V. APPENDICES

§A. Appendix A. Some notation and prerequisites

We denote by $\mathbb{Z}$ the integers, by $\mathbb{N}$ the positive integers and by $\mathbb{N}_0$ the nonnegative integers. $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+$ and $\mathbb{R}_-$ the positive, resp. negative real numbers. $\mathbb{R}^n$ is the n-dimensional real Euclidean space, with points $x = (x_1, \dots, x_n)$ and distance $\operatorname{dist}(x, y) = |x - y|$, where $|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$. $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ denote the subsets, respectively,

$$\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n \mid x_n \gtrless 0\} , \tag{A.1}$$

whose boundary $\{x \in \mathbb{R}^n \mid x_n = 0\}$ is identified with $\mathbb{R}^{n-1}$. The points in $\mathbb{R}^{n-1}$ are then often denoted x',

$$x' = (x_1, \dots, x_{n-1}), \tag{A.2}$$

so that $x = (x', x_n)$.

We denote

$$\begin{aligned} \{t \in \mathbb{R} \mid a \le t \le b\} &= [a, b], \quad \{t \in \mathbb{R} \mid a < t \le b\} =]a, b], \\ \{t \in \mathbb{R} \mid a \le t < b\} &= [a, b[\,, \quad \{t \in \mathbb{R} \mid a < t < b\} =]a, b[\end{aligned}$$

(to avoid conflict between the use of (x, y) for an open interval and for a scalar product).

The space of complex numbers is denoted $\mathbb{C}$; $\mathbb{C}_\pm$ denote the complex numbers with positive resp. negative imaginary part:

$$\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \operatorname{Im} z \gtrless 0\}. \tag{A.3}$$

$\mathbb{C}^n$ denotes the n-dimensional complex Euclidean space. The functions we consider are usually functions on (subsets of) $\mathbb{R}^n$ taking values in $\mathbb{C}$. (Vector valued functions, valued in $\mathbb{C}^N$, can also occur, or we can consider real functions.)

Set inclusions are denoted by $\subset$ (whether or not the sets are equal).

Differentiation of functions on $\mathbb{R}$ is indicated by $\frac{d}{dx}$, ∂_x or ∂. Moreover, we write $\frac{1}{i}\frac{d}{dx} = D_x$ or D (here i is the imaginary unit $i = \sqrt{-1}$); the factor

$\dfrac{1}{i}$ is included for convenience in the use of the Fourier transformation later on. Partial differentiation of functions on $\mathbb{R}^n$ is indicated by

$$\frac{\partial}{\partial x_j} = \partial_{x_j} \text{ or } \partial_j \,; \quad \frac{1}{i}\frac{\partial}{\partial x_j} = D_{x_j} \text{ or } D_j \,. \tag{A.5}$$

In more complicated expressions we use multi-index notation: When $\alpha \in \mathbb{N}_0^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$, then

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} \,, \quad D^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n} = (-i)^{|\alpha|} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} \,, \tag{A.5}$$

here $|\alpha| = \alpha_1 + \dots + \alpha_n$. The notation is used for instance for functions having continuous partial derivatives up to order $|\alpha|$, such that differentiations in different directions (up to that order) are interchangeable. Using the conventions

$$\begin{aligned} \alpha \le \beta \text{ means } \alpha_1 \le \beta_1, \dots, \alpha_n \le \beta_n \,, \\ \alpha! = \alpha_1! \dots \alpha_n! \\ \alpha \pm \beta = (\alpha_1 \pm \beta_1, \dots, \alpha_n \pm \beta_n) \,, \end{aligned} \tag{A.6}$$

we have for u and v with continuous derivatives up to order N the *Leibniz formula*

$$\begin{aligned} \partial^\alpha (uv) &= \sum_{\beta \le \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} \partial^\beta u \, \partial^{\alpha - \beta} v \,, \quad \text{for } |\alpha| \le N \,, \\ D^\alpha (uv) &= \sum_{\beta \le \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} D^\beta u \, D^{\alpha - \beta} v \,, \quad \text{for } |\alpha| \le N \,, \end{aligned} \tag{A.7}$$

and the *Taylor formula*

$$u(x+y) = \sum_{|\alpha| < N} \frac{y^\alpha}{\alpha!} \partial^\alpha u(x) + \sum_{|\alpha| = N} \frac{N}{\alpha!} y^\alpha \int_0^1 (1-\theta)^{N-1} \partial^\alpha u(x + \theta y) d\theta \tag{A.8}$$

(this is an exact version from which the other well-known formulations can be deduced).

When $x \in \mathbb{R}^n$ or $\mathbb{C}^n$, we write

$$\begin{aligned} x^\alpha &= x_1^{\alpha_1} \dots x_n^{\alpha_n} \,, \text{ and} \\ x \cdot y &= x_1 y_1 + \dots + x_n y_n \,, \quad |x| = (x \cdot \overline{x})^{\frac{1}{2}} \,. \end{aligned}$$

The norm $|x|$ (the Euclidean norm) makes $\mathbb{R}^n$ and $\mathbb{C}^n$ Hilbert spaces over $\mathbb{R}$ resp. $\mathbb{C}$, with scalar product $x \cdot \overline{y}$. (The overline indicates complex conjugation.)

We also define

$$\begin{gathered}\langle x\rangle = \sqrt{1+|x|^2},\text{ which satisfies, for } m\in\mathbb{N}: \\ \textstyle\sum_{|\alpha|\le m} x^{2\alpha} \le (1+|x|^2)^m = \sum_{|\alpha|\le m} C_{m,\alpha}x^{2\alpha};\end{gathered} \tag{A.9}$$

here $C_{m,\alpha} = \frac{m!}{\alpha!(m-|\alpha|)!}$, it is integer ≥ 1.

When X and Y are topological spaces, $X\times Y$ denotes the product space, consisting of pairs $\{x,y\}$ where $x\in X$ and $y\in Y$, provided with the product topology (having as a subbasis the sets $U\times V$ where U resp. V run through a subbasis of the topology of X resp. Y). When X and Y are vector spaces, $X\times Y$ is a vector space in the obvious way. If X and Y are normed spaces, one can provide $X\times Y$ by the norm

$$\|\{x,y\}\|_{X\times Y} = \|x\|_X + \|y\|_Y\ , \tag{A.10}$$

making $X\times Y$ a normed space. When X and Y are Hilbert spaces, it is more convenient to use the equivalent norm

$$\|\{x,y\}\|_{X\oplus Y} = (\|x\|_X^2 + \|y\|_Y^2)^{\frac12}\ , \tag{A.11}$$

associated with the scalar product

$$(\{x,y\}\,,\{x',y'\})_{X\oplus Y} = (x,x')_X + (y,y')_Y\ , \tag{A.12}$$

with which $X\times Y$ is a Hilbert space, denoted $X\oplus Y$. We use this notation also for the direct sum of two orthogonal closed subspaces X and Y of a Hilbert space H. For L_p-spaces it can be convenient to use $(\|x\|^p + \|y\|^p)^{\frac1p}$ as the norm on the product space.

We generally define

$$\begin{gathered} X\pm Y = \{x\pm y \mid x\in X \text{ and } y\in Y\} \\ \Omega X = \{\alpha x \mid \alpha\in\Omega \text{ and } x\in X\}\end{gathered} \tag{A.13}$$

when X and Y are subsets of a vector space V with scalar field $\mathbb{L}$ ($\mathbb{L}=\mathbb{R}$ or $\mathbb{C}$), and $\Omega\subset\mathbb{L}$. In particular, we write

$$\begin{gathered}\{x\}+Y = x+Y \\ \{\alpha\}Y = \alpha Y\end{gathered} \tag{A.14}$$

when $x\in X$ and $\alpha\in\mathbb{L}$. When X and Y are subspaces of a vector space V, $X+Y$ is denoted $X\dot{+}Y$ if X and Y are linearly independent. (There is also the notation $X\oplus Y$ for orthogonal closed subspaces of a Hilbert space.)

When X is a closed subspace of a Hilbert space H, the orthogonal complement is denoted $H \ominus X$.

Integration by parts in one variable is generalized to functions of several variables by the Gauss and Green formulas, which we briefly recall:

Let $\Omega \subset \mathbb{R}^n$ be an open set with C^1 boundary $\partial\Omega$ and let $\nu(x)$ denote the interior unit normal vector field at $\partial\Omega$.

To explain this further: Ω is said to have a C^1 boundary, when every boundary point has a neighborhood V such that — after a relabelling of the coordinates if necessary —

$$\Omega \cap V = \{\, (x_1, \dots, x_n) \in V \mid x_n > f(x_1, \dots, x_{n-1}) \,\}, \tag{A.15}$$

where $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a C^1-function. Here

$$\partial\Omega \cap V = \{\, x \in V \mid x_n = f(x_1, \dots, x_{n-1}) \,\}, \tag{A.16}$$

and the interior unit normal vector at the point $x \in \partial\Omega \cap V$ equals (with the notation (A.4))

$$\nu(x', f(x')) = \frac{(-\partial_1 f(x'), \dots, -\partial_{n-1} f(x'), 1)}{\sqrt{(\partial_1 f(x'))^2 + \dots + (\partial_{n-1} f(x'))^2 + 1}}. \tag{A.17}$$

For a C^1 function u defined on a neighborhood of $\overline{\Omega}$ one has the *Gauss formula* (when u has compact support or the integrability is assured in some other way):

$$\int_\Omega \partial_k u \, dx = -\int_{\partial\Omega} \nu_k(x) u(x) \, d\sigma, \quad k = 1 \dots, n, \tag{A.18}$$

where $d\sigma$ is the surface measure on $\partial\Omega$. In the situation of (A.16),

$$d\sigma = \frac{1}{|\nu_n(x)|} \, dx = \sqrt{(\partial_1 f)^2 + \dots + (\partial_{n-1} f)^2 + 1} \, dx_1 \dots dx_{n-1}; \tag{A.19}$$

and the formula (A.18) is for $k = n$ verified for functions supported in V simply by the change of coordinates $x = (x', x_n) \mapsto (x', x_n - f(x'))$ that replaces $\partial\Omega \cap V$ with a subset of $\mathbb{R}^{n-1}$. From the Gauss formula one derives several other formulas, usually called *Green's formulas*, when u and v are suitably differentiable:

$$
\begin{aligned}
&\int_\Omega \partial_k u\,\overline{v}\,dx = -\int_\Omega u\,\overline{\partial_k v}\,dx - \int_{\partial\Omega} \nu_k(x)u(x)\overline{v}(x)\,d\sigma,\\
&\int_\Omega D_k u\,\overline{v}\,dx = \int_\Omega u\,\overline{D_k v}\,dx + i\int_{\partial\Omega} \nu_k(x)u(x)\overline{v}(x)\,d\sigma,\\
&\int_\Omega (-\Delta u)\,\overline{v}\,dx = \sum_{k=1,\dots,n}\int_\Omega \partial_k u\overline{\partial_k v}\,dx + \int_{\partial\Omega}\frac{\partial u}{\partial\nu}\overline{v}\,d\sigma, \qquad \text{(A.20)}\\
&\int_\Omega (-\Delta u)\,\overline{v}\,dx - \int_\Omega u\overline{(-\Delta v)}\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}\overline{v}\,d\sigma - \int_{\partial\Omega} u\overline{\frac{\partial v}{\partial\nu}}\,d\sigma;\\
&\text{where } \frac{\partial u}{\partial\nu} = \textstyle\sum_{k=1}^n \nu_k\partial_k u,
\end{aligned}
$$

the interior normal derivative. Here Δ is the Laplace operator $\partial_1^2 + \cdots + \partial_n^2$.

The signs are chosen with later applications in mind (it is the operator $-\Delta$ that is "positive").

Let $p \in [1,\infty]$. For a Lebesgue measurable subset M of $\mathbb{R}^n$, $L_p(M)$ denotes the vector space of equivalence classes of measurable functions $f\colon M \to \mathbb{C}$ with finite norm

$$
\begin{aligned}
\|f\|_{L_p(M)} &= \Big(\int_M |f(x)|^p dx\Big)^{1/p} \ \text{ if } p < \infty\ ,\\
\|f\|_{L_\infty(M)} &= \operatorname*{ess\,sup}_M |f| \ \text{ if } p = \infty\ .
\end{aligned} \qquad \text{(A.21)}
$$

It is a Banach space with this norm. (The equivalence classes consist of functions that are equal almost everywhere (a.e.); we use the customary "abuse of notation" where one calls the equivalence class a function, denoting the class containing f by f again. If the class contains a continuous function — necessarily unique if M is an open set or the closure of an open set — we use the continuous function as representative. Note that $C^0(M)$ identifies with a subset of $L_1(M)$ when M is the closure of a bounded open set.) We recall that for a real measurable function u on M,

$$
\operatorname*{ess\,sup}_M u = \inf\{\, a \mid u(x) \le a \text{ a.e. in } M\,\}. \qquad \text{(A.22)}
$$

When $p = 2$ we get a Hilbert space, where the norm is associated with the scalar product

$$
(f,g)_{L_2(M)} = \int_M f(x)\overline{g}(x)dx\ . \qquad \text{(A.23)}
$$

Hölder's inequality

$$\Big|\int_M f(x)g(x)dx\Big| \le \|f\|_{L_p(M)}\|g\|_{L_{p'}(M)}\,, \quad \frac{1}{p}+\frac{1}{p'}=1\,, \tag{A.24}$$

holds for $f \in L_p(M)$ and $g \in L_{p'}(M)$; it is the *Cauchy-Schwarz inequality* in the case $p = 2$. Note that $L_p(\Omega) = L_p(\overline{\Omega})$ when for example Ω has C^1 boundary.

When the measure of M is finite, we have an inclusion

$$L_p(M) \subset L_q(M) \quad \text{for } 1 \le q \le p \le \infty. \tag{A.25}$$

Recall that the proof for $p < \infty$ consists of observing that for $f \in L_p(M)$ one has, with $r = p/q$, $1/r + 1/r' = 1$, by the Hölder inequality:

$$\begin{aligned} \|f\|_{L_q(M)} = \Big(\int_M |f(x)|^q dx\Big)^{1/q} &= \Big(\int_M |f(x)|^{p/r}\cdot 1dx\Big)^{1/q} \\ &\le \Big(\int_M |f(x)|^p\Big)^{1/rq}\Big(\int_M 1dx\Big)^{1/r'q} \\ &= \|f\|_{L_p(M)}\,\mathrm{vol}(M)^{1/q-1/p} \end{aligned} \tag{A.26}$$

where $\mathrm{vol}(M) = \int_M 1dx$ is the volume (measure) of M.

When $M \subset V$ for some set V, we denote by 1_M the function on V defined by

$$1_M(x) = \begin{cases} 1 & \text{for } x \in M, \\ 0 & \text{for } x \in V \setminus M. \end{cases} \tag{A.27}$$

When $f \in L_p(\mathbb{R}^n)$, $g \in L_q(\mathbb{R}^n)$, and

$$1 \le p \le \infty, \quad 1 \le q \le \infty, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \ge 0, \tag{A.28}$$

then the convolution $(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy$ defines a function $f * g$ in $L_r(\mathbb{R}^n)$, and

$$\|f * g\|_{L_r(\mathbb{R}^n)} \le \|f\|_{L_p(\mathbb{R}^n)}\|g\|_{L_q(\mathbb{R}^n)}; \tag{A.29}$$

Young's inequality. In particular, if $f \in L_1(\mathbb{R}^n)$ and $g \in L_2(\mathbb{R}^n)$, then $f * g \in L_2(\mathbb{R}^n)$, and

$$\|f * g\|_{L_2(\mathbb{R}^n)} \le \|f\|_{L_1(\mathbb{R}^n)}\|g\|_{L_2(\mathbb{R}^n)}. \tag{A.30}$$

When Ω is an open subset of $\mathbb{R}^n$, we denote by $L_{p,\mathrm{loc}}(\Omega)$ the set of funtions on Ω whose restrictions to compact subsets $K \subset \Omega$ are in $L_p(K)$. In view of (A.25), one has that

$$L_{p,\mathrm{loc}}(\Omega) \subset L_{q,\mathrm{loc}}(\Omega) \quad \text{for } 1 \le q \le p \le \infty. \tag{A.31}$$

In particular, $L_{1,\mathrm{loc}}(\Omega)$ is the space of locally integrable functions on Ω (containing all the other spaces $L_{p,\mathrm{loc}}(\Omega)$).

The lower index p on L_p-spaces (instead of an upper index) reflects the fact that p is placed in this way in the modern literature on function spaces, such as L_p-types of Sobolev spaces H^s_p, B^s_p (and their numerous generalizations), where the upper index s is reserved for the number of well-defined derivatives.

Exercises for Chapter 1.

A.1. Show the general Leibniz formulas (A.7).

A.2. (a) Let $f \in C^1(\mathbb{R}^n)$. Show for any $x, y \in \mathbb{R}^n$ that the function $g(\theta) = f(x + \theta y)$ $(\theta \in \mathbb{R})$ satisfies:

$$\frac{d}{d\theta} g(\theta) = \sum_{j=1}^{n} \partial_j f(x + \theta y) y_j,$$

and conclude from this that

$$f(x + y) = f(x) + \sum_{j=1}^{n} y_j \int_0^1 \partial_j f(x + \theta y)\, d\theta.$$

(b) Show Taylor's formula (A.8) for arbitrary N.

A.3. Deduce the formulas in (A.20) from (A.18).
(*Hint*. Apply (A.18) to $\partial_k(u\overline{v})$.)