§B. Appendix B. Topological vector spaces

B.1. Fréchet spaces.

In this appendix we go through the definition of Fréchet spaces and their inductive limits, such as they are used for definitions of function spaces in Chapter 2. A reader who just wants an orientation about Fréchet spaces will not have to read every detail, but need only consider Definition B.4, Theorem B.5, Remark B.6, Lemma B.7, Remark B.7a and Theorem B.8, skipping the proofs of Theorems B.5 and B.8. For inductive limits of such spaces, Section B.2 gives an overview (the proofs are established in a succession of exercises), and all that is needed for the definition of the space $C_0^\infty(\Omega)$ is collected in Theorem 2.5.

We recall that a topological space S is a space provided with a collection τ of subsets (called the *open* sets), satisfying the rules: S is open, $\emptyset$ is open, the intersection of two open sets is open, the union of any collection of open sets is open. The closed sets are then the complements of the open sets. A neighborhood of $x \in S$ is a set containing an open set containing x. Recall also that when S and S_1 are topological spaces, and f is a mapping from S to S_1, then f is continuous at $x \in S$ when there for any neighborhood V of $f(x)$ in S_1 exists a neighborhood U of x in S such that $f(U) \subset V$.

Much of the following material is also found in the book of Rudin [R 1974], which was an inspiration for the formulations here.

Definition B.1. *A topological vector space (t.v.s.) over the scalar field* $\mathbb{L} = \mathbb{R}$ *or* $\mathbb{C}$ *(we most often consider* $\mathbb{C}$*), is a vector space* X *provided with a topology* τ *having the following properties:*

(i) *A set consisting of one point* $\{x\}$ *is closed.*

(ii) *The maps*

$$\begin{aligned} &\{x,y\} \mapsto x+y \quad \textit{from} \;\; X \times X \quad \textit{into} \;\; X \\ &\{\lambda, x\} \mapsto \lambda x \qquad \textit{from} \;\; \mathbb{L} \times X \quad \textit{into} \;\; X \end{aligned} \tag{B.1}$$

are continuous.

In this way, X is in particular a Hausdorff space, cf. Exercise B.4. (We here follow the terminology of [R 1974] and [P 1989] where (i) is included in the definition of a t.v.s.; all spaces that we shall meet have this property. In part of the literature, (i) is not included in the definition and one speaks of Hausdorff topological vector spaces when it holds.)

$\mathbb{L}$ is considered with the usual topology for $\mathbb{R}$ or $\mathbb{C}$.

Definition B.2. *A set $Y \subset X$ is said to be*

a) convex, *when $y_1, y_2 \in Y$ and $t \in]0,1[$ imply*

$$ty_1 + (1-t)y_2 \in Y \ ,$$

b) balanced, *when $y \in Y$ and $|\lambda| \le 1$ imply $\lambda y \in Y$;*

c) **bounded** *(with respect to τ), when there for every neighborhood U of 0 exists $t > 0$ so that $Y \subset tU$.*

Note that boundedness is defined without reference to "balls" or the like.

Lemma B.3. *Let X be a topological vector space.*

1° *Let $a \in X$ and $\lambda \in \mathbb{L} \setminus \{0\}$. The maps from X to X*

$$\begin{aligned} T_a &: x \mapsto x - a \\ M_\lambda &: x \mapsto \lambda x \end{aligned} \tag{B.2}$$

are continuous with continuous inverses T_{-a} resp. $M_{1/\lambda}$.

2° *For any neighborhood V of 0 there exists a balanced neighborhood W of 0 such that $W + W \subset V$.*

3° *For any convex neighborhood V of 0 there exists a convex balanced neighborhood W of 0 so that $W \subset V$.*

Proof. 1° follows directly from the definition of a topological vector space.

For 2° we appeal to the continuity of the two maps in (B.2) as follows: Since $\{x, y\} \mapsto x + y$ is continuous at $\{0, 0\}$, there exist neighborhoods W_1 and W_2 of 0 so that $W_1 + W_2 \subset V$. Since $\{\lambda, x\} \mapsto \lambda x$ is continuous at $\{0, 0\}$ there exist balls $B(0, r_1)$ and $B(0, r_2)$ in $\mathbb{L}$ with $r_1, r_2 > 0$ and neighborhoods W_1' and W_2' of 0 in X, such that

$$B(0, r_1) W_1' \subset W_1 \text{ and } B(0, r_2) W_2' \subset W_2.$$

Let $r = \min\{r_1, r_2\}$ and let $W = B(0, r)(W_1' \cap W_2')$, then W is a balanced neighborhood of 0 with $W + W \subset V$.

For 3° we first choose W_1 as under 2°, so that W_1 is a balanced neighborhood of 0 with $W_1 \subset V$. Let $W = \bigcap_{\alpha \in \mathbb{L}, |\alpha| = 1} \alpha V$. Since W_1 is balanced, $\alpha^{-1} W_1 = W_1$ for all $|\alpha| = 1$, hence $W_1 \subset W$. Thus W is a neighborhood of 0. It is convex, as an intersection of convex sets. That W is balanced is seen as follows: For $\lambda = 0$, $\lambda W \subset W$. For $0 < |\lambda| \le 1$,

$$\lambda W = |\lambda| \frac{\lambda}{|\lambda|} \bigcap_{\alpha \in \mathbb{L}, |\alpha| = 1} \alpha V = |\lambda| \bigcap_{\beta \in \mathbb{L}, |\beta| = 1} \beta V \subset W$$

(with $\beta = \alpha\lambda/|\lambda|$); the last inclusion follows from the convexity since $0 \in W$. $\square$

Note that in 3° the interior W° of W is an open convex balanced neighborhood of x.

The lemma implies that the topology of a t.v.s. X is translation invariant, i.e., $E \in \tau \iff a + E \in \tau$ for all $a \in X$. The topology is therefore determined from the system of neighborhoods of 0. Here it suffices to know a *local basis* for the neighborhood system at 0, i.e., a system $\mathcal{B}$ of neighborhoods of 0 such that every neighborhood of 0 contains a set $U \in \mathcal{B}$.

X is said to be *locally convex*, when it has a local basis of neighborhoods at 0 consisting of convex sets.

X is said to be *metrizable*, when it has a metric d such that the topology on X is identical with the topology defined by this metric; this happens exactly when the balls $B(x, \frac{1}{n})$, $n \in \mathbb{N}$, are a local basis for the system of neighborhoods at x for any $x \in X$. Here $B(x, r)$ denotes as usual the open ball

$$B(x, r) = \{y \in X \mid d(x, y) < r\} ,$$

and we shall also use the notation $\underline{B}(x, r) = \{y \in X \mid d(x, y) \leq r\}$ for the closed ball. Such a metric need not be translation invariant, but it will usually be so in the cases we consider; translation invariance (also just called *invariance*) here means that

$$d(x + a, y + a) = d(x, y) \text{ for } x, y, a \in X .$$

One can show, see e.g. [R 1974, Th. 1.24], that when a t.v.s. is metrizable, then the metric can be chosen to be translation invariant.

A *Cauchy sequence* in a t.v.s. X is a sequence $(x_n)_{n\in\mathbb{N}}$ with the property: For any neighborhood U of 0 there exists an $N \in \mathbb{N}$ so that $x_n - x_m \in U$ for n and $m \geq N$.

In a metric space (M, d), Cauchy sequences — let us here call them *metric Cauchy sequences* — are usually defined as sequences (x_n) for which $d(x_n, x_m) \to 0$ in $\mathbb{R}$ for n and $m \to \infty$. This property need not be preserved if the metric is replaced by another equivalent metric (defining the same topology). We have however for t.v.s. that if the topology in X is given by an *invariant* metric d, then the general concept of Cauchy sequences for X gives just the metric Cauchy sequences (Exercise B.2).

A metric space is called *complete*, when every metric Cauchy sequence is convergent. More generally we call a t.v.s. *sequentially complete*, when every Cauchy sequence is convergent.

Banach spaces and Hilbert spaces are of course complete metrizable topological vector spaces. The following more general type is also important:

Definition B.4. *A topological vector space is called a Fréchet space, when X is metrizable with a translation invariant metric, is complete, and is locally convex.*

The local convexity is mentioned explicitly because the balls belonging to a given metric need not be convex, cf. Exercise B.1. (One has however that if X is metrizable *and* locally convex, then there exists a metric for X with convex balls, cf. [R 1974, Th. 1.24].) Note that the balls defined from a *norm* are convex.

It is also possible to define Fréchet space topologies (and other locally convex topologies) by use of seminorms; we shall now take a closer look at this method to define topologies.

Recall that a seminorm on a vector space X is a function $p : X \to \overline{\mathbb{R}}_+$ with the properties

$$\begin{aligned} &\text{(i)} \quad p(x+y) \le p(x) + p(y) \quad \text{for} \quad x, y \in X \quad \text{(subadditivity)}\,, \\ &\text{(ii)} \quad p(\lambda x) = |\lambda| p(x) \quad \text{for} \quad \lambda \in \mathbb{L} \quad \text{and} \quad x \in X \quad \text{(multiplicativity)}\,. \end{aligned} \tag{B.3}$$

A family $\mathcal{P}$ of seminorms is called *separating*, when for every $x_0 \in X \setminus \{0\}$ there is a $p \in \mathcal{P}$ such that $p(x_0) > 0$.

Theorem B.5. *Let X be a vector space and let $\mathcal{P}$ be a separating family of seminorms on X. Define a topology on X by taking, as a local basis $\mathcal{B}$ for the system of neighborhoods at 0, the convex balanced sets*

$$V(p, \varepsilon) = \{x \mid p(x) < \varepsilon\}\,, \quad p \in \mathcal{P} \ \ \textit{and} \ \ \varepsilon > 0\,, \tag{B.4}$$

together with their finite intersections

$$W(p_1, \dots, p_N; \varepsilon_1, \dots, \varepsilon_N) = V(p_1, \varepsilon_1) \cap \dots \cap V(p_N, \varepsilon_N)\,; \tag{B.5}$$

and letting a local basis for the system of neighborhoods at each $x \in X$ consist of the translated sets $x + W(p_1, \dots, p_N; \varepsilon_1, \dots, \varepsilon_N)$. (It suffices to let $\varepsilon_j = 1/n_j$, $n_j \in \mathbb{N}$.)

With this topology, X is a topological vector space. The seminorms $p \in \mathcal{P}$ are continuous maps of X into $\mathbb{R}$. A set $E \subset X$ is bounded if and only if $p(E)$ is bounded in $\mathbb{R}$ for all $p \in \mathcal{P}$.

Proof. That a topology on X is defined in this way, follows from the observation: When $x \in (x_1 + W_1) \cap (x_2 + W_2)$, then there is a basis-neighborhood W such that $x + W \subset (x_1 + W_1) \cap (x_2 + W_2)$. Moreover, it is clear that the topology is invariant under translation and under multiplication by a scalar $\neq 0$, and that the sets with $\varepsilon_j = 1/n_j$, $n_j \in \mathbb{N}$, form a local basis. The continuity of the p's follows from the subadditivity: For each $x_0 \in X$ and

$\varepsilon > 0$, $V(p, x_0, \varepsilon) = x_0 + V(p, \varepsilon)$ is mapped into the neighborhood $B(p(x_0), \varepsilon)$ of $p(x_0)$ in $\mathbb{R}$, since

$$p(x) - p(x_0) \leq p(x - x_0) + p(x_0) - p(x_0) < \varepsilon,$$
$$p(x_0) - p(x) \leq p(x_0 - x) + p(x) - p(x) < \varepsilon,$$

for $x \in V(p, x_0, \varepsilon)$.

The point set $\{0\}$ (and then also any other point) is closed, since every $x_0 \neq 0$ has a neighborhood disjoint from $\{0\}$, namely $V(p, x_0, \varepsilon)$, where p is chosen so that $p(x_0) \neq 0$ and $\varepsilon < p(x_0)$. We shall now show the continuity of addition and multiplication. For the continuity of addition we must show that there for any neighborhood $W + x + y$ exist neighborhoods $W_1 + x$ of x and $W_2 + y$ of y (recall the notation (1.23)–(1.24)) so that

$$W_1 + x + W_2 + y \subset W + x + y\ , \qquad \text{i.e., so that } W_1 + W_2 \subset W. \tag{B.6}$$

Here we simply use that when $W = W(p_1, \dots, p_N\,; \varepsilon_1, \dots, \varepsilon_n)$ is a basis-neighborhood at 0, and we set

$$W' = \tfrac{1}{2} W(p_1, \dots, p_N; \varepsilon_1, \dots, \varepsilon_N) = W(p_1, \dots, p_N; \tfrac{1}{2}\varepsilon_1 \dots, \tfrac{1}{2}\varepsilon_N)\ ,$$

then $W' + W' \subset W$ because of the subadditivity. For the continuity of multiplication we must show that for any neighborhood $W + \alpha x$ of αx there exist neighborhoods $B(0, r) + \alpha$ (of α in $\mathbb{L}$) and $W' + x$ (of x in X) such that

$$(B(0, r) + \alpha)(W' + x) \subset W + \alpha x\ . \tag{B.7}$$

Here

$$(B(0, r) + \alpha)(W' + x) \subset B(0, r)W' + \alpha W' + B(0, r)x + \alpha x\ .$$

Let $W = W(p_1, \dots, p_N; \varepsilon_1, \dots, \varepsilon_N)$, and let $W' = \delta W$, $\delta > 0$. For c taken larger than $\varepsilon_j^{-1} p_j(x)$, $j = 1, \dots, N$, we have that $x \in cW$ and hence

$$B(0, r)x \subset rcW\ .$$

Moreover,

$$B(0, r)W' \subset rW' = r\delta W$$

and

$$\alpha W' \subset |\alpha| W' = |\alpha| \delta W\ .$$

Now we first take δ so small that $|\alpha|\delta < \frac{1}{3}$; next we choose r so small that $r\delta < \frac{1}{3}$ and $rc < \frac{1}{3}$. Then

$$B(0,r)W' + \alpha W' + B(0,r)x \subset \tfrac{1}{3}W + \tfrac{1}{3}W + \tfrac{1}{3}W \subset W \ ,$$

whereby (B.7) is satisfied. All together we find that the topology on X makes X a topological vector space.

Finally we shall describe the bounded sets. Assume first that E is bounded. Let $p \in \mathcal{P}$, then there exists by assumption (see Definition B.2) a $t > 0$, so that $E \subset tV(p,1) = V(p,t)$. Then $p(x) \in [0,t[$ for $x \in E$, so that $p(E)$ is bounded. Conversely, let E be a set such that for every p there is a $t_p > 0$ with $p(x) < t_p$ for $x \in E$. When $W = W(p_1,\dots,p_N;\varepsilon_1,\dots,\varepsilon_N)$, we then have that $E \subset tW$ for $t \geq \max\{t_{p_1}/\varepsilon_1,\dots,t_{p_N}/\varepsilon_N\}$. □

Note that the sets (B.5) are open, convex and balanced in view of (B.3).

Note that x_k converges to x in X if and only if $p(x_k - x) \to 0$ for every seminorm $p \in \mathcal{P}$; and that a linear operator $T \colon Y \to X$, where Y is a t.v.s., is continuous if and only if all the maps $p \circ T$ are continuous.

Remark B.6. We remark that in the definition of the topology in Theorem B.5, the sets (B.4) constitute a *local subbasis* for the neighborhood system at 0, i.e., a system $\mathcal{B}'$ of neighborhoods of 0 such that the finite intersections of elements from $\mathcal{B}'$ constitute a local basis for the neighborhood system at 0. If the given family of seminorms $\mathcal{P}$ has the following property (which we shall call the *max-property*):

$$\forall p_1, p_2 \in \mathcal{P}\ \exists p \in \mathcal{P},\ c > 0 : p \geq c\max\{p_1, p_2\} \ , \tag{B.8}$$

then each of the basis sets (B.5) contains a set of the form (B.4), namely one with $p \geq c\max\{p_1,\dots,p_N\}$ and $\varepsilon \leq \min c\{\varepsilon_1,\dots,\varepsilon_N\}$; here (B.4) is in itself a *local basis* for the topology. For a given family of seminorms $\mathcal{P}$ one can always supply the family to obtain one that has the max-property and *defines the same topology.* For, one can simply replace $\mathcal{P}$ by $\mathcal{P}'$, consisting of $\mathcal{P}$ and all seminorms of the form

$$p = \max\{p_1,\dots,p_N\} \ ,\ p_1,\dots,p_N \in \mathcal{P} \ , \quad N \in \mathbb{N} \ . \tag{B.9}$$

In the following we can often obtain that $\mathcal{P}$ is *ordered* (when p and $p' \in \mathcal{P}$ then either $p(x) \leq p'(x)\,\forall x \in X$ or $p'(x) \leq p(x)\,\forall x \in X$); then $\mathcal{P}$ has the max-property.

When the max-property holds, we have a simple reformulation of continuity properties:

Lemma B.7. 1° *When the topology on X is given by Theorem* B.5 *and $\mathcal{P}$ has the max-property, then a linear functional Λ on X is continuous if and only if there exists a $p \in \mathcal{P}$ and a constant $c > 0$ so that*

$$|\Lambda(x)| \le cp(x) \quad \text{for all } x \in X \, . \tag{B.10}$$

2° *When X and Y are topological vector spaces with topologies given as in Theorem* B.5 *by separating families $\mathcal{P}$ resp. $\mathcal{Q}$ having the max-property, then a linear operator T from X to Y is continuous if and only if: For each $q \in \mathcal{Q}$ there exists a $p \in \mathcal{P}$ and a constant $c > 0$ so that*

$$|q(Tx)| \le cp(x) \quad \text{for all } x \in X \, . \tag{B.10a}$$

Proof. 1°. Continuity of Λ holds when it holds at 0. Since the neighborhoods (B.4) form a basis, the continuity can be expressed as the property: For any $\varepsilon > 0$ there is a $p \in \mathcal{P}$ and a $\delta > 0$ such that $(*)$ $p(x) < \delta \implies |\Lambda(x)| < \varepsilon$.

If (B.10) holds, then we can for a given ε obtain $(*)$ by taking $\delta = \varepsilon/c$. Conversely, assume that $(*)$ holds. We then claim that (B.10) holds with $c = \varepsilon/\delta$. For, if $p(x) = 0$, $\Lambda(x)$ must equal 0, for otherwise $|\Lambda(tx)| = t|\Lambda(x)| \to \infty$ for $t \to \infty$ whereas $p(tx) = 0$ for $t > 0$, contradicting $(*)$. If $p(x) > 0$, let $0 < \delta' < \delta$, then $p(\frac{\delta'}{p(x)}x) = \delta' < \delta$ so $|\Lambda(\frac{\delta'}{p(x)}x)| < \varepsilon$, hence $|\Lambda(x)| < \frac{\varepsilon}{\delta'}p(x)$. Letting $\delta' \to \delta$, we conclude that $|\Lambda(x)| \le \frac{\varepsilon}{\delta}p(x)$. This shows 1°.

2°. Continuity of T holds when it holds at 0; here it can be expressed as the property: For any $q \in \mathcal{Q}$ and $\varepsilon > 0$ there is a $p \in \mathcal{P}$ and a $\delta > 0$ such that $(**)$ $p(x) < \delta \implies |q(Tx)| < \varepsilon$.

If (B.10a) holds, then we can for a given ε obtain $(**)$ by taking $\delta = \varepsilon/c$. Conversely, when $(**)$ holds, we find that (B.10a) holds with $c = \varepsilon/\delta$ in a similar way as in the proof of 1°. $\square$

Remark B.7a. If the given family of seminorms is not known to have the max-property, the lemma holds when $\mathcal{P}$ and $\mathcal{Q}$ are replaced by $\mathcal{P}'$, $\mathcal{Q}'$, defined as indicated in Remark B.6. We can express this in another way: Without assumption of the max-property, the lemma is valid when p and q in (B.10) and (B.10a) are replaced by expressions as in (B.9).

Note in particular that, whether the max-property holds or not, *the existence of a $p \in \mathcal{P}$, $c > 0$,* such that (B.10) holds, is *sufficient* to assure that the linear functional Λ is continuous.

We observe, as an outcome of the lemma, the general principle: *Continuity of linear mappings is shown by proving inequalities.* This is familiar for normed spaces, and we see that it holds also for spaces with topologies defined by seminorms.

The family of seminorms $\mathcal{P}$ may in particular be derived from a vector space Y of linear functionals on X, by taking $|\Lambda(x)|$ as a seminorm when $\Lambda \in Y$. An important example is where $Y = X^*$, the space of continuous linear functionals on X. Here the topology defined by the family of seminorms $x \mapsto |x^*(x)|$, $x^* \in X^*$, is called the weak*-topology on X. Much more can be said about this important case (and the aspects of weak topology and weak* topology in connection with Banach spaces); for this we refer to textbooks on functional analysis.

Theorem B.8. *When X is a t.v.s. where the topology is defined by a countable separating family of seminorms $\mathcal{P} = (p_k)_{k \in \mathbb{N}}$, then X is metrizable and the topology on X is determined by the invariant metric*

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x-y)}{1+p_k(x-y)} \,. \tag{B.11}$$

the balls $B(0,r)$ in this metric are balanced.

Proof. The series (B.11) is clearly convergent for all $x, y \in X$. Consider the function $f(a) = a/(1+a)$, $a \geq 0$. We want to show that it satisfies

$$f(a) \leq f(a+b) \leq f(a) + f(b). \tag{B.12}$$

Here the first inequality follows since $f(a) = 1 - 1/(1+a)$ with $1/(1+a)$ decreasing. For the second inequality we use this together with the formula $f(a)/a = 1/(1+a)$ for $a > 0$, which gives that

$$f(a)/a \geq f(a+b)/(a+b), \quad f(b)/b \geq f(a+b)/(a+b), \quad a, b > 0,$$

hence alltogether $f(a) + f(b) \geq f(a+b)(a+b)/(a+b) = f(a+b)$. ((B.12) is immediately verified in cases where a or b is 0.) Define $d_0(x)$ by

$$d_0(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x)}{1+p_k(x)}; \tag{B.13}$$

then $d(x,y) = d_0(x-y)$. We first show

$$\begin{aligned} d_0(x) \geq 0, \quad d_0(x) = 0 &\iff x = 0; \\ d_0(x+y) \leq d_0(x) + d_0(y); \\ d_0(\lambda x) \leq d_0(x) \quad \text{for } |\lambda| \leq 1; \end{aligned}$$

these properties assure that $d(x, y)$ is a metric with balanced balls. For the first line, we note that $d_0(x) > 0$ for $x \neq 0$ follows from the fact that $\mathcal{P}$ is separating. The second line is obtained by use of (B.12):

$$\begin{aligned} d_0(x+y) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x+y)}{1+p_k(x+y)} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x)+p_k(y)}{1+p_k(x)+p_k(y)} \\ &\leq \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x)}{1+p_k(x)} + \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(y)}{1+p_k(y)} = d_0(x) + d_0(y) \ . \end{aligned}$$

The third line is obtained using that for $0 < |\lambda| \leq 1$,

$$\begin{aligned} d_0(\lambda x) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(\lambda x)}{1+p_k(\lambda x)} = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\lambda| p_k(x)}{1+|\lambda| p_k(x)} \\ &= \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x)}{1/|\lambda| + p_k(x)} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x)}{1+p_k(x)} = d_0(x). \end{aligned}$$

The metric is clearly translation invariant. We shall now show that the hereby defined topology is the same as the topology defined by the seminorms. Note that on one hand, one has for $\varepsilon \in]0,1[$ and $N \in \mathbb{N}$:

$$\begin{aligned} d_0(x) \leq \varepsilon 2^{-N} &\implies \frac{1}{2^k} \frac{p_k(x)}{1+p_k(x)} \leq \varepsilon 2^{-N} \text{ for all } k \\ &\implies \frac{p_k(x)}{1+p_k(x)} \leq \varepsilon \text{ for } k \leq N \implies p_k(x) \leq \frac{\varepsilon}{1-\varepsilon} \text{ for } k \leq N; \end{aligned}$$

on the other hand one has, if $2^{-N} < \varepsilon/2$,

$$\begin{aligned} & p_k(x) \leq \varepsilon/2 \text{ for } k \leq N \implies \\ & d_0(x) = \sum_{k=1}^{N} \frac{1}{2^k} \frac{p_k(x)}{1+p_k(x)} + \sum_{k=N+1}^{\infty} \frac{1}{2^k} \frac{p_k(x)}{1+p_k(x)} \\ & \qquad \leq \frac{\varepsilon}{2} \sum_{k=1}^{N} \frac{1}{2^k} + \frac{1}{2^N} \leq \varepsilon. \end{aligned}$$

Therefore we have the following inclusions between basis-neighborhoods $W(p_1, \dots, p_N; \delta, \dots, \delta)$ and balls $B(0, r)$ (with $\delta > 0$, $N \in \mathbb{N}$, $r > 0$):

$$B(0, r) \subset W(p_1, \dots, p_N; \delta, \dots, \delta),$$

when δ and N are given, $\varepsilon > 0$ is chosen so that $\varepsilon/(1-\varepsilon) < \delta$, and r is taken $= \varepsilon 2^{-N}$;

$$W(p_1, \dots, p_N; \delta, \dots, \delta) \subset B(0, r),$$

when r is given, N is chosen so that $2^{-N} \le r/2$, and δ is taken $= r/2$. This shows that $B(0, r)$ for $r > 0$ is a system of basis-neighborhoods at 0 for the topology defined by $\mathcal{P}$. $\square$

When a space X topologized by Theorem B.8 is a *complete* metric space, then we have a Fréchet space, cf. Definition B.4. Observe that the space is locally convex because of the definition of the topology by use of seminorms, but that the ball defined by the metric (B.11) need not be convex, cf. Exercise B.1. (However, there does exist, as mentioned after Definition B.4, another compatible metric with convex balls.) It will in general be most convenient to calculate on the basis of the seminorms rather than a metric, although its existence has useful consequences.

Remark B.9. One can ask conversely whether the topology of an arbitrary locally convex t.v.s. X can be defined by a separating family of seminorms? The answer is yes: To a neighborhood basis at 0 consisting of convex, balanced open sets V one can associate the Minkowski functionals μ_V defined by

$$\mu_V(x) = \inf\{t > 0 \mid \frac{x}{t} \in V\} ; \tag{B.14}$$

then one can show that the μ_V's are a separating family of seminorms defining the topology on X. If X in addition is metrizable, the topology can be defined from a *countable* family of seminorms, since X then has a neighborhood basis at 0 consisting of a sequence of open, convex balanced sets. See e.g. [R 1974, Ch. 1] and [P 1989].

Like in Banach spaces there are some convenient rules characterizing the continuous linear operators on Fréchet spaces. By a bounded operator we mean an operator sending bounded sets into bounded sets.

Theorem B.10. *Let X be a Fréchet space and let Y be a topological vector space. When T is a linear map of X into Y, the following four properties are equivalent:*

(a) *T is continuous.*

(b) *T is bounded.*

(c) *$x_n \to 0$ in X for $n \to \infty \implies (Tx_n)_{n\in\mathbb{N}}$ is bounded in Y;*

(d) *$x_n \to 0$ in $X \implies Tx_n \to 0$ in Y.*

Proof. (a) $\implies$ (b). Let E be bounded in X, and let $F = T(E)$. Let V be a neighborhood of 0 in Y. Since T is continuous, there is a neighborhood U

of 0 in X such that $T(U) \subset V$. Since E is bounded, there is a number $t > 0$ such that $E \subset tU$. Then $F \subset tV$; hence F is bounded.

(b) $\implies$ (c). When $x_n \to 0$, the set $\{x_n \mid n \in \mathbb{N}\}$ is bounded. Then $\{Tx_n \mid n \in \mathbb{N}\}$ is bounded according to (b).

(c) $\implies$ (d). Let d denote a translation invariant metric defining the topology. The triangle inequality then gives that

$$d(0, nx) \leq d(0, x) + d(x, 2x) + \cdots + d((n-1)x, nx) = nd(0, x) \;, \qquad \text{(B.15)}$$

for $n \in \mathbb{N}$. When $x_n \to 0$, then $d(x_n, 0) \to 0$, so there exists a strictly increasing sequence of indices n_k so that $d(x_n, 0) < k^{-2}$ for $n \geq n_k$. Then $d(kx_n, 0) \leq kd(x_n, 0) \leq k^{-1}$ for $n \geq n_k$; hence also the sequence $t_n x_n$ goes to 0 for $n \to \infty$, where t_n is defined By

$$t_n = k \;\text{ for }\; n_k \leq n \leq n_{k+1} - 1 \,.$$

According to (c), the sequence $t_n T x_n$ is bounded. It follows (cf. Exercise B.5) that $Tx_n \to 0$ for $n \to \infty$.

(d) $\implies$ (a). If T is not continuous, there exists a neighborhood V of 0 in Y and for any n an x_n with $d(x_n, 0) < \frac{1}{n}$ and $Tx_n \notin V$. Here x_n goes to 0, whereas Tx_n does not go to 0, i.e., (d) does not hold. □

Remark B.11. It is seen from the proof that the assumption in Theorem B.10 that X is a Fréchet space can be replaced with the assumption that X is a metrizable topological vector space with an invariant metric.

We see in particular that a functional Λ on X is continuous if and only if $x_n \to 0$ in X implies $\Lambda x_n \to 0$ in $\mathbb{L}$.

Since "bounded" and "continuous" are synonymous for linear operators in the situation described in Theorem B.10, we may, like for operators in Banach spaces, denote the set of continuous linear operators from X to Y by $\mathbf{B}(X, Y)$.

One can also consider unbounded operators, and operators which are not everywhere defined, by conventions like those in Chapter 11.

We end this presentation by mentioning briefly that the well-known basic principles for Banach spaces which build on the theorem of Baire (on complete metric spaces), are easily generalized to Fréchet spaces.

Theorem B.12 (The Banach-Steinhaus theorem). *Let X be a Fréchet space and Y a topological vector space, and consider a family $(T_\lambda)_{\lambda \in \Lambda}$ of continuous operators $T_\lambda \in \mathbf{B}(X, Y)$. If the set $\{T_\lambda x \mid \lambda \in \Lambda\}$ (called the "orbit" of x) is bounded in Y for each $x \in X$, then the family T_λ is equicontinuous, i.e., when V is a neighborhood of 0 in Y, then there exists a neighborhood W of 0 in X so that $\bigcup_{\lambda \in \Lambda} T_\lambda(W) \subset V$.*

Corollary B.13. *Let X be a Fréchet space and Y a topological vector space, and consider a sequence $(T_n)_{n\in\mathbb{N}}$ of continuous operators $T_n \in \mathbf{B}(X,Y)$. If $T_n x$ has a limit in Y for each $x \in X$, then the map $T\colon x \mapsto Tx = \lim_{n\to\infty} T_n x$ is a continuous map of X into Y. Moreover, the sequence $(T_n)_{n\in\mathbb{N}}$ is equicontinuous.*

We leave to the reader to formulate a similar corollary for nets.

Theorem B.14 (The open mapping principle). *Let X and Y be Fréchet spaces. If $T \in \mathbf{B}(X,Y)$ is surjective, then T is open (i.e., sends open sets into open sets).*

Theorem B.15 (The closed graph theorem). *Let X and Y be Fréchet spaces. If T is a linear map of X into Y and the graph of T is closed in $X \times Y$, then T is continuous.*

B.2. Inductive limits of Fréchet spaces.

When defining the topology on $C_0^\infty(\Omega)$ for an open set $\Omega \subset \mathbb{R}^n$, we shall need the following generalization of Fréchet spaces:

Theorem B.16. *Let*

$$X_1 \subset X_2 \subset \cdots \subset X_j \subset \dots \tag{B.16}$$

be a sequence of Fréchet spaces such that for every $j \in \mathbb{N}$, X_j is a subspace of X_{j+1} and the topology on X_j is the topology induced from X_{j+1}. Let

$$X = \bigcup_{j=1}^{\infty} X_j \ , \tag{B.17}$$

and consider the sets W that satisfy: $W \cap X_j$ is an open, convex, balanced neighborhood of 0 in X_j for all j. Then X has a unique locally convex vector space topology whose open, convex, balanced neighborhoods of 0 are precisely the sets W.

With this topology, all the injections $X_j \subset X$ are continuous.

The topology determined in this way is called the inductive limit topology, and spaces of this kind are called $\mathcal{LF}$ spaces (short for: inductive limits of Fréchet spaces). There is more information on this topology in Exercises B.15 and B.16, where it is shown how it can also be described in terms of seminorms.

Theorem B.17. *Let $X = \bigcup_{j\in\mathbb{N}} X_j$ be an inductive limit of Fréchet spaces.*

(a) *A set E in X is bounded if and only if there exists a j_0 so that E lies in X_{j_0} and is bounded there.*

(b) *If a sequence* $(u_k)_{k\in\mathbb{N}}$ *is a Cauchy sequence in* X*, then there exists a* j_0 *such that* $u_k \in X_{j_0}$ *for all* k*, and* $(u_k)_{k\in\mathbb{N}}$ *is convergent in* X_{j_0} *and in* X*.*

(c) *Let* Y *be a locally convex topological vector space. A linear map* T *of* X *into* Y *is continuous if and only if* $T\colon X_j \to Y$ *is continuous for every* $j \in \mathbb{N}$*.*

A variant of the theorem is (essentially) shown [R 1974, Ch. 6]; see also Exercise B.16.

The fundamental property of the inductive limit topology that we use, is that important concepts such as convergence and continuity in connection with the topology on X can be referred to the corresponding concepts for one of the simpler spaces X_j. This is seen from Theorem B.17, and we observe the following further consequences.

Corollary B.18. *Hypotheses as in Theorem* B.17*.*

(a) *A linear map* $T\colon X \to Y$ *is continuous if and only if one has for each* $j \in \mathbb{N}$ *that when* $(u_k)_{k\in\mathbb{N}}$ *is a sequence in* X_j *with* $u_k \to 0$ *in* X_j *for* $k \to \infty$*, then* $Tu_k \to 0$ *in* Y*.*

(b) *Assume that the topology in each* X_j *is given by a family* $\mathcal{P}_j$ *of seminorms with the max-property (cf. Remark* B.6*). A linear functional* $\Lambda\colon X \to \mathbb{L}$ *is continuous if and only if there exists, for each* j*, a seminorm* $p_j \in \mathcal{P}_j$ *and a constant* $c_j > 0$ *so that*

$$|\Lambda(x)| \le c_j p_j(x) \quad \text{for } x \in X_j \; ; \tag{B.18}$$

Proof. (a) and (b) follow from Theorem B.17 combined with, respectively, Theorem B.10 and Lemma B.7. $\square$

Exercises for Appendix B

B.1. Define d_1 and d_2 for $x, y \in \mathbb{R}$ by

$$d_1(x,y) = |x-y| \ , \quad d_2(x,y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right| \ .$$

Show that d_1 and d_2 are metrics on $\mathbb{R}$ which induce the same topology on $\mathbb{R}$, and that $(\mathbb{R}, d_1)$ is a complete metric space whereas $(\mathbb{R}, d_2)$ is not a complete metric space.

B.2. Let X be a t.v.s. and assume that the topology is defined by a translation invariant metric d. Show that (x_n) is a Cauchy sequence in X if and only if (x_n) is a metric Cauchy sequence in (X, d).

B.3. Consider the family of seminorms on $C^0(\mathbb{R})$ defined by

$$p_k(f) = \sup\{\, |f(x)| \mid x \in [-k,k]\,\} \ , \quad \text{for } \ k \in \mathbb{N} \ ,$$

and the metric defined by this as in Theorem B.8.
Define

$$f(x) = \max\{0, 1-|x|\} \ , \ g(x) = 100 f(x-2) \ , \ h(x) = \tfrac{1}{2}(f(x)+g(x)) \ ,$$

and show that

$$d(f,0) = \frac{1}{2} \ , \quad d(g,0) = \frac{50}{101} \ , \quad d(h,0) = \frac{1}{6} + \frac{50}{102} \ .$$

Hence the ball $B(0, \frac{1}{2})$ is not convex. Is $B(0,r)$ convex for any $r < 1$?

B.4. Let X be a topological vector space.
(a) Show Lemma B.3 1°.
(b) Show that X is a Hausdorff space, i.e., for $x \neq y$ there exist neighborhoods V_1 of x and V_2 of y so that $V_1 \cap V_2 = \varnothing$.

B.5. Let X be a topological vector space.
(a) Show that $E \subset X$ is bounded if and only if, for any neighborhood V of 0 there exists a $t > 0$ so that $E \subset sV$ for $s \geq t$.
(b) Show that the only bounded subspace of X is $\{0\}$.

B.6. Consider $\mathbb{R}^2$, provided with the usual topology, and let $A \subset \mathbb{R}^2$, $B \subset \mathbb{R}^2$.

(a) Show that $2A \subset A + A$, and find an example where $2A \neq A + A$.
(b) Show that if A is closed and B is compact, then $A + B$ is closed.
(c) Show that $\overline{A} + \overline{B} \subset \overline{A + B}$ in general, and find an example where $\overline{A} + \overline{B} \neq \overline{A + B}$.

B.7. Show Lemma B.7.

B.8. Show that when X and Y are topological vector spaces, then the product space $X \times Y$ is a topological vector space.

B.9. Give proofs of Theorems B.12–B.15.

The following exercises refer to the notation of Chapter 2.

B.10. Show that $C^\infty([a,b])$ and $C_K^\infty(\Omega)$ are Fréchet spaces.

B.11. Show that D^α is a continuous operator in $C^\infty(\Omega)$ and in $C_K^\infty(\Omega)$. Let $f \in C^\infty(\Omega)$ and show that the operator $M_f : u \mapsto f \cdot u$ is continuous in $C^\infty(\Omega)$ and in $C_K^\infty(\Omega)$.

B.12. Let $f \in L_2(\Omega)$ and show that the functional

$$\Lambda : u \mapsto \int_\Omega u\, f\, dx$$

is continuous on $C_K^\infty(\Omega)$ for every compact $K \subset \Omega$.

B.13. Show that $C_K^\infty(\Omega)$ is a closed subspace of $C^\infty(\Omega)$.

B.14. Let A be a convex subset of a topological vector space E. Let x_0 be an interior point of A and x a point in the closure $\overline{A}$ of A. Show that all points $u \neq x$ on the segment

$$[x_0, x] = \{\, \lambda x_0 + (1-\lambda)x \mid \lambda \in [0,1] \,\}$$

are interior in A.
(*Hint.* Begin with the case where $x \in A$.)
Use this to show that when $A \subset E$, A is convex then:

$$\mathring{A} \neq \varnothing \Rightarrow \overline{\mathring{A}} = \overline{A} \text{ and } \mathring{\overline{A}} = \mathring{A}\,.$$

Show by an example that $\mathring{A} \neq \varnothing$ is a necessary condition.

B.15. Let E be a locally convex topological vector space and let $M \subset E$ be a subspace. Let U be a convex balanced neighborhood of 0 in M (with the topology induced from E).
(a) Show that there exists a convex balanced neighborhood V of 0 in E so that $V \cap M = U$.
(b) Assume that $x_0 \in E \setminus \overline{M}$. Show that V in (a) may be chosen so that $x_0 \notin V$.
(c) Let p_0 be a continuous seminorm on M. Show that there exists a continuous seminorm p on E so that $p|_M = p_0$.

B.16. The inductive limit topology. Let X be a vector space and $E_1 \subsetneqq E_2 \subsetneqq E_3 \subsetneqq \cdots \subsetneqq E_j \subsetneqq \dots$ a strictly increasing sequence of subspaces of X so that $X = \bigcup_{j=1}^{\infty} E_j$. Assume that for all $j \in \mathbb{N}$, (E_j, τ_j) is a locally convex topological vector space; and assume that for $j < k$, the topology induced by τ_k on E_j is the same as τ_j (as in Theorem B.16).
(a) Define $\mathcal{P}$ by

$$\mathcal{P} = \{ \text{ seminorms } p \text{ on } X \mid p|_{E_j} \text{ is } \tau_j\text{-continuous } \forall j \in \mathbb{N} \}.$$

Show that $\mathcal{P}$ is separating. (One can use Exercise B.15 (c).)
(b) Thus $\mathcal{P}$ defines a locally convex vector space topology τ on X. Show that the topology induced by τ on E_j equals τ_j. (Cf. Exercise B.15 (a); show that τ has properties as in Theorem B.16.)
(c) Assume that Λ is a linear map of X into a locally convex topological vector space Y. Show that Λ is τ-continuous if and only if $\Lambda|_{E_j}$ is continuous $E_j \to Y$ for every $j \in \mathbb{N}$.
(d) Show that a set $B \subset X$ is bounded (w.r.t. τ) if and only if $\exists j \in \mathbb{N} : B \subset \overline{E}_j$ and B is bounded in $\overline{E}_j$. (cf. Exercise B.15 (b).)
(e) Example. $X = C_0^0(\mathbb{R})$ (cf. (C.7)). Take $E_j = C_{[-j,j]}^0(\mathbb{R})$, with the sup-norm topology. Show that the topology τ on $C_0^0(\mathbb{R})$ determined from τ_j as in (b) is strictly finer than the sup-norm topology on $C_0^0(\mathbb{R})$.

B.17. Show that when X is as in Theorem B.11, then

$$d_1(x, y) = \sum_{k=1}^{\infty} \min\{2^{-k}, p_k(x - y)\}$$

is likewise an invariant metric defining the topology.