

# DISTRIBUTIONS AND OPERATORS

## LECTURE NOTES

GERD GRUBB

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### Part I. DISTRIBUTION SPACES

#### 1. Motivation and notation

- 1.1 Introduction.
  - 1.2 On the definition of distributions.
  - 1.3 Some notation and prerequisites.
- Exercises for Chapter 1.

#### 2. Function spaces and approximation

- 2.1 The space of test functions.
  - 2.2 Some other function spaces.
  - 2.3 Approximation theorems.
  - 2.4 Partitions of unity.
  - 2.5 Boundary behavior.
- Exercises for Chapter 2.

#### 3. Distributions. Examples and rules of calculus

- 3.1 Distributions.
  - 3.2 Rules of calculus for distributions.
  - 3.3 Distributions with compact support.
  - 3.4 Coordinate changes and convolutions.
  - 3.5 The calculation rules and the weak\* topology on  $\mathcal{D}'$ .
- Exercises for Chapter 3.

## Part II. EXTENSIONS AND APPLICATIONS

### 4. Realizations and Sobolev spaces

- 4.1 Realizations of differential operators.
  - 4.2 Sobolev spaces.
  - 4.3 The one-dimensional case.
  - 4.4 Boundary value problems in higher dimensions.
- Exercises for Chapter 4.

### 5. Fourier transformation of distributions

- 5.1 Rapidly decreasing functions.
  - 5.2 Temperate distributions.
  - 5.3 The Fourier transform on  $\mathcal{S}'$ .
  - 5.4 Homogeneity.
  - 5.5 Applications to the Laplace operator.
  - 5.6 Distributions associated with non-integrable functions.
- Exercises for Chapter 5.

### 6. Applications to differential operators. The Sobolev theorem

- 6.1 Differential and pseudodifferential operators on  $\mathbb{R}^n$ .
  - 6.2 Sobolev spaces of arbitrary real order. The Sobolev theorem.
  - 6.3 Dualities between Sobolev spaces. The Structure Theorem.
  - 6.4 Regularity theory for elliptic differential equations.
- Exercises for Chapter 6. (Includes old examination problems.)

## Part III. PSEUDODIFFERENTIAL METHODS

### 7. Pseudodifferential operators on open sets

- 7.1 Symbols and operators.
- 7.2 Negligible operators.
- 7.3 Composition rules.
- 7.4 Elliptic pseudodifferential operators.

### 8. Pseudodifferential operators on manifolds, index of elliptic operators

- 8.1 Coordinate changes.
- 8.2 Operators on manifolds.
- 8.3 Fredholm theory.

### 9. Boundary value problems in a constant-coefficient case

- 9.1 Boundary maps for the half-space.
- 9.2 The Dirichlet problem for  $I - \Delta$  on the half-space.
- 9.3 The Neumann problem for  $I - \Delta$  on the half-space.

- 9.4 Other realizations of  $I - \Delta$ .
- 9.5 Variable-coefficient cases, higher orders, systems.
- Exercises for Chapter 9.

## **Part IV. SUPPLEMENT. OPERATORS IN HILBERT SPACE**

- 10. **Unbounded linear operators**
  - 10.1 Unbounded operators in Banach spaces.
  - 10.2 Unbounded operators in Hilbert spaces.
  - 10.3 Symmetric, selfadjoint and semibounded operators.
  - 10.4 Operators associated with sesquilinear forms.
  - 10.5 The Friedrichs extension.
  - 10.6 More on variational operators.
  - Exercises for Chapter 10.
- 11. **Families of extensions**
  - 11.1 A general parametrization of extensions.
  - 11.2 The symmetric case.
- 12. **Semigroups of operators**
  - 12.1 Evolution operators.
  - 12.2 Contraction semigroups in Banach spaces.
  - 12.3 Contraction semigroups in Hilbert spaces.
  - 12.4 Applications.
  - Exercises for Chapter 12.

## **Appendix**

- A. **Appendix. Topological vector spaces**
  - A.1 Fréchet spaces.
  - A.2 Inductive limits of Fréchet spaces.
  - Exercises for the Appendix.

## **References**