. The mapping K_γ can be extended still further down to "negative Sobolev spaces". One can define $H^s(\mathbb{R}^n_+)$ for all orders (including noninteger and negative values, and consistently with the definitions for $s \in \mathbb{N}_0$) by

$$\begin{aligned}
&H^s(\mathbb{R}^n_+) = \{u \in \mathscr{D}'(\mathbb{R}^n_+) \mid u = U|_{\mathbb{R}^n_+} \text{ for some } U \in H^s(\mathbb{R}^n)\}, \\
&\|u\|_{s,\wedge} = \inf_{\text{such } U} \|U\|_{s,\wedge};
\end{aligned} \tag{9.18}$$

and there is for any $s \in \mathbb{R}$ a continuous linear extension mapping $p_s : H^s(\mathbb{R}^n_+) \to H^s(\mathbb{R}^n)$ such that r^+p_s is the identity on $H^s(\mathbb{R}^n_+)$. Then one can show that K_γ extends to map $H^{s-\frac12}(\mathbb{R}^{n-1})$ continuously into $H^s(\mathbb{R}^n_+)$ for any $s \in \mathbb{R}$.

On the other hand, the mapping γ_0 *does not* extend to negative Sobolev spaces. More precisely, one can show that γ_0 *makes sense on* $H^s(\mathbb{R}^n_+)$ *if and only if* $s > \frac12$. To show the sufficiency, we use the inequality $|v(0)| \le \int_{\mathbb{R}} |\hat v(t)| \, d\!\!{}^-t$, and the calculation, valid for $s > \frac12$,

$$\int \langle \xi \rangle^{-2s} \, d\!\!{}^-\xi_n = \langle \xi' \rangle^{-2s+1} \int (1 + |\xi_n/\langle \xi' \rangle|^2)^{-s} \, d\!\!{}^-(\xi_n/\langle \xi' \rangle) = c_s \langle \xi' \rangle^{-2s+1}. \tag{9.19}$$

Then we have for $u \in \mathscr{S}(\mathbb{R}^n)$:

$$\begin{aligned}
\|u(x',0)\|^2_{s-\frac12,\wedge} &= \int_{\mathbb{R}^{n-1}} |\acute u(\xi',0)|^2 \langle \xi' \rangle^{2s-1} \, d\!\!{}^-\xi' \\
&\le c \int_{\mathbb{R}^{n-1}} \langle \xi' \rangle^{2s-1} \Big(\int_{\mathbb{R}} |\hat u(\xi)| \, d\!\!{}^-\xi_n \Big)^2 d\!\!{}^-\xi' \\
&\le c' \int_{\mathbb{R}^{n-1}} \langle \xi' \rangle^{2s-1} \Big(\int_{\mathbb{R}} |\hat u(\xi)|^2 \langle \xi \rangle^{2s} \, d\!\!{}^-\xi_n \Big) \Big(\int_{\mathbb{R}} \langle \xi \rangle^{-2s} \, d\!\!{}^-\xi_n \Big) d\!\!{}^-\xi'
\end{aligned}$$

$$= c'' \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 \, d\!\!{}^-\xi = c_3 \|u\|^2_{s,\wedge}. \tag{9.20}$$

By density, this defines a bounded mapping $\tilde{\gamma}_0$ from $H^s(\mathbb{R}^n)$ to $H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$, and it follows that γ_0 extends to a bounded mapping from $H^s(\mathbb{R}^n_+)$ to $H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. The necessity of $s > \frac{1}{2}$ is seen in case $n = 1$ as follows: Assume that $\gamma_0 : u \mapsto u(0)$ is bounded from $H^s(\mathbb{R})$ to $\mathbb{C}$ when $u \in \mathscr{S}(\mathbb{R})$. Write

$$u(0) = \int_{\mathbb{R}} \hat{u}(t) \, d\!\!{}^-t = \int_{\mathbb{R}} \langle t \rangle^{-s} \langle t \rangle^{s} \hat{u}(t) \, d\!\!{}^-t = \langle \langle t \rangle^{-s}, g(t) \rangle,$$

where $g(t) = \frac{1}{2\pi} \langle t \rangle^s \hat{u}(t)$ likewise runs through $\mathscr{S}(\mathbb{R})$. The boundedness implies that

$$|u(0)| = |\langle \langle t \rangle^{-s}, g(t) \rangle| \leq c\|u\|_{s,\wedge} = c'\|g\|_{L_2(\mathbb{R})},$$

which can only hold when $\langle t \rangle^{-s} \in L_2$, i.e., $s > \frac{1}{2}$. The general case is treated in [LM68, Th. I 4.3].

Nevertheless, γ_0 does have a sense, for any $s \in \mathbb{R}$, on the *subset* of distributions $u \in H^s(\mathbb{R}^n_+)$ for which $(I - \Delta)u = 0$. We shall show this below for $s = 0$. Variable-coefficient A's are treated in Lions and Magenes [LM68]; a general account is given in Theorem 11.4 later.

To show the surjectiveness of the Cauchy data map $\varrho_{(m)}$, one can construct a right inverse as a linear combination of operators defined as in (9.16) with $x_n^j e^{-\langle \xi' \rangle x_n}$, $j = 0, \dots, m-1$, inserted. Here follows another choice that is easy to check (and has the advantage that it could be used for $\mathbb{R}^n_-$ too). It is called a *lifting operator*.

Theorem 9.5. *Let $\psi \in \mathscr{S}(\mathbb{R})$ with $\psi(t) = 1$ on a neighborhood of 0. Define the Poisson operator $\mathcal{K}_{(m)}$ from $\varphi = \{\varphi_0, \dots, \varphi_{m-1}\} \in \mathscr{S}(\mathbb{R}^{n-1})^m$ to $\mathcal{K}_{(m)}\varphi \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ by*

$$\begin{aligned} &\mathcal{K}_{(m)}\varphi = \mathcal{K}_0\varphi_0 + \dots + \mathcal{K}_{m-1}\varphi_{m-1}, \ \textit{where} \\ &\mathcal{K}_j\varphi_j = \mathscr{F}^{-1}_{\xi' \to x'}\big(\tfrac{(ix_n)^j}{j!} \psi(\langle \xi' \rangle x_n) \hat{\varphi}_j(\xi')\big), \ \textit{for } j = 0, \dots, m-1. \end{aligned} \tag{9.21}$$

It satisfies

1° *$\varrho_{(m)}\mathcal{K}_{(m)}\varphi = \varphi$ for $\varphi \in \mathscr{S}(\mathbb{R}^{n-1})^m$.*

2° *$\mathcal{K}_{(m)}$ extends to a continuous mapping (likewise denoted $\mathcal{K}_{(m)}$) from $\Pi_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\mathbb{R}^{n-1})$ to $H^m(\mathbb{R}^n_+)$, and the identity in 1° extends to $\varphi \in \Pi_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\mathbb{R}^{n-1})$.*

Proof. Since $\psi(0) = 1$ and $D^j_{x_n}\psi(0) = 0$ for $j > 0$, and $[D^k_{x_n}(\frac{1}{j!} i^j x_n^j)]_{x_n=0} = \delta_{kj}$ (the Kronecker delta),

$$\gamma_k \mathcal{K}_j \varphi_j = (D^k_{x_n} \tfrac{(ix_n)^j}{j!})|_{x_n=0} \varphi_j = \delta_{kj}\varphi_j; \tag{9.22}$$

showing 1°.

For the estimates in 2°, we observe the formulas that arise from replacing x_n by $t = \langle \xi' \rangle x_n$ in the integrals:

$$\int_0^\infty |D_{x_n}^k (x_n^j \psi(\langle \xi' \rangle x_n))|^2 \, dx_n \\ = \langle \xi' \rangle^{2k-2j-1} \int_0^\infty |D_t^k (t^j \psi(t))|^2 \, dt \equiv \langle \xi' \rangle^{2k-2j-1} c_{kj}. \quad (9.23)$$

Then

$$\begin{aligned} &\|\mathcal{K}_j \varphi_j\|_{m,\prime}^2 \\ &= \int_{\mathbb{R}^{n-1}} \int_0^\infty \sum_{k \le m} \langle \xi' \rangle^{2m-2k} |D_{x_n}^k [\tfrac{(ix_n)^j}{j!} \psi(\langle \xi' \rangle x_n)] \hat{\varphi}_j(\xi')|^2 \, d\!\!{}^-\xi' dx_n \\ &= \int_{\mathbb{R}^{n-1}} \sum_{k \le m} \langle \xi' \rangle^{2m-2k} \langle \xi' \rangle^{2k-2j-1} \tfrac{c_{kj}}{(j!)^2} |\hat{\varphi}_j(\xi')|^2 \, d\!\!{}^-\xi' \\ &= c_j \|\varphi_j\|_{m-j-\frac{1}{2},\wedge}^2, \end{aligned}$$

which shows the desired boundedness estimate. Now the identity in 1° extends by continuity. □

We know from Theorem 4.25 that the space $H_0^1(\mathbb{R}_+^n)$, defined as the closure of $C_0^\infty(\mathbb{R}_+^n)$ in $H^1(\mathbb{R}_+^n)$, is exactly the space of elements u of $H^1(\mathbb{R}_+^n)$ for which $\gamma_0 u = 0$. This fact extends to higher Sobolev spaces. As usual, $H_0^m(\mathbb{R}_+^n)$ is defined as the closure of $C_0^\infty(\mathbb{R}_+^n)$ in $H^m(\mathbb{R}_+^n)$.

Theorem 9.6. *For all $m \in \mathbb{N}$,*

$$H_0^m(\mathbb{R}_+^n) = \{u \in H^m(\mathbb{R}_+^n) \mid \gamma_0 u = \cdots = \gamma_{m-1} u = 0\}. \quad (9.24)$$

This is proved by a variant of the proof of Theorem 4.25, or by the method described after it, and will not be written in detail here. We observe another interesting fact concerning $H_0^m(\mathbb{R}_+^n)$:

Theorem 9.7. *For each $m \in \mathbb{N}_0$, $H_0^m(\mathbb{R}_+^n)$ identifies, by extension by zero on $\mathbb{R}_-^n$, with the subspace of $H^m(\mathbb{R}^n)$ consisting of the functions supported in $\overline{\mathbb{R}}_+^n$.*

Proof (indications). When $u \in H_0^m(\mathbb{R}_+^n)$, it is the limit of a sequence of function $u_k \in C_0^\infty(\mathbb{R}_+^n)$ in the m-norm. Extending the u_k by 0 (to $e^+ u_k$), we see that $e^+ u_k \to e^+ u$ in $H^m(\mathbb{R}^n)$ with $e^+ u$ supported in $\overline{\mathbb{R}}_+^n$. All such functions are reached, for if $v \in H^m(\mathbb{R}^n)$ has support in $\overline{\mathbb{R}}_+^n$, we can approximate it in m-norm by functions in $C_0^\infty(\mathbb{R}_+^n)$ by 1) truncation, 2) translation *into* $\mathbb{R}_+^n$ and 3) mollification. □

For any $s \in \mathbb{R}$, one can define the closed subspace of $H^s(\mathbb{R}^n)$:

$$H_0^s(\overline{\mathbb{R}}^n_+) = \{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\mathbb{R}}^n_+\}; \tag{9.25}$$

it identifies with $H_0^m(\mathbb{R}^n_+)$ when $s = m \in \mathbb{N}_0$. For negative s (more precisely, when $s \le -\frac{1}{2}$), this is *not* a space of distributions on $\mathbb{R}^n_+$. It serves as a dual space: For any s one can show that $H^s(\mathbb{R}^n_+)$ and $H_0^{-s}(\overline{\mathbb{R}}^n_+)$ are dual spaces of one another, with a duality extending the scalar product in $L_2(\mathbb{R}^n_+)$, similarly to Theorem 6.15. (The duality is shown e.g. in [H63, Sect. 2.5].)

9.2 The Dirichlet problem for $I - \Delta$ on the half-space

We now consider the elliptic operator $A = I - \Delta$ on $\mathbb{R}^n_+$. Recall the definition of $A_{\max}$ and $A_{\min}$ from Chapter 4; note that by Theorem 6.24, $D(A_{\min}) = H_0^2(\mathbb{R}^n_+)$.

First we will show that γ_0 can be extended to $D(A_{\max})$. An important ingredient is the following denseness result (adapted from [LM68]):

Theorem 9.8. *The space $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ is dense in $D(A_{\max})$.*

Proof. This follows if we show that when ℓ is a continuous antilinear (conjugate linear) functional on $D(A_{\max})$ which vanishes on $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, then $\ell = 0$. So let ℓ be such a functional; it can be written

$$\ell(u) = (f, u)_{L_2(\mathbb{R}^n_+)} + (g, Au)_{L_2(\mathbb{R}^n_+)} \tag{9.26}$$

for some $f, g \in L_2(\mathbb{R}^n_+)$. We know that $\ell(\varphi) = 0$ for $\varphi \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$. Any such φ is the restriction to $\mathbb{R}^n_+$ of a function $\Phi \in C_0^\infty(\mathbb{R}^n)$, and in terms of such functions we have

$$\ell(r^+\Phi) = (e^+f, \Phi)_{L_2(\mathbb{R}^n)} + (e^+g, (I - \Delta)\Phi)_{L_2(\mathbb{R}^n)} = 0, \text{ all } \Phi \in C_0^\infty(\mathbb{R}^n). \tag{9.27}$$

Now use that $I - \Delta$ on $\mathbb{R}^n$ has the formal adjoint $I - \Delta$, so the right-hand side equations in (9.27) imply

$$\langle e^+f + (I - \Delta)e^+g, \overline{\Phi}\rangle = 0, \text{ all } \Phi \in C_0^\infty(\mathbb{R}^n),$$

i.e.,

$$e^+f + (I - \Delta)e^+g = 0, \text{ or } (I - \Delta)e^+g = -e^+f, \tag{9.28}$$

as distributions on $\mathbb{R}^n$. Here we know that e^+g and e^+f are in $L_2(\mathbb{R}^n)$, and Theorem 6.12 then gives that $e^+g \in H^2(\mathbb{R}^n)$. Since it has support in $\overline{\mathbb{R}}^n_+$, it identifies with a function in $H_0^2(\mathbb{R}^n_+)$ by Theorem 9.7, i.e., $g \in H_0^2(\mathbb{R}^n_+)$. Then by Theorem 6.24, g is in $D(A_{\min})$! And (9.28) implies that $Ag = -f$. But then, for any $u \in D(A_{\max})$,

$$\ell(u) = (f, u)_{L_2(\mathbb{R}^n_+)} + (g, Au)_{L_2(\mathbb{R}^n_+)} = -(Ag, u)_{L_2(\mathbb{R}^n_+)} + (g, Au)_{L_2(\mathbb{R}^n_+)} = 0,$$

since $A_{\max}$ and $A_{\min}$ are adjoints. □

In the study of second-order operators, it is customary to omit the factor $-i$ on the normal derivative. So we define the Cauchy data as

$$\varrho u = \begin{pmatrix} \gamma_0 u \\ \nu u \end{pmatrix}, \text{ where } \nu u = \frac{\partial u}{\partial \nu} = \gamma_0(\partial_{x_n} u) = i\gamma_1 u. \tag{9.29}$$

Lemma 9.9. *For u and v in $H^2(\mathbb{R}^n_+)$ one has Green's formula*

$$(Au, v)_{L_2(\mathbb{R}^n_+)} - (u, Av)_{L_2(\mathbb{R}^n_+)} = (\nu u, \gamma_0 v)_{L_2(\mathbb{R}^{n-1})} - (\gamma_0 u, \nu v)_{L_2(\mathbb{R}^{n-1})}. \tag{9.30}$$

Proof. For functions in $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, the formula follows directly from (A.20). (It is easily verified by integration by parts.) Since γ_0 and ν by Theorem 9.2 map $H^2(\mathbb{R}^n_+)$ to spaces continuously injected in $L_2(\mathbb{R}^{n-1})$, the equation extends by continuity to $u, v \in H^2(\mathbb{R}^n_+)$. □

Now we can show:

Theorem 9.10. *Let $A = I - \Delta$ on $\mathbb{R}^n_+$. The Cauchy trace operator $\varrho = \{\gamma_0, \nu\}$, defined on $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, extends by continuity to a continuous mapping from $D(A_{\max})$ to $H^{-\frac{1}{2}}(\mathbb{R}^{n-1}) \times H^{-\frac{3}{2}}(\mathbb{R}^{n-1})$. Here Green's formula* (9.30) *extends to the formula*

$$(Au, v)_{L_2(\mathbb{R}^n_+)} - (u, Av)_{L_2(\mathbb{R}^n_+)} = \langle \nu u, \overline{\gamma_0 v} \rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}} - \langle \gamma_0 u, \overline{\nu v} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}, \tag{9.31}$$

for $u \in D(A_{\max})$, $v \in H^2(\mathbb{R}^n_+)$.

Moreover, $\gamma_0 K_\gamma = I$ on $H^{-\frac{1}{2}}(\mathbb{R}^{n-1})$.

Proof. Let $u \in D(A_{\max})$. In the following, we write $H^s(\mathbb{R}^{n-1})$ as H^s. We want to define $\varrho u = \{\gamma_0 u, \nu u\}$ as a continuous antilinear functional on $H^{\frac{1}{2}} \times H^{\frac{3}{2}}$, depending continuously (and of course linearly) on $u \in D(A_{\max})$. For a given $\varphi = \{\varphi_0, \varphi_1\} \in H^{\frac{1}{2}} \times H^{\frac{3}{2}}$, we can use Theorem 9.5 to define

$$w_\varphi = \mathcal{K}_0 \varphi_1 + i\mathcal{K}_1 \varphi_0; \text{ then } \gamma_0 w_\varphi = \varphi_1,\ \nu w_\varphi = -\varphi_0.$$

Now we set

$$\begin{aligned} &\ell_u(\varphi) = (Au, w_\varphi) - (u, Aw_\varphi), \text{ noting that} \\ &|\ell_u(\varphi)| \le C\|u\|_{D(A_{\max})} \|w_\varphi\|_{H^2(\mathbb{R}^n_+)} \le C'\|u\|_{D(A_{\max})} \|\varphi\|_{H^{\frac{1}{2}} \times H^{\frac{3}{2}}}. \end{aligned} \tag{9.32}$$

So, ℓ_u is a continuous antilinear functional on $\varphi \in H^{\frac{1}{2}} \times H^{\frac{3}{2}}$, hence defines an element $\psi = \{\psi_0, \psi_1\} \in H^{-\frac{1}{2}} \times H^{-\frac{3}{2}}$ such that

$$l_u(\varphi) = \langle \psi_0, \overline{\varphi_0} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} + \langle \psi_1, \overline{\varphi_1} \rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}}.$$

Moreover, it depends continuously on $u \in D(A_{\max})$, in view of the estimates in (9.32). If u is in $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, the defining formula in (9.32) can be rewritten using Green's formula (9.30), which leads to

$$l_u(\varphi) = (Au, w_\varphi) - (u, Aw_\varphi) = (\nu u, \gamma_0 w_\varphi) - (\gamma_0 u, \nu w_\varphi) = (\gamma_0 u, \varphi_0) + (\nu u, \varphi_1),$$

for such u. Since φ_0 and φ_1 run through full Sobolev spaces, it follows that $\psi_0 = \gamma_0 u$, $\psi_1 = \nu u$, when $u \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, so the functional ℓ_u is consistent with $\{\gamma_0 u, \nu u\}$ then. Since $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ is dense in $D(A_{\max})$, we have found the unique continuous extension.

(9.31) is now obtained in general by extending (9.30) by continuity from $u \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, $v \in H^2(\mathbb{R}^n_+)$.

For the last statement, let $\eta \in H^{-\frac{1}{2}}(\mathbb{R}^{n-1})$. Note that we have already shown in Theorem 9.3 that K_γ maps $\eta \in H^{-\frac{1}{2}}$ into a function in $L_2(\mathbb{R}^n_+)$ satisfying $(I - \Delta)K_\gamma \eta = 0$. But then $K_\gamma \eta \in D(A_{\max})$, so γ_0 can be applied in the new sense. The formula $\gamma_0 K_\gamma \eta = \eta$ is then obtained by extension by continuity from the case $\eta \in \mathscr{S}(\mathbb{R}^{n-1})$. $\square$

The definition was made via a choice w_φ of the "lifting" of the vector φ, but the final result shows that the definition is independent of this choice.

Consider now the Dirichlet problem (with a scalar φ)

$$Au = f \text{ in } \mathbb{R}^n_+, \quad \gamma_0 u = \varphi \text{ on } \mathbb{R}^{n-1}, \tag{9.33}$$

and its two semihomogeneous versions

$$Au = f \text{ in } \mathbb{R}^n_+, \quad \gamma_0 u = 0 \text{ on } \mathbb{R}^{n-1}, \tag{9.34}$$

$$Az = 0 \text{ in } \mathbb{R}^n_+, \quad \gamma_0 z = \varphi \text{ on } \mathbb{R}^{n-1}. \tag{9.35}$$

For (9.34), we find by application of the variational theory in Section 12.4 to the triple $(L_2(\mathbb{R}^n_+), H^1_0(\mathbb{R}^n_+), a(u,v))$ with $a(u,v) = (u,v)_1$, the variational operator A_γ, which is selfadjoint with lower bound 1. It is seen as in Section 4.4 (cf. Exercise 4.23), that A_γ is a realization of A with domain $D(A_\gamma) = D(A_{\max}) \cap H^1_0(\mathbb{R}^n_+)$; the Dirichlet realization. Thus problem (9.34) has for $f \in L_2(\mathbb{R}^n_+)$ the unique solution $u = A_\gamma^{-1} f$ in $D(A_{\max}) \cap H^1_0(\mathbb{R}^n_+)$.

For (9.35), we have just shown that when $m \in \mathbb{N}_0$, it has a solution in $H^m(\mathbb{R}^n_+)$ for any $\varphi \in H^{m-\frac{1}{2}}(\mathbb{R}^{n-1})$, namely, $K_\gamma \varphi$, by Theorem 9.3 and the last statement in Theorem 9.10. We have not yet investigated the uniqueness of the latter solution; it will be obtained below.

We can improve the information on (9.34) by use of K_γ and the structure of $I - \Delta$: To solve (9.34) for a given $f \in L_2(\mathbb{R}^n_+)$, introduce

$$v = r^+ Q e^+ f, \quad Q = \operatorname{Op}(\langle \xi \rangle^{-2}). \tag{9.36}$$

Since Q maps $L_2(\mathbb{R}^n)$ homeomorphically onto $H^2(\mathbb{R}^n)$ (Section 6.2), $r^+ Q e^+$ is continuous from $L_2(\mathbb{R}^n_+)$ to $H^2(\mathbb{R}^n_+)$. Application of the differential operator

$A = I - \Delta$ to v gives that

$$Av = Ar^+Qe^+f = r^+(I - \Delta)Qe^+f = f,$$

so when u solves (9.34), the function $z = u - v$ must solve (9.35) with $\varphi = -\gamma_0 r^+Qe^+f$; here φ lies in $H^{\frac{3}{2}}(\mathbb{R}^{n-1})$. The problem for z has a solution $z = K_\gamma\varphi = -K_\gamma\gamma_0 r^+Qe^+f \in H^2(\mathbb{R}^n_+)$. Writing $u = v + z$, we finally obtain a solution of (9.34) of the form

$$u = r^+Qe^+f - K_\gamma\gamma_0 r^+Qe^+f. \tag{9.37}$$

It lies in $H^2(\mathbb{R}^n_+) \subset D(A_{\max})$, and in $H^1_0(\mathbb{R}^n_+)$ since $\gamma_0 u = 0$, so it must equal the unique solution already found. This improves the regularity! We have shown:

Theorem 9.11. *The solution operator A_γ^{-1} of (9.34) equals*

$$A_\gamma^{-1} = r^+Qe^+ - K_\gamma\gamma_0 r^+Qe^+, \text{ also denoted } R_\gamma. \tag{9.38}$$

It maps $L_2(\mathbb{R}^n_+)$ continuously into $H^2(\mathbb{R}^n_+)$, hence

$$D(A_\gamma) = H^2(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+). \tag{9.39}$$

For the special case we are considering, this gives the optimal regularity statement for the problem (9.34) with $f \in L_2(\mathbb{R}^n_+)$. And not only that; it also gives a *solution formula* (9.38) which is more constructive than the existence-and-uniqueness statement we had from the variational theory.

Remark 9.12. With further efforts it can be shown that r^+Qe^+ maps $H^k(\mathbb{R}^n_+)$ into $H^{k+2}(\mathbb{R}^n_+)$ also for $k \geq 1$; this is not clear from its form, since e^+ does not map $H^k(\mathbb{R}^n_+)$ into $H^k(\mathbb{R}^n)$ for $k \geq 1$, but hinges on the so-called transmission property of Q. Then one can obtain that u defined by (9.37) is in $H^{k+2}(\mathbb{R}^n_+)$, when $f \in H^k(\mathbb{R}^n_+)$: *higher elliptic regularity*. A systematic result of this kind is shown in Chapter 10.

As a corollary, we obtain a uniqueness statement for (9.35) in a regular case:

Corollary 9.13. *For any $\varphi \in H^{\frac{3}{2}}(\mathbb{R}^{n-1})$, $K_\gamma\varphi$ is the unique solution in $H^2(\mathbb{R}^n_+)$ of (9.35).*

Proof. Let $v = K_\gamma\varphi$, then $v \in H^2(\mathbb{R}^n_+)$ with $Av = 0$ and $\gamma_0 v = \varphi$. So a function z solves (9.35) if and only if $w = z - v$ solves

$$Aw = 0, \quad \gamma_0 w = 0.$$

Here if $z \in H^2(\mathbb{R}^n_+)$, also $w \in H^2(\mathbb{R}^n_+)$, so by the uniqueness of solutions of (9.34) in $H^2(\mathbb{R}^n_+)$, w must equal 0. □

The argument can be extended to cover $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$, $z \in H^1(\mathbb{R}^n_+)$, but since we are aiming for $z \in L_2(\mathbb{R}^n_+)$ we go on toward that case (and include the H^1 case afterwards).

Define, for $k \in \mathbb{N}_0$,

$$Z^k(A) = \{z \in H^k(\mathbb{R}^n_+) \mid Az = 0\}, \text{ closed subspace of } H^k(\mathbb{R}^n_+); \tag{9.40}$$

note that $Z^0(A) = Z(A_{\max})$.

Corollary 9.13 and Theorem 9.3 together imply:

Corollary 9.14. *The mappings*

$$\gamma_0 : Z^2(A) \to H^{\frac{3}{2}}(\mathbb{R}^{n-1}) \text{ and } K_\gamma : H^{\frac{3}{2}}(\mathbb{R}^{n-1}) \to Z^2(A)$$

are inverses of one another.

Proof. The mappings are well-defined according to Theorem 9.3, which also shows the identity $\gamma_0 K_\gamma = I$ on $H^{\frac{3}{2}}(\mathbb{R}^{n-1})$; it implies surjectiveness of γ_0 and injectiveness of K_γ. Corollary 9.13 implies that when $z \in Z^2(A)$, $K_\gamma \gamma_0 z = z$, so $K_\gamma : H^{\frac{3}{2}}(\mathbb{R}^{n-1}) \to Z^2(A)$ is surjective. □

Now we can show:

Proposition 9.15. *$Z^2(A)$ is dense in $Z^0(A)$.*

Proof. Let $z \in Z^0(A)$. By Theorem 9.8 there exists a sequence $u_k \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ such that $u_k \to z$ in $L_2(\mathbb{R}^n_+)$, $Au_k \to 0$. Let $v_k = A_\gamma^{-1} A u_k$, then $v_k \to 0$ in $H^2(\mathbb{R}^n_+)$ by Theorem 9.11, so also $z_k = u_k - v_k \to z$ in $D(A_{\max})$, with $z_k \in H^2(\mathbb{R}^n_+)$. Here $Au_k = Av_k$ by definition, so indeed $z_k \in Z^2(A)$. □

Then we finally obtain:

Theorem 9.16. *The mappings*

$$\gamma_0 : Z^0(A) \to H^{-\frac{1}{2}}(\mathbb{R}^{n-1}) \text{ and } K_\gamma : H^{-\frac{1}{2}}(\mathbb{R}^{n-1}) \to Z^0(A)$$

are inverses of one another.

Proof. We already have the identity $\gamma_0 K_\gamma \varphi = \varphi$ for $\varphi \in H^{-\frac{1}{2}}(\mathbb{R}^{n-1})$ from Theorem 9.10, and the other identity $K_\gamma \gamma_0 z = z$ for $z \in Z^0(A)$ now follows from the identity valid on $Z^2(A)$ by extension by continuity, using Proposition 9.15. □

Corollary 9.17. *Let $m \in \mathbb{N}_0$. The mappings*

$$\gamma_0 : Z^m(A) \to H^{m-\frac{1}{2}}(\mathbb{R}^{n-1}) \text{ and } K_\gamma : H^{m-\frac{1}{2}}(\mathbb{R}^{n-1}) \to Z^m(A) \tag{9.41}$$

are inverses of one another.

Proof. That the mappings are well-defined follows from Theorem 9.3 and its extension in Theorem 9.10. Then the statements $\gamma_0 K_\gamma \varphi = \varphi$ and $K_\gamma \gamma_0 z = z$ for the relevant spaces follow by restriction from Theorem 9.16. □

Theorem 9.11 and Corollary 9.17 give highly satisfactory results on the solvability of the semihomogeneous problems (9.34) and (9.35). We collect some consequences for the nonhomogeneous problem (9.33):

Theorem 9.18. *The Dirichlet problem* (9.33) *is uniquely solvable in* $D(A_{\max})$ *for* $f \in L_2(\mathbb{R}^n_+)$, $\varphi \in H^{\frac{3}{2}}(\mathbb{R}^{n-1})$*; the solution belongs to* $H^2(\mathbb{R}^n_+)$ *and is defined by the formula*

$$u = r^+ Q e^+ f - K_\gamma \gamma_0 r^+ Q e^+ f + K_\gamma \varphi = R_\gamma f + K_\gamma \varphi; \tag{9.42}$$

cf. (9.38)*. In particular, the mapping*

$$\begin{pmatrix} A \\ \gamma_0 \end{pmatrix} : H^2(\mathbb{R}^n_+) \to \begin{matrix} L_2(\mathbb{R}^n_+) \\ \times \\ H^{\frac{3}{2}}(\mathbb{R}^{n-1}) \end{matrix} \quad \textit{has inverse} \quad \begin{pmatrix} R_\gamma & K_\gamma \end{pmatrix} : \begin{matrix} L_2(\mathbb{R}^n_+) \\ \times \\ H^{\frac{3}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to H^2(\mathbb{R}^n_+). \tag{9.43}$$

The above gives a complete clarification of the basic solvability properties for the Dirichlet problem (9.33) in the very special case of the simple constant-coefficient operator $I - \Delta$ on $\mathbb{R}^n_+$.

The deduction is of general interest. In fact, one can to some extent treat Dirichlet problems for strongly elliptic operators A (those where the real part of the principal symbol is positive) on bounded smooth domains Ω according to the same scheme. For such operators one can again define a variational realization A_γ representing the Dirichlet condition (this is done at the end of Chapter 7). One can add a constant to A to obtain that A_γ is invertible.

By use of localizations as in Appendix C and Section 4.2, one can establish mapping properties of the Cauchy trace operator analogous to (9.15). Moreover, one can establish a lifting operator $\mathcal{K}$ (a Poisson operator) with properties similar to those in Theorem 9.5. The denseness of $C^\infty(\overline{\Omega})$ in the maximal domain, the generalized trace operators and the extended Green's formula can be obtained much as in Theorems 9.8 and 9.10.

It is somewhat harder to extend Theorem 9.3 which defines the Poisson operator K_γ mapping into the nullspace, to the general situation, though; one can easily get a first approximation, but from then on there is as much work in it as in the general treatment of (9.33). The strategy in Lions and Magenes [LM68] is to begin with the regularity theory for the fully nonhomogeneous boundary value problem, next to pass to an adjoint situation in negative Sobolev spaces and finally use interpolation theory. (Their method works for far more general boundary conditions too.) Methods for constructing K_γ more directly can be based on the theory of Boutet de Monvel [B71] (see Chapters 10 and 11).

Note that in the case of $I-\Delta$ we could use the straightforward inverse $Q = \mathrm{Op}(\langle\xi\rangle^{-2})$ on $\mathbb{R}^n$; in general cases it will be replaced by more complicated expressions, pseudodifferential parametrices.

In the calculus of pseudodifferential boundary problems initiated by Boutet de Monvel, operators like r^+Qe^+ are called *truncated pseudodifferential operators*, operators like K_γ and $\mathcal{K}_{(m)}$ are called *Poisson operators* (we have already used this name) and operators like $K_\gamma\gamma_0$ and $K_\gamma\gamma_0 r^+Qe^+$, acting on functions on $\mathbb{R}^n_+$ but not of the form r^+Pe^+ with a ψdo P, are called *singular Green operators*. More on such operators in Chapter 10.

9.3 The Neumann problem for $I-\Delta$ on the half-space

For the special operator $I-\Delta$ it is now also easy to discuss the Neumann problem on $\mathbb{R}^n_+$. First we have, similarly to Theorem 9.3:

Theorem 9.19. *Define the Poisson operator K_ν from $\mathscr{S}(\mathbb{R}^{n-1})$ to $\mathscr{S}(\overline{\mathbb{R}}^n_+)$ by*

$$K_\nu : \varphi(x') \mapsto \mathscr{F}^{-1}_{\xi'\to x'}(-\langle\xi'\rangle^{-1}e^{-\langle\xi'\rangle x_n}\hat\varphi(\xi')). \tag{9.44}$$

It satisfies

1° $\nu K_\nu\varphi = \varphi$ *for* $\varphi \in \mathscr{S}(\mathbb{R}^{n-1})$.

2° $(I-\Delta)K_\nu\varphi = 0$ *for* $\varphi \in \mathscr{S}(\mathbb{R}^{n-1})$.

3° K_ν *extends to a continuous mapping (likewise denoted K_ν) from $H^{m-\frac32}(\mathbb{R}^{n-1})$ to $H^m(\mathbb{R}^n_+)$ for any $m \in \mathbb{N}_0$; the identity in 1° extends to $\varphi \in H^{\frac12}(\mathbb{R}^{n-1})$, and the identity in 2° extends to $\varphi \in H^{-\frac32}(\mathbb{R}^{n-1})$.*

Proof. 1° is seen from

$$\nu K_\nu\varphi = \big[\mathscr{F}^{-1}_{\xi'\to x'}(-\partial_{x_n}\langle\xi'\rangle^{-1}e^{-\langle\xi'\rangle x_n}\hat\varphi(\xi'))\big]_{x_n=0} = \mathscr{F}^{-1}_{\xi'\to x'}(\hat\varphi(\xi')) = \varphi.$$

2° holds since

$$\begin{aligned}\mathscr{F}_{x'\to\xi'}((I-\Delta)K_\nu\varphi) &= (\langle\xi'\rangle^2 - \partial^2_{x_n})(-\langle\xi'\rangle^{-1}e^{-\langle\xi'\rangle x_n}\hat\varphi(\xi'))\\ &= (-\langle\xi'\rangle + \langle\xi'\rangle)e^{-\langle\xi'\rangle x_n}\hat\varphi(\xi') = 0.\end{aligned}$$

For the continuity statements in 3°, we calculate as in (9.17), with the modification that the extra factor $-\langle\xi'\rangle^{-1}$ changes the norm on φ to be the norm in $H^{m-\frac32}(\mathbb{R}^{n-1})$.

Since ν is well-defined as a continuous operator from $H^2(\mathbb{R}^n_+)$ to $H^{\frac12}(\mathbb{R}^{n-1})$ by Theorem 9.2, and K_ν is continuous in the opposite direction, the identity in 1° extends by continuity to $H^{\frac12}(\mathbb{R}^{n-1})$. The proof of the extension of 2° goes in the same way as in Theorem 9.3. □

Now consider the Neumann problem

$$Au = f \text{ in } \mathbb{R}^n_+, \quad \nu u = \varphi \text{ on } \mathbb{R}^{n-1}, \tag{9.45}$$

and its two semihomogeneous versions

$$Au = f \text{ in } \mathbb{R}^n_+, \quad \nu u = 0 \text{ on } \mathbb{R}^{n-1}, \tag{9.46}$$
$$Az = 0 \text{ in } \mathbb{R}^n_+, \quad \nu z = \varphi \text{ on } \mathbb{R}^{n-1}. \tag{9.47}$$

For (9.46) we find by application of the variational theory in Section 12.4 to the triple $(L_2(\mathbb{R}^n_+), H^1(\mathbb{R}^n_+), a(u,v))$ with $a(u,v) = (u,v)_1$, the variational operator A_ν, which is selfadjoint with lower bound 1. It is seen as in Section 4.4 (cf. Exercise 4.23), that A_ν is a realization of A with domain

$$D(A_\nu) = \{u \in D(A_{\max}) \cap H^1(\mathbb{R}^n_+) \mid (Au, v) = a(u,v) \text{ for all } v \in H^1(\mathbb{R}^n_+)\}; \tag{9.48}$$

the so-called Neumann realization. The smooth elements of $D(A_\nu)$ satisfy the Neumann condition $\nu u = 0$, but the interpretation of the conditions in (9.48) in the general case was not fully clarified (we do so below). However, in this generalized sense, problem (9.46) has for $f \in L_2(\mathbb{R}^n_+)$ the unique solution $u = A_\nu^{-1} f$ in $D(A_\nu)$.

For the problem (9.47), Theorem 9.19 shows that when $m \geq 2$, it has the solution $K_\nu \varphi \in H^m(\mathbb{R}^n_+)$ for any $\varphi \in H^{m-\frac{3}{2}}(\mathbb{R}^{n-1})$. To include the value $m = 0$, we use the statement from Theorem 9.10 that ν extends to a continuous operator from $D(A_{\max})$ to $H^{-\frac{3}{2}}$. The identity $\varphi = \nu K_\nu \varphi$, valid for $\varphi \in H^{\frac{1}{2}}$, then extends by continuity to $H^{-\frac{3}{2}}$. We also know from Theorem 9.19 that $(1 - \Delta) K_\nu \varphi = 0$ for $\varphi \in H^{-\frac{3}{2}}$. Thus $K_\nu \varphi$ is indeed a solution of (9.47) when $\varphi \in H^{-\frac{3}{2}}$.

Now the information on (9.46) can be improved by our knowledge of K_ν. Define again v by (9.36) and subtract it from u in (9.46), then calculations as after (9.36) with γ_0, K_γ replaced by ν, K_ν lead to the formula

$$u = r^+ Q e^+ f - K_\nu \nu \, r^+ Q e^+ f, \tag{9.49}$$

giving a solution in $H^2(\mathbb{R}^n_+)$ of (9.46). It belongs to the domain of A_ν, since the "halfways Green's formula"

$$(Au, v) - a(u,v) = (\nu u, \gamma_0 v), \tag{9.50}$$

known from (A.20) for smooth functions, extends by continuity (using Theorem 9.2) to $u \in H^2(\mathbb{R}^n_+)$ and $v \in H^1(\mathbb{R}^n_+)$, where νu and $\gamma_0 v$ are in $H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \subset L_2(\mathbb{R}^{n-1})$.

Then u defined by (9.49) must be equal to $A_\nu^{-1} f$, and we conclude, similarly to Theorem 9.11:

Theorem 9.20. *The solution operator A_ν^{-1} of* (9.46) *equals*

$$A_\nu^{-1} = r^+ Q e^+ - K_\nu \nu \, r^+ Q e^+, \text{ also denoted } R_\nu. \tag{9.51}$$

It maps $L_2(\mathbb{R}^n_+)$ *continuously into* $H^2(\mathbb{R}^n_+)$*, hence*

$$D(A_\nu) = \{u \in H^2(\mathbb{R}^n_+) \mid \nu u = 0\}. \tag{9.52}$$

Having thus obtained optimal regularity for the Neumann problem (9.46), we can also clear up (9.47) completely. The proofs of Corollaries 9.13 and 9.14 generalize immediately to show:

Corollary 9.21. *For any* $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$*,* $K_\nu \varphi$ *is the unique solution in* $H^2(\mathbb{R}^n_+)$ *of* (9.47)*. The mappings*

$$\nu : Z^2(A) \to H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \text{ and } K_\nu : H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \to Z^2(A)$$

are inverses of one another.

Then we get, using the denseness of $Z^2(A)$ in $Z^0(A)$ shown in Proposition 9.15:

Theorem 9.22. *Let* $m \in \mathbb{N}_0$*. The mappings*

$$\nu : Z^m(A) \to H^{m-\frac{3}{2}}(\mathbb{R}^{n-1}) \text{ and } K_\nu : H^{m-\frac{3}{2}}(\mathbb{R}^{n-1}) \to Z^m(A) \tag{9.53}$$

are inverses of one another.

This is shown just as in the proofs of Theorem 9.16 and Corollary 9.17.

We collect some facts from Theorem 9.20 and Corollary 9.21 in a theorem on the fully nonhomogeneous problem:

Theorem 9.23. *The Neumann problem* (9.45) *is uniquely solvable in* $D(A_{\max})$ *for* $f \in L_2(\mathbb{R}^n_+)$*,* $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$*; the solution belongs to* $H^2(\mathbb{R}^n_+)$ *and is defined by the formula*

$$u = r^+Qe^+f - K_\nu \nu\, r^+Qe^+f + K_\nu\varphi = R_\nu f + K_\nu \varphi; \tag{9.54}$$

cf. (9.51)*. In particular, the mapping*

$$\begin{pmatrix} A \\ \nu \end{pmatrix} : H^2(\mathbb{R}^n_+) \to \begin{matrix} L_2(\mathbb{R}^n_+) \\ \times \\ H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} \text{ has inverse } \begin{pmatrix} R_\nu & K_\nu \end{pmatrix} : \begin{matrix} L_2(\mathbb{R}^n_+) \\ \times \\ H^{\frac{1}{2}}(\mathbb{R}^{n-1}) \end{matrix} \to H^2(\mathbb{R}^n_+). \tag{9.55}$$

9.4 Other realizations of $I - \Delta$

Besides the Dirichlet and the Neumann boundary conditions, it is of interest to study *Neumann-type* conditions:

$$\nu u + B\gamma_0 u = \varphi, \tag{9.56}$$

when B is a differential operator on $\mathbb{R}^{n-1}$ of order 1. This type of condition is called *normal*, since the γ_j with highest j appears with an invertible coefficient.

To keep things simple, we shall here only discuss constant-coefficient operators B. On the other hand, we shall allow the operators B to be pseudodifferential — however, just with x'-independent symbols as in Chapter 6. We take them closely related to the example $\langle\xi'\rangle$, to avoid reference to the general definition of the order of a ψdo at this moment.

The problem

$$\begin{aligned} Au &= f \text{ in } \mathbb{R}^n_+, \\ \nu u + B\gamma_0 u &= \varphi \text{ on } \mathbb{R}^{n-1}, \end{aligned} \tag{9.57}$$

will for some B behave like the Neumann problem (having $H^2(\mathbb{R}^n_+)$-regularity of solutions), for other B not, in particular when complex coefficients or pseudodifferential terms are allowed.

Still more general problems can be considered, for example with a Dirichlet boundary condition on part of the boundary and a Neumann-type condition on the rest of the boundary; then further complications arise.

An interesting question is, how all this fits into the abstract analysis established in Chapter 13, when it is applied with $A_0 = A_{\min}$ and $A_1 = A_{\max}$: How do the concrete boundary conditions fit into the picture? Can all the closed operators $\widetilde{A} \in \mathcal{M}$ be interpreted as representing boundary conditions? We shall deal with this question in the following, assuming that the reader is familiar with Sections 13.1–13.2.

With $A = I - \Delta$ on $\mathbb{R}^n_+$, consider the operators in $H = L_2(\mathbb{R}^n_+)$:

$$A_0 = A_{\min}, \quad A_1 = A_{\max}, \quad A_\gamma = \text{ the Dirichlet realization;} \tag{9.58}$$

then we have a setup as in Section 13.2 with

$$D(A_\gamma) = H^2(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+), \quad Z(A_1) = Z(A_{\max}) = Z^0(A), \tag{9.59}$$

and the positive selfadjoint operator A_γ has lower bound 1. We shall use that $Z(A_{\max})$ is isomorphic with $H^{-\frac{1}{2}}(\mathbb{R}^{n-1})$, as established in Theorem 9.16. In view of (9.17) we have in fact an isometry:

$$2^{\frac{1}{2}} K_\gamma : H^{-\frac{1}{2}}(\mathbb{R}^{n-1}) \to Z(A_{\max}) \text{ is a surjective isometry.} \tag{9.60}$$

The inverse of K_γ here acts like γ_0 and will be denoted γ_Z.

In the following, we generally omit the indication "$(\mathbb{R}^{n-1})$" from the boundary Sobolev spaces. Recall from Section 6.3 for the spaces $H^{-\frac{1}{2}}$ and $H^{\frac{1}{2}}$ that although they are Hilbert spaces provided with well-defined norms, and are of course self-dual, we put greater emphasis on their identification as dual spaces of one another with respect to the extension $\langle\varphi,\overline{\psi}\rangle_{H^{-\frac12},H^{\frac12}}$ of the scalar product in $H^0 = L_2(\mathbb{R}^{n-1})$ (consistent with the distribution duality):

$$H^{\frac12} \subset H^0 \subset H^{-\frac12}, \quad H^{-\frac12} \simeq (H^{\frac12})^*, \quad H^{\frac12} \simeq (H^{-\frac12})^*. \tag{9.61}$$

We take the same point of view for subspaces of them:

Let X be a closed subspace of $H^{-\frac12}$. Then we make very little use of the identification of the dual space X^* (the space of antilinear continuous functionals) with X, but regard X^* as a separate object. Since X is not dense in $H^{-\frac12}$ when different from $H^{-\frac12}$, there is not a natural inclusion between X^* and $H^{\frac12}$ (except what comes from identifying them with their duals). But there is a surjective mapping:

Definition 9.24. For $\psi \in H^{\frac12}$ we define the element $\psi|_X$ of X^* as the functional acting like ψ on X:

$$\langle\psi|_X,\overline{\varphi}\rangle_{X^*,X} = \langle\psi,\overline{\varphi}\rangle_{H^{\frac12},H^{-\frac12}} \text{ for } \varphi \in X. \tag{9.62}$$

If $\psi \in H^{\frac12}$ and $\eta \in X^*$, the identity $\psi|_X = \eta$ may also be expressed as

$$\psi = \eta \text{ on } X. \tag{9.63}$$

The functional $\psi|_X$ is continuous on X since the norm on this space is inherited from $H^{-\frac12}$. All elements of X^* are obtained in this way. (For example, if η is in X^*, one can extend it to an element $\tilde\eta$ of $H^{\frac12}$ by defining it to be zero on $H^{-\frac12} \ominus X$. Other choices of complement of X in $H^{-\frac12}$ will give other extensions.) The perhaps slightly abusive formulation (9.63) is standard for functions.

The following operator will play an important role:

Definition 9.25. Define the Dirichlet-to-Neumann operator P by

$$P\varphi = \nu\, K_\gamma\varphi; \tag{9.64}$$

it is continuous from $H^{m-\frac12}$ to $H^{m-\frac32}$ for all $m \in \mathbb{N}_0$.

For $m \geq 2$ and for $m = 0$, the continuity follows from the continuity of K_γ from $H^{m-\frac12}$ to $Z^m(A)$ by Theorem 9.3, and the continuity of $\nu = i\gamma_1$ from $H^m(\mathbb{R}^n_+)$ to $H^{m-\frac32}$ for $m \geq 2$ in Theorem 9.2 and from $D(A_{\max})$ to $H^{-\frac32}$ for $m = 0$ by Theorem 9.10 ($m = 1$ is easily included). In fact, we can say more about P: From the exact form of K_γ in (9.16) (for smooth functions) we see that P is the pseudodifferential operator (in the sense of Definition 6.1)

$$P = \operatorname{Op}(-\langle\xi'\rangle), \tag{9.65}$$

and it is clearly an isometry of H^s onto H^{s-1} for all $s \in \mathbb{R}$ (cf. (6.17)). Moreover, P is formally selfadjoint and $-P$ is positive with lower bound 1 in $L_2(\mathbb{R}^{n-1})$, since the symbol $-\langle\xi'\rangle$ is real and ≤ -1; cf. Theorem 6.3.

Next, we define a somewhat strange trace operator:

Definition 9.26. The trace operator μ is defined on $D(A_{\max})$ by

$$\mu u = \nu u - P\gamma_0 u. \tag{9.66}$$

At first sight, it ranges in $H^{-\frac{3}{2}}$, since ν and $P\gamma_0$ do so. But this information can be improved:

Proposition 9.27. *The trace operator μ satisfies*

$$\mu u = \nu A_\gamma^{-1} Au, \text{ for } u \in D(A_{\max}), \tag{9.67}$$

hence maps $D(A_{\max})$ continuously into $H^{\frac{1}{2}}$. The following Green's formula holds **for all** $u, v \in D(A_{\max})$*:*

$$(Au, v) - (u, Av) = \langle \mu u, \overline{\gamma_0 v}\rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}} - \langle \gamma_0 u, \overline{\mu v}\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}. \tag{9.68}$$

Furthermore,

$$\mu z = 0 \text{ for } z \in Z(A_{\max}), \tag{9.69}$$

and

$$(Au, z) = \langle \mu u, \overline{\gamma_0 z}\rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}} \text{ when } u \in D(A_{\max}), z \in Z(A_{\max}). \tag{9.70}$$

Proof. When $u \in D(A_{\max})$, we decompose it in

$$u = u_\gamma + u_\zeta, \quad u_\gamma = A_\gamma^{-1} Au \in D(A_\gamma),$$

where $u_\zeta \in Z(A_{\max})$ (as in Lemma 13.1). Then since $\gamma_0 u_\gamma = 0$,

$$\gamma_0 u = \gamma_0 u_\zeta, \text{ so } P\gamma_0 u = P\gamma_0 u_\zeta = \nu u_\zeta \tag{9.71}$$

by definition of P, and hence

$$\nu u - P\gamma_0 u = \nu u - \nu u_\zeta = \nu u_\gamma = \nu A_\gamma^{-1} Au.$$

This shows (9.67), and the continuity follows from the continuity properties of the three factors.

Consider Green's formula (9.30) for functions $u, v \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ and subtract from it the identity $(P\gamma_0 u, \gamma_0 v) - (\gamma_0 u, P\gamma_0 v) = 0$, then we get

$$(Au, v) - (u, Av) = (\nu u - P\gamma_0 u, \gamma_0 v) - (\gamma_0 u, \nu v - P\gamma_0 v),$$

for functions in $\mathscr{S}(\overline{\mathbb{R}}^n_+)$; this shows (9.68) for smooth functions. It extends by continuity to $u, v \in D(A_{\max})$ in view of the continuity of μ shown above and the continuity of γ_0 shown in Theorem 9.10.

The equation (9.69) is clear from the fact that $\nu z = P\gamma_0 z$ when $z \in Z(A_{\max})$, and (9.70) follows from (9.68) with $v = z$ since Az and μz vanish. $\square$

Recall from Section 13.2 that any closed realization $\widetilde{A}$ corresponds (in a unique way) to a closed, densely defined operator $T : V \to W$, where V and W are closed subspaces of $Z(A_{\max})$. We carry this over to a situation with spaces over the boundary as follows:

Definition 9.28. Let V and W be closed subspaces of $Z(A_{\max})$, and let $T : V \to W$ be closed, densely defined. The corresponding setup over the boundary is then defined by letting

$$X = \gamma_0 V,\ Y = \gamma_0 W,\ \text{closed subspaces of } H^{-\frac{1}{2}}, \tag{9.72}$$

and defining $L : X \to Y^*$ by

$$\begin{aligned} D(L) &= \gamma_0 D(T), \\ \langle L\gamma_0 v, \overline{\gamma_0 w}\rangle_{Y^*,Y} &= (Tv, w),\ \text{for all } v \in D(T), w \in W. \end{aligned} \tag{9.73}$$

Since γ_0 equals the invertible operator γ_Z in all the formulas in this definition, T is determined from L, and when $T : V \to W$ runs through all choices of closed subspaces V, W of $Z(A_{\max})$ and closed densely defined operators from V to W, then $L : X \to Y^*$ runs through all choices of closed subsaces X, Y of $H^{-\frac{1}{2}}$ and closed densely defined operators from X to Y^*. This is obvious if we do provide X and Y with the norm in $H^{-\frac{1}{2}}$, identify them with their duals and use that $2^{-\frac{1}{2}}\gamma_Z$ is an isometry. (So the norm comes in useful here, but in general we focus on properties that are expressed without reference to a choice of norm.)

Note that T and L have similar properties: They are simultaneously injective, or surjective. The nullspace of L is $Z(L) = \gamma_Z Z(T)$. The adjoint of T, $T^* : W \to V$, corresponds to the adjoint L^* of L defined as an operator from Y to X^* with domain $D(L^*) = \gamma_0 D(T^*)$:

$$\begin{aligned} (v, T^* w) = (Tv, w) = \langle L\gamma_0 v, \overline{\gamma_0 w}\rangle_{Y^*,Y} &= \langle \gamma_0 v, \overline{L^*\gamma_0 w}\rangle_{X,X^*}, \\ & v \in D(T), w \in D(T^*). \end{aligned}$$

(It is understood that $\langle \varphi, \overline{\psi}\rangle_{X,X^*} = \overline{\langle \psi, \overline{\varphi}\rangle}_{X^*,X}$.) Note that $Y = \overline{D(L^*)}$, the closure in $H^{-\frac{1}{2}}$. When $V = W$, i.e., $X = Y$, T is selfadjoint if and only if L is so. Also lower boundedness is preserved in the correspondence: When $V \subset W$, then $X \subset Y$, and

$$\operatorname{Re}(Tv, v) \geq c\|v\|^2 \text{ for } v \in D(T) \tag{9.74}$$

holds if and only if

$$\operatorname{Re}\langle L\varphi, \overline{\varphi}\rangle_{Y^*,Y} \geq c' \|\varphi\|_Y^2 \text{ for } \varphi \in D(L), \tag{9.75}$$

for some c' whose *value* depends on the choice of norm in $H^{-\frac{1}{2}}$, but whose *sign* (positive, negative or zero) is the same as that of c, independently of the choice of norm.

We can now interpret the general realizations of A by boundary conditions.

Theorem 9.29. *Consider a closed realization $\widetilde{A}$ of A, corresponding to $T : V \to W$ as in Theorem 13.7 (with $A_1 = A_1' = A_{\max}$), and let $L : X \to Y^*$ be the corresponding operator introduced in Definition 9.28. Then $D(\widetilde{A})$ consists of the functions $u \in D(A_{\max})$ for which*

$$\gamma_0 u \in D(L), \quad \mu u|_Y = L\gamma_0 u. \tag{9.76}$$

In this correspondence, $D(L) = \gamma_0 D(\widetilde{A})$.

Proof. We have from Theorem 13.5 that the elements of $D(\widetilde{A})$ are characterized by the two conditions:

$$u_\zeta \in D(T), \quad (Au, w) = (Tu_\zeta, w) \text{ for all } w \in W. \tag{9.77}$$

We just have to translate this to boundary conditions. In view of the definition of L, we have for any $u \in D(A_{\max})$, since $\gamma_0 u_\gamma = 0$, that

$$u_\zeta \in D(T) \iff \gamma_0 u_\zeta \in D(L) \iff \gamma_0 u \in D(L), \tag{9.78}$$

showing that the first conditions in (9.76) and (9.77) are equivalent. When $w \in W \subset Z(A_{\max})$, we have in view of (9.70):

$$(Au, w) = \langle \mu u, \overline{\gamma_0 w}\rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}},$$

whereas

$$(Tu_\zeta, w) = \langle L\gamma_0 u, \overline{\gamma_0 w}\rangle_{Y^*,Y},$$

in view of (9.71) and (9.73). Then the second conditions in (9.76) and (9.77) are equivalent, in view of Definition 9.24.

The last assertion follows from (9.78), since $D(T) = \operatorname{pr}_\zeta D(\widetilde{A})$; cf. Theorem 13.5. $\square$

Let us consider some examples.

Example 9.30. For A_γ itself, $V = W = \{0\}$, and T is trivial (zero) then. So $X = Y = \{0\}$ and $L = 0$. The boundary condition is

$$\gamma_0 u = 0, \tag{9.79}$$

as we know very well.

Example 9.31. Consider A_M from Example 13.10, the von Neumann (or Kreĭn) extension; it is the realization with domain

$$D(A_M) = D(A_{\min}) \dot{+} Z(A_{\max}) = H_0^2(\mathbb{R}^n_+) \dot{+} Z^0(A). \tag{9.80}$$

Here $V = W = Z^0(A)$, and $T = 0$ on $D(T) = Z^0(A)$. So $X = Y = H^{-\frac{1}{2}}$ with $L = 0$ on $D(L) = H^{-\frac{1}{2}}$. The boundary condition is

$$\mu u = 0, \text{ i.e., } \nu u - P\gamma_0 u = 0. \tag{9.81}$$

This is a normal boundary condition (cf. (9.56)ff.), with $B = -P$. The realization, it defines, has *no regularity:* The domain is not in $H^m(\mathbb{R}^n_+)$ for any $m > 0$, since it contains $Z^0(A)$. (By Corollary 9.17, the $Z^m(A)$ are strictly different for different m, since the $H^{m-\frac{1}{2}}$ are so.)

Example 9.32. The Neumann realization is defined by the boundary condition

$$\nu u = 0. \tag{9.82}$$

We know from Section 9.3 that the domain of the hereby defined realization A_ν equals $\{u \in H^2(\mathbb{R}^n_+) \mid \nu u = 0\}$.

Since $\varrho = \{\gamma_0, \nu\}$ is surjective from $H^2(\mathbb{R}^n_+)$ to $H^{\frac{3}{2}} \times H^{\frac{1}{2}}$ (cf. Theorem 9.5), $\gamma_0 u$ runs through $H^{\frac{3}{2}}$ when u runs through the domain, so $D(L) = H^{\frac{3}{2}}$. Moreover, $X = H^{-\frac{1}{2}}$ since $H^{\frac{3}{2}}$ is dense in $H^{-\frac{1}{2}}$. Since A_ν is selfadjoint, also $Y = H^{-\frac{1}{2}}$. Then $D(A_\nu)$ must be characterized by a condition of the form

$$\mu u - L\gamma_0 u = 0, \text{ for } \gamma_0 u \in H^{\frac{3}{2}},$$

so since $\mu u = \nu u - P\gamma_0 u$,

$$\nu u - P\gamma_0 u - L\gamma_0 u = 0, \text{ for } \gamma_0 u \in H^{\frac{3}{2}},$$

when $u \in D(A_\nu)$. Since $\nu u = 0$ there, $L + P = 0$ on $H^{\frac{3}{2}}$. This shows that

$$L \text{ acts like } -P, \quad D(L) = H^{\frac{3}{2}}, \tag{9.83}$$

in the case of A_ν.

Let us now consider more general realizations defined by Neumann-type boundary conditions

$$\nu u + B\gamma_0 u = 0. \tag{9.84}$$

Here we let B be a pseudodifferential operator with symbol

$$b(\xi') = ib_1\xi_1 + \cdots + ib_{n-1}\xi_{n-1} + c\langle \xi' \rangle, \text{ some } c \in \mathbb{R}; \tag{9.85}$$

i.e., it is the sum of a differential operator B_1 and a multiple of P:

$$B = \mathrm{Op}(b) = B_1 - cP, \quad B_1 = b_1\partial_{x_1} + \cdots + b_{n-1}\partial_{x_{n-1}}. \tag{9.86}$$

(9.84) determines the realization $\widetilde{A}$ with domain

$$D(\widetilde{A}) = \{u \in D(A_{\max}) \mid \nu u + B\gamma_0 u = 0\}. \tag{9.87}$$

Since the trace operator $\nu + B\gamma_0$ is continuous from $D(A_{\max})$ to $H^{-\frac{3}{2}}$, the domain is closed.

Clearly,

$$D(\widetilde{A}) \supset \{u \in H^2(\mathbb{R}^n_+) \mid \nu u + B\gamma_0 u = 0\} = D(\widetilde{A}) \cap H^2(\mathbb{R}^n_+),$$

but we do not know a priori whether this is an equality; in fact, we shall look for criteria for whether it is so.

Since $\varrho = \{\gamma_0, \nu\}$ is surjective from $H^2(\mathbb{R}^n_+)$ to $H^{\frac{3}{2}} \times H^{\frac{1}{2}}$, $\gamma_0 u$ runs through $H^{\frac{3}{2}}$ when u runs through $D(\widetilde{A}) \cap H^2(\mathbb{R}^n_+)$, so $D(L) \supset H^{\frac{3}{2}}$. Since the latter space is dense in $H^{-\frac{1}{2}}$, $X = H^{-\frac{1}{2}}$.

Observe that in view of Corollary 9.17 and the decomposition of $D(A_{\max})$ into $D(A_\gamma)$ and $Z(A_{\max})$,

$$D(L) = H^{\frac{3}{2}} \iff D(\widetilde{A}) = D(\widetilde{A}) \cap H^2(\mathbb{R}^n_+). \tag{9.88}$$

Concerning the adjoint $\widetilde{A}^*$, we have from the general Green's formula (9.31) that when $v \in D(\widetilde{A}^*)$ and $u \in D(\widetilde{A}) \cap H^2(\mathbb{R}^n_+)$, then

$$\begin{aligned} 0 = (Av, u) - (v, Au) &= \langle \nu v, \overline{\gamma_0 u}\rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}} - \langle \gamma_0 v, \overline{\nu u}\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \\ &= \langle \nu v, \overline{\gamma_0 u}\rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}} - \langle \gamma_0 v, -\overline{B\gamma_0 u}\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \\ &= \langle \nu v + B^*\gamma_0 v, \overline{\gamma_0 u}\rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}}. \end{aligned}$$

This shows that

$$v \in D(\widetilde{A}^*) \implies \nu v + B^*\gamma_0 v = 0,$$

since $\gamma_0 u$ runs freely in $H^{\frac{3}{2}}$. It was used here that $B : H^s \to H^{s-1}$ has the adjoint $B^* : H^{-s+1} \to H^{-s}$, any s, where B^* acts as the ps.d.o. with symbol $\overline{b}(\xi')$. On the other hand, if $v \in H^2(\mathbb{R}^n_+)$ with $\nu v + B^*\gamma_0 v = 0$, then an application of (9.31) with $u \in D(\widetilde{A})$ shows that

$$\begin{aligned} (Au, v) - (u, Av) &= \langle \nu u, \overline{\gamma_0 v}\rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}} - \langle \gamma_0 u, \overline{\nu v}\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \\ &= \langle \nu u, \overline{\gamma_0 v}\rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}} - \langle \gamma_0 u, -\overline{B^*\gamma_0 v}\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \\ &= \langle \nu u + B\gamma_0 u, \overline{\gamma_0 v}\rangle_{H^{-\frac{3}{2}}, H^{\frac{3}{2}}} = 0, \end{aligned}$$

which implies that $v \in D(\widetilde{A}^*)$. So we conclude that

$$\{v \in H^2(\mathbb{R}^n_+) \mid \nu v + B^*\gamma_0 v = 0\} \subset D(\widetilde{A}^*) \subset \{v \in D(A_{\max}) \mid \nu v + B^*\gamma_0 v = 0\}. \tag{9.89}$$

This gives us the information that $D(L^*)$ contains $H^{\frac{3}{2}}$ in its domain, so since Y equals the closure of $D(L^*)$ in $H^{-\frac{1}{2}}$, $Y = H^{-\frac{1}{2}}$.

It follows that L is an operator from $D(L) \subset H^{-\frac{1}{2}}$ to $Y^* = H^{\frac{1}{2}}$. We can now use that $L\gamma_0 u = \mu u$ on $D(\widetilde{A})$ to find how L acts, when (9.84) holds:

$$0 = L\gamma_0 u - \mu u = L\gamma_0 u - \nu u + P\gamma_0 u = L\gamma_0 u + B\gamma_0 u + P\gamma_0 u,$$

so

$$L = -B - P \text{ on } D(L). \tag{9.90}$$

Similarly,

$$L^* = -B^* - P \text{ on } D(L^*). \tag{9.91}$$

A further precision of L can be made when L is elliptic. Note that

$$\begin{aligned} &L \text{ acts like } -B_1 + cP - P = \operatorname{Op}(l(\xi')), \\ &l(\xi') = -ib_1\xi_1 - \cdots - ib_{n-1}\xi_{n-1} + (1-c)\langle\xi'\rangle. \end{aligned} \tag{9.92}$$

The symbol $\langle\xi'\rangle$ has a series expansion

$$\langle\xi'\rangle = (1 + |\xi'|^2)^{\frac{1}{2}} = |\xi'|(1 + |\xi'|^{-2})^{\frac{1}{2}} = |\xi'| + |\xi'| \sum_{k\geq 1} \tbinom{\frac{1}{2}}{k} |\xi'|^{-2k},$$

converging for $|\xi'| > 1$. (Here $\binom{\frac{1}{2}}{k} = \frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)/k!$.) With a notation borrowed from Chapter 7, see Definition 7.2 ff. and Definition 7.17, we have that the *principal part* of $l(\xi')$ is

$$l_1(\xi') = -ib_1\xi_1 - \cdots - ib_{n-1}\xi_{n-1} + (1-c)|\xi'|, \tag{9.93}$$

and l is *elliptic* precisely when $l_1(\xi') \neq 0$ for $\xi' \neq 0$. In this case the requirement for Theorem 6.22 with $m = 1$ is satisfied.

Thus in the elliptic case we can conclude from Theorem 6.22 that $\operatorname{Op}(l)\varphi \in H^{\frac{1}{2}}$ implies $\varphi \in H^{\frac{3}{2}}$. Since L acts like $\operatorname{Op}(l)$, has $D(L) \supset H^{\frac{3}{2}}$ and $R(L) \subset H^{\frac{1}{2}}$, we conclude that $D(L) = H^{\frac{3}{2}}$. This shows:

Theorem 9.33. *Let $\widetilde{A}$ be the realization defined by the boundary condition* (9.84). *If $l(\xi')$ in* (9.92) *is elliptic, then*

$$D(\widetilde{A}) = \{u \in H^2(\mathbb{R}^n_+) \mid \nu u + B\gamma_0 u = 0\}.$$

In this case also

$$D(\widetilde{A}^*) = \{u \in H^2(\mathbb{R}^n_+) \mid \nu u + B^*\gamma_0 u = 0\}.$$

The last statement follows since the adjoint symbol $\bar{l}(\xi')$ is then likewise elliptic.

Ellipticity clearly holds if

$$c \neq 1,\ b_1, \dots, b_{n-1} \in \mathbb{R}, \tag{9.94}$$

in particular when $c = 0$ and the b_j are real. So we have as a corollary:

Corollary 9.34. *When B is a differential operator with real coefficients, the boundary condition* (9.84) *defines a realization with domain in $H^2(\mathbb{R}^n_+)$.*

Nonelliptic examples are found when $c = 1$ — this is so in Example 9.30 — or if some of the coefficients b_j are nonreal, e.g., if one of them equals $\pm i(1-c)$, so that there are points $\xi' \neq 0$ with $l_1(\xi') = 0$. One can also show that ellipticity of $l(\xi')$ is *necessary* for having $D(L) \subset H^{\frac{3}{2}}$, i.e., $D(\widetilde{A}) \subset H^2(\mathbb{R}^n_+)$.

We can also discuss lower boundedness, in view of the results in Section 13.2 and the equivalence of (9.74) and (9.75). The most immediate results are:

Theorem 9.35. *Let $\widetilde{A}$ be the realization determined by the boundary condition* (9.84).

1° *If $\widetilde{A}$ is lower bounded, so is L, with a similar sign of the lower bound.*
2° *If L has positive or zero lower bound, so has $\widetilde{A}$.*

Proof. We use the equivalence of (9.74) and (9.75). Note that we are in a case where we know beforehand that $V = W = Z(A_1)$. The first statement follows from Theorem 13.15. The second statement follows from Theorem 13.17. □

Again this applies easily to the differential case (where $c = 0$ in (9.85)):

Corollary 9.36. *When B is a differential operator with real coefficients, the realization defined by the boundary condition* (9.84) *is variational with positive lower bound.*

Proof. In this case, $\operatorname{Re} l(\xi') = \langle \xi' \rangle \geq 1$ and $D(L) = H^{\frac{3}{2}}$, so L has lower bound 1, as an operator in $L_2 = H^0$. The same holds for L^*. Moreover,

$$\begin{aligned} &|\operatorname{Im} l(\xi')| = |b_1\xi_1 + \cdots + b_{n-1}\xi_{n-1}| \leq C|\xi'|, \text{ so} \\ &|\operatorname{Re} l(\xi')| = \langle \xi' \rangle \geq C^{-1} |\operatorname{Im} l(\xi')|, \end{aligned} \tag{9.95}$$

which imply similar inequalities for L and L^* (as in Theorems 12.13, 6.4 and Exercises 12.35 and 12.36). It follows in view of (9.73) that T and T^* have their numerical ranges in an angular set

$$\mu = \{ \lambda \in \mathbb{C} \mid |\operatorname{Im} \lambda| \leq C' \operatorname{Re} \lambda,\ \operatorname{Re} \lambda \geq c' \}$$

with C' and $c' > 0$. Now Theorem 13.17 applies to show that $\widetilde{A}$ and $\widetilde{A}^*$ have their numerical ranges in a similar set, and the variational property follows from Theorem 12.26. □

We observe in general that if L has lower bound > 0, then it is elliptic, and both $\widetilde{A}$ and $\widetilde{A}^*$ are lower bounded. A closer look at $l_1(\xi')$ will show that then necessarily $c < 1$; moreover, $\widetilde{A}$ and $\widetilde{A}^*$ are variational.

Conditions for the coerciveness inequality

$$\operatorname{Re}(Au, u) \geq c_1\|u\|_1^2 - c_0\|u\|_0^2 \text{ for } u \in D(\widetilde{A}), \tag{9.96}$$

can be fully analyzed, and have been done so in the litterature (more on this in [G71] and subsequent works).

It is also possible to analyze the resolvent of $\widetilde{A}$ and its relation to λ-dependent operators over $\mathbb{R}^{n-1}$ by use of the results of Sections 13.3 and 13.4, for this we refer to [BGW08].

Example 9.37. Assume that $n \geq 3$. Consider $\widetilde{A}$ defined by the boundary condition (9.84) with

$$B = \chi(x_1)\partial_{x_1} - P, \tag{9.97}$$

where χ is as defined in Section 2.1 (here we allow a variable coefficient in the differential operator). By (9.90)ff.,

$$L = -\chi(x_1)\partial_{x_1},$$

in this case. It is not elliptic on $\mathbb{R}^{n-1}$. Moreover, L has a large nullspace, containing the functions that are constant in x_1 for $|x_1| \leq 2$. The nullspace is clearly infinite dimensional, and it is not contained in $H^1(\mathbb{R}^{n-1})$ (let alone $H^{\frac{3}{2}}$), since it contains for example all products of L_2-functions of $(x_2, \dots, x_{n-1})$ with L_2-functions of x_1 that are constant on $[-2, 2]$. We have from Theorem 13.8 that the nullspace of $\widetilde{A}$ equals that of T, and it equals $K_\gamma Z(L)$, cf. (9.62)ff.

So this is an example of a realization with low regularity and a large infinite dimensional nullspace.

9.5 Variable-coefficient cases, higher orders, systems

The case of $I - \Delta$ on the half-space is just a very simple example, having the advantage that precise results can be easily obtained. Let us give some remarks on more general boundary value problems.

Consider a differential operator of a general order $d > 0$ with C^∞ coefficients on an open set $\Omega \subset \mathbb{R}^n$ with smooth boundary:

$$A = \sum_{|\alpha| \leq d} a_\alpha(x) D^\alpha; \tag{9.98}$$

its principal symbol is $a_d(x,\xi) = \sum_{|\alpha|=d} a_\alpha(x)\xi^\alpha$ (cf. (6.5), (6.42)), also called $a^0(x,\xi)$. A is elliptic when $a^0(x,\xi) \neq 0$ for $\xi \neq 0$ (and all x); and A is said to be *strongly elliptic* when

$$\operatorname{Re} a^0(x,\xi) > 0 \text{ for } \xi \neq 0, \text{ all } x. \tag{9.99}$$

It is not hard to see that strongly elliptic operators are necessarily of *even order*, $d = 2m$.

There are a few scalar odd-order elliptic operators, for example the Cauchy-Riemann operator on $\mathbb{R}^2$, of order 1 (cf. Exercise 5.8). But otherwise, odd-order elliptic operators occur most naturally when we consider *systems* — matrix-formed operators — where the $a_\alpha(x)$ are matrix functions (this is considered for pseudodifferential operators in Section 7.4). Ellipticity here means that the principal symbol $a^0(x,\xi)$ is an invertible matrix for all $\xi \neq 0$, all x. Then it must be a square matrix, and the ellipticity means that the determinant of $a^0(x,\xi)$ is nonzero for $\xi \neq 0$.

Examples of first-order systems that are of current interest are Dirac operators, used in Physics and Geometry.

The strongly elliptic *systems* are those for which

$$a^0(x,\xi) + a^0(x,\xi)^* \text{ is positive definite when } \xi \neq 0. \tag{9.100}$$

Here the order must be even, $d = 2m$.

In the strongly elliptic case where Ω is bounded and (9.100) holds for all $x \in \overline{\Omega}$, there is (as shown at the end of Chapter 7) a variational realization A_γ of the Dirichlet problem, representing the boundary condition $\gamma u = 0$, where

$$\gamma u = \{\gamma_0 u, \gamma_1 u, \dots, \gamma_{m-1} u\}. \tag{9.101}$$

The domain is $D(A_\gamma) = D(A_{\max}) \cap H_0^m(\Omega)$. It is a deeper result to show that in fact

$$D(A_\gamma) = H^{2m}(\Omega) \cap H_0^m(\Omega), \tag{9.102}$$

and that A_γ is a Fredholm operator (cf. Section 8.3) from its domain to $L_2(\Omega)$. Moreover, one can consider the fully nonhomogeneous problem

$$Au = f \text{ in } \Omega, \quad \gamma u = \varphi \text{ on } \partial\Omega, \tag{9.103}$$

showing that it defines a Fredholm operator

$$\begin{pmatrix} A \\ \gamma \end{pmatrix} : H^{2m}(\Omega) \to \begin{matrix} L_2(\Omega) \\ \times \\ \Pi_{j=0}^{m-1} H^{2m-j-\frac{1}{2}}(\partial\Omega) \end{matrix}. \tag{9.104}$$

In the case of bijectiveness, the semihomogeneous problem

$$Au = 0 \text{ in } \Omega, \quad \gamma u = \varphi \text{ on } \partial\Omega, \tag{9.105}$$

has a solution operator K_γ (a so-called Poisson operator) which is bijective:

$$K_\gamma : \Pi_{j=0}^{m-1} H^{s-j-\frac{1}{2}}(\partial\Omega) \xrightarrow{\sim} \{u \in H^s(\Omega) \mid Au = 0\}, \tag{9.106}$$

for all $s \in \mathbb{R}$, with inverse γ (defined in a generalized sense for low s). When the system in (9.104) is merely Fredholm, (9.106) holds modulo finite dimensional subspaces.

A classical method to show the regularity (9.102) of the solutions to (9.103) with $\varphi = 0$ relies on approximating the derivatives with difference quotients, where the inequalities resulting from the ellipticity hypothesis can be used. The method is due to Nirenberg [N55], and is also described e.g. in Agmon [A65] (including some other variational cases), and in Lions and Magenes [LM68] for general normal boundary problems with nonhomogeneous boundary condition. The difference quotient method allows keeping track of how much smoothness of the coefficients $a_\alpha(x)$ is needed for a specific result. It enters also in modern textbooks such as e.g. Evans [E98].

Another important reference in this connection is the paper of Agmon, Douglis and Nirenberg [ADN64], treating general systems by reduction to local coordinates and complex analysis.

In the case of C^∞ coefficients and domain, there is a modern strategy where the problems are solved in a pseudodifferential framework. Here we need other operators than just ψdo's on open sets of $\mathbb{R}^n$, namely, trace operators T going from functions on Ω to functions on $\partial\Omega$, Poisson operators K going from functions on $\partial\Omega$ to functions on Ω, and composed operators KT, called singular Green operators. Such a theory has been introduced by Boutet de Monvel [B71] (and further developed e.g. in [G84], [G90], [G96]). The use of this framework makes the solvability discussion more operational; one does not just obtain regularity of solutions, but one constructs in a concrete way the operator that maps the data into the solution (obtaining general versions of (9.38), (9.51)).

We give an introduction to the theory of psedodifferential boundary operators (ψdbo's) in Chapter 10, building on the theory of ψdo's on open sets explained in Chapters 7 and 8. In Chapter 11 we introduce the Calderón projector ([C63], [S66], [H66]), which is an efficient tool for the discussion of the most general boundary value problems for elliptic differential operators.

We shall not here go into the various theories for nonsmooth boundary value problems that have been developed through the times.

Exercises for Chapter 9

9.1. An analysis similar to that in Section 9.2 can be set up for the Dirichlet problem for $-\Delta$ on the circle, continuing the notation of Exercise 4.5; you are asked to investigate this. The Sobolev spaces over the boundary can here be defined in terms of the coefficient series (c_k) of Fourier expansions as in Section 8.2, H^s being provided with the norm $(\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s}|c_k|^2)^{\frac{1}{2}}$.

9.2. Consider $l(\xi')$ defined in (9.92). Show that if $\operatorname{Re} l(\xi') > 0$ for all ξ', then $c < 1$. Show that L is then variational.

9.3. Let $B = c_0 + ic_1\partial_{x_1} + \cdots + ic_{n-1}\partial_{x_{n-1}}$, with real numbers $c_0, \dots, c_{n-1}$. Show that if $c_1^2 + \cdots + c_{n-1}^2 < 1$, then the realization $\widetilde{A}$ defined by (9.84) with this B is selfadjoint and has $D(\widetilde{A}) \subset H^2(\mathbb{R}^n_+)$.

9.4. Let $A = (1-\Delta)^2$ on $\mathbb{R}^n_+$, and consider the Dirichlet problem

$$Au = f \text{ in } \mathbb{R}^n_+, \quad \gamma_0 u = \varphi_0 \text{ on } \mathbb{R}^{n-1}, \quad \gamma_1 u = \varphi_1 \text{ on } \mathbb{R}^{n-1}. \tag{9.107}$$

(a) Show that for any $\xi' \in \mathbb{R}^{n-1}$, the equation

$$(\langle\xi'\rangle^2 - \partial_{x_n}^2)^2 u(x_n) = 0 \text{ on } \mathbb{R}_+$$

has the following bounded solutions on $\mathbb{R}_+$:

$$c_0 e^{-\langle\xi'\rangle x_n} + c_1 x_n e^{-\langle\xi'\rangle x_n},$$

where $c_0, c_1 \in \mathbb{C}$.

(b) Show that one can find a linear transformation

$$C(\xi') : (\hat\varphi_0(\xi'), \hat\varphi_1(\xi')) \mapsto (c_0(\xi'), c_1(\xi'))$$

for each ξ', such that the operator K defined by

$$K : (\varphi_0(x'), \varphi_1(x')) \mapsto \mathscr{F}^{-1}_{\xi'\to x'}(c_0(\xi')e^{-\langle\xi'\rangle x_n} + c_1(\xi')x_n e^{-\langle\xi'\rangle x_n}),$$

goes from $\mathscr{S}(\mathbb{R}^{n-1}) \times \mathscr{S}(\mathbb{R}^{n-1})$ to $\mathscr{S}(\overline{\mathbb{R}}^n_+)$ and solves (9.107) with $f = 0$, φ_0, φ_1 given in $\mathscr{S}(\mathbb{R}^{n-1})$. Determine $C(\xi')$.

(c) Show that K is continuous from $H^{m-\frac{1}{2}}(\mathbb{R}^{n-1}) \times H^{m-\frac{3}{2}}(\mathbb{R}^{n-1})$ to $H^m(\mathbb{R}^n_+)$ for all $m \in \mathbb{N}_0$, and deduce from this a more general theorem on the solution of the problem (9.107) with $f = 0$.

9.5. For the boundary value problem considered in Exercise 9.4, set up an analysis (possibly with some parts only sketched) that leads to a theorem analogous to Theorem 9.18 in the present situation. One can use here that a variational construction with

$$H = L_2(\mathbb{R}^n_+),\ V = H_0^2(\mathbb{R}^n_+),\ a(u,v) = ((1-\Delta)u, (1-\Delta)v)_{L_2(\mathbb{R}^n_+)}$$

gives an invertible realization A_γ of $A = (I-\Delta)^2$ with $D(A_\gamma) = D(A_{\max}) \cap H_0^2(\mathbb{R}^n_+)$.