Differential operators and Fourier methods

Gerd Grubb,
Copenhagen University

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Introduction

It is a very great honor for me to give this talk, and to receive tomorrow the promotion to Honorary Doctor in Mathematics at the University of Lund.

As such, I have some very fine predecessors:

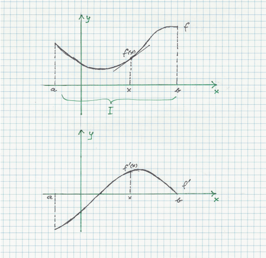
- Ivan Georgievich Petrovskiĭ (Steklov Institute in Moscow) in 1968;
- Laurent Schwartz (Ecole Polytechnique in Paris) in 1981;
- Peter Lax (Courant Institute in New York) in 2008.

In the following I shall try to describe some ingredients of the development in the area of Mathematics that has always ocuupied me.

1. Functions, spaces, operators

First a few mathematical concepts, that I hope will be reasonably well explained.

It is common knowledge what a real **function** is: $y = f(x)$, as in the graphic illustration here:



If there is a **derivative** $f'(x)$ (the slope of the tangent) at each x, we have another function $y = f'(x)$ that we can also draw:

Now we go to a higher abstract level.

Let us view functions as elements of a **function space**. Consider, instead of a single function f, a *collection of functions*, which together form a space. For example: The space $C^0(I)$ of all functions $f(x)$ defined for x in an interval I and *continuous* there. It is a *linear* space, meaning that when f and g belong there, so does any *linear combination*: $af + bg$,

$$(af + bg)(x) = af(x) + bg(x).$$

Another example is $C^1(I)$, the space of functions on I that moreover have a *continuous derivative* f'. It is also a linear space. One moreover defines $C^k(I)$ as the space where repeated differentiation — up to k times — gives continuous functions. $C^\infty(I)$ is the space of functions that have continuous derivatives of *all* orders.

We can also pass to functions of several variables $f(x_1, x_2, \dots, x_n)$ where $x = (x_1, \dots, x_n)$ runs in a suitable subset Ω of $\mathbb{R}^n$. Here one can study derivatives with respect to each coordinate x_j, the **partial derivatives**. And one can define the space $C^0(\Omega)$ of continuous functions on Ω, and spaces $C^k(\Omega)$ of functions with continuous derivatives on Ω up to order k.

There are lots of other interesting function spaces used by mathematicians. For example $L_p(\Omega)$, consisting of functions f on Ω such that $\int_\Omega |f(x)|^p \, dx$ exists. (Here Lebesgue's measure theory is needed.)

And then one can define further spaces, consisting of those functions f in $L_p(\Omega)$ that have some derivatives in $L_p(\Omega)$ in a generalized ("weak") sense. This was used already in the early half of the 1900's, but became more transparent when Laurent Schwartz developed *Distribution Theory* around 1950.

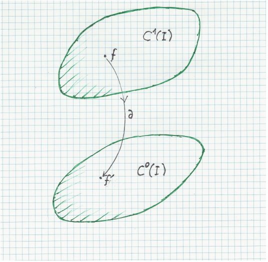
The Distribution Theory allowed for still more general spaces, and gave an extremely useful setting for the application of Fourier Transformation, invented in the 1800's and particularly appreciated by physicists.

As you see, there are lots of examples of function spaces (collections of single functions). Now I turn to the next abstract level, introducing: **Operators between function spaces**.

Basic example: When f is in $C^1(I)$, the passage from f to the derived function f' can be viewed as *a mapping* ∂ *from* $C^1(I)$ *to* $C^0(I)$:

$$\partial \colon f \mapsto f' \text{ goes from } C^1(I) \text{ to } C^0(I).$$

This mapping respects the linearity: $\partial(af + bg) = a\partial f + b\partial g$. Therefore it is called a **linear operator**.



Also in general, we can define linear operators going from one function space defined over I to another, or from one function space defined over Ω to another.

Important examples are the differential operators built up from ∂, for example the powers ∂^k, where one applies ∂ again, k times. Combinations of such operators are called **ordinary differential operators**; the *order* of the operator is the highest power that occurs.

For functions on $\Omega \subset \mathbb{R}^n$. we have the partial derivatives $\partial_1, \dots, \partial_n$, where ∂_j stands for the derivative with respect to the variable x_j. The combinations of such operators are called **partial differential operators**.

A very important example is *the Laplace operator*

$$\Delta \colon u \mapsto \Delta u = \partial_1^2 u + \cdots + \partial_n^2 u.$$

It enters in three basic equations:

$$\begin{aligned} -\Delta u(x) &= f(x) \text{ on } \Omega, && \text{the Laplace equation,} \\ \partial_t u(x,t) - \Delta u(x,t) &= f(x,t) \text{ on } \Omega \times \mathbb{R}, && \text{the heat equation,} \\ \partial_t^2 u(x,t) - \Delta u(x,t) &= f(x,t) \text{ on } \Omega \times \mathbb{R}, && \text{the wave equation,} \end{aligned}$$

describing physical problems. One wants to find *solutions* u for given f. For example, the heat equation describes how the temperature develops in a container, but it is also used in financial theory, wrapped up in a stochastic formulation.

One asks for: *existence of solution*? *Uniqueness?* Moreover if f has a *singularity*, for example a jump, how does this effect u? *Spectral questions, eigenvalues?*

Another type of examples of operators are **integral operators**, both for functions on I and for functions on Ω,

$$(Qu)(x) = \int_\Omega K(x,y)u(y)\,dy.$$

Differential equations often have integral operators as solution operators.

I shall now leave the detailed explanation, and tell the continued story more as an overview, with "namedropping", both concerning mathematical concepts and the people involved in developing them.

2. The Fourier transform and pseudo-differential operators

A very important operator is the **Fourier transformation** $\mathcal{F}$, it is an integral operator. It is invertible from the space $L_2(\mathbb{R}^n)$ onto itself, and the inverse operator has very much the same structure. It has been used much, both in physics and mathematics, and the mathematical rigor was perfected when Schwartz' Distribution Theory came into the picture. The success of $\mathcal{F}$ is due to the fact that **it turns differential operators into multiplication operators**.

When $u(x) \in L_2(\mathbb{R}^n)$, we denote $\mathcal{F}u = \hat{u}(\xi)$, again a function on $\mathbb{R}^n$, where the variable is now denoted $\xi = (\xi_1, \dots, \xi_n)$.

The differential operator ∂_j is turned into multiplication by $i\xi_j$:

$$\mathcal{F}(\partial_j u) = i\xi_j \hat{u}(\xi) \qquad \text{(here } i = \sqrt{-1}\text{)}.$$

For example, $\mathcal{F}(\Delta u) = -(\xi_1^2 + \dots + \xi_n^2)\hat{u}(\xi)$, and therefore:

$$-\Delta u = f \iff (\xi_1^2 + \dots + \xi_n^2)\hat{u} = \hat{f}.$$

Similarly, the differential operator $P = \sum_{j,k=1}^n a_{jk}\partial_j\partial_k$ is turned into the multiplication by the polynomial $p(\xi) = -\sum a_{jk}\xi_j\xi_k$;

$$\mathcal{F}(Pu) = p(\xi)\hat{u}(\xi).$$

$p(\xi)$ is called the *symbol* of P. Multiplication operators are *much easier* to work with than differential operators.

The idea can be generalized: Take a function $p(\xi)$ that is not necessarily a polynomial, and *define* the **pseudodifferential operator** $P = p(D)$ by

$$p(D)(u) = \mathcal{F}^{-1}(p(\xi)\hat{u}(\xi)), \text{ also called } \operatorname{Op}(p)u.$$

This is the simple basic idea of pseudodifferential operators!!!

But there are of course complications in the usage. The equation

$$p(\xi)\hat{u}(\xi) = \hat{f}(\xi) \text{ has the solution } \hat{u}(\xi) = \tfrac{1}{p(\xi)}\hat{f}(\xi);$$

therefore

$$p(D)u(x) = f(x) \text{ is solved by } u(x) = \operatorname{Op}(\tfrac{1}{p(\xi)})f(x).$$

This is OK when $p(\xi) \neq 0$ for all ξ. But how is it understood when $p(\xi)$ is zero for some values of ξ?

When p is not zero anywhere, the solution operator $\operatorname{Op}(\frac{1}{p(\xi)})$ is itself a pseudodifferential operator. This gives a unified theory where *both the operator and its inverse* belong to the class of operators; both differential operators and integral operators belong there.

When p does have zeroes, it is still possible to do something. There are refined interpretations of $\operatorname{Op}(\frac{1}{p(\xi)})$ (so-called fundamental solutions) in very general cases.

The next question is: What if we let p depend on x also? This is natural, since differential operators in general have x-dependent coefficients. This was answered in the 1960's and further developed through the rest of the century, by the introduction of **pseudodifferential operators**, applied to increasingly difficult problems. The general formula is

$$\begin{aligned} p(x,D)u = \operatorname{Op}(p(x,\xi))u &= \tfrac{1}{(2\pi)^n} \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\,d\xi \\ &= \tfrac{1}{(2\pi)^n} \int\int e^{i(x-y)\cdot\xi} p(x,\xi)u(y)\,dyd\xi, \end{aligned}$$

under suitable requirements on the symbols $p(x,\xi)$.

The construction is due to Kohn and Nirenberg (New York) and Hörmander (Lund) around 1965 (with preceding insights by Mihlin, Calderon, Zygmund and others).

Ps.d.o.s are particularly useful when p is not zero on a large set. This goes especially for *elliptic* operators, generalizing the Laplacian Δ.
But for the *hyperbolic* case, for problems generalizing the wave problem

$$\begin{aligned} \partial_t^2 u(x,t) - \Delta u(x,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad t = 0, \end{aligned}$$

the ps.d.o. construction was not enough; here the symbol $-\tau^2 + |\xi|^2$ of the operator vanishes on infinite conical surfaces, and $1/(-\tau^2 + |\xi|^2)$ is too singular.
For this, Hörmander introduced **Fourier Integral Operators**

$$Au(x) = \tfrac{1}{(2\pi)^n} \int\int e^{i\varphi(x,y,\xi)} a(x,y,\xi) u(y)\, dyd\xi,$$

with phase functions $\varphi(x,y,\xi)$ generalizing $(x-y)\cdot\xi$.
The questions one asks are not just about existence and uniqueness of solutions, but also for example about regularity properties of the solutions.
For the wave problem, a singularity (for example a jump or kink) in the initial function $u_0(x)$, leads to singularities in the solution $u(x,t)$ on cones in space-time.

Fourier Integral Operators are able to detect how singularities in the data lead to singularities in the solutions. This was perfected with the concept of the **Wavefront set**, a kind of spectral resolution of the singularity.

The Fourier Integral Operators were created around 1971 by Hörmander, who also introduced the Wavefront set. At the same time there was a related construction by Japanese mathematicians Sato, Kawai and Kashiwara in complex function theory. Also Duistermaat (coauthor of Hörmander) and Maslov (Russian mathematician with related ideas) should be mentioned.

Through the next many years, large classes of equations were solved, and operators were classified and analysed in detail. This very successful theory, often called **Microlocal Analysis**, was developed much further in many directions; it involved a large number of researchers throughout the world, including a strong group here at Lund University.

3. Nonlinear problems

The Microlocal Analysis was particularly successful for *linear* partial differential equations (PDE). But there is an even larger field of *nonlinear* PDE, where the unknown function and its derivatives are tied together in a nonlinear way; and here many other tricks are needed.

For a given nonlinear equation, there is often a simplified linear version, whose solvability is a first step in the analysis of the given problem. Sometimes, Microlocal Analysis is useful, sometimes other methods.

Nonlinear equation:	$F(u, \Delta u) = 0,$
for example	$-\Delta u = f(u),$
linearized as	$-\Delta u = cu.$

For example, I worked in the '90s with a Russian colleague Solonnikov on the Navier-Stokes equations (describing how fluids move); this was based on a crucial use of pseudodifferential methods in the linearized case. Another example is: Nonlinear wave equations.

On the other hand, on *nonsmooth domains*, direct integration methods (not passing via the Fourier transform) could sometimes handle problems more efficiently. For example in *free boundary problems*, where one has to find an unknown interface between two regions, it is vital to allow nonsmoothness.

Now there is a vast diversity of nonlinear problems, and some people preferred problems that could be attacked “walking in from the street”. (I remember a conversation with Vazquez in the mid-80s.)
This is also a way to attract young researchers, without forcing them to start learning the full microlocal machinery.

It seems to be a tendency in the recent past to start research on a less theoretical level. Important textbooks preparing graduate students for working in PDE leave out (or hide) the use of Distribution theory, and mention Fourier transformation just as a side remark.

Pseudodifferential operators and Fourier Integral Operators did not (as I would have expected) become part of “what everyone should know” – at least not in many research communities.

4. Nonlocal operators

I have mentioned two NON-issues: **non-linearity** and **non-smoothness**. There has recently arisen a vivid interest in one more, namely in certain **non-local** operators on $\mathbb{R}^n$. They are used in the same role at differential operators but are more difficult since they "smear out" the result to the full space (in contrast to the way a differential operator acts on a function $u(x)$ such that the effect at a point x is only felt in the neighborhood of the point).

A nonlocal operator that has attracted much attention is *the fractional Laplacian* $(-\Delta)^a$, $0 < a < 1$, for instance the square root $(-\Delta)^{\frac{1}{2}}$. They come up in geometry and physics, and also in finance and probability theory: They are generators of the so-called Lévy processes, which are of interest since they apparently give better predictions for stock options.

The standard heat equation $\partial_t u - \Delta u = 0$ is here replaced by the fractional "heat equation" $\partial_t u + (-\Delta)^a u = 0$.

The fractional Laplacian can be defined as an integral operator:

$$(-\Delta)^a u(x) = c_{n,a} PV \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n+2a}}\, dy, \qquad (1)$$

but it also belongs to pseudodifferential operators:

$$(-\Delta)^a u = \mathrm{Op}(|\xi|^{2a})u = \mathcal{F}^{-1}(|\xi|^{2a}\hat{u}(\xi)). \qquad (2)$$

Working with (1), one can stay within *real functions* and *real* integral operator methods. Working with (2) necessarily requires that one ventures into *complex functions* and Fourier transform methods.

In the papers after 2000 on this operator, everybody in nonlinear PDE used (1). There was an celebrated trick of Caffarelli and Silvestre 2007 relating $(-\Delta)^a$ to a differential operator in one more variable.

I got interested when they treated a question on bounded subsets Ω of $\mathbb{R}^n$, the "homogeneous Dirichlet problem"

$$(-\Delta)^a u = f \text{ on } \Omega, \quad u = 0 \text{ on } \mathbb{R}^n \setminus \Omega. \qquad (3)$$

Two young Spaniards Ros-Oton and Serra showed in 2012 that the solution u for bounded f has the form $d(x)^a v(x)$, where $v \in C^t(\overline{\Omega})$ for a small $t > 0$, and $d(x) = \mathrm{dist}(x, \partial\Omega)$. They claimed that this regularity, important for nonlinear problems where f is replaced by $f(u)$, had not been shown before.

The story, very briefly: They were essentially right. From the point of view of (2), there was a pseudodifferential theory of Vishik and Eskin from the '60s, but it does not cover their result.

In 2013 I was reading some unpublished notes of Hörmander from '65 on boundary problems. For the problem (3) with a smooth Ω, they contained some central arguments for showing the form $u = d^a v$ with higher regularity of v when f is more regular.

I set out to develop and complete the theory, combining it with my own work on pseudodifferential boundary problems through the years.

Ros-Oton and Serra contacted me when they saw my work, and we have had many good discussions. They say that their methods can only handle relatively low regularity situations.

In addition, my methods allow x-dependent operators, for which not much has been done in their community.

I am in touch with many more people now, give talks on the theory at international meetings, and lectured recently on it at a PhD-school in Germany.

The latest development is integration-by-parts formulas.
Just to exhibit a piece of math, let me show you the Pohozaev identity, for real solutions u of the equation (3) with $(-\Delta)^a$ replaced by a more general operator $P = \mathrm{Op}(p(x,\xi))$, and f replaced by a nonlinear right-hand side $f(u)$. Let $F = \int f\,dx$.

$$-2n\int_\Omega F(u)\,dx + n\int_\Omega f(u)\,u\,dx = \Gamma(1+a)^2\int_{\partial\Omega}(x\cdot\nu)\,s_0\gamma_0(\tfrac{u}{d^a})^2\,d\sigma$$
$$+\int_\Omega\big(\mathrm{Op}(\xi\cdot\nabla_\xi p) - \mathrm{Op}(x\cdot\nabla_x p)\big)u\,u\,dx.$$

It is nice to produce new results, but my hope is most of all that such efforts can bridge the gap between the various communities: Making tools from Microlocal Analysis generally available, rather than see them being left out (in view of their complicatedness).
The more tools we have, the better problems we can solve.

To end the talk, I will show a nice graphical illustration of Mathematics that I have always liked. It was drawn by Anatol Fomenko, a Russian mathematician in algebraic geometry, showing the formidable constructions that challenge those who work in his field: