Encounters with Spectral Theory

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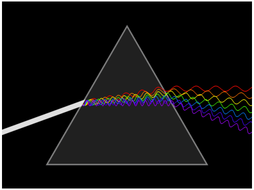
Spectrum



Prism with light ray

A spectrum

A light ray is resolved in many different-colour components

A specter

can also mean a ghostly apparition!

What is spectral theory?

In mathematics, finding the *spectrum* of an operator means decomposing it into small pieces of a simple form. Consider

- a vector space X;
- a linear operator A going from X to X; $A(au+bv) = aAu + bAv$;
- a vector $u \neq 0$ and a number λ such that $Au = \lambda u$; then u is called *an eigenvector* with *eigenvalue* λ.

Note that A acts on u simply by *multiplication by* λ. For $c \neq 0$, $c \cdot u$ is also an eigenvector with eigenvalue λ.
Sometimes one can find many linearly independent eigenvectors

$$u_1, u_2, u_3, \dots$$

with eigenvalues

$$\lambda_1, \lambda_2, \lambda_3, \dots.$$

If the eigenvectors fill out the whole space X, in the way that any vector v in X can be written as a sum of eigenvectors, then

$$v = u_1 + u_2 + u_3 + \dots \text{ leads to } Av = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \dots,$$

and we have found a *spectral decomposition of A*.

Example: The vibrating string

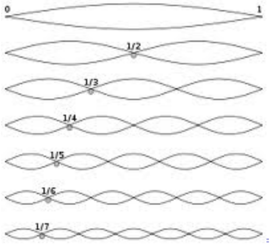
In this example, the vector space X consists of the functions $u(x)$ on the interval $[0, \pi]$ that can be differentiated infinitely many times.
The operator A is $-d^2/dx^2$, the second derivative with a minus; we apply it to functions that are 0 at the endpoints of the interval.
Here we have lots of eigenvectors, for example all the sine functions $\sin x$, $\sin 2x$, $\sin 3x$, ...

$$A(\sin x) = \sin x, A(\sin 2x) = 4\sin 2x, A(\sin 3x) = 9\sin 3x, \dots ;$$

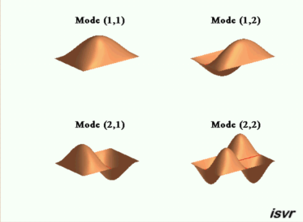
collected in one formula:

$$A(\sin kx) = k^2 \sin kx, \text{ for } k = 1, 2, 3, \dots .$$

This A enters in the differential equation for a vibrating string, and the sine functions are special solutions, "fundamental vibrations", also called "standing waves". You see the first 7 of them here (adapted to the interval $[0, 1]$). General vibrations are superpositions of multiples of these fundamental waves.



Also for n-dimensional domains $\Omega \subset \mathbb{R}^n$, $A = -\Delta = -\partial^2/\partial x_1^2 - \cdots - \partial^2/\partial x_n^2$ has a system of eigenfunctions that are 0 on the boundary of Ω. Here are the first few for a rectangle in $\mathbb{R}^2$, a vibrating membrane:



Ideally, the question in spectral theory is to find all the eigenvalues of an operator A. Not always possible; then search for qualitative information. In the string example, we had the exact formula

$$\lambda_k = k^2.$$

In more general situations, one can sometimes show that the eigenvalues "behave like a power of k", namely, there are constants c and a such that

$$\lambda_k / k^a \to c \text{ for } k \to \infty. \text{ We say that } \lambda_k \sim c k^a.$$

This is called a *spectral asymptotic estimate*.
Such one holds for the vibrating membrane (for bounded sets $\Omega \subset \mathbb{R}^2$):

$$\lambda_k \sim \frac{4\pi}{\operatorname{area}(\Omega)} k, \qquad \text{by Hermann Weyl in 1912.}$$

People and projects

In the following I will talk about my own research and how it connected to spectral theory, but here I will refrain from giving further mathematical explanations, just mention the words as they are used, and the people I met *(word-dropping*, and *name-dropping)*.

Aarhus 1959-63

After my bachelor studies at Copenhagen University, **Svend Bundgård** invited me to do the second part of my studies at the new Math. Inst. at Århus University. **Ebbe Thue Poulsen** was my advisor towards my magisterkonferens 1963. I enjoyed the guest lectures by **Jacques-Louis Lions** (Papa Lions) on elliptic partial differential equations.

I studied Russian in order to read **M. G. Krein**'s important 1947 article on selfadjoint extensions of semibounded operators (never translated!), and found that it contained a more complete analysis than reproduced in the western literature. As late as in 2009 I helped **Fritz Gesztesy** by sending him the translated piece I made for the thesis.

The thesis gave an account of semigroup theory, selfadjoint extensions and the use of Fourier transformation in these questions.

Stanford 1963-66

After Århus I studied for a Ph.D. at Stanford University 1963-66 with **Ralph Phillips** as advisor. At that time, most of his students were doing Scattering Theory, but I ended up writing "A characterization of the nonlocal boundary value problems associated with an elliptic operator". It extends the abstract works of **Krein** '47, **Vishik** '52 and **Birman** '56, and its concrete application relies on a new theory of distribution solutions of elliptic operators introduced by **Lions** and **Magenes**.

There is a 1–1 correspondence, preserving important properties, between closed realizations $\widetilde{A}$ of a second-order elliptic operator A on Ω and closed operators $L\colon X \to Y^*$ over $\partial\Omega$:

$$\widetilde{A} \text{ closed realization} \longleftrightarrow \begin{cases} X, Y \text{ closed subspaces of } H^{-\frac{1}{2}}(\partial\Omega), \\ L\colon X \to Y^* \text{ closed, densely defined.} \end{cases}$$

At Stanford I also learned about a new analysis tool, *pseudodifferential operators*, which I found very difficult, but have worked with ever since, from **Lars Hörmander, Louis Nirenberg**. These operators include partial differential operators (of positive order), as well as the integral operators solving elliptic PDE (they are of negative order).

Copenhagen from 1966 on

After the Ph.D. I became universitetsadjunkt in Copenhagen 1966-69, followed by a permanent position.

The year 1968-69 was spent in Paris, where I met and became friends with a lot of young (then) French mathematicians, **Bardos, Bony, Goulaouic, Baouendi, Métivier, Unterberger**, and later **Helffer, Sjöstrand** and others. For many years we had a French-Danish-Swedish cooperation in organizing a yearly meeting in PDE on the atlantic coast.

At a summer school in Sweden 1969 I met **Louis Boutet de Monvel** and **Giuseppe Geymonat**, the latter encouraged me to learn the former's theory of *pseudodifferential boundary operators*, ps.d.b.o.s. Pippo and I applied it together to find the *essential spectrum* of multi-order elliptic systems on bounded domains in '73.

The essential spectrum of A_B defined from an elliptic system consists exactly of the points λ where $\{A - \lambda, B\}$ fails to be elliptic.

The Stanford thesis was followed up in 5 publications in 1968–74. It was well-known that Weyl's asymptotic estimate extends to elliptic $2m$-order operators A, with suitable boundary conditions B,

$$\lambda_k(A_B^{-1}) \sim c(A)k^{-2m/n}.$$

One of the results of the '74 paper is that the *difference* between the inverses of the general A_B and the Dirichlet case A_γ satisfies

$$\lambda_k(A_B^{-1} - A_\gamma^{-1}) \sim c\, k^{-2m/(n-1)}, \quad (*)$$

a stronger estimate that picks up the boundary dimension $n-1$.
This improves a result of **Birman** '62, which gives an upper bound

$$\lambda_k(A_B^{-1} - A_\gamma^{-1}) \leq Ck^{-2m/(n-1)},$$

in the case $m = 1$. It should be noted that he moreover included exterior domains with bounded boundary.
$(*)$ was also proved by **Birman** and **Solomiak** in '78 - '80. They did not know my work, but got acquainted with it when I visited St. Petersburg in 1981, invited by **Solonnikov**.

An interesting ingredient in the calculus of ps.d.b.o.s is the *singular Green operators* G, which include the resolvent differences $A_B^{-1} - A_{B'}^{-1}$. In 1984, I found the asymptotic estimate for general G's of order $-t < 0$:

$$\lambda_k(G) \sim c(G)k^{-t/(n-1)}.$$

The next effort was to apply the calculus to *time-dependent* (parabolic) problems, which made it necessary to develop a theory of ps.d.b.o.s depending on a *spectral parameter*. The subject was swelling up so much that I had to write a book to give a full account:

"Functional calculus of pseudodifferential boundary problems", Birkhäuser Verlag 1986, 2nd edition 1996.

In the book, resolvents and their applications to: 1) heat operators, 2) geometric coefficients, 3) index formulas, 4) spectral asymptotic formulas and 5) singular perturbations, were worked out.

The book also completed some points in Boutet de Monvel's original, somewhat sketchy presentation.

The research took a new turn when I met **Solonnikov**, who was working on the nonlinear *Navier-Stokes equations* describing motion in fluids. The principles of the parameter-dependent calculus turned out to be useful here. We reduced the *degenerate parabolic* differential problem to a truly parabolic, but pseudodifferential, problem. This led to solution constructions improving the known results on how the regularity of solutions depends on the regularity of data (1987–95).

Extensions of the pseudodifferential boundary operator calculus to general L_p-based function spaces were also developed, partly in cooperation with **Niels-Jørgen Kokholm** '93.

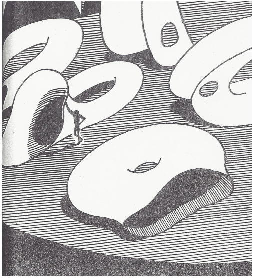
There are several "schools" on pseudodifferential boundary problems (with a certain amount of competition), namely also **Bert-Wolfgang Schulze's** ps.d.o.s (without transmission property) on manifolds with edges and cones, and **Richard Melrose's** (e.g. with **Rafe Mazzeo**) on singular spaces; we have learned from each other through the years.

The NON-tendency

It is customary in mathematics, when a theory has reached a certain maturity, that people turn to work on theories with converse assumptions, "non-theories".
For example, many people turned away from elliptic problems to non-elliptic problems that required other tools.
I myself had focused on non-local boundary conditions in addition to the local ones.
Schulze turned away from the **Boutet de Monvel** calculus by including non-transmission cases.
The Navier-Stokes equations have a non-linearity that makes them harder to treat than linear problems.

In recent years, the pseudodifferential theory has been developed from theories with smooth symbols to theories with non-smooth symbols; here **Helmut Abels** (who did part of his Ph.D.-studies with me) has worked out a non-smooth pseudodifferential boundary operator theory in '05.

My experience with non-local boundary conditions and ps.d.o.-techniques led to studies of the **Atiyah-Patodi-Singer** index problem for a Dirac operator D with a pseudodifferential boundary condition, defining D_B.



(Drawing by A.T. Fomenko)

The **A-P-S** formula shown in '77 when there is a cylindrical structure at the boundary:

$$\text{index}(D_B) = c(D) + \eta.$$

Index formula in the general case '92:

$$\text{index}(D_B) = c(D) + \eta + b.$$

Peter Gilkey had posed the question to me, and this made me study the coefficients in associated *heat trace expansions*, up to and including the index term c_0:

$$\text{Tr}(e^{-tD_B^* D_B}) \sim c_{-n} t^{\frac{-n}{2}} + c_{1-n} t^{\frac{1-n}{2}} + \cdots + c_0 + O(t^{\frac{3}{8}}) \text{ for } t \to 0+ .$$

This was the beginning of many works on *trace expansions* and their coefficients, which represent *geometric invariants*. I had a fruitful cooperation with **Bob Seeley**, where we developed a refined parameter-dependent calculus, and found a full expansion ('95 and '96):

$$\mathrm{Tr}(\varphi e^{-tD_B^* D_B}) \sim \sum_{j=-n}^{-1} c_j t^{j/2} + \sum_{j=0}^{\infty} (c_j t^{j/2} + c_j' t^{j/2} \log t)$$

where c_0 enters in *index* formulas, c_0' in *noncommutative residues*.

The search for geometric invariants in more general situations was pursued in joint works with **Elmar Schrohe** 2001 and 2004. Here we investigated the *noncommutative residue* and the *canonical trace* (regularized trace) for general ps.d.b.o's.

In particular there was the question of whether a certain residue is zero, connected with *sectorial projections*. This is carried out in papers up to 2008, one of them together with **Anders Gaarde**. A main result:

c_0-coefficient of $\mathrm{Tr}((P_+ + G)e^{-tP_{1,T}})$ is a sum of 5 different integrals,

over Ω and $\partial\Omega$.

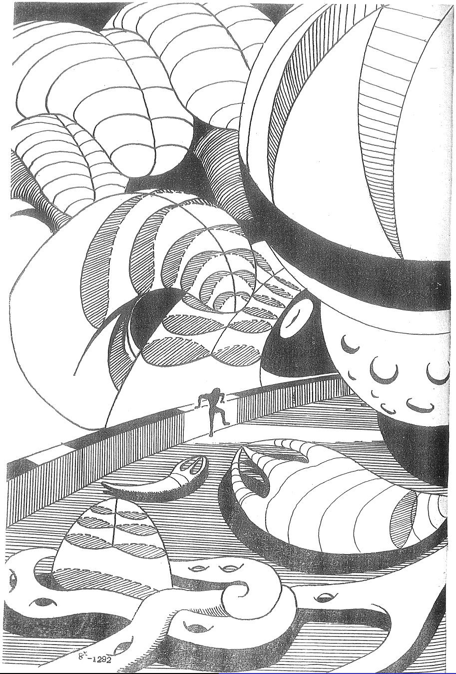
In more detail:

Theorem

Let $P_+ + G$ be of order m, and let $P_{1,T}$ be defined from an auxiliary elliptic system $\{P_1, T\}$ with a differential operator P_1 of even order. Then $C_0(P_+ + G, P_{1,T})$ is a sum of terms, calculated in local coordinates:

$$\begin{aligned} C_0(P_+ + G, P_{1,T}) = &\int_X [\mathrm{TR}_x\, P - \tfrac{1}{m}\operatorname{res}_{x,0}(P \log P_1)]\, dx \\ &+ \tfrac{1}{m}\operatorname{res}(L(P, \log P_1)) \\ &+ \int_{X'} [\mathrm{TR}_{x'}\, \mathrm{tr}_n\, G - \tfrac{1}{m}\operatorname{res}_{x',0}\, \mathrm{tr}'_n(G(\log P_1)_+)]\, dx' \\ &- \tfrac{1}{m}\operatorname{res}(P_+ G_1^{\log}) \\ &- \tfrac{1}{m}\operatorname{res}(G G_1^{\log}). \end{aligned}$$

Each line is an invariant defined from P, G, $\log P_1$ and $G_1^{\log}$.



Return to old fields

In 2005 I read a survey paper in Bulletin American Math. Soc. by two old gentlemen, **W. N. Everitt** and **L. Marcus**, on operators connected with ODE and PDE.
They had a long experience with ODE problems and the various boundary conditions (where there is a finite-dimensional choice), and had more recently gotten interested in elliptic PDE and their boundary conditions (where the selection is infinite dimensional).
In particular they were praising the mysteries of what they called *the Harmonic operator*, defined from the Laplacian on a ball, saying that

"...the Harmonic operator remains elusively unspecified by any kind of boundary evaluations."

BUT: This operator is well-known as an extreme example of a realization of an elliptic operator. It was introduced abstractly by **von Neumann** in '29 and enters in the systematic treatment by **Krein** '47 as the "soft extension".
I had described it in terms of a nonlocal nonelliptic boundary condition in my '68 paper, and in '83 I had shown the asymptotic behavior of the eigenvalues outside of 0 (here 0 is in the essential spectrum).

I wrote a short note in Bulletin Amer. Math. Soc. 2006 commenting on their paper, explaining the knowledge that has been around for many years, referring to **Krein, Vishik, Hörmander, Lions** and **Magenes**, **Seeley** and others, and to my own early works.

E-M didn't take it well, but otherwise the response has been positive. I got in touch with two English mathematicians **Malcolm Brown** and **Ian Wood**, who published a paper (jointly with **Marletta** and **Naboko**) in '08 making a good use of the nonlocal boundary operators from my early papers. Their work was formulated within the so-called *boundary triples theory*, that I learned had been around since the mid-seventies, mainly aimed at ODE. It was developed in particular by a Russian school that had evolved in Ukraine among successors to **Krein**.

After this, **Brown, Wood** and I wrote a joint paper in '09 connecting the boundary triples theory with my old work from 1968-74.
A central aspect is that one has a *Krein resolvent formula*, concretizing the resolvent difference as a composition of three factors:

$$(A_B-\lambda)^{-1}-(A_\gamma-\lambda)^{-1}=K^{\lambda}_{\gamma,X}(L^{\lambda})^{-1}(K'^{\bar\lambda}_{\gamma,Y})^*;$$

here $L^\lambda\colon X\to Y^*$ is the operator corresponding to $A_B-\lambda$ in the characterization from 1968.

I included both the theory of ps.d.b.o.s and the characterization of extensions in a new book

"Distributions and Operators", Springer Verlag, 2009,

to make them easily accessible. The elementary part of the book gives an introduction to Distribution Theory and its application to operators describing concrete PDE situations, as taught through the years in Mat3, now placed in the 4'th study year. (Thanks to colleagues **J. Johnsen, M. Pedersen, J.P. Solovej.**)

Since the Bulletin paper, I have met a score of people active in the field. In particular the Ukrainians (such as **Malamud** and **Arlinskii**), at first bearing an element of suspicion, but we are on good terms now. And many more people in England, Germany, Holland, USA, all working on spectral theory for ODE or PDE.

There are still open problems to solve! In particular for non-smooth domains, and non-smooth coefficients. I have recently completed works on spectral asymptotics formulas for problems with jumps in the boundary condition, and for problems with a rough boundary (the latter building on a complicated analysis worked out together with **Abels** and **Wood**). In all these works, suitable versions of the Krein formula are an essential ingredient.

Retirement

When my 70'th birthday in 2009 approached, I was hit by the general expectation that one simply stops functioning as a thinking mathematician on that date, and I almost felt obliged to fulfill it in self-destruction.
But a new law lifted the pressure, and I was given a deal with less teaching and less salary, that allowed me to pursue the new issues that I had just found. This has been a lot of fun in the last $3\frac{1}{2}$ years. Now I replace the low salary by a better pension, and stop teaching altogether.

I hope to go on with the research work, at the pace that my health and other conditions will allow.

Thank you to all the people that have meant something to me, both in friendship and science! There are many more than those I have managed to include in this talk.

Thank you to all colleagues and friends here, and to my splendid family.

THANK YOU!