

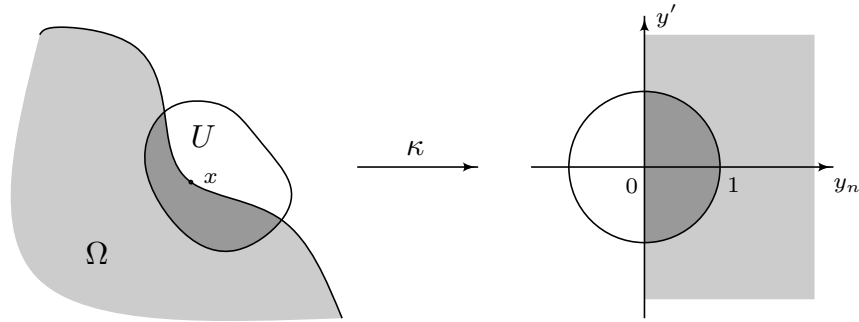
§C. Appendix C. Function spaces

We here define some additional function spaces, supplying those introduced in Chapter 2. The new types of spaces are associated with sets Ω where the boundary has a certain smoothness.

Definition C.1. *An open set $\Omega \subset \mathbb{R}^n$ is said to be (of class) C^m for some $m \leq \infty$ when every boundary point $x \in \partial\Omega$ has an open neighborhood U with an associated C^m -diffeomorphism κ (a bijective C^m mapping with C^m inverse) such that κ maps U onto the unit ball $B(0, 1) \subset \mathbb{R}^n$, and*

$$\begin{aligned}\kappa(x) &= 0 \\ \kappa(U \cap \Omega) &= B(0, 1) \cap \mathbb{R}_+^n \\ \kappa(U \cap \partial\Omega) &= B(0, 1) \cap \mathbb{R}^{n-1}.\end{aligned}\tag{C.1}$$

We then also say that $\bar{\Omega}$ is C^m (in detail: closed and C^m).



In the case $m = \infty$ one often says instead that Ω is *smooth*. In some texts, the above definition is taken to mean that $\partial\Omega$ is C^m . But in fact the boundary of Ω can be a C^m manifold also when Definition C.1 is not verified — think for example of the set $\Omega = \{x \in \mathbb{R}^n \mid |x| \neq 1\}$. The definition assures that Ω is locally “on one side of $\partial\Omega$ ”; this property involves Ω as well as $\partial\Omega$.

Examples: \mathbb{R}_+^n is smooth. Balls in \mathbb{R}^n are smooth. The set

$$\{x \in \mathbb{R}^2 \mid x_2 > 0, x_1 < \sqrt{x_2}\}$$
(C.2)

is C^1 but not C^2 .

For a function u on $\bar{\mathbb{R}}_+^n$ it makes good sense to speak of the partial derivatives on $\bar{\mathbb{R}}_+^n$, and they exist up to order m when the partial derivatives of u on \mathbb{R}_+^n up to order m extend to continuous functions on $\bar{\mathbb{R}}_+^n$. This carries over to smooth sets Ω , since one can define the derivatives on sets of the form $U \cap \bar{\Omega}$

C.2

in Definition C.1 by use of κ and the chain rule. [More precisely: κ maps x over to $y = (\kappa_1(x), \dots, \kappa_n(x))$, so if we write $u(x) = u(\kappa^{-1}(y)) = \tilde{u}(y)$, we have that

$$(\partial_{x_j} u)(x) = \sum_{l=1}^n \frac{\partial \kappa_l(x)}{\partial x_j} \partial_{y_l} \tilde{u}(y)|_{y=\kappa(x)}. \quad (\text{C.3})$$

In other words, the differential operator ∂_{x_j} on $U \cap \overline{\Omega}$ corresponds to the differential operator $\sum_{l=1}^n a_{jl} \partial_{y_l}$ on $B(0, 1) \cap \overline{\mathbb{R}_+^n}$ with C^∞ coefficients $a_{jl}(y) = (\partial_{x_j} \kappa_l)(\kappa^{-1}(y))$.]

Regardless of the smoothness of Ω we denote by $C^k(\overline{\Omega})$ the vector space of functions on $\overline{\Omega}$ such that the partial derivatives up to order m defined on Ω are uniformly continuous on bounded subsets of Ω (hence extend to continuous functions on $\overline{\Omega}$). When $\overline{\Omega}$ is compact, this is a Banach space with the norm

$$\|u\|_{C^k(\overline{\Omega})} = \sup \{ |\partial^\alpha u(x)| \mid x \in \overline{\Omega}, |\alpha| \leq k \}. \quad (\text{C.4})$$

If $\overline{\Omega}$ is not assumed to be compact, we can define a Fréchet topology on $C^k(\overline{\Omega})$ by the family of seminorms (2.9) where k is fixed and K_j for example runs through the increasing sequence of compact sets $K_j = \overline{\Omega} \cap \underline{B}(0, j)$ satisfying $\bigcup_{j \in \mathbb{N}} K_j = \overline{\Omega}$. (In the compact case, the sequence $(K_j)_{j \in \mathbb{N}}$ can be replaced by one set $\overline{\Omega}$ and the seminorm is a norm.)

We can also define

$$\begin{aligned} C^\infty(\overline{\Omega}) &= \bigcap_{k=0}^{\infty} C^k(\overline{\Omega}), \\ C_K^k(\overline{\Omega}) &= \{ u \in C^k(\overline{\Omega}) \mid \text{supp } u \subset K \}, \quad C_K^\infty(\overline{\Omega}) = \bigcap_{k=0}^{\infty} C_K^k(\overline{\Omega}), \\ C_{(0)}^k(\overline{\Omega}) &= \{ u \in C^k(\overline{\Omega}) \mid \text{supp } u \text{ compact} \subset \overline{\Omega} \}, \\ C_{(0)}^\infty(\overline{\Omega}) &= \bigcap_{k=0}^{\infty} C_{(0)}^k(\overline{\Omega}). \end{aligned} \quad (\text{C.5})$$

The first space in this list is a Fréchet space with the topology determined by the family of seminorms (2.9) (k runs through all integers ≥ 0 , the sequence (K_j) may be replaced by one set $\overline{\Omega}$ if this is compact). The space $C_K^k(\overline{\Omega})$ is a Banach space with norm $\sup \{ |\partial^\alpha u(x)| \mid |\alpha| \leq k, x \in K \}$, while $C_K^\infty(\overline{\Omega})$ is a Fréchet space. Finally, $C_{(0)}^k(\overline{\Omega})$ equals the Banach space $C^k(\overline{\Omega})$ when $\overline{\Omega}$ is compact; otherwise it is an \mathcal{LF} space (considered as $\bigcup_{K_j \subset \overline{\Omega}} C_{K_j}^k(\overline{\Omega})$ where $\bigcup_j K_j = \overline{\Omega}$ as above); $C_{(0)}^\infty(\overline{\Omega})$ is likewise \mathcal{LF} when $\overline{\Omega}$ is not compact. (We use the index (0) to avoid confusion with $C_0^\infty(\Omega)$.)

C.3

One can show that

$$C_{(0)}^\infty(\overline{\mathbb{R}}_+^n) = \{ u|_{\overline{\mathbb{R}}_+^n} \mid u \in C_0^\infty(\mathbb{R}^n) \}, \quad (\text{C.6a})$$

and that when Ω is smooth,

$$C_{(0)}^\infty(\overline{\Omega}) = \{ u|_{\overline{\Omega}} \mid u \in C_0^\infty(\mathbb{R}^n) \}. \quad (\text{C.6})$$

The proof of (C.6a) is found in Seeley's paper [S 1964], and then (C.6) is deduced from this by use of diffeomorphisms as in Definition C.1. (When Ω is smooth and bounded, the localization arguments are similar to those used in Theorems 4.10–12.)

For an open set Ω and a compact subset K we define $C_K^k(\Omega)$ and $C_0^k(\Omega)$ by

$$\begin{aligned} C_K^k(\Omega) &= \{ u \in C^k(\Omega) \mid \text{supp } u \subset K \} \\ C_0^k(\Omega) &= \{ u \in C^k(\Omega) \mid \text{supp } u \text{ compact } \subset \Omega \}, \end{aligned} \quad (\text{C.7})$$

the former is a Banach space (like $C_K^k(\overline{\Omega})$) and the latter is an \mathcal{LF} space, namely $\bigcup_{K_j \subset \Omega} C_{K_j}^k(\Omega)$.

Finally, we mention that one may need a notation for the following spaces (where M is open, or closed and C^k resp. C^∞):

$$\begin{aligned} C_{L_p}^k(M) &= \{ u \in C^k(M) \mid \partial^\alpha u \in L_p(M) \text{ for } |\alpha| \leq k \}, \\ C_{L_p}^\infty(M) &= \bigcap_{k=0}^{\infty} C_{L_p}^k(M). \end{aligned} \quad (\text{C.8})$$

They coincide with $C^k(M)$ resp. $C^\infty(M)$ when M is compact. The spaces $C_{L_p}^k(M)$ ($1 \leq p \leq \infty$) may be provided with norms

$$\left(\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L_p(M)}^p \right)^{\frac{1}{p}} \text{ for } p < \infty, \quad \sup_{|\alpha| \leq k} \|\partial^\alpha u\|_{L_\infty(M)} \text{ for } p = \infty, \quad (\text{C.9})$$

which for $p = 2$ take the form

$$\|u\|_k = \left(\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L_2(M)}^2 \right)^{\frac{1}{2}}. \quad (\text{C.10})$$

This makes them normed but in general not complete spaces. Similarly, $C_{L_p}^\infty(M)$ may be topologized by a system of seminorms.

C.4

A special case is $C_{L^\infty}^k(\mathbb{R}^n)$; it is the space of C^k -functions on \mathbb{R}^n with bounded derivatives up to order k . It is well-known that this is a Banach space with the norm

$$\|u\|_{C_{L^\infty}^k(\mathbb{R}^n)} = \sup\{|\partial^\alpha u(x)| \mid x \in \mathbb{R}^n, |\alpha| \leq k\}. \quad (\text{C.11})$$

Exercises for Appendix C

C.1. Let $M = \underline{B}(0, 1)$ in \mathbb{R}^n . Show that $C^1(M)$ is complete with respect to the C^1 -norm (cf. (C.4)), but not complete with respect to the norm (C.10) with $k = 1$.