CHAPTER I

STANDARD PSEUDODIFFERENTIAL BOUNDARY PROBLEMS AND THEIR REALIZATIONS

1.1 Introductory remarks

The typical class of boundary problems that we shall be concerned with lies fairly close to boundary problems for differential operators, but it is large enough to include interesting nonlocal phenomena, as in the following example.

1.1.1 Example. Let Ω be a smooth open subset of $\mathbb{R}^n$ with boundary $\partial\Omega = \Gamma$ and consider the problem

$$\begin{aligned} -\Delta u + Gu &= f \text{ in } \Omega, \\ Tu &= \varphi \text{ at } \Gamma. \end{aligned} \tag{1.1.1}$$

Here Δ is the Laplace operator $\partial_{x_1}^2 + \cdots + \partial_{x_n}^2$, and for G we take an operator of the form

$$Gu = K_0\gamma_0 u + K_1\gamma_1 u + G'u, \tag{1.1.2}$$

where $\gamma_j u = (-i\partial_n)^j u|_\Gamma$ (∂_n being the interior normal derivative), K_0 and K_1 are (integral) operators going from Γ to Ω, and G' is an integral operator over Ω. The trace operator T can be taken either as a variant of the Dirichlet trace operator

$$T_0 u = \gamma_0 u + T_0' u, \tag{1.1.3}$$

where T_0' is an integral operator going from Ω to Γ, or of the Neumann trace operator

$$T_1 u = \gamma_1 u + S_0\gamma_0 u + T_1' u, \tag{1.1.4}$$

where S_0 acts on functions on Γ, and T_1' is an integral operator going from Ω to Γ. Boundary conditions like (1.1.3) and (1.1.4), containing interior contributions, were considered already in, for example, Phillips [Ph59], where they were called "lateral conditions".

The terms $K_0\gamma_0$ and $K_1\gamma_1$ in (1.1.2) are sometimes called "boundary feedback" terms (information from the boundary is fed back to the interior). Conditions like (1.1.3) and (1.1.4) can be seen as "boundary renewal" conditions, with a terminology inspired by population theory, where the condition

$$u(0) = \int_0^\infty u(t)f(t)dt,$$

expressing the number $u(0)$ of newborn individuals as a function of the age profile $u(t)$, is a special case of the homogeneous condition (in case $\Omega = \mathbb{R}_+$)

$$\gamma_0 u + T_0' u = 0, \quad u \in \mathcal{S}(\overline{\mathbb{R}}_+),$$

with $-T_0'u = \int_0^\infty u(t)f(t)dt$. It is also easy to think of physical situations with nonlocal phenomena of this kind. Consider for example the temperature distribution in a house, where the walls have temperatures imposed from the exterior (changing with the time of the day or year). Here one could set up further wall heating governed by a temperature measuring system distributed in the interior in such a way that the boundary temperature is modified proportionally to an integral (weighted average) over the interior. It should be noted that the theory allows integral operators that are rather "singular" near the boundary.

The solvability properties of the system (1.1.1) can be discussed within the framework of the Boutet de Monvel theory [BM66,69,71], when the operators belong to his class (wider classes have been considered by, e.g., Vishik and Eskin in [V-E66,67], [E81], by Rempel and Schulze in [R-S83,84], and in later works by Schulze and coauthors).

We shall generally replace $-\Delta$ in (1.1.1) by a pseudodifferential operator P or, more precisely, a certain restriction (truncation) to Ω, called P_Ω or P_+, of a ps.d.o. P defined in $\mathbb{R}^n$. Also, G and T will be taken to be more general but usually adapted to the order of P.

Let us consider the evolution equation ("heat equation")

$$(1.1.5)\quad \begin{aligned} &\text{(i)} & \partial_t u(x,t) + P_+ u(x,t) + Gu(x,t) &= f(x,t) \text{ for } x \in \Omega,\ t > 0,\\ &\text{(ii)} & Tu(x,t) &= \varphi(x,t) \text{ for } x \in \Gamma,\ t > 0,\\ &\text{(iii)} & u(x,0) &= u_0(x) \quad \text{for } x \in \Omega,\ t = 0. \end{aligned}$$

(For applications in control theory, Nambu [Nm79] and Triggiani [Trg79] consider various cases of this problem, developed further in Pedersen [Pe91]. The time-dependent Stokes problem can be written in this form; see [G-So87–91].) As is usual in this kind of problem, one may start out by considering cases where part of the data are zero, e.g., where f and φ are

0. The problem can then be written more abstractly

$$\begin{aligned} &\partial_t u(t) + Bu(t) = 0 \text{ for } t > 0, \\ &u(0) = u_0, \end{aligned} \tag{1.1.6}$$

where B is an operator acting like $P_+ + G$ and with a domain defined by the boundary condition $Tu = 0$ (B is a so-called realization of P). It is natural to search for the solution operator of (1.1.6) in the form

$$u(t) = \exp(-tB)u_0, \tag{1.1.7}$$

with a suitable definition of the operator function $\exp(-tB)$.

One of the purposes of the functional calculus that we shall set up here will be precisely to give a good sense to $\exp(-tB)$ under reasonable hypotheses on B. Of course, $\exp(-tB)$ has a meaning in the traditional theory of semigroups (Hille and Phillips [H-P57]) when B has certain elementary properties. What we want to do here is to go much further: to investigate the expression for $\exp(-tB)$ in terms of symbolic calculus and pseudodifferential (and Fourier integral) operator techniques, in such a way that we can get detailed information both on the solutions in terms of their data and on the kernel of the solution operator and its trace.

The basic tool here will be an analysis of the resolvent $R_\lambda = (B - \lambda I)^{-1}$, and this analysis will permit the investigation also of other functions of B, defined by a Cauchy integral formula

$$f(B) = \tfrac{i}{2\pi} \int_C f(\lambda) R_\lambda d\lambda \tag{1.1.8}$$

where the integration is along a curve C in the complex plane going around the spectrum of B.

The road to the precise analysis of R_λ is quite long. Before going deeply into the λ-dependent calculus we shall use, we give an account of the local *parameter-independent* calculus, with a special aim towards those operators that do admit a sensible resolvent. The present chapter is concerned with that. First we recall the essential ingredients in the Boutet de Monvel calculus (Section 1.2), explained with relatively few technicalities (the complete story is told in connection with parameter-dependent symbols later anyway). The standard terminology and definitions of function spaces, manifolds, etc., are collected in the Appendix.

In the rest of this chapter, the focus is on pseudodifferential operators P admitting solvable *realizations* B in L_2, i.e., for which P can be supplied with a singular Green operator G and a system T of trace operators so that the operator B in L_2, acting like $P_+ + G$ and defined for functions u

with $Tu = 0$, is close to being a bijection of $D(B)$ onto L_2. This holds (for compact manifolds) if the system

$$A = \begin{pmatrix} P_+ + G \\ T \end{pmatrix}$$

is elliptic in Boutet de Monvel's sense. Here the systems where T is *normal* (Section 1.4) are of special interest, on the one hand because the corresponding class of realizations is closed under composition and passage to adjoints (in the sense of *unbounded* operators, not just for the bounded case considered in [Sl79], [R-S82]), and on the other hand because the normality is necessary for the property of "ellipticity with a parameter" that we use in the present work. The concepts of parameter-ellipticity and parabolicity are discussed in Section 1.5. In Section 1.6 we analyze the normal realizations more deeply, determining the adjoint, giving criteria for selfadjointness, and presenting some basic examples.

An interesting subset of the systems satisfying the requirement of parameter-ellipticity consists of those that define *positive* realizations (or m-coercive realizations, satisfying the "Gårding inequality"). Because of their importance for the applications of the theory, we study these systems at some length in Section 1.7, showing how certain ideas from Grubb [G68–74] generalize to the present case. The discussion is carried far enough to explain how the Dirichlet-like and Neumann-like boundary conditions (1.1.3) and (1.1.4) fit into the context; in fact, we often return to Example 1.1.1 to illustrate the general principles.

Occasionally, the later theory returns to parameter-independent cases that do not need the strong assumptions presented in Section 1.5, e.g., the index formula in Section 4.3 and some results on spectral estimates in Section 4.5.

Let us finally mention that certain versions (or variants) of the integro-differential problem (1.1.1) have been studied by Bony, Courrège and Priouret [B-C-P68] and Cancelier [Ca85]; they consider second-order cases where they can obtain a maximum principle (Cancelier also treats cases without ellipticity), and they discuss the associated semigroup in spaces of continuous functions. Widom [Wi85] presents a functional calculus for ps.d.o.s of negative order, restricted to a bounded domain; here a certain extendability condition replaces in some sense the need for boundary conditions.

1.1.2 Remark. One of the immediate questions one could ask concerning the boundary problems considered in Example 1.1.1 is whether the nonlocalness could be eliminated by a change of the independent variable. Indeed, it is possible to remove the nonlocal terms from the boundary conditions (1.1.3), resp. (1.1.4), by insertion of $u = \Lambda^{-1}v$ for a suitable bijective operator Λ; cf. Lemma 1.6.8 below. This leads to a change of the first line of (1.1.1) to another equation

$$-\Delta v + G_1 v = f$$

with another G_1 (nonzero in general). Although such a change of variables can be useful for some questions (e.g., the discussion of positivity at the end of Section 1.7), in general it does not simplify our problem, since the nonlocal terms that can occur in the first and second line of (1.1.1) have a closely related nature — one might even say that the G terms are more complicated than the T terms. And of course, in questions of operational calculus, the replacement of u by v would lead to the construction of functions of another operator than the given one.

1.2 The calculus of pseudodifferential boundary problems

In this section, we briefly recall the essential ingredients in the Boutet de Monvel calculus of Green operators [BM66–71]. Accurate details are given in, e.g., [G84] and [R-S82], and we shall later in this book give a precise account of the parameter-dependent case, where the original operator classes are included as those operators that are independent of the parameter μ. So for the moment, an outline will suffice. (The reader who wants a self-contained presentation of the theory can actually find this in Chapters 2–3, by reading the parameter μ as a constant and neglecting the complications caused by the presence of μ.) Notation is explained in the Appendix.

The basic notions are defined relative to $\mathbb{R}^n$ and the subset $\overline{\mathbb{R}}^n_+ \subset \mathbb{R}^n$; then they are carried over to manifold situations by use of local coordinate systems.

Pseudodifferential operators (ps.d.o.s) were invented as a class of operators that was rich enough to encompass both differential operators and those (singular) integral operators that appear as inverse operators (parametrices) for elliptic differential operators. One can write a differential operator on $\mathbb{R}^n$ of order m

$$A(x, D_x) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$$

by the help of the Fourier transformation as follows:

$$A(x, D_x)u(x) = \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\, đ\xi \quad (đ\xi = (2\pi)^{-n} d\xi),$$

where $a(x,\xi)$ is the function

$$a(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x)\xi^\alpha,$$

called the symbol. The definition of pseudodifferential operators simply allows a more general function (symbol) to be put in the place of $a(x,\xi)$ — for example, the inverse of the operator $1-\Delta$ on $\mathbb{R}^n$ is expressed by

$$(1-\Delta)^{-1}u = \int e^{ix\cdot\xi}\frac{1}{1+|\xi|^2}\hat{u}(\xi)\,đ\xi$$

(where $u \in \mathcal{S}(\mathbb{R}^n)$, or the formula is generalized to larger spaces). Various classes of ps.d.o.s are obtained, according to which conditions are imposed on the function entering as $a(x,\xi)$. We shall work here only with (parameter-dependent generalizations of) standard ps.d.o. symbols because the boundary conditions create sufficient complications, but we note in passing that the same questions are of great interest also for more general symbol classes (e.g., $S_{\varrho,\delta}$ and Weyl generalizations).

The pseudodifferential operator P with symbol $p(x,\xi)$ is the operator defined by

$$(Pu)(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}p(x,\xi)u(y)\,dyđ\xi; \tag{1.2.1}$$

one also denotes $P = \mathrm{OP}(p(x,\xi))$. The formula is valid for $u \in \mathcal{S}(\mathbb{R}^n)$ and extends by continuity (or by consideration of the adjoint) to more general u, when $p(x,\xi)$ satisfies suitable hypotheses. We shall be concerned mainly with *the case where* $p \in S^d_{1,0}$, for some $d \in \mathbb{R}$, i.e., p is a C^∞-function satisfying

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le c(x)\langle\xi\rangle^{d-|\alpha|} \text{ for all indices } \alpha \text{ and } \beta \tag{1.2.2}$$

(with a continuous function $c(x)$ depending on the indices and $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$), and the *polyhomogeneous case*, i.e., the case where p is in $S^d_{1,0}$ and moreover has an asymptotic expansion

$$p(x,\xi) \sim \sum_{l\in\mathbb{N}} p_{d-l}(x,\xi), \tag{1.2.3}$$

where the p_{d-l} are C^∞-functions that are homogeneous of degree $d-l$ in ξ for $|\xi| \ge 1$, and $p - \sum_{l<M} p_{d-l} \in S^{d-M}_{1,0}$ for each $M \in \mathbb{N}$. The class of polyhomogeneous symbols is called S^d. P and p are said to be of order d, and p_d (also denoted p^0) is called the principal symbol (of order d). The function $p(x,\xi)$ is called the symbol of P, sometimes denoted $\sigma(P)$. The *strictly homogeneous* function coinciding with p^0 for $|\xi| \ge 1$ will be called $p^h(x,\xi)$.

When $c(x)$ in (1.2.2) can be taken independent of x, we speak of *uniformly estimated* symbols, and we use the notation $\dot{\le}$ (see (2.1.1)ff. or

Section A.1). The class of symbols satisfying (1.2.2) is usually denoted $S^d_{1,0}(\mathbb{R}^n, \mathbb{R}^n)$, whereas the subset of uniformly estimated symbols is denoted $S^d_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ (as in Hörmander [H85, Sect. 18.1]). In this introductory chapter we mainly refer to the locally estimated symbol spaces, since the original Boutet de Monvel theory used these, which suffice for the study of operators on compact manifolds. From Chapter 2 on, we consider uniformly estimated symbol spaces that allow the definition of operators on suitable noncompact manifolds, as defined in Section A.5.

The function p in formula (1.2.1) can be allowed to depend on (x, y) instead of just x, in which case the operator is denoted $P = \mathrm{OP}(p(x, y, \xi))$. $p(x, y, \xi)$ is again called a symbol of P (with a loose terminology, since $\sigma(P)(x, \xi)$ is essentially determined from P, whereas $p(x, y, \xi)$ is not).

The $S^d_{1,0}$ (resp. polyhomogeneous) classes of symbols $p(x, y, \xi)$ are defined as above, with x replaced by (x, y).

The operator $P = \mathrm{OP}(p(x, y, \xi))$ has the distribution kernel

$$K_p(x, y) = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} p(x, y, \xi)\, đ\xi = \mathcal{F}^{-1}_{\xi \to z} p(x, y, \xi)\big|_{z=x-y}. \tag{1.2.4}$$

The middle expression here is understood in the sense of oscillatory integrals, as introduced by Hörmander in [H71]. The kernel $K_p(x, y)$ is a C^∞ function of (x, y) outside the diagonal $\{x = y\}$, for when $2M > n + d + 2N$ we have that $|z|^{2M}(1 - \Delta_z)^N \mathcal{F}^{-1}_{\xi \to z} p = \mathcal{F}^{-1}_{\xi \to z}(-\Delta_\xi)^M \langle \xi \rangle^{2N} p$, the inverse Fourier transform of an L^1 function in ξ, so that $(1 - \Delta_z)^N \mathcal{F}^{-1}_{\xi \to z} p$ is continuous in z for $z \neq 0$. This calculation also shows that $\mathcal{F}^{-1}_{\xi \to z} p$ is $O(|z|^{-2M})$ for $M > n + d$, so $\mathcal{F}^{-1}_{\xi \to z} p$ is rapidly decreasing for $|z| \to \infty$. However, the behavior of $\mathcal{F}^{-1}_{\xi \to z} p$ at $z = 0$ depends on the order. If $d < -n$, $p(x, \xi)$ is integrable in ξ so the kernel is continuous at 0; if $d < -n - M$, the derivatives up to order M are continuous. But for higher d there is usually a singularity at $z = 0$.

Similarly, if we just take the inverse Fourier transform in one of the variables, ξ_n, we get a function

$$\tilde{p}_{\alpha,\beta}(x', x_n, \xi', z_n) = \mathcal{F}^{-1}_{\xi_n \to z_n} D^\beta_x D^\alpha_\xi p(x', x_n, \xi', \xi_n) \tag{1.2.5}$$

which is C^∞ in z_n for $z_n \in \mathbb{R}\setminus\{0\}$ and even rapidly decreasing for $|z_n| \to \infty$, but which usually has an important singularity at $z_n = 0$.

Symbols $p(x, y, \xi)$ in the space $S^{-\infty}_{1,0} = \bigcap_{d \in \mathbb{R}} S^d_{1,0}$ define ps.d.o.s that are in fact integral operators with C^∞ kernel; such operators are called *negligible* (the symbols are also sometimes called negligible). They are smoothing operators, i.e., they send distribution spaces into C^∞ spaces, and so do their adjoints.

About the calculus of ps.d.o.s, which is amply treated elsewhere in articles and textbooks (see also our Chapter 2), let us for the moment mention just the following facts.

1.2.1 Proposition. 1° *When $p \in S^d_{1,0}$, the associated operator is continuous*

$$\mathrm{OP}(p)\colon H^s_{\mathrm{comp}}(\mathbb{R}^n) \to H^{s-d}_{\mathrm{loc}}(\mathbb{R}^n),$$

for any $s \in \mathbb{R}$.

2° *When P is a ps.d.o., the adjoint $P^*\colon H^{d-s}_{\mathrm{comp}}(\mathbb{R}^n) \to H^{-s}_{\mathrm{loc}}(\mathbb{R}^n)$ is likewise a ps.d.o., and if $P = \mathrm{OP}(p(x,y,\xi))$ then $P^* = \mathrm{OP}(\overline{p}(y,x,\xi))$. When P is polyhomogeneous and has the principal symbol $p^0(x,\xi)$, P^* is also polyhomogeneous and has the principal symbol $\overline{p^0(x,\xi)}$.*

3° *Any ps.d.o. P can be written as a sum $P = P_1 + \mathcal{R}$, where $\mathcal{R}$ is negligible and P_1 is* proper, *i.e., P_1 and its adjoint P_1^* map compactly supported functions into compactly supported functions.*

4° *When P and P' are ps.d.o.s of order d, resp. d', and one of them is proper, then $P'' = PP'$ is a ps.d.o. of order $d + d'$. If P and P' are polyhomogeneous, so is P'', and the principal symbols satisfy*

$$p''^0(x,\xi) = p^0(x,\xi)p'^0(x,\xi).$$

In the classical theory of elliptic differential operators A, the theory becomes particularly interesting when the operators are considered on domains with boundary, so that *boundary conditions* (representing various physical situations) have to be adjoined to get wellposed problems. The calculus of Boutet de Monvel [BM66–71] is a solution to the problem of establishing a class of operators encompassing the elliptic boundary value problems as well as their solution operators; moreover, his class of operators is closed under composition (it is an "algebra").

In the case of a differential operator A, the analysis of the various boundary conditions is usually based on the polynomial structure of the symbol of A; in particular the roots of the polynomial $a^0(x', 0, \xi', \xi_n)$ in ξ_n (in the situation where the domain is $\mathbb{R}^n_+$) play a role. When pseudodifferential operators P are considered, the principal symbol p^0 is generally not a polynomial. It may be a rational function (this happens naturally when one makes reductions in a system of differential operators), in which case one can consider the roots and poles with respect to ξ_n. But then, even when P is elliptic, there is much less control over how these behave than when a^0 is a polynomial; roots and poles may cancel each other or reappear, as the coordinate ξ' varies. For a workable theory, a more universal point of view is needed.

Vishik and Eskin (see [V-E67] and [E81]) based a theory on factorization of symbols. This works well in the scalar case but can be problematic in the case of matrix-formed operators (since the factorization here is generally only piecewise continuous in ξ'). They mainly consider ps.d.o.s of a general kind, with fewer conditions on the ξ-dependence than (1.2.2) states.

Boutet de Monvel worked out a calculus (that we shall describe below) for a special class of ps.d.o.s; one of the advantages of that theory is that it replaces the factorization by a *projection* procedure that works equally well for scalar and matrix formed operators (depends smoothly on ξ'); it is linked in a natural way with the projections of $L_2(\mathbb{R})$ onto $L_2(\mathbb{R}_+)$ and $L_2(\mathbb{R}_-)$ (in the x_n-variable) obtained by restriction to $\mathbb{R}_\pm$ and extension by 0 on $\mathbb{R}_\mp$. The description that now follows is given in relation to the latter projections, and the Fourier-transformed version (of interest in connection with symbols) will be taken up in full detail in Chapter 2.

When P is a ps.d.o. on $\mathbb{R}^n$, its truncation (or "restriction") to the subset $\Omega = \mathbb{R}^n_+$ is defined by

$$P_+u = r^+Pe^+u, \text{ also denoted } P_{\mathbb{R}^n_+}u \text{ or } P_\Omega u, \tag{1.2.6}$$

where r^+ and e^+ are the restriction and extension-by-zero operators; cf. (A.30–31). (The notation with the subset instead of $+$ is used when one wants to mention it explicitly.) When P is proper and of order d, the operator P_+ is continuous from $L_{2,\mathrm{comp}}(\mathbb{R}^n_+)$ to $H^{-d}_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+)$ but in general does not map $H^m_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+)$ into $H^{m-d}_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+)$ for $m > 0$; the discontinuity of e^+u at $x_n = 0$ causes a singularity. Boutet de Monvel singled out a class of ps.d.o.s where the mapping properties of P_+ are nice, namely the ps.d.o.s having the transmission property.

P is said to have *the transmission property* with respect to $\mathbb{R}^n_+$ when $P_{\mathbb{R}^n_+}$ "preserves C^∞ up to the boundary", i.e., $P_{\mathbb{R}^n_+}$ maps $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ into $C^\infty(\overline{\mathbb{R}}^n_+)$. Boutet de Monvel showed in [BM66] a necessary and sufficient condition for a polyhomogeneous symbol $p(x,\xi) \sim \sum_{l\in\mathbb{N}} p_{d-l}(x,\xi)$ to define an operator $\mathrm{OP}(p)$ with the transmission property w.r.t. $\mathbb{R}^n_+$. It states that the homogeneous terms p_{d-l} have the symmetry property

$$D^\beta_x D^\alpha_\xi p_{d-l}(x',0,0,-\xi_n) = e^{i\pi(d-l-|\alpha|)} D^\beta_x D^\alpha_\xi p_{d-l}(x',0,0,\xi_n), \tag{1.2.7}$$

for $|\xi_n| \geq 1$ and all indices α, β, l.

When d is integer and the equations (1.2.7) hold, then they also hold for the symbol $p(x,-\xi)$; this implies that also $P_{\mathbb{R}^n_-}$ preserves C^∞ up to the boundary in $\overline{\mathbb{R}}^n_-$. So P then in fact has a *two-sided* transmission property; it holds both with respect to $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$.

When d is not integer, (1.2.7) implies a "wrong" kind of symmetry for $p(x,-\xi)$, and P will not in general preserve C^∞ in $\overline{\mathbb{R}}^n_-$, but maps the functions in $C^\infty_{(0)}(\overline{\mathbb{R}}^n_-)$ into functions with a specific singularity at $x_n = 0$.

There is a complete discussion of necessary and sufficient conditions for the transmission property, extending the analysis to $S^d_{\varrho,\delta}$ symbols with

$\varrho > \delta$, in Grubb and Hörmander [G-H90]. Earlier studies were made by Vishik and Eskin (see [V-E67], [E81]) and in [H85, Sect. 18.2].

It is shown in [BM71] that the properties (1.2.7) in the case $d \in \mathbb{Z}$ may be rewritten as an expansion property of $p(x,\xi)$ and its derivatives at $x_n = 0$:

$$D_x^\beta D_\xi^\alpha p(x',0,\xi) \sim \sum_{-\infty<l\le d-|\alpha|} s_{l,\alpha,\beta}(x',\xi')\xi_n^l, \tag{1.2.8}$$

where the $s_{l,\alpha,\beta}(x',\xi')$ are *polynomials* in ξ' of degree $d-|\alpha|-l$, for all α and $\beta \in \mathbb{N}^n$. (We prove this in Theorem 2.2.5 below.) We usually denote $s_{l,0,0} = s_l$. (1.2.8) has the advantage that it makes sense for $S^d_{1,0}$ symbols also, and it is called "the transmission property" in [BM71].

Formula (1.2.8) means that $p(x',0,\xi)$ (and similarly the derivatives $D_x^\beta D_\xi^\alpha p(x',0,\xi)$) has an expansion in integer powers of ξ_n such that, for any $m \in \mathbb{N}$,

$$\xi_n^m p(\xi_n) - \sum_{-m<l\le d} s_l \xi_n^{l+m}$$

has the same limit (namely s_{-m}) for $\xi_n \to +\infty$ as for $\xi_n \to -\infty$. The C^∞ functions of ξ_n with this property form the important space $\mathcal{H}$ that we study in Section 2.2 below, as a basis for other interesting expansion properties.

In the present chapter we do not want to go into details with the complex analysis involved in studying $\mathcal{H}$; instead we mention a more "real" point of view. As noted around (1.2.5), the partial inverse Fourier transform $\tilde{p}(x',0,\xi',z_n) = \mathcal{F}^{-1}_{\xi_n\to z_n} p(x',0,\xi)$ generally has a singularity at $z_n = 0$. Now the transmission property holds with respect to $\mathbb{R}^n_+$ *if and only if*

$$\tilde{p}_{\alpha,\beta}(x',0,\xi',z_n) \text{ for } (x',z_n) \in \mathbb{R}_+ \quad \text{extends to a } C^\infty \text{ function of } (x',z_n) \in \overline{\mathbb{R}}^n_+, \tag{1.2.9}$$

for all α and β. This is shown for $S^d_{\varrho,\delta}$ symbols with $\varrho > \delta$ in [G-H90]; for polyhomogeneous symbols it follows from [BM66]. In the case $d \in \mathbb{Z}$ for polyhomogeneous symbols, the symmetry of (1.2.8) in $\pm\xi_n$ assures that p *moreover* has the two-sided property

$$\tilde{p}_{\alpha,\beta}(x',0,\xi',z_n) \text{ for } (x',z_n) \in \mathbb{R}_\pm \quad \text{extends to } C^\infty \text{ functions of } (x',z_n) \in \overline{\mathbb{R}}^n_\pm. \tag{1.2.10}$$

The limits for $z_n \to 0+$ and $z_n \to 0-$ are in general different, and the condition does not exclude singularities supported in $\{z_n = 0\}$. (For example, a differential operator $P = \sum a_\alpha(x) D^\alpha$, which clearly has the two-sided transmission property, has $\tilde{p} = \sum a_\alpha(x', 0)\xi'^{\alpha'}\delta(z_n)^{\alpha_n}$ entirely supported in $\{z_n = 0\}$.)

In the following, we consider (like [BM71]) polyhomogeneous as well as $S^d_{1,0}$ symbols $p(x, \xi)$ of integer order d. For the polyhomogeneous symbols, we have seen that the transmission property is equivalent with any of the conditions (1.2.7), (1.2.8), (1.2.9) and (1.2.10). For the $S^d_{1,0}$ symbols *we prescribe* any of the equivalent conditions (1.2.8) and (1.2.10), i.e., a generalization of the two-sided transmission property. It assures that the adjoint operator is also in the calculus, and it is basic for the setup of [BM71]. But for $S^d_{1,0}$ symbols, it is not *necessary* for the one-sided transmission property, so we shall henceforth carefully speak of this as a transmission *condition*. We shall say that $p \in S^d_{1,0}(\mathbb{R}^n, \mathbb{R}^n)$ ($d \in \mathbb{Z}$) *satisfies the transmission condition* at $x_n = 0$ when (1.2.8) (or equivalently (1.2.10)) holds.

All transmission conditions used in this book are two-sided, so the words "two-sided" will usually be left out. For an example of an ps.d.o. in OP $S^d_{1,0}$ with $d \in \mathbb{Z}$ and only the one-sided transmission property, one can take the sum of an operator satisfying (1.2.8) and a polyhomogeneous operator of noninteger order d', for which the transmission property w.r.t. $\mathbb{R}^n_+$ prevents the transmission property w.r.t. $\mathbb{R}^n_-$.

An introductory explanation is given in [G91]. A different kind of characterization of (1.2.8) (for uniformly estimated symbols) is given in Schrohe [Sr94], in terms of commutators with x_j and D_{x_j}.

When (1.2.8) holds, the operator $P_+ = r^+ \operatorname{OP}(p)e^+$ can be shown to be continuous:

$$\begin{aligned} &P_+ \colon H^s_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+) \to H^{s-d}_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+) \text{ for } s > -\tfrac{1}{2}, \\ &P_+ \colon H^{(s,t)}_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+) \to H^{(s-d,t)}_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+) \text{ for } s > -\tfrac{1}{2},\ t \in \mathbb{R}. \end{aligned} \tag{1.2.11}$$

Details are given in Section 2.5. (P_+ also maps $\overset{\circ}{\mathcal{E}}{}'(\overline{\mathbb{R}}^n_+)$ into $\overset{\circ}{\mathcal{D}}{}'(\overline{\mathbb{R}}^n_+)$ if $d \le 0$.) This suffices for the consideration of nonparametrized operators on compact manifolds, but later in this book we shall need operators where (1.2.11) is replaced by global versions (valid for $H^s(\overline{\mathbb{R}}^n_+)$) and parameter-dependent versions (valid for $H^{s,\mu}(\overline{\mathbb{R}}^n_+)$ with control over the μ-dependence of norms). Such properties are assured by suitable variants of (1.2.8). Also here we are careful to speak of *transmission conditions*, since these are just *sufficient* conditions for the properties we want.

Note that we have as a consequence of (1.2.8)

$$p(x', 0, \xi) = \sum_{0 \le l \le d} s_l(x', \xi')\xi_n^l + p'(x', \xi), \tag{1.2.12}$$

where p' is $O(\langle \xi_n \rangle^{-1})$, s_l is polynomial in ξ' of order $d - l$, and the first coefficient s_d is a *function of* x',

$$s_d(x', \xi') = s_d(x'). \tag{1.2.13}$$

(One has in fact that $p'(x',\xi)$ is $O(\langle\xi'\rangle^{d+1}\langle\xi\rangle^{-1})$.) Note that *if p is constant in x_n*, P is the sum of a differential operator and an operator that preserves L_2 with respect to x_n. Generally, p can be quite nasty for $x_n \neq 0$, but here a Taylor expansion in x_n gives a number of good terms behaving as in (1.2.12) and a term with a factor x_n^M that makes it harmless (there is a precise statement in Lemma 1.3.1 further below).

We denote by $p(x,\xi',D_n)$ or $\mathrm{OP}_n(p(x,\xi))$ the operator on $\mathbb{R}$ where (1.2.1) is applied with respect to (x_n,ξ_n) only, and by $\mathrm{OP}'(p(x,\xi))$ the operator on $\mathbb{R}^{n-1}$ where (1.2.1) is applied with respect to (x',ξ').

On the basis of the transmission condition, one can build up a theory of boundary value problems for P with many of the same features as the theory for differential operators (and some differences); much more will be said about this in the rest of the present chapter.

Let us now introduce the other ingredients in the Boutet de Monvel calculus. (The precise symbol definitions are listed in Definition 2.3.13.)

Let P be $N'\times N$-matrix formed. Along with P_+, which operates on $\mathbb{R}^n_+$, we shall consider operators going to and from the boundary, forming together with P a system

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ \\ T & S \end{pmatrix} : \begin{matrix} C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)^N \\ \times \\ C^\infty_0(\mathbb{R}^{n-1})^M \end{matrix} \quad \rightarrow \quad \begin{matrix} C^\infty(\overline{\mathbb{R}}^n_+)^{N'} \\ \times \\ C^\infty(\mathbb{R}^{n-1})^{M'} \end{matrix} . \tag{1.2.14}$$

Here T is a so-called *trace operator*, going from $\mathbb{R}^n_+$ to $\mathbb{R}^{n-1}$; K is a so-called *Poisson operator*, going from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n_+$; S is a *pseudodifferential operator on* $\mathbb{R}^{n-1}$; and G is an operator on $\mathbb{R}^n_+$ called a *singular Green operator*, a non-pseudodifferential term that has to be included in order to have good composition rules. The full system $\mathcal{A}$ is called a *Green operator*. We shall usually take $N = N'$, whereas the dimensions M and M' can have all values, including zero. When P is a differential operator, it is classical to study systems of the form

$$\mathcal{A} = \begin{pmatrix} P_+ \\ T \end{pmatrix}; \tag{1.2.15}$$

here $M = 0$ and $M' > 0$. The terms in (1.2.14) will now be explained.

The *trace operators* we consider are defined in such a way that

$$T = \gamma_0 P_+ \tag{1.2.16}$$

is a trace operator whenever P is a ps.d.o. satisfying the transmission condition. Here, it is found for the right-hand side that when $P = \mathrm{OP}(p(x,\xi))$

then $\gamma_0 P_+ u = \gamma_0 r^+ \operatorname{OP}(p(x',0,\xi))e^+ u$ where we can insert (1.2.12). This gives a sum of differential trace operators plus a term where the symbol is $O(\langle \xi_n \rangle^{-1})$. (Symbol calculus for (1.2.16) is given in Example 2.3.1 and Proposition 2.4.1.) The general definition goes as follows:

A trace operator of order d $(\in \mathbb{R})$ *and class* r $(\in \mathbb{N})$ is an operator of the form

$$Tu = \sum_{0 \le j \le r-1} S_j \gamma_j + T', \tag{1.2.17}$$

where γ_j denotes the standard trace operator $(\gamma_j u)(x') = D^j_{x_n} u(x',0)$; the S_j are ps.d.o.s in $\mathbb{R}^{n-1}$ of order $d-j$, and T' is an operator of the form

$$(T'u)(x') = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{t}'(x',y_n,\xi')\acute{u}(\xi',y_n)dy_n \, đ\xi', \tag{1.2.18}$$

with $\tilde{t}' \in \mathcal{S}(\overline{\mathbb{R}}_+)$ as a function of x_n, satisfying estimates for all indices:

$$\| x_n^l D^{l'}_{x_n} D^\beta_{x'} D^\alpha_{\xi'} \tilde{t}'(x',x_n,\xi') \|_{L_{2,x_n}(\mathbb{R}_+)} \le c(x') \langle \xi' \rangle^{d+\frac{1}{2}-l+l'-|\alpha|}; \tag{1.2.19}$$

here we denote $\mathcal{F}_{x'\to\xi'} u = \acute{u}$; cf. (A.18).

This is the definition for the $S_{1,0}$ type of operator, and one can show (by use of the Seeley extension [Se64]) that for any such T' there exists a ps.d.o. Q of $S_{1,0}$ type, satisfying the transmission condition at $x_n = 0$, such that

$$T' = \gamma_0 Q_+. \tag{1.2.20}$$

The subclass of *polyhomogeneous* trace operators are those where the S_j are polyhomogeneous, and $\tilde{t}'$ has an asymptotic expansion

$$\tilde{t}'(x',x_n,\xi') \sim \sum_{l \in \mathbb{N}} \tilde{t}'_{d-l}(x',x_n,\xi'), \tag{1.2.21}$$

where each $\tilde{t}'_{d-l}$ is C^∞ and quasi-homogeneous in the sense that

$$\tilde{t}'_{d-l}(x', \tfrac{1}{\lambda}x_n, \lambda\xi') = \lambda^{d-l+1}\, \tilde{t}'_{d-l}(x',x_n,\xi') \text{ for } \lambda \ge 1 \text{ and } |\xi'| \ge 1, \tag{1.2.22}$$

$\tilde{t}' - \sum_{l<M} \tilde{t}'_{d-l}$ satisfying the estimates (1.2.19) with d replaced by $d-M$, for any $M \in \mathbb{N}$. The function (distribution when $r > 0$)

$$\tilde{t}(x',x_n,\xi') = \sum_{0 \le j < r} s_j(x',\xi') D^j_{x_n} \delta(x_n) + \tilde{t}'(x',x_n,\xi') \tag{1.2.23}$$

(understood as extended by 0 to $\mathbb{R}_-$ if relevant) is called the *symbol-kernel* of T; its Fourier transform

$$t(x',\xi) = \mathcal{F}_{x_n\to\xi_n} e^+\tilde{t}(x,\xi') \tag{1.2.23$'$}$$

is the *symbol* of T. (The symbol spaces and symbol-kernel spaces are taken up again in Chapter 2; see, e.g., Definition 2.3.13. There we also allow negative class $r<0$.) We also set $t'(x',\xi) = \mathcal{F}_{x_n\to\xi_n} e^+\tilde{t}'(x,\xi')$. In the polyhomogeneous case, t has an expansion corresponding to (1.2.21)

$$\begin{aligned} t(x',\xi) &\sim \sum_{l\in\mathbb{N}} t_{d-l}(x',\xi), \quad \text{with} \\ t_{d-l}(x',\lambda\xi) &= \lambda^{d-l}\, t_{d-l}(x',\xi) \quad \text{for } \lambda\ge 1,\ |\xi'|\ge 1. \end{aligned} \tag{1.2.24}$$

We often denote t_d and $\tilde{t}_d$ by t^0, resp. $\tilde{t}^0$, the *principal* symbol and symbol-kernel. Application of the operator definition with respect to only the x_n variable gives the *boundary symbol operator* $t(x',\xi',D_n)$ (resp. *principal* boundary symbol operator $t^0(x',\xi,D_n)$) from $\mathcal{S}(\overline{\mathbb{R}}_+)$ to $\mathbb{C}$,

$$t(x',\xi',D_n)u = \sum_{0\le j<r} s_j(x',\xi')\gamma_j u + \int_0^\infty \tilde{t}'(x',y_n,\xi')u(y_n)dy_n; \tag{1.2.25}$$

it is also denoted $\mathrm{OPT}_n(t)$. We can then write

$$Tu = \mathrm{OP}'(t(x',\xi',D_n))u, \quad \text{also denoted } \mathrm{OPT}(t(x',\xi))u, \tag{1.2.26}$$

where OP' as usual stands for application of the ps.d.o. definition with respect to the (x',ξ')-variables. Here $T'=\mathrm{OPT}(t')$ is *of class* 0. The trace operator (1.2.16) has precisely the symbol-kernel $\tilde{p}(x',0,\xi',x_n)\big|_{x_n>0}$ (cf. (1.2.5)).

A trace operator T of order d and class r defines continuous mappings (cf. Section 2.5)

$$\begin{aligned} &T\colon H^s_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+) \to H^{s-d-\frac12}_{\mathrm{loc}}(\mathbb{R}^{n-1}) \quad \text{for } s>r-\tfrac12, \\ &T\colon H^{(s,t)}_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+) \to H^{s+t-d-\frac12}_{\mathrm{loc}}(\mathbb{R}^{n-1}) \text{ for } s>r-\tfrac12,\ \text{all } t\in\mathbb{R}; \end{aligned} \tag{1.2.27}$$

it maps $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ into $C^\infty(\mathbb{R}^{n-1})$, and if $r=0$, $\overset{\circ}{\mathcal{E}}{}'(\overline{\mathbb{R}}^n_+)$ into $\mathcal{D}'(\mathbb{R}^{n-1})$.

Just as the general definition of trace operators was motivated by (1.2.16), one can introduce the class of *Poisson operators* with the motivation that K defined by

$$(Kv)(x) = r^+P(v(x')\otimes\delta(x_n)) \tag{1.2.28}$$

should be in this class whenever P is a ps.d.o. satisfying the transmission condition. (Symbol calculus for (1.2.28) is given in Example 2.3.1 and Proposition 2.4.1.) In general, a *Poisson operator of order* d is defined by the formula

$$(Kv)(x',x_n)=\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}\tilde{k}(x',x_n,\xi')\hat{v}(\xi')\,d\!\!^-\xi' \tag{1.2.29}$$

where the *symbol-kernel* $\tilde{k}$ satisfies just the same estimates as $\tilde{t}'$ in (1.2.19), except that d is replaced by $d-1$. This gives the $S_{1,0}$ type of operators; the *polyhomogeneous* symbol-kernels $\tilde{k}$ moreover have asymptotic expansions in quasi-homogeneous terms, and the corresponding symbols

$$k(x',\xi)=\mathcal{F}_{x_n\to\xi_n}e^+\tilde{k}(x,\xi') \tag{1.2.30}$$

have expansions in homogeneous terms in ξ (for $|\xi'|\geq 1$) of degree $d-1-l$. We often denote $\tilde{k}_{d-1}=\tilde{k}^0$ and $k_{d-1}=k^0$, the *principal* symbol-kernel or symbol. Again, one can view K defined in (1.2.29) (also denoted $\mathrm{OPK}(k)$ or $\mathrm{OPK}(\tilde{k})$) as an operator

$$K=\mathrm{OP}'(k(x',\xi',D_n)) \tag{1.2.31}$$

where $k(x',\xi',D_n)$ is the *boundary symbol operator* from $\mathbb{C}$ to $\mathcal{S}(\overline{\mathbb{R}}_+)$

$$k(x',\xi',D_n)a=\tilde{k}(x',x_n,\xi')\cdot a \text{ for } a\in\mathbb{C}, \tag{1.2.32}$$

also denoted $\mathrm{OPK}_n(k)$.

One can show (by use of [Se64]) that any $S_{1,0}$-type Poisson operator can be written in the form (1.2.28) with a ps.d.o. P of $S_{1,0}$-type, satisfying the transmission condition.

The Poisson operators of order d are continuous (cf. Section 2.5)

$$\begin{aligned} K&\colon H^s_{\mathrm{comp}}(\mathbb{R}^{n-1})\to H^{s-d+\frac12}_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+)\\ K&\colon H^s_{\mathrm{comp}}(\mathbb{R}^{n-1})\to H^{(m,s-d-m+\frac12)}_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+)\end{aligned} \tag{1.2.33}$$

for any $s\in\mathbb{R}$, any $m\geq 0$, besides mapping $C_0^\infty(\mathbb{R}^{n-1})$ (resp. $\mathcal{E}'(\mathbb{R}^{n-1})$) into $C^\infty(\overline{\mathbb{R}}^n_+)$ (resp. $\mathcal{D}'(\overline{\mathbb{R}}^n_+)$).

The order convention may seem a bit strange (polyhomogeneous Poisson operators of order d have principal symbols homogeneous of degree $d-1$), but this fits the purpose that the composition of two operators of order d, resp. d', will be of order $d+d'$ (valid, e.g., for the ps.d.o. TK on $\mathbb{R}^{n-1}$).

Now there is a very important observation concerning *adjoints*: The trace operators T' of class 0 (and order d) have as adjoints precisely the Poisson operators (of order $d+1$), and vice versa:

$$\text{(1.2.34)} \qquad T' \colon L_{2,\mathrm{comp}}(\mathbb{R}^n_+) \to H^{-d-\frac12}_{\mathrm{loc}}(\mathbb{R}^{n-1}) \text{ and}$$
$$K = T'^* \colon H^{d+\frac12}_{\mathrm{comp}}(\mathbb{R}^{n-1}) \to L_{2,\mathrm{loc}}(\mathbb{R}^n_+) \text{ are adjoints of each other.}$$

This is quite obvious on the boundary-symbol-operator level, and it follows in general after application of Proposition 1.2.1 2° to OP'. In fact, when the OP' notion is extended to symbols depending on (x', y') instead of x', then if T' has symbol-kernel $\tilde{t}'(x, \xi')$, T'^* has symbol-kernel $\overline{\tilde{t}'}(y', x_n, \xi')$ (the complex conjugate). Trace operators of class $r > 0$ do not have adjoints within the present calculus.

We now get to the more peculiar element G of $\mathcal{A}$ in (1.2.14). A singular Green operator (s.g.o.) G arises, for instance, when we compose a Poisson operator K with a trace operator T as $G = KT$; this operator acts in $\Omega = \mathbb{R}^n_+$ but is not a P_+. Another situation where s.g.o.s enter is when we compose two ps.d.o.s P_+ and Q_+ (satisfying the transmission condition); then the "leftover" operator

$$\text{(1.2.35)} \qquad L(P,Q) \equiv (PQ)_+ - P_+Q_+ = r^+PQe^+ - r^+Pe^+r^+Qe^+$$
$$= r^+P(I - e^+r^+)Qe^+$$

is an operator, acting in Ω, that is not a ps.d.o. It turns out that these cases are covered by operators of the following form (on the boundary symbol level it is a completed tensor product of Poisson and trace symbols).

A singular Green operator G of order d $(\in \mathbb{R})$ and class r $(\in \mathbb{N})$ is an operator

$$\text{(1.2.36)} \qquad G = \sum_{0 \le j \le r-1} K_j \gamma_j + G',$$

where the K_j are Poisson operators of order $d-j$, the γ_j are standard trace operators, and G' is an operator of the form

$$\text{(1.2.37)} \qquad (G'u)(x) = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{g}'(x', x_n, y_n, \xi')\acute{u}(\xi', y_n)\,dy_n\,đ\xi'.$$

Here $\tilde{g}'$, the *symbol-kernel* of G', is in $\mathcal{S}(\overline{\mathbb{R}}^2_{++}) = r^+_{x_n} r^+_{y_n} \mathcal{S}(\mathbb{R}^2)$ as a function of (x_n, y_n) satisfying estimates for all indices

$$\text{(1.2.38)} \qquad \|x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} D_{x'}^\beta D_{\xi'}^\alpha \tilde{g}'(x', x_n, y_n, \xi)\|_{L_{2,x_n,y_n}(\mathbb{R}^2_{++})}$$
$$\le c(x')\langle\xi'\rangle^{d-k+k'-m+m'-|\alpha|}.$$

This is the definition for operators of type $S_{1,0}$; for the subclass of *polyhomogeneous* s.g.o.s, it is furthermore required that the K_j are polyhomogeneous, and the symbol-kernel $\tilde{g}'$ of G' has an asymptotic expansion in quasi-homogeneous terms

$$\tilde{g}'(x', x_n, y_n, \xi') \sim \sum_{l\in\mathbb{N}} \tilde{g}'_{d-1-l}(x', x_n, y_n, \xi') \tag{1.2.39}$$

where

$$\tilde{g}'_{d-1-l}(x', \tfrac{1}{\lambda}x_n, \tfrac{1}{\lambda}y_n, \lambda\xi') = \lambda^{d+1-l}\, \tilde{g}'_{d-1-l}(x', x_n, y_n, \xi') \quad \text{for } \lambda \geq 1 \text{ and } |\xi'| \geq 1, \tag{1.2.40}$$

corresponding to an expansion of the corresponding *symbol* g'

$$g'(x', \xi', \xi_n, \eta_n) = \mathcal{F}_{x_n\to\xi_n}\overline{\mathcal{F}}_{y_n\to\xi_n} e^+_{x_n} e^+_{y_n} \tilde{g}'(x', x_n, y_n, \xi') \tag{1.2.41}$$

in homogeneous terms

$$g'_{d-1-l}(x', \lambda\xi', \lambda\xi_n, \lambda\eta_n) = \lambda^{d-1-l}\, g'_{d-1-l}(x, \xi', \xi_n, \eta_n) \quad \text{for } \lambda \geq 1 \text{ and } |\xi'| \geq 1. \tag{1.2.42}$$

The expansion holds in the sense that $\tilde{g}' - \sum_{l<M} \tilde{g}'_{d-1-l}$ satisfies (1.2.38) with d replaced by $d - M$, for any $M \in \mathbb{N}$. The symbol-kernel and symbol of G itself are

$$\begin{aligned} \tilde{g}(x', x_n, y_n, \xi') &= \sum_{0\leq j<r} \tilde{k}_j(x', x_n, \xi') D^j_{y_n}\delta(y_n) + \tilde{g}'(x', x_n, y_n, \xi') \\ g(x', \xi', \xi_n, \eta_n) &= \sum_{0\leq j<r} k_j(x', \xi)\eta_n^j + g'(x', \xi', \xi_n, \eta_n). \end{aligned} \tag{1.2.43}$$

The *principal* symbol-kernel and symbol are $\tilde{g}^0 = \tilde{g}_{d-1}$ resp. $g^0 = g_{d-1}$. Singular Green symbols are taken up again in Section 2.3, where also negative class ($r < 0$) is allowed.

We define the *boundary symbol operator* $g(x', \xi', D_n)$ from $\tilde{g}$ by

$$g(x', \xi', D_n)u(x_n) = \sum_{0\leq j<r} \tilde{k}_j(x', x_n, \xi')\gamma_j u + \int_0^\infty \tilde{g}'(x', x_n, y_n, \xi')u(y_n)dy_n, \tag{1.2.44}$$

also called $\mathrm{OPG}_n(g)$; then G (also called $\mathrm{OPG}(g)$ or $\mathrm{OPG}(\tilde{g})$) can be viewed as

$$G = \mathrm{OP}'(g(x', \xi', D_n)). \tag{1.2.45}$$

An s.g.o. G of order d and class r defines continuous mappings (cf. Section 2.5)

$$(1.2.46)\qquad \begin{aligned} &G\colon H^s_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+) \to H^{s-d}_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+) && \text{for } s > r - \tfrac12, \\ &G\colon H^{(s,t)}_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+) \to H^{(m,s-d-m+t)}_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+) && \text{for } s > r - \tfrac12,\ t \in \mathbb{R}, \end{aligned}$$

and maps $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ into $C^\infty(\overline{\mathbb{R}}^n_+)$ and, if $r = 0$, $\mathring{\mathcal{E}}'(\overline{\mathbb{R}}^n_+)$ into $\mathring{\mathcal{D}}'(\overline{\mathbb{R}}^n_+)$.

It is important to observe that s.g.o.s of class 0 have *adjoints* of the same kind: Allowing symbols depending on (x', y') we have that

$$(1.2.47)\qquad \begin{aligned} G &= \mathrm{OPG}(\tilde g(x', x_n, y_n, \xi')) && \text{implies} \\ G^* &= \mathrm{OPG}(\overline{\tilde g}(y', y_n, x_n, \xi')) && \text{(complex conjugate),} \end{aligned}$$

when G is of class 0. S.g.o.s of class $r > 0$ do not have adjoints within the present calculus.

By an expansion of $\tilde g'$ in a double Laguerre series with respect to x_n and y_n (as explained, e.g., in [G84]; see also Section 2.2), we can write G in the form

$$(1.2.48)\qquad G = \sum_{m=1}^{\infty} K_m T_m,$$

where the K_m and T_m are sequences of Poisson operators of order 0, resp. trace operators of order d and class r, such that $K_m T_m$ is rapidly decreasing (with respect to all symbol norms, as in (1.2.38)). This shows that any singular Green operator can be written as the product of a Poisson operator (the row $\{K_m\}_{m<M}$) and a trace operator (the column $\{T_m\}_{m<M}$) plus a small s.g.o., a point of view that is fruitful in, e.g., the study of inverses of boundary problems. Moreover, the K_m and T_m can be written in a form generated from ps.d.o.s as in (1.2.28), resp. (1.2.17, 20).

The special s.g.o. $L(P, Q)$ in (1.2.35) is generated from ps.d.o.s in a somewhat different way also, by a point of view introduced in [G84]. In fact, $L(P, Q)$ can be broken up in simpler terms

$$(1.2.49)\qquad L(P, Q) = \sum_{0 \le j < d'} K_j \gamma_j + G^+(P) G^-(Q),$$

where the K_j are Poisson operators derived from P and the "differential operator part" of Q (cf. (1.2.12), and $G^+(P)$ and $G^-(Q)$ are special s.g.o.s of class 0, defined by the formulas

$$(1.2.50)\qquad \begin{aligned} G^+(P) &= r^+ P e^- J \\ G^-(Q) &= J r^- Q e^+ = G^+(Q^*)^*, \end{aligned}$$

where J is the reflection operator

$$J\colon u(x', x_n) \mapsto u(x', -x_n). \tag{1.2.51}$$

To the above operators defined by Fourier integral formulas, one adds the negligible operators of each type, defined as operators of the form (1.2.17), (1.2.34), (1.2.36) with S_j, T', K, K_j and G' replaced by integral operators with C^∞ kernels (up to the boundary) over the respective domains. They are of class r when they contain trace operators γ_j for $j \le r-1$.

The various operator classes defined above are *invariant* under coordinate changes in $\overline{\mathbb{R}}^n_+$ preserving the boundary $\{x_n = 0\}$; this holds both for the polyhomogeneous classes and the $S_{1,0}$ classes. This is stated in [BM71], with an indication of how to conclude the invariance for $S_{1,0}$ Poisson operators once it is shown for $S_{1,0}$ ps.d.o.s satisfying the transmission condition. [R-S82] proves the invariance under coordinate changes in x' alone, where the rules for ps.d.o.s in x' apply. A complete proof, with formulas for the symbols of the transformed operators, is given below in Section 2.4 (see also Theorem 2.2.13), including parameter-dependence.

Having defined the ingredients in $\mathcal{A}$ in (1.2.14) we shall now look at composition rules. When $\mathcal{A}'$ is another such system, going from $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)^{N'} \times C^\infty_0(\mathbb{R}^{n-1})^{M'}$ to $C^\infty(\overline{\mathbb{R}}^n_+)^{N''} \times C^\infty(\mathbb{R}^{n-1})^{M''}$, and one of the operators is proper, the composition equals

$$\mathcal{A}'' = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} \begin{pmatrix} P'_+ + G' & K' \\ T' & S' \end{pmatrix} = \tag{1.2.52}$$
$$\begin{pmatrix} (PP')_+ - L + P_+G' + GP'_+ + GG' + KT' & P_+K' + GK' + KS' \\ TP'_+ + TG' + ST' & TK' + SS' \end{pmatrix}.$$

One can now show that $\mathcal{A}''$ again has the structure of a Green operator, which really amounts to showing 14 different composition rules (cf. [BM71], [R-S82] and [G84]); these will be taken up in detail in Sections 2.6 and 2.7.

It is also of interest to see whether $\mathcal{A}$ has an adjoint within the Green operator calculus. In view of the preceding information, this occurs when G and T are of class 0 and P is of order ≤ 0 (it can also hold in certain positive order cases; cf. Lemma 1.3.1 later). There is a technical device worth mentioning here, which can transform the system $\mathcal{A}$ into one that does have an adjoint, namely, that there exists a family of ps.d.o.s Λ^m_-, $m \in \mathbb{Z}$, such that $\Lambda^m_{-,\mathbb{R}^n_+}$ maps $H^s(\overline{\mathbb{R}}^n_+)$ homeomorphically onto $H^{s-m}(\overline{\mathbb{R}}^n_+)$ for all s. They are called order-reducing operators. (Such operators having the property for $s > -\frac12$ are constructed in [BM71], [R-S82], [G86]; the full scale of s is obtained in [Fra86], [G90]; details allowing also uniformly estimated operators on noncompact manifolds are given below in Section

3.2; see Remark 3.2.15.) If P is of order $d > 0$ and T and G are of class $r > 0$, we can compose $\mathcal{A}$ to the right with

$$\begin{pmatrix} \Lambda^{-m}_{-,\mathbb{R}^n_+} & 0 \\ 0 & I \end{pmatrix}, \tag{1.2.52$'$}$$

where $m = \max\{r, d\}$, which reduces $\mathcal{A}$ to a system of class 0 with ps.d.o. part of order ≤ 0. This is useful, e.g., in the study of index problems.

For the study of invertible elements in the "algebra", we need to define the concept of *ellipticity*. This really consists of two conditions. One is that P is elliptic on $\overline{\mathbb{R}}^n_+$, i.e., the principal symbol $p^0(x,\xi)$ is an *invertible* function (or square matrix) when $|\xi| \ge c > 0$. The other condition is that the (x_n-independent) *principal boundary symbol operator* for $\mathcal{A}$,
(1.2.53)

$$a^0(x',\xi',D_n) = \begin{pmatrix} p^0(x',0,\xi',D_n)_+ + g^0(x',\xi',D_n) & k^0(x',\xi',D_n) \\ t^0(x',\xi',D_n) & s^0(x',\xi') \end{pmatrix},$$

defines a *bijection* from $\mathcal{S}(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^M$ to $\mathcal{S}(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^{M'}$, for all x', all $|\xi'| \ge c > 0$. It is an important point in the theory, that this hypothesis suffices to assure that the inverse operator $(a^0(x',\xi',D_n))^{-1}$ is again the principal boundary symbol operator $c^0(x',\xi',D_n)$ for a Green system $\mathcal{C}^0$ belonging to the theory, which is not at all obvious from the mere bijectiveness. When the ellipticity holds, it is possible to construct a Green operator $\mathcal{C}$ (with the same principal boundary symbol operator as $\mathcal{C}^0$) which is a *parametrix* of $\mathcal{A}$, in the sense that

$$\mathcal{A}\,\mathcal{C} - I \text{ and } \mathcal{C}\,\mathcal{A} - I \text{ are negligible.} \tag{1.2.54}$$

(Note here that since $\mathcal{A}$ and $\mathcal{C}$ are not necessarily of class 0, the negligible operators in (1.2.54) need not be of class 0.)

The above description is for polyhomogeneous symbols. Occasionally, it is convenient to extend the definition slightly. For example, in the boundaryless case, if $P = I - P_1$ where P_1 has norm $\le \delta < 1$ and symbol $|p_1(x,\xi)| \le \delta$, one can construct a parametrix symbol $q(x,\xi) \sim I + \sum_{k=1}^\infty p_1(x,\xi)^{\circ k}$ and an inverse $Q = I + \sum_{k=1}^\infty P_1^k$ such that $Q - \mathrm{OP}(q)$ is negligible; hence Q belongs to the calculus. We may call such operators P and Q elliptic too. Similar considerations can be made for Green systems.

For a complete description, we also have to define the operators as acting in bundles over manifolds. The details can be left out here since we do it carefully for parameter-dependent operators later on (in Section 2.4). There we allow noncompact manifolds, but at present we just *consider the compact case.* $\overline{\Omega}$ is an n-dimensional compact C^∞ manifold with boundary

$\partial\Omega = \Gamma$, smoothly imbedded in a neighboring n-dimensional manifold Σ, and $\widetilde{E}$ and $E = \widetilde{E}|_{\overline{\Omega}}$, resp. F and F', are vector bundles of dimension N, resp. M and M', over Σ and $\overline{\Omega}$, resp. Γ (described by local coordinates and trivializations in Section A.5). P is a ps.d.o. in $\widetilde{E}$ over Σ satisfying the transmission condition at Γ and the Green operators considered in connection with P are of the form

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ & \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(E) \\ \times \\ C^\infty(F) \end{matrix} \quad \rightarrow \quad \begin{matrix} C^\infty(E) \\ \times \\ C^\infty(F') \end{matrix} ; \tag{1.2.55}$$

here P_+ is defined by (1.2.6), (A.66). The terms T, K and S (and sometimes even $P_+ + G$) are often given as block matrices with different orders for different entries (fitting together as in Douglis-Nirenberg elliptic systems). The continuity statements (1.2.11), (1.2.27), (1.2.33) and (1.2.46) carry over to E and the respective bundles over Γ, when we replace $H^s_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+)$ and $H^s_{\mathrm{comp}}(\overline{\mathbb{R}}^n_+)$ by $H^s(E)$, replace the boundary spaces similarly, and replace the $H^{(s,t)}$ spaces by $H^{(s,t)}(E_{\Sigma'_+})$ (where Σ'_+ is a neighborhood of Γ of the form $\Gamma \times [0, 1[\,$), inserting cutoff functions supported near Γ in the formulas containing the $H^{(s,t)}$-spaces. (A range bundle $E' \neq E$ can be allowed.)

When P is elliptic, it can be shown (cf. Lemma 3.1.1) that $p^0(x', \xi', D_n)_+$ is a Fredholm operator in $\mathcal{S}(\overline{\mathbb{R}}_+)^N$ (and between suitable Sobolev spaces over $\overline{\mathbb{R}}_+$, the nullspace and range complement being the same as for $\mathcal{S}(\overline{\mathbb{R}}_+)^N$), with an index depending continuously on (x', ξ'). A necessary condition for the ellipticity of the full system $\mathcal{A}$ is that $M' - M = \operatorname{index} p^0$. When ellipticity holds, $\mathcal{A}$ itself is a Fredholm operator, both as an operator between the C^∞ spaces in (1.2.55) and as an operator between suitable Sobolev spaces; the nullspace and range complement are the same C^∞ spaces as in the situation (1.2.55). More details are given for a particular case in Section 1.4, and for the parameter-dependent case in Chapter 3. The index of $\mathcal{A}$ is studied in [BM71] and in [R-S82]. In the present work, our analysis of the trace of the heat operator will lead to a new formula for the index of normal boundary problems (see Section 4.3), which involves slightly fewer symbol data than the general formula of S. Rempel (see [R-S82] or [Re80]).

1.2.2 Remark. Throughout this work we use a convention for polyhomogeneous symbols $s \sim \sum_{l \geq 0} s_{d-l}$, where the individual terms s_{d-l} are only assumed to be homogeneous for $|\xi| \geq 1$ (in the ps.d.o. case), resp. for $|\xi'| \geq 1$, and belong to the respective symbol classes themselves. Occasionally we also need to refer to the associated *strictly homogeneous* symbols, s^h_{d-l}, defined by extension by homogeneity; these are generally irregular at $\xi = 0$ resp. $\xi' = 0$, whereas the smooth terms s_{d-l} have the advantage of being directly used in the operator definitions.

This convention is perhaps not the most usual for ps.d.o.s, but it is very useful in the study of boundary problems (as introduced in [BM71]), where ξ' and ξ_n (resp. ξ', ξ_n and η_n) enter together in an intricate way. This is even moreso when we include the parameter μ later in this book, considering symbols that are homogeneous in (ξ', ξ_n, μ) (or in $(\xi', \xi_n, \eta_n, \mu)$) for $|\xi'| \geq 1$ but with a complicated behavior near $\xi' = 0$ that is better described by estimates.

In the following the operators are always taken to belong to the calculus, meaning in particular that the ps.d.o. on the n-dimensional manifold is of integer order and satisfies the transmission condition at the boundary (this is only occasionally stated explicitly).

1.3 Green's formula

In the present calculus, the ps.d.o.s of positive order satisfy a Green's formula quite similar to the one in the differential operator case. We first observe

Lemma 1.3.1. *Let P be a ps.d.o. of order d in $\widetilde{E}$, satisfying the transmission condition at Γ. Then P can be written*

$$P = A + P', \tag{1.3.1}$$

where A is a differential operator of order d of the form

$$A = \sum_{l=0}^{d} S_l(x, D') D_n^l \tag{1.3.2}$$

with tangential differential operators S_l of order $d - l$ supported near Γ and D_n denoting a normal derivative (cf. (A.64)ff.*), and where P' is a ps.d.o. of order d satisfying*

$$((P')_+ u, v)_\Omega = (u, (P'^*)_+ v)_\Omega \text{ for } u, v \in C^\infty_{(0)}(E). \tag{1.3.3}$$

(1.3.3) *holds for P itself, when $d \leq 0$. At Γ, the coefficient of D_n^d equals $s_d(x')$ in* (1.2.12–13) *(as determined in the chosen coordinates).*

Proof. Consider the situation in local coordinates, where Ω, Σ and Γ are replaced by $\mathbb{R}^n_+, \mathbb{R}^n$ and $\mathbb{R}^{n-1}$ and the vector bundles are trivial. Let P be a properly supported operator with symbol $p(x, \xi)$ (a negligible correction will always satisfy (1.3.3)). The statements are obvious for $d < 0$, so we assume $d \geq 0$; (1.3.3) clearly also holds for P when $d = 0$. We expand $p(x, \xi)$ in two ways: the Taylor expansion at $x_n = 0$

$$p(x, \xi) = \sum_{0 \leq j < d} \tfrac{1}{j!} x_n^j \partial_{x_n}^j p(x', 0, \xi) + x_n^d r_d(x, \xi), \tag{1.3.4}$$

and the expansion of each $\partial_{x_n}^j p$ due to the transmission condition

$$\partial_{x_n}^j p(x',0,\xi) = \sum_{0\le l\le d} s_{l,j}(x',\xi')\xi_n^l + p_j(x',\xi) \tag{1.3.5}$$

(with polynomials $s_{l,j}$ in ξ' of degree $d-l$). Then we can write, taking $\eta(x_n)\in C_0^\infty(\mathbb{R})$, equal to 1 near 0:

$$\begin{aligned} p(x,\xi) &= \sum_{0\le l\le d}\sum_{0\le j<d} \tfrac{1}{j!}\eta(x_n)x_n^j s_{l,j}(x',\xi')\xi_n^l + p^{(1)} + p^{(2)} + p^{(3)},\\ p^{(1)}(x,\xi) &= \sum_{0\le j<d} \tfrac{1}{j!}\eta(x_n)x_n^j p_j(x',\xi),\\ p^{(2)}(x,\xi) &= \eta(x_n)x_n^d r_d(x,\xi),\\ p^{(3)}(x,\xi) &= (1-\eta(x_n))p(x,\xi). \end{aligned} \tag{1.3.6}$$

Here the sum over l is the symbol of the differential operator A. For the other terms we can in each case justify the calculation for $u,v\in C_{(0)}^\infty(\overline{\mathbb{R}}_+^n)$:

$$\begin{aligned} (\mathrm{OP}(p^{(i)})_+u,v)_{\mathbb{R}_+^n} &= (\mathrm{OP}(p^{(i)})e^+u,e^+v)_{\mathbb{R}^n}\\ &= (e^+u,\mathrm{OP}(p^{(i)})^*e^+v)_{\mathbb{R}^n} = (u,\mathrm{OP}(p^{(i)})_+^*v)_{\mathbb{R}_+^n}. \end{aligned}$$

For $p^{(1)}$ this holds since the symbol satisfies $D_x^\beta D_\xi^\alpha p^{(1)} = O(\langle\xi'\rangle^{d-|\alpha|}\langle\xi\rangle^{-1})$, which implies that $\mathrm{OP}(p^{(1)})e^+$ as well as $\mathrm{OP}(p^{(1)})^*e^+$ map $C_{(0)}^\infty(\overline{\mathbb{R}}_+^n)$ into $L_2(\mathbb{R}^n)$. For $p^{(2)}$ it holds because of the factor x_n^d; likewise for $p^{(3)}$ since it can be written with a factor x_n^d. Here we use that when $Q=\mathrm{OP}(q(x,\xi))$ is of order d (with compact x-support), then $Qx_n^de^+$ maps $C_{(0)}^\infty(\overline{\mathbb{R}}_+^n)$ into $L_2(\mathbb{R}^n)$ since $x_n^de^+u\in H^d(\mathbb{R}^n)$ when $u\in C_{(0)}^\infty(\overline{\mathbb{R}}_+^n)$, and we can use Proposition 1.2.1 1° for $s=d$; $x_n^dQe^+$ does so, since it equals $\sum_{k=0}^d\binom{d}{k}\mathrm{OP}(\overline{D}_{\xi_n}^k q(x,\xi))x_n^{d-k}e^+$, where the preceding argument can be applied to each term.

The decomposition carries over to the manifold situation, by a partition of unity and local coordinates. □

The operator A indicated in the proof is not the only possible choice; see Remark 1.3.3.

1.3.2 Proposition. *Let P be as in Lemma* 1.3.1. *The following Green's formula holds for u and $v\in C_{(0)}^\infty(E)$*

$$(P_+u,v)_\Omega - (u,P_+^*v)_\Omega = (\mathfrak{A}\rho u,\rho v)_\Gamma, \tag{1.3.7}$$

where $\mathfrak{A}$ *is a (uniquely determined) matrix (the Green's matrix)*

$$\mathfrak{A} = (\mathfrak{A}_{jk})_{j,k=0,\dots,d-1}$$

of differential operators $\mathfrak{A}_{jk}$ *in* $E\big|_\Gamma$ *of orders* $d-j-k-1$, *with*

$$\mathfrak{A}_{jk}(x', D') = iS_{j+k+1}(x', 0, D') + \text{lower-order terms} \tag{1.3.8}$$

(zero if $j+k+1 > d$*), and* ρ *denotes the Cauchy boundary operator*

$$\rho u = \{\gamma_0 u, \dots, \gamma_{d-1} u\}, \quad \textit{where } \gamma_j u = (D_n^j u)|_\Gamma. \tag{1.3.9}$$

Proof. In view of Lemma 1.3.1, one can write

$$(P_+ u, v)_\Omega - (u, P_+^* v)_\Omega = (Au, v)_\Omega - (u, A^* v)_\Omega, \tag{1.3.10}$$

with a differential operator A, (1.3.2). This satisfies the usual Green's formula, $(Au, v)_\Omega - (u, A^* v)_\Omega = (\mathfrak{A}\rho u, \rho v)_\Gamma$, which follows from (1.3.2) and the formula

$$(D_n u, v)_\Omega - (u, D_n v)_\Omega = i(\gamma_0 u, \gamma_0 v)_\Gamma;$$

cf. (A.71). (Further details are in, e.g., [Se66], [G73].) $\mathfrak{A}$ is uniquely determined since ρ is *surjective* from $C^\infty_{(0)}(E)$ to $\prod_{j=0}^{d-1} C_0^\infty(E_\Gamma)$. $\square$

The trace operator ρ and the formulas extend to Sobolev spaces as described, e.g., in Lions and Magenes [L-M68]. Here ρ is continuous

$$\rho\colon H^d(E) \to \prod_{j=0}^{d-1} H^{d-j-\frac12}(E_\Gamma),$$

and (1.3.7) is valid for u and $v \in H^d(E)$.

We observe that $\mathfrak{A}$ has a skew-triangular character

$$\mathfrak{A} = \mathfrak{A}^0 + \mathfrak{A}' = i \begin{pmatrix} S_1^0 & \cdots & S_{d-1}^0 & S_d^0 \\ S_2^0 & \cdots & S_d^0 & 0 \\ \vdots & & \vdots & \vdots \\ S_d^0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} \text{lower} & & \dots & 0 \\ \text{order} & \dots & 0 & \\ \vdots & & & \vdots \\ 0 & & \dots & 0 \end{pmatrix}. \tag{1.3.11}$$

It is sometimes advantageous to write

$$\mathfrak{A} = I^\times \mathfrak{A}^\times \text{ and } \mathfrak{A}^0 = I^\times \mathfrak{A}^{0\times} \tag{1.3.12}$$

where $I^\times$ is the "skew-unit matrix"

$$I^\times = (\delta_{j,d-j-k-1})_{0 \le j,k < d}, \tag{1.3.13}$$

and $\mathfrak{A}^\times$ and $\mathfrak{A}^{0\times}$ are ordinary triangular matrices, e.g.

$$(1.3.14) \qquad \mathfrak{A}^{0\times} = i \begin{pmatrix} S_d^0 & 0 & \cdots & 0 \\ S_{d-1}^0 & S_d^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S_1^0 & S_2^0 & \cdots & S_d^0 \end{pmatrix}.$$

The terms in the diagonal of $\mathfrak{A}^\times$ and $\mathfrak{A}^{0\times}$, resp. second diagonal of $\mathfrak{A}$ and $\mathfrak{A}^0$, equal $iS_d(x', 0, D') = is_d(x')$, where $s_d(x')$ is the coefficient of ξ_n^d in the expansion (1.2.12) of $p(x, \xi)$ (and also of $p^0(x, \xi)$ in the polyhomogeneous case); it is a zero-order differential operator and hence a multiplication operator (a morphism in E_Γ). The operator $\mathfrak{A}$ (and also $\mathfrak{A}^0$) is then *invertible* if and only if $s_d(x')$ is a *bijective* morphism for each $x' \in \Gamma$. In the polyhomogeneous case that holds exactly when Γ is *noncharacteristic* for P, for the latter condition means that $p^0(x', 0, 0, \xi_n)$ is invertible for $\xi_n \neq 0$, and by (1.2.12) and the homogeneity of p^0 for $|\xi| \geq 1$,

$$(1.3.15) \qquad p^0(x', 0, 0, \xi_n) = s_d(x')\xi_n^d \text{ for } |\xi_n| \geq 1.$$

We can (and shall) take invertibility of $s_d(x')$ as the *definition* of noncharacteristicness of Γ in the case of symbols in $S_{1,0}^d$.

1.3.3 Remark. When P is decomposed as in Lemma 1.3.1 in two different ways,

$$P = A + P' = A_1 + P_1',$$

the difference $A' = A - A_1$ must be such that

$$(A'u, v)_\Omega - (u, A'^* v)_\Omega = 0 \quad \text{ for all } u, v \in C_{(0)}^\infty(E),$$

i.e., the corresponding matrix $\mathfrak{A}_{A'}$ in Green's formula must be zero. So A is uniquely determined at Γ up to a summand with Green's matrix zero. Such summands are the differential operators that are of the form

$$A' = \sum_{0 \leq l \leq d} x_n^l A'_{d-l}(x, D') D_n^l,$$

the A'_{d-l} being of order $d - l$ with smooth coefficients. (Melrose [M81] calls these differential operators totally characteristic.)

The following observation for the noncharacteristic case will be useful later on.

1.3.4 Lemma. *Let P be a ps.d.o. of order $d>0$ satisfying the transmission condition at Γ. When Γ is noncharacteristic for P, i.e., $s_d(x')$ is invertible, the boundary symbol operator $p(x',0,\xi',D_n)_+$ (as well as the principal boundary symbol operator $p^0(x',0,\xi',D_n)_+$ in the polyhomogeneous case) has the (hypoellipticity) property at each (x',ξ') with $|\xi'|\geq 1$:*

$$\text{When } u\in L_2(\mathbb{R}_+)^N \text{ and } p_+u\in L_2(\mathbb{R}_+)^N, \text{ then } u\in H^d(\overline{\mathbb{R}}_+)^N.$$

Proof. Let u and $p_+u\in L_2(\mathbb{R}_+)^N$. In view of the proof of Lemma 1.3.1 and (1.3.15) (for constant coefficient operators) we can write

$$p_+u=s_d(x')D_n^du+\textstyle\sum_{0\leq j<d}s_j(x',\xi')D_n^ju+p'_+u,$$

where $p'_+u\in L_2(\mathbb{R}_+)^N$. It follows that

$$a(D_n)u\equiv D_n^du+\textstyle\sum_{0\leq j<d}s_d^{-1}s_jD_n^ju+cu\in L_2(\mathbb{R}_+)^N$$

for any constant c; we can choose c so large that

$$|a(\xi_n)|=|\xi_n^d+\textstyle\sum_{0\leq j<d}s_d^{-1}s_j\xi_n^j+c|\geq c_1\langle\xi_n\rangle^d,$$

for some $c_1>0$. The problem is then reduced to the wellknown differential operator case; for completeness we give a full proof. It is seen by successive integration that u is in H^d on bounded intervals $I=\,]0,t[\,$; here $D_n(D_n^{d-1}u+\sum_{1\leq j<d}s_d^{-1}s_j^0D_n^{j-1}u)\in L_2(I)^N$ implies $D_n^{d-1}u+\Sigma s_d^{-1}s_j^0D_n^{j-1}u\in H^1(I)^N$, etc. Thus for $\chi\in C_{(0)}^\infty(\overline{\mathbb{R}}_+)$ with $\chi=1$ near 0, $\chi u\in H^d(\overline{\mathbb{R}}_+)^N$, so also $a(D_n)[(1-\chi)u]=a(D_n)u-a(D_n)\chi u\in L_2(\mathbb{R}_+)^N$. Extending $(1-\chi)u$ by zero on $\mathbb{R}_-$, we have that $\|\langle\xi_n\rangle^d\mathcal{F}[(1-\chi)u]\|_{L_2}\leq c_1^{-1}\|a(\xi_n)\mathcal{F}[(1-\chi)u]\|_{L_2}<\infty$, so all together, $u\in H^d(\overline{\mathbb{R}}_+)^N$.

A similar proof works for p^0. $\square$

1.3.5 Example. As an application of the calculus described in Sections 1.2 and 1.3, let us briefly explain the classical *Calderón projectors* (as introduced in [C63], [Se66], [H66]).

Let A be an elliptic differential operator of order d in $\widetilde{E}$ over the compact manifold Σ, and assume for simplicity that $A\colon C^\infty(\widetilde{E})\to C^\infty(\widetilde{E})$ is invertible with inverse Q; then also $A\colon H^s(\widetilde{E})\to H^{s-d}(\widetilde{E})$ is invertible for $s\in\mathbb{R}$. (Otherwise, the construction works modulo smoothing operators.) Q is a polyhomogeneous ps.d.o. of order $-d$, and since A is local,

$$AQ_+=I \text{ on } E. \tag{1.3.16}$$

Q satisfies the transmission condition since its symbol is a sequence of rational functions whose denominators are powers of $\det a^0(x,\xi)$. Denote

$\Omega = \Omega_+$ and $\Sigma \setminus \overline{\Omega} = \Omega_-$; they have the common boundary Γ. We shall use the notation $r^\pm$, $e^\pm$, $\gamma_j^\pm$ as in (A.30,31,51) with $\mathbb{R}^n_\pm$ replaced by $\Omega_\pm$, and let $\widetilde{\gamma}_j$ denote the "two-sided" trace operator from Σ to Γ; all are of the form $u(x', x_n) \mapsto D_n^j u(x', 0)$ for the *same* operator D_n defined near Γ (cf. also (A.64)ff.). We also set $\varrho^\pm = \{\gamma_0^\pm, \dots, \gamma_{d-1}^\pm\}$ and $\widetilde{\varrho} = \{\widetilde{\gamma}_0, \dots, \widetilde{\gamma}_{d-1}\}$. We denote $r^\pm \widetilde{E} = E_\pm$ (so $E = E_+$). Proposition 1.3.2 shows the Green's formula

$$(Au, v)_{\Omega_+} - (u, A^* v)_{\Omega_+} = (\mathfrak{A}\rho^+ u, \rho^+ v)_\Gamma, \tag{1.3.17}$$

and since $(Au, v)_\Sigma - (u, A^* v)_\Sigma = 0$, we also have

$$(Au, v)_{\Omega_-} - (u, A^* v)_{\Omega_-} = -(\mathfrak{A}\rho^- u, \rho^- v)_\Gamma. \tag{1.3.18}$$

The adjoint of the two-sided trace operator

$$\widetilde{\varrho}\colon H^s(\widetilde{E}) \to \textstyle\prod_{0 \le j < d} H^{s-j-\frac12}(E_\Gamma), \quad s > d - \tfrac12, \tag{1.3.19}$$

is the operator

$$\widetilde{\varrho}^* = (I, D_n^*, \dots, D_n^{d-1^*})\tilde{\gamma}_0^*\colon \textstyle\prod_{0 \le j < d} H^{-s+j+\frac12}(E_\Gamma) \to H^{-s}(\widetilde{E}), \quad s > d - \tfrac12; \tag{1.3.20}$$

here $\tilde{\gamma}_0^* \varphi = \varphi(x') \otimes \delta(x_n)$ when Ω is replaced by $\mathbb{R}^n_+$, and it is a suitable generalization otherwise (it is a distribution supported in Γ). Now define

$$K = Q\widetilde{\varrho}^*\mathfrak{A}, \quad K^\pm = \mp r^\pm K = \mp r^\pm Q \widetilde{\varrho}^* \mathfrak{A}. \tag{1.3.21}$$

The mapping properties of Q, $\mathfrak{A}$ and $\widetilde{\varrho}^*$ (cf. (1.3.20)) assure that

$$\begin{aligned} K&\colon \textstyle\prod_{0 \le j < d} H^{s-j-\frac12}(E_\Gamma) \to H^s(\widetilde{E}), \quad \text{when } s < \tfrac12; \\ K^\pm&\colon \textstyle\prod_{0 \le j < d} H^{s-j-\frac12}(E_\Gamma) \to H^s(E_\pm), \quad \text{when } s < \tfrac12. \end{aligned} \tag{1.3.22}$$

Observe, moreover, that $K^\pm$ in fact map into

$$Z^s_\pm = \{\, u \in H^s(E_\pm) \mid Au = 0 \,\}, \tag{1.3.23}$$

since

$$AK\varphi = \widetilde{\varrho}^* \mathfrak{A}\varphi \quad \text{is supported in } \Gamma. \tag{1.3.24}$$

The operators $K^\pm$ are *Poisson operators*, as in the basic example (1.2.28); so by the general results, the mapping properties in the second line of (1.3.22) extend to all $s \in \mathbb{R}$ (this is not true for the operator K itself that gives an important singularity at $x_n = 0$).

Let us denote

$$\varrho^{\pm} Z_{\pm}^{s} = N_{\pm}^{s} \subset \prod_{0 \le j < d} H^{s-j-\frac{1}{2}}(E_\Gamma).$$

The application of $\varrho^{\pm}$ to $Z_{\pm}^{s}$ makes sense for $s > d - \frac{1}{2}$; it can be shown to extend to all $s \in \mathbb{R}$, either by appealing to a generalization of $\varrho^{\pm}$ in Lions and Magenes [L-M68], or as a by-product of what we show below.

One has that

$$\begin{aligned} K^{+} \varrho^{+} z &= z, \text{ for } z \in Z_{+}^{d}, \\ K^{-} \varrho^{-} z &= z, \text{ for } z \in Z_{-}^{d}; \end{aligned} \tag{1.3.25}$$

this is seen as follows: Let $w \in L_2(E_+)$ and let $v = Q^* e^+ w$, which lies in $H^d(\widetilde{E})$. Then by Green's formula,

$$\begin{aligned} -(z, w)_{\Omega_+} &= (Az, r^+ v)_{\Omega_+} - (z, r^+ A^* v)_{\Omega_+} \\ &= (\mathfrak{A} \varrho^+ z, \varrho^+ v)_\Gamma = (\mathfrak{A} \varrho^+ z, \widetilde{\varrho} v)_\Gamma \\ &= \langle \widetilde{\varrho}^* \mathfrak{A} \varrho^+ z, \overline{Q^* e^+ w} \rangle_\Sigma = (Q \widetilde{\varrho}^* \mathfrak{A} \varrho^+ z, e^+ w)_\Sigma \\ &= -(K^+ \varrho^+ z, w)_{\Omega_+}. \end{aligned}$$

Since w was arbitrary, it follows that $z = K^+ \varrho^+ z$. There is a similar proof with minus. The properties (1.3.25) extend in fact to $Z_{\pm}^{s}$ for all $s \in \mathbb{R}$.

Now we can define

$$P^{\pm} = \varrho^{\pm} K^{\pm}, \tag{1.3.26}$$

the *Calderón projectors* associated with A. They are ps.d.o.s in $\bigoplus_{0 \le j < d} E_\Gamma$, by the general calculus (1.2.52)ff. (or by special verification as in [Se66], [H66,85 20.1]), with the continuity property

$$P^{\pm} \colon \prod_{0 \le j < d} H^{s-j-\frac{1}{2}}(E_\Gamma) \to \prod_{0 \le j < d} H^{s-j-\frac{1}{2}}(E_\Gamma), \text{ all } s \in \mathbb{R}. \tag{1.3.27}$$

The projection property follows since, for example,

$$(P^+)^2 = \varrho^+ K^+ \varrho^+ K^+ = \varrho^+ K^+ = P^+$$

in view of (1.3.25) (here it holds for smooth sections and extends by continuity to general distribution sections). Moreover,

$$P^{\pm} \text{ maps } \prod_{0 \le j < d} H^{s-j-\frac{1}{2}}(E_\Gamma) \text{ into } N_{\pm}^{s}, \tag{1.3.28}$$

respectively, since $K^\pm$ maps into $Z^s_\pm$, respectively. In particular,

$$P^\pm\varphi = \varphi, \text{ for } \varphi \in N^d_\pm,$$

in view of (1.3.25).

Now P^+ and P^- are in fact *complementing projections*,

$$P^+ + P^- = I, \tag{1.3.29}$$

which we see as follows: Let $\varphi \in \prod_{0\le j<d} H^{d-j-\frac12}(E_\Gamma)$, and let $z^\pm = K^\pm\varphi$. For each $\psi \in C^\infty(\bigoplus_{0\le j<d} E_\Gamma)$, choose a section $v \in C^\infty(\widetilde E)$ with $\widetilde\varrho v = \psi$. Then we have, using the Green's formulas (1.3.17–18) "backwards", together with the fact that $z^\pm \in Z^d_\pm$,

$$\begin{aligned}
&(\mathfrak{A}(P^+ + P^-)\varphi, \psi)_\Gamma = (\mathfrak{A}\varrho^+ z^+, \varrho^+ r^+ v) + (\mathfrak{A}\varrho^- z^-, \varrho^- r^- v)\\
&= (Az^+, r^+v)_{\Omega_+} - (z^+, r^+A^*v)_{\Omega_+} - (Az^-, r^-v)_{\Omega_-} + (z^-, r^-A^*v)_{\Omega_-}\\
&= -(K^+\varphi, r^+A^*v)_{\Omega_+} + (K^-\varphi, r^-A^*v)_{\Omega_-}\\
&= (Q\widetilde\varrho^*\mathfrak{A}\varphi, A^*v)_\Sigma = \langle \widetilde\varrho^*\mathfrak{A}\varphi, \overline{v}\rangle_\Sigma\\
&= (\mathfrak{A}\varphi, \psi)_\Gamma.
\end{aligned}$$

Since φ and ψ were arbitrary and $\mathfrak{A}$ is invertible, (1.3.29) follows.

One then concludes furthermore that $P^+P^- = P^-P^+ = 0$, and that

$$\textstyle\prod_{0\le j<d} H^{s-j-\frac12}(E_\Gamma) = N^s_+ \dot{+} N^s_-$$

(topological direct sum) for all $s \in \mathbb{R}$, with $N^s_\pm$ *equal to the range of* $P^\pm$ in (1.3.28). Moreover, the $K^\pm$ define *homeomorphisms*

$$K^\pm \colon N^s_\pm \xrightarrow{\sim} Z^s_\pm, \text{ with inverse } \varrho^\pm. \tag{1.3.30}$$

This gives us a parametrization of the nullspace $Z^s_\pm$ by its Cauchy data.

(1.3.30) and $P^\pm$ are very useful for the discussion of boundary value problems

$$Au = f \text{ in } \Omega_+, \quad B\varrho^+u = \eta \text{ on } \Gamma. \tag{1.3.31}$$

First we eliminate the nonzero right-hand side in the equation $Au = f$ by replacing u by $z = u - Q_+f$; hereby the boundary condition becomes $B\varrho^+z = \psi$, where $\psi = \eta - B\varrho^+Q_+f$. Thus we have to consider the problem of finding z such that

$$Az = 0 \text{ in } \Omega_+, \quad B\varrho^+z = \psi. \tag{1.3.32}$$

Now $z \in Z_+^s$ is in one-to-one correspondence with $\varphi = \varrho^+ z \in N_+^s$ (where $z = K^+\varphi$), so that (1.3.32) can be reduced to *either* of the problems on Γ:

$$B\varphi = \psi, \quad P^-\varphi = 0, \tag{1.3.33}$$

or

$$BP^+\chi = \psi \text{ (here we take } \varphi = P^+\chi). \tag{1.3.34}$$

The formulation (1.3.33) is good for the discussion of injectiveness and regularity of solutions (based on possible injectiveness of the system $\begin{pmatrix} B \\ P^- \end{pmatrix}$), and the formulation (1.3.34) is good for discussing surjectiveness (based on possible surjectiveness of the system BP^+). Ellipticity of the problem (1.3.31) holds precisely when $\begin{pmatrix} B \\ P^- \end{pmatrix}$ and BP^+ are injectively (resp. surjectively) elliptic, i.e., their principal symbols $\begin{pmatrix} b^0(x',\xi') \\ {p^-}^0(x',\xi') \end{pmatrix}$ and $b^0(x',\xi'){p^+}^0(x',\xi')$ are injective (resp. surjective) for $|\xi'| \geq 1$. (More precisely, one finds that ${p^+}^0(x',\xi')$ has rank m^+, where m^+ is the number of roots τ of $\det a^0(x',0,\xi',\tau)$ with $\operatorname{Im}\tau > 0$. The ellipticity then requires that $b^0(x',\xi')$ has m^+ rows, and when this holds, surjectiveness of $\begin{pmatrix} b^0(x',\xi') \\ {p^-}^0(x',\xi') \end{pmatrix}$ is equivalent with injectiveness of $b^0(x',\xi'){p^+}^0(x',\xi')$.)

In this way, the discussion of a boundary problem for an elliptic *differential* operator can be reduced to a discussion of matrix formed ps.d.o.s over the boundary, which is a compact boundaryless manifold. Further information — such as parametrix constructions — can be found, e.g., in [G77′] (when specialized to the single-order case, so that the auxiliary matrices $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ in [G77′] are void).

1.4 Realizations and normal boundary conditions

Consider a ps.d.o. P of order $d > 0$ in $\widetilde{E}$, satisfying the transmission condition at Γ (this is tacitly understood in the following). For simplicity of formulations we take $\overline{\Omega}$ compact here, as in the rest of Chapter 1. As it is customary for differential operators, one can define the maximal and minimal realizations of P in $L_2(E)$ as the operators acting like P_+ and determined by

$$\begin{aligned} D(P_{\max}) &= \{u \in L_2(E) \mid P_+u \in L_2(E)\}, \\ P_{\min} &= \text{ closure of } P_+\big|_{C_0^\infty(\Omega,E)} \text{ in } L_2(E) \end{aligned} \tag{1.4.1}$$

(that $P_{\max}$ is closed follows from the continuity of $P\colon L_2(\widetilde{E}) \to H^{-d}_{\mathrm{loc}}(\widetilde{E})$; hence $P_{\min} \subset P_{\max}$). When P is *elliptic*, the graph norm and the H^d-norm are equivalent on $C_0^\infty(\Omega, E)$, so

$$D(P_{\min}) = H_0^d(E). \tag{1.4.2}$$

One has the usual identity

$$P_{\max} = (P^*_{\min})^*, \text{ as operators in } L_2(E), \tag{1.4.3}$$

where P^* is the formal adjoint of P on $\widetilde{E}$. To see this, note that the domain of the adjoint S of $(P^*)_{\min}$ consists of those functions $u \in L_2(E)$ for which there exists $f \in L_2(E)$ so that

$$(u, (P^*)_+\varphi)_\Omega = (f, \varphi)_\Omega \text{ for all } \varphi \in C_0^\infty(\Omega, E),$$

and then $f = Su$. But when this holds,

$$(u, (P^*)_+\varphi)_\Omega = (u, r^+P^*\varphi)_\Omega = \langle e^+u, \overline{P^*\varphi}\rangle_\Sigma = \langle r^+Pe^+u, \overline{\varphi}\rangle_\Omega,$$

so $u \in D(P_{\max})$ with $f = P_{\max}u$. Conversely, when $r^+Pe^+u = f \in L_2(E)$, it is seen that $u \in D(S)$ with $Su = f$, so all together $S = P_{\max}$.

Note that $H^d(E) \subset D(P_{\max})$ (the inclusion is strict when $n > 1$).

Realizations of P would in the differential operator case be defined as the operators $\widetilde{P}$ acting like P and with domains lying between $D(P_{\min})$ and $D(P_{\max})$. Here one would, in the elliptic case, be particularly interested in the realizations $\widetilde{P}$ with

$$H_0^d(E) \subset D(\widetilde{P}) \subset H^d(E),$$

which includes the realizations defined by boundary conditions satisfying the Shapiro-Lopatinski-condition (*elliptic* boundary conditions).

The present problems require a wider concept of realization. For one thing, it will be allowed to add a singular Green term to P, which changes the *action* of the operator. Another thing is that we include trace operators with nonlocal terms, which may very well violate the usual inclusion of $D(P_{\min})$ in the domain of all realizations.

Before we give the precise definition, we shall explain the notation we use for trace operators. By a *system of trace operators* (or *a trace operator) T associated with the order d*, we understand a (column) vector of trace operators $T = \{T_0, T_1, \dots, T_{d-1}\}$ (void if $d \le 0$), where each T_k is of order k and class $\le d$, going from E to F_k; here $\{F_0, F_1, \dots, F_{d-1}\}$ is a given system of vector bundles over Γ, with $\dim F_k = M_k \ge 0$. We denote

$$F = F_0 \oplus \cdots \oplus F_{d-1} \text{ and } M = \textstyle\sum_{0 \le k < d} M_k. \tag{1.4.4}$$

For notational convenience, we have here included all orders k between 0 and $d-1$, but this encompasses those systems of trace operators that pick out certain orders, *since we have allowed bundles F_k with dimension*

0; when M_k is zero, T_k is trivial and could be omitted. When P is scalar, it is customary to consider only scalar trace operators, i.e., $M_k = 1$ or 0, and one could label the nontrivial trace operators by a subset J of the set $\{0, 1, \dots, d-1\}$. The statements will sometimes be specified for this notation. (For example, for (1.1.3) (resp. (1.1.4)), J is the subset $\{0\}$ (resp. $\{1\}$) of $\{0, 1\}$. In the general notation (1.4.4), $\{M_0, M_1\}$ here equals $\{1, 0\}$ (resp. $\{0, 1\}$).)

When T is as above, it maps $H^d(E)$ continuously into

$$H_F^d = \prod\nolimits_{0\le k<d} H^{d-k-\frac{1}{2}}(F_k). \tag{1.4.5}$$

Note also that since each T_k is of class $\le d$, it may be written

$$T_k = \sum_{0\le j<d} S_{kj}\gamma_j + T'_k,$$

where the S_{kj} are ps.d.o.s from E_Γ to F_k of order $k-j$, and T'_k is of class 0, so T may be written in short form (cf. (1.3.9))

$$T = S\varrho + T', \tag{1.4.6}$$

where $S = (S_{kj})_{\substack{0\le k<d \\ 0\le j<d}}$, and T' is of class 0.

1.4.1 Definition. *Let P be a ps.d.o. of order $d \in \mathbb{N}$ in $\widetilde{E}$, satisfying the transmission condition at Γ; let G be a singular Green operator in E of class $\le d$ and order d, and let $T = \{T_0, \dots, T_{d-1}\}$ be a system of trace operators T_k from E to F_k, associated with the order d (as defined around* (1.4.4)*). The operator $B = (P+G)_T$ in $L_2(E)$, acting like $P_+ + G$ and with domain*

$$D((P+G)_T) = \{u \in H^d(E) \mid Tu = 0\} \tag{1.4.7}$$

is called the H^d-realization (or just realization) of P defined by the system $\{P_+ + G, T\}$. The realization is called **elliptic** *when the system $\{P_+ + G, T\}$ is elliptic.*

The reasons for excluding trace operators of class $> d$ will be discussed in Section 1.5. (The *orders* can in principle always be adapted by composition with ps.d.o.s over Γ.)

Observe that even when $G = 0$, one may have that

$$D(P_{\min}) \not\subset D(P_T), \tag{1.4.8}$$

namely whenever T contains integrals over the interior of Ω (as in Example 1.1.1 above). Here the operator P_T does *act like* P_+, but the adjoint (when it exists) will in general not act like P_+^*, in view of (1.4.8),

$$(P_T)^* \not\subset (P^*)_{\max}. \tag{1.4.9}$$

For the class of *normal boundary problems* to be introduced further below, however, $(P_T)^*$ will be of the form $(P^*+G')_{T'}$ for suitable G' and T', when considered on $H^d(E)$; cf. Theorem 1.6.9 below. See also Example 1.4.5.

It will be useful to make some observations on elliptic realizations here. For one thing, the ellipticity hypothesis, which states that the boundary symbol operator (defined in local coordinates)

$$\begin{pmatrix} p^0(x',0,\xi',D_n)_+ + g^0(x',\xi',D_n) \\ \\ t^0(x',\xi',D_n) \end{pmatrix} : \mathcal{S}(\overline{\mathbb{R}}_+)^N \to \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+)^N \\ \times \\ \mathbb{C}^M \end{matrix} \tag{1.4.10}$$

is bijective for all x', all $|\xi'| \ge 1$, is *equivalent with the property*:

$$\begin{pmatrix} p_+^0 + g^0 \\ \\ t^0 \end{pmatrix} : H^d(\overline{\mathbb{R}}_+)^N \to \begin{matrix} L_2(\mathbb{R}_+)^N \\ \times \\ \mathbb{C}^M \end{matrix} \tag{1.4.11}$$

is bijective for all x', all $|\xi'| \ge 1$. This hinges on the fact that $p_+^0 + g^0 \colon H^d(\overline{\mathbb{R}}_+)^N \to L_2(\mathbb{R}_+)^N$ has nullspace (kernel) and co-range (co-kernel) in $\mathcal{S}(\overline{\mathbb{R}}_+)^N$, *coinciding* with the nullspace, resp. co-range, of $p_+^0 + g^0 \colon \mathcal{S}(\overline{\mathbb{R}}_+)^N \to \mathcal{S}(\overline{\mathbb{R}}_+)^N$ (cf. Lemma 3.3.1 below). In the present case, ellipticity means that $p_+^0 + g^0$ is surjective, its kernel has dimension M, and t^0 provides a bijection of the kernel onto $\mathbb{C}^M$; this can be expressed equally well by the bijectiveness of (1.4.10) or of (1.4.11).

Let us also introduce the *boundary symbol realization* $b^0(x',\xi',D_n)$, defined for each (x',ξ') as the operator in $L_2(\mathbb{R}_+)^N$ acting like $p^0(x',0,\xi',D_n)_+ + g^0(x',\xi',D_n)$ and with domain

$$D(b^0(x',\xi',D_n)) = \{u \in H^d(\overline{\mathbb{R}}_+)^N \mid t^0(x',\xi',D_n)u = 0\}. \tag{1.4.11$'$}$$

Ellipticity implies that b^0 is bijective from $D(b^0)$ to $L_2(\mathbb{R}_+)^N$; the converse holds when t^0 has a right inverse, as in Lemma 1.6.6 below.

Now the ellipticity implies a Fredholm solvability of the problem

$$\begin{aligned} (P_+ + G)u &= f, \\ Tu &= \varphi, \end{aligned} \tag{1.4.12}$$

where f and φ are given in $L_2(E)$ resp. H^d_F (cf. (1.4.5)), and u is sought in $H^d(E)$. Basically, this follows from the fact that

$$\mathcal{A} = \begin{pmatrix} P_+ + G \\ T \end{pmatrix} \text{ has a parametrix } \mathcal{C} = \begin{pmatrix} R & K \end{pmatrix} \tag{1.4.13}$$

within the calculus. A good analysis of the solution operator can be given by use of the fact (mentioned after (1.2.52), proved in Remark 3.2.15) that there exists a family of ("order-reducing") elliptic ps.d.o.s Λ^m_- in $\widetilde{E}$ such that $\Lambda^m_{-,\Omega}$ maps $H^s(E)$ homeomorphically onto $H^{s-m}(E)$ for all $s \in \mathbb{R}$. There is also a pseudodifferential homeomorphism Λ_Γ of H^d_F onto $L_2(F)$. Composing $\mathcal{A}$ to the right with $\Lambda^{-d}_{-,\Omega}$ and to the left with $\left(\begin{smallmatrix} I & 0 \\ 0 & \Lambda_\Gamma \end{smallmatrix}\right)$, we reduce it to a system

$$\widetilde{\mathcal{A}} = \begin{pmatrix} P_+ + G \\ \\ \Lambda_\Gamma T \end{pmatrix} \Lambda^{-d}_{-,\Omega} = \begin{pmatrix} Q_+ + \widetilde{G} \\ \\ \widetilde{T} \end{pmatrix} : L_2(E) \to \begin{matrix} L_2(E) \\ \times \\ L_2(F) \end{matrix}, \tag{1.4.14}$$

where $Q = P\Lambda^{-d}_-$ and $\widetilde{G}$ have order 0, $\widetilde{T}$ has order $-\frac{1}{2}$, and $\widetilde{G}$ and $\widetilde{T}$ have class 0. The ellipticity holds again for the system $\widetilde{\mathcal{A}}$, so $\widetilde{\mathcal{A}}$ has a parametrix $\widetilde{\mathcal{C}} = \begin{pmatrix} \widetilde{R} & \widetilde{K} \end{pmatrix}$, with

$$\begin{aligned} &\widetilde{\mathcal{A}}\widetilde{\mathcal{C}} = I + \mathcal{S} = \begin{pmatrix} I & 0 \\ \\ 0 & I \end{pmatrix} + \begin{pmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ \\ \mathcal{S}_{21} & \mathcal{S}_{22} \end{pmatrix} \text{ in } \begin{matrix} L_2(E) \\ \times \\ L_2(F) \end{matrix}, \\ &\widetilde{\mathcal{C}}\,\widetilde{\mathcal{A}} = I + \mathcal{S}' \text{ in } L_2(E), \end{aligned} \tag{1.4.15}$$

$\mathcal{S}$ and $\mathcal{S}'$ being negligible. The latter are of class 0, so they are in fact integral operators with C^∞ kernels. They are compact operators, which implies that $\widetilde{\mathcal{A}}$ has closed range and a finite-dimensional nullspace and co-range. The nullspace $Z(\widetilde{\mathcal{A}})$ is contained in $Z(I + \mathcal{S}')$, so it is in $C^\infty(E)$, and the co-range $R(\widetilde{\mathcal{A}})^\perp = Z(\widetilde{\mathcal{A}}^*)$ is of the same kind, since

$$\widetilde{\mathcal{C}}^*\widetilde{\mathcal{A}}^* = (\widetilde{\mathcal{A}}\widetilde{\mathcal{C}})^* = I + \mathcal{S}^*,$$

where $\mathcal{S}^*$ has C^∞ kernel. Returning to $\mathcal{A}$ by use of Λ_Γ^{-1} and $(\Lambda^{-d}_{-,\Omega})^{-1}$, one gets a parametrix $\mathcal{C} = \begin{pmatrix} R & K \end{pmatrix}$ for the problem (1.4.12). In particular, there is a solution for any $\{f, \varphi\} \in L_2(E) \times H^d_F$ satisfying a finite set of orthogonality relations (with smooth g_l and ψ_l)

$$(f, g_l)_{L_2(E)} + (\varphi, \psi_l)_{L_2(F)} = 0, \text{ for } l = 1, \dots, l_0, \tag{1.4.16}$$

and the solution is unique modulo a finite-dimensional C^∞ subspace of $H^d(E)$.

All this is quite wellknown, and satisfactory for the nonhomogeneous problem (1.4.12). In particular, the problem

$$\begin{aligned}(P_+ + G)u &= f\\ Tu &= 0\end{aligned} \tag{1.4.17}$$

is solved by Rf (cf. (1.4.13)) in a parametrix sense, in that $(P_+ + G)R - I$ and TR are smoothing operators. But R is not of much help when we consider the realization B, for R need not map into $D(B)$ (where Tu must *equal* zero). On the other hand, the abovementioned Fredholm properties imply that there exists a more abstractly defined operator $R_1 \colon L_2(E) \to D(B)$ such that $BR_1 - I = K_1$ and $R_1 B - I = K_2$ have finite-dimensional C^∞ range. We shall now show that one can in fact find a parametrix R_0 *within the calculus* that has all these properties (by a modification of Grubb and Geymonat [G-G77, Th. 5.3]).

1.4.2 Proposition. *Let $B = (P+G)_T$ be an elliptic realization of order $d \geq 0$, as introduced in Definition* 1.4.1. *There exists a parametrix R_0 with the properties:*

(i) *$R_0 = (P^{-1})_+ + G_0$, where P^{-1} is a parametrix of P on $\widetilde{E}$ and G_0 is a singular Green operator of order $-d$ and class* 0*;*
(ii) *R_0 maps $L_2(E)$ into $D(B)$;*
(iii) *$BR_0 = I + \mathcal{S}$, where $\mathcal{S}$ is an integral operator with C^∞ kernel and finite rank. Also $R_0 B - I$ on $D(B)$ has finite-dimensional C^∞ range.*

Proof. The point is to modify the parametrix $\widetilde{\mathcal{C}}$ of $\widetilde{\mathcal{A}}$ given above. Since we know from the above analysis that $\widetilde{\mathcal{A}}$ has a smooth nullspace Z and that its closed range has a smooth orthogonal complement Z', we can choose $\widetilde{\mathcal{C}}'$ as the operator acting like the inverse of $\widetilde{\mathcal{A}}$ from $(L_2(E) \times L_2(F)) \ominus Z'$ to $L_2(E) \ominus Z$ and mapping Z' into 0. Then

$$\begin{aligned}\widetilde{\mathcal{C}}'\widetilde{\mathcal{A}} &= I - \operatorname{pr}_Z \ \text{ on } L_2(E),\\ \widetilde{\mathcal{A}}\widetilde{\mathcal{C}}' &= I - \operatorname{pr}_{Z'} \text{ on } L_2(E) \times L_2(F),\end{aligned} \tag{1.4.18}$$

where pr_X denotes orthogonal projection onto X; since Z and Z' are smooth finite-dimensional, pr_Z and $\operatorname{pr}_{Z'}$ are operators with C^∞ kernel (of the form $\sum_{j \leq j_0} u_j(x)\overline{u}_j(y)$). Now (cf. (1.4.15))

$$\widetilde{\mathcal{C}}' - \widetilde{\mathcal{C}} = (\widetilde{\mathcal{C}}\,\widetilde{\mathcal{A}} - \mathcal{S}')(\widetilde{\mathcal{C}}' - \widetilde{\mathcal{C}}) = \widetilde{\mathcal{C}}(I - \operatorname{pr}_{Z'}) - \widetilde{\mathcal{C}}(I + \mathcal{S}) - \mathcal{S}'(\widetilde{\mathcal{C}}' - \widetilde{\mathcal{C}})$$

maps $L_2(E) \times L_2(F)$ into $C^\infty(E)$, and its adjoint

$$(\widetilde{\mathcal{C}}' - \widetilde{\mathcal{C}})^* = (\widetilde{\mathcal{C}}^* \widetilde{\mathcal{A}}^* - \mathcal{S}^*)(\widetilde{\mathcal{C}}'^* - \widetilde{\mathcal{C}}^*) = \widetilde{\mathcal{C}}^*(I - \mathrm{pr}_Z) - \widetilde{\mathcal{C}}^*(I + \mathcal{S}'^*) - \mathcal{S}^*(\widetilde{\mathcal{C}}' - \widetilde{\mathcal{C}})^*$$

maps $L_2(E)$ into $C^\infty(E) \times C^\infty(F)$. Thus $\widetilde{\mathcal{C}}' - \widetilde{\mathcal{C}}$ maps $\overset{\circ}{\mathcal{E}}{}'(E) \times \mathcal{E}'(F)$ into $C^\infty(E) \times C^\infty(F)$ (cf. (A.35)ff.) and hence, by the Schwartz kernel theorem (see, e.g., [H85, Th. 5.2.6]) is an integral operator with C^∞ kernel, so $\widetilde{\mathcal{C}}'$ belongs to the Green operator calculus. In view of the structure of $\widetilde{\mathcal{C}}$, we have that

$$\widetilde{\mathcal{C}}' = \begin{pmatrix} \widetilde{R}' & \widetilde{K}' \end{pmatrix}, \text{ with } \widetilde{R}' = (Q^{-1})_+ + \widetilde{G}',$$

Q^{-1} being a parametrix of Q, $\widetilde{G}'$ an s.g.o. of order and class 0, and $\widetilde{K}'$ a Poisson operator. $\widetilde{R}'$ is a parametrix of the realization $\widetilde{B} = (Q + \widetilde{G})_{\widetilde{T}}$.

Let us write $\mathrm{pr}_{Z'}$ as

$$\mathrm{pr}_{Z'} = \begin{pmatrix} \widetilde{\mathcal{S}}_{11} & \widetilde{\mathcal{S}}_{12} \\ \widetilde{\mathcal{S}}_{21} & \widetilde{\mathcal{S}}_{22} \end{pmatrix} \text{ in } \begin{matrix} L_2(E) \\ \times \\ L_2(F) \end{matrix};$$

then each of the entries is smoothing, with finite rank. Let $Z'' = R(\widetilde{\mathcal{S}}_{21}^*) = Z(\widetilde{\mathcal{S}}_{21})^\perp$, it is a finite-dimensional C^∞ subspace of $L_2(E)$. Let $\mathrm{pr}_{Z''}$ be the orthogonal projection onto Z'' in $L_2(E)$. Then we have

$$\begin{aligned} \begin{pmatrix} P_+ + G \\ \Lambda_\Gamma T \end{pmatrix} \Lambda^{-d}_{-,\Omega} (\widetilde{R}' \quad \widetilde{K}') \begin{pmatrix} I - \mathrm{pr}_{Z''} \\ 0 \end{pmatrix} &= \widetilde{\mathcal{A}} \widetilde{\mathcal{C}}' \begin{pmatrix} I - \mathrm{pr}_{Z''} \\ 0 \end{pmatrix} \\ = \begin{pmatrix} I - \widetilde{\mathcal{S}}_{11} & -\widetilde{\mathcal{S}}_{12} \\ -\widetilde{\mathcal{S}}_{21} & I - \widetilde{\mathcal{S}}_{22} \end{pmatrix} \begin{pmatrix} I - \mathrm{pr}_{Z''} \\ 0 \end{pmatrix} &= \begin{pmatrix} (I - \widetilde{\mathcal{S}}_{11})(I - \mathrm{pr}_{Z''}) \\ 0 \end{pmatrix}, \end{aligned}$$

which shows that the operator

$$R_0 = \Lambda^{-d}_{-,\Omega} \widetilde{R}'(I - \mathrm{pr}_{Z''}) \tag{1.4.19}$$

satisfies $\Lambda_\Gamma T R_0 = 0$ (and hence $TR_0 = 0$), and $(P_+ + G)R_0 = I + \mathcal{S}'''$, with an integral operator $\mathcal{S}'''$ with C^∞ kernel and finite rank. Thus R_0 maps into $D(B)$, and $BR_0 - I = \mathcal{S}'''$. By the rules of calculus, R_0 satisfies (i), and $R_0 B - I$ maps $D(B)$ into $C^\infty(E)$. By comparison with R_1 mentioned before the proposition,

$$\begin{aligned} R_0 B - I &= (R_0 - R_1)B + R_1 B - I \\ &= ((R_1 B - K_2)R_0 - R_1(BR_0 - \mathcal{S}'''))B + K_2 \\ &= -K_2 R_0 B + R_1 \mathcal{S}''' B + K_2 \end{aligned}$$

has finite rank. This ends the proof. □

We now specify an interesting class of boundary conditions, the *normal* boundary conditions.

1.4.3 Definition. *Let $T = \{T_0, \dots, T_{d-1}\}$ be a system of trace operators T_k from E to F_k, associated with the order d. Then T is said to be* **normal**, *when each T_k is of the form (void if $M_k = 0$)*

$$T_k = \sum_{0 \le j \le k} S_{kj}\gamma_j + T'_k, \tag{1.4.20}$$

with S_{kk} being a **surjective morphism** *from E_Γ to F_k, S_{kj} a ps.d.o. from E_Γ to F_k of order $k - j$ for $j < k$ and T'_k a trace operator from E to F_k of order k and class 0. (In particular, $M_k = \dim F_k$ is $\le N$ for each k, and T_k is of class $k + 1$.)*

The boundary condition $Tu = \varphi$ is then also called normal.

With the notation of Definition 1.4.1, the system $\{P_+ {+} G, T\}$, the boundary value problem

$$\begin{aligned} (P_+ + G)u &= f \\ Tu &= \varphi, \end{aligned}$$

and the realization $B = (P + G)_T$, are said to be **normal**, *when Γ is noncharacteristic for P and T is normal.*

It will be proved further below that normal realizations have dense domains in $L_2(E)$, so that they have welldefined adjoints. When $T = \gamma_0 Q_+$ for a ps.d.o. Q of order $k \ge 0$ (satisfying, as always, the transmission condition), normality means that Γ is noncharacteristic for Q (cf. (1.3.15)ff).

Observe that when T is written in the short form $T = S\varrho + T'$ as in (1.4.6), the normality means precisely that S is triangular

$$S = \begin{pmatrix} S_{00} & 0 & \cdots & 0 \\ S_{01} & S_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S_{0,d-1} & S_{1,d-1} & \cdots & S_{d-1,d-1} \end{pmatrix} \tag{1.4.21}$$

with surjective morphisms $S_{kk}(x')$ in the diagonal. The surjectiveness of the diagonal elements S_{kk} implies surjectiveness of the trace operator $S\varrho$ from $C^\infty(E)$ to $C^\infty(F)$ (cf. [G73] or Lemma 1.6.1 below), and we show in Section 1.6 that the full trace operator $T = S\varrho + T'$ is surjective.

1.4.4 Remark. The definition can be written in a (perhaps) simpler way, when P is *scalar* (E is a one-dimensional bundle). Here the M_k are necessarily 0 or 1; when $M_k = 0$ we can omit the T_k, and when $M_k = 1$, $S_{kk}(x')$ is an invertible function so a multiplication by $1/S_{kk}$ reduces T_k to the form

$$T_k = \gamma_k + \sum_{j<k} S_{kj}\gamma_j + T'_k \text{ (for each } k \text{ with } M_k = 1). \tag{1.4.22}$$

Sometimes one uses a formulation here where T is written as $\{T_k\}_{k\in J}$, J being the subset of indices $k \in J_\varrho \equiv \{0, 1, \dots, d-1\}$ for which F_k is nonzero. (One could also write $T = \{T_{m_j}\}_{j=1,\dots,j_0}$ with mutually distinct m_j, but this gives other notational complications.) This formulation can also be used in general when

$$F_k = E_\Gamma \text{ for } k \in J,\qquad F_k = 0 \quad \text{for } k \in J_\varrho \setminus J, \tag{1.4.23}$$

for some index set $J \subset J_\varrho = \{0, 1, \dots, d-1\}$.

1.4.5 Example. Consider the system in Example 1.1.1, where we denote $-\Delta = P$ and take $G' = 0$ for simplicity; $E = \overline{\Omega} \times \mathbb{C}$. Each of the boundary conditions (1.1.3) and (1.1.4) is normal (one can take $\{F_0, F_1\} = \{E_\Gamma, 0\}$ (resp. $\{0, E_\Gamma\}$), adding an empty first-order (resp. zero-order) boundary condition). Let us consider the adjoint of the realization P_{T_0} defined by the boundary condition (1.1.3). For $u \in D(P_{T_0})$ and $v \in H^d(E)$ we have, in view of Green's formula for $-\Delta$ (where $\mathfrak{A} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathfrak{A}^\times = i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$),

$$\begin{aligned}(P_{T_0}u, v)_\Omega = (-\Delta u, v)_\Omega &= i(\gamma_1 u, \gamma_0)_\Gamma + i(\gamma_0 u, \gamma_1 v)_\Gamma + (u, -\Delta v)_\Omega \\ &= i(\gamma_1 u, \gamma_0 v)_\Gamma + i(-T_0' u, \gamma_1 v)_\Gamma + (u, -\Delta v)_\Omega \\ &= i(\gamma_1 u, \gamma_0 v)_\Gamma + (u, -\Delta v + iT_0'^{*}\gamma_1 v)_\Omega.\end{aligned}$$

It follows that $(P_{T_0})^*$ *contains* the realization

$$(P_{T_0})^* \supset (-\Delta + iT_0'^{*}\gamma_1)_{\gamma_0}, \tag{1.4.24}$$

an operator acting like $-\Delta$ plus the singular Green operator $iT_0'^{*}\gamma_1$, and with domain defined by a *local* boundary condition $\gamma_0 v = 0$. As we shall see later, there is equality in (1.4.24) when P_{T_0} is elliptic. When $T_0' \not\equiv 0$, this is an example where (1.4.8)–(1.4.9) hold. The operator P is local and the boundary condition $T_0 u = 0$ nonlocal, whereas this situation is reversed for the adjoint $(P_{T_0})^*$.

Normal boundary conditions are of interest here, not only because they define convenient realizations (that behave nicely under composition and formation of adjoints; cf. Theorems 1.4.6 and 1.6.9 later), but also because normality is practically necessary for the parameter-ellipticity that we need to assume for the resolvent construction (see the discussion in Section 1.5).

1.4.6 Theorem. *Let $B_1 = (P_1 + G_1)_{T_1}$ and $B_2 = (P_2 + G_2)_{T_2}$ be realizations of ps.d.o.s P_1 and P_2 on $\widetilde{E}$ of order $d_1, d_2 \in \mathbb{N}$, as defined in*

Definition 1.4.1*, and denote the composition $B_1 \circ B_2$ by C; i.e., C is the operator acting like*

$$Cu = (P_{1,+} + G_1)(P_{2,+} + G_2)u, \tag{1.4.25}$$

with domain

$$\begin{aligned}&D(C) = \\ &\{u \in H^{d_2}(E) \mid T_2 u = 0, (P_{2,+} + G_2)u \in H^{d_1}(E), T_1(P_{2,+} + G_2)u = 0\}.\end{aligned} \tag{1.4.26}$$

Then C is an extension of the realization $B_3 = (P_3 + G_3)_{T_3}$, where

$$\begin{aligned} P_3 &= P_1 P_2 \text{ on } \widetilde{E}, \text{ of order } d_3 = d_1 + d_2, \\ G_3 &= -L(P_1, P_2) + P_{1,+}G_2 + G_1 P_{2,+} + G_1 G_2, \\ T_3 &= \begin{pmatrix} T_2 \\ T_1(P_{2,+} + G_2) \end{pmatrix}. \end{aligned} \tag{1.4.27}$$

When $(P_2 + G_2)_{T_2}$ is elliptic, C equals $(P_3 + G_3)_{T_3}$. When both $(P_1 + G_1)_{T_1}$ and $(P_2 + G_2)_{T_2}$ are elliptic, then so is $(P_3 + G_3)_{T_3}$. When $(P_1 + G_1)_{T_1}$ and $(P_2 + G_2)_{T_2}$ are normal in the sense of Definition 1.4.3*, then so is $(P_3 + G_3)_{T_3}$.*

Proof. It is obvious from the rules of calculus that C is an extension of $(P_3 + G_3)_{T_3}$ defined by (1.4.27); the latter acts like $P_{3,+} + G_3$ and has the domain

$$D((P_3 + G_3)_{T_3}) = \{u \in H^{d_1+d_2}(E) \mid T_3 u = 0\}.$$

Moreover, when B_2 is elliptic, $B_2 u \in H^{d_1}(E)$ implies $u \in H^{d_1+d_2}(E)$, so here $D(C) = D((P_3 + G_3)_{T_3})$.

When B_1 and B_2 are elliptic, then so is B_3, for bijectiveness of the boundary symbol operators

$$\begin{pmatrix} p^0_{1,+} + g^0_1 \\ t^0_1 \end{pmatrix} \text{ and } \begin{pmatrix} p^0_{2,+} + g^0_2 \\ t^0_2 \end{pmatrix}$$

implies the bijectiveness of the composed operator

$$\begin{pmatrix} p^0_{1,+} + g^0_1 & 0 \\ 0 & I \\ t^0_1 & 0 \end{pmatrix} \begin{pmatrix} p^0_{2,+} + g^0_2 \\ t^0_2 \end{pmatrix} = \begin{pmatrix} (p^0_{1,+} + g^0_1)(p^0_{2,+} + g^0_2) \\ t^0_2 \\ t^0_1(p^0_{2,+} + g^0_2) \end{pmatrix}.$$

Now assume that B_1 and B_2 are normal realizations. Γ is noncharacteristic for P_3, since the top coefficient in (1.2.12) for P_3 is the product of those for P_1 and P_2. The normality of T_3 is seen as follows: There are given the trace operators $T_i = \{T_{i,0}, \dots, T_{i,d_i-1}\}$ going from E to $F_i = \bigoplus_{0\le k<d_i} F_{i,k}$ for $i = 1, 2$, and we define $T_3 = \{T_{3,0}, \dots, T_{3,d_3-1}\}$ going from E to $F_3 = \bigoplus_{0\le k<d_3} F_{3,k}$ by

$$\begin{aligned} &F_{3,k} = F_{2,k} && \text{and } T_{3,k} = T_{2,k} && \text{for } 0 \le k < d_2 \\ &F_{3,k} = F_{1,k-d_2} && \text{and } T_{3,k} = T_{1,k-d_2}(P_{2,+} + G_2) && \text{for } d_2 \le k < d_3. \end{aligned} \tag{1.4.28}$$

The normality condition is obviously satisfied by the $T_{3,k}$ with $k < d_2$. For $k \ge d_2$, we use that near Γ

$$P_{2,+} = S_{d_2}(x)D_n^{d_2} + Q_+,$$

where Q is a sum of terms that either contain a factor x_n in front or have symbols that are $O(\langle\xi_n\rangle^{d_2-1})$ (cf. Lemma 1.3.1), and

$$G_2 = \textstyle\sum_{0\le j<d_2} K_j\gamma_j + G_2',$$

with G_2' of class 0. Writing

$$T_{1,l} = S_{ll}(x')\gamma_l + \{\text{terms of class } \le l\},$$

we then have, since $\gamma_0(x_n u) = 0$, $\gamma_j(x_n u) = \gamma_0 D_n^j(x_n u) = j\gamma_{j-1}u$ for $j > 0$, and $T_{1,l}G_2$ is of class $\le d_2$,

$$\begin{aligned} T_{3,l+d_2} &= T_{1,l}(P_{2,+} + G_2) \\ &= S_{ll}(x')\gamma_l S_{d_2}(x)D_n^{d_2} + \{\text{terms of class } \le l + d_2\}, \end{aligned}$$

for $0 \le l < d_3 - d_2$. Here the coefficient $S_{ll}(x')S_{d_2}(x', 0)$ is a surjective morphism from E_Γ to $F_{1,l}$ since S_{ll} is one and $S_{d_2}(x', 0)$ is bijective in E_Γ. Then all the terms $T_{3,k}$ in (1.4.28) satisfy the condition for normality. $\square$

The normal boundary conditions and realizations are studied in further detail in Sections 1.6 and 1.7, and the motivation for their use in resolvent problems is given in Section 1.5.

1.5 Parameter-ellipticity and parabolicity

In this section, P is always assumed to be *polyhomogeneous*.

It is wellknown that a boundary value problem does not have to be normal in order to have a nice solution operator (or parametrix); ellipticity suffices for this. However, it turns out that in the study of the *resolvent* $R_\lambda = (B - \lambda I)^{-1}$ of a realization $B = (P + G)_T$, normality of T plays a crucial role. We shall now introduce the concept of *parameter-ellipticity* for polyhomogeneous systems $\{P_+ + G - \lambda I, T\}$ (for λ on a ray in $\mathbb{C}$), with a discussion of the restrictions that it places on P, G and T.

Concerning *examples* of systems $\{P_+ + G, T\}$ for which $\{P_+ + G - \lambda I, T\}$ has the parameter-ellipticity property, let us mention first of all that one can show that the Dirichlet problem for a strongly elliptic ps.d.o. P (with the transmission condition, as always here) belongs to this class, just as in the classical case of differential operators; see Theorem 1.7.2 later on. More generally, there is large class of realizations that we call *m-coercive* (the realizations satisfying the "Gårding inequality"; see Section 1.7), which have the parameter-ellipticity property. In fact, one has the following hierarchy for normal boundary value problems:

(1.5.1)
$$
\begin{aligned}
&(P+G)_T \text{ is } m\text{-coercive}\\
&\Downarrow\\
&\begin{pmatrix} P_+ + G - \lambda I \\ T \end{pmatrix} \begin{array}{l}\text{is parameter-elliptic on all rays } \lambda = -e^{i\theta} r\\ \text{with } \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \text{ (parabolicity)}\end{array}\\
&\Downarrow\\
&\begin{pmatrix} P_+ + G - \lambda I \\ T \end{pmatrix} \text{ is parameter-elliptic on a ray.}
\end{aligned}
$$

An interesting observation here is that whereas the condition for parameter-ellipticity (or the condition for parabolicity) is a bijectiveness condition on certain principal symbols and hence is stable under "small" or lower-order perturbations, the condition for m-coerciveness is not generally stable in this way (even perturbations of the Dirichlet problem may fail to be m-coercive, when we allow pseudodifferential elements). Much more is said about this in Section 1.7, to which the interested reader is referred.

For differential operators, the property of parameter-ellipticity was formulated explicitly in Agmon [Ag62,65] and in Agranovich and Vishik [A-V63], where it was called, respectively, the condition for having a "ray of minimal growth" and "ellipticity with a parameter". The condition was called "Agmon's condition" in Seeley [Se69′, 69″]. "Agmon's condition" for differential operators *implies normality*, but this was not always observed: [Ag62,65] assumes normality on beforehand, whereas [A-V63] state that they are free from this assumption; [Se69′] later carries out a general resolvent construction without assuming normality, remarking however at the end of the paper that normality is a consequence of the "Agmon condition" assumed throughout (so that a certain projection P_0 that complicated the presentation, was trivial). The observation is repeated in [Se69″] and [Se71] where it is credited to T. Burak.

The reason that "Agmon's condition" for differential operators implies normality is that the condition includes (in a natural way) the invertibility of a certain limit boundary symbol operator for $\xi' \to 0$, where only the top normal-order terms in the trace operator survive. In the pseudodifferential operator case, one can generalize "Agmon's condition" in various ways. There is a weak generalization (consisting of Conditions (I) and (II) in Definition 1.5.5 below) that does not require normality nor the existence of a limit operator for $\xi' \to 0$; this allows interesting nonstandard estimates in constant coefficient cases, as analyzed in, e.g., [R-S83] and Frank and Wendt [F-W82,84]. However, we do not think that these phenomena were completely established in the variable-coefficient cases where there can be a lack of control of lower-order terms (we return to this problem in Remarks 1.5.16, 1.7.17, 2.1.24, 3.2.19), so we settle in this book for a definition of parameter-ellipticity that moreover requires the existence of an invertible limit operator for $\xi' \to 0$ (Condition (III) in Definition 1.5.5 below). This requirement *implies* (in the scalar case it *is*) *normality.* (Our first announcements [G81,81′] were deficient on this point; they were corrected in [G82].)

The following explanation of the concepts gives a motivation for the general symbol classes and parameter-dependent operators studied systematically in Chapters 2 and 3. Some technicalities are only sketchily described, since they are taken up again in detail later.

Consider first the case of an operator $P = \mathrm{OP}(p(x,\xi))$ of order $d > 0$ on a boundaryless manifold (or on $\mathbb{R}^n$). In the differential operator case, parameter-ellipticity of $P - \lambda$ (for λ on the ray $\lambda = e^{i\theta}r, r \geq 0$, for some fixed θ) simply means that the principal symbol $p^0(x,\xi)$ avoids the values $\lambda = e^{i\theta}r$ (has no eigenvalues on the ray $\lambda = e^{i\theta}r$, in the matrix case). The same definition is adequate for ps.d.o.s of order $d > 0$ (see below). Let us see how the property is *used* in the differential operator case: We write

$$-\lambda = \omega\mu^d, \text{ where } \mu = |\lambda|^{1/d} \text{ and } \omega = \exp(i(\theta - \pi)). \tag{1.5.2}$$

Then $p^0(x,\xi) - \lambda = p^0(x,\xi) + \omega\mu^d$ is a homogeneous polynomial of degree d in (ξ,μ) with C^∞ coefficients in x, so μ can in principle be considered as another cotangent variable. This is a very fruitful point of view, for the parameter-ellipticity now simply means that

$$\det(p^0(x,\xi) + \omega\mu^d) \neq 0 \text{ for } (\xi,\mu) \in \overline{\mathbb{R}}^{n+1}_+ \setminus 0, \tag{1.5.3}$$

i.e., the symbol $p(x,\xi) + \omega\mu^d$ is elliptic in the usual sense, for each x, when considered as a function of the $n+1$ cotangent variables (ξ,μ). (It is an inessential restriction that μ runs on $\overline{\mathbb{R}}_+$, as long as μ is used mainly as a parameter. When d is even, one can let $\mu \in \mathbb{R}$.) The wellknown techniques for usual elliptic operators can then be applied; for example there is a straightforward construction of a parametrix symbol; moreover, in the even-order case, there are standard techniques to pass from informations on the operator $P+\omega D_{n+1}^d$ to $P+\omega\mu^d$; cf. Agmon [Ag62].

In the case of pseudodifferential operators, there are certain difficulties with this point of view, primarily because the symbol $p^0(x,\xi)+\omega\mu^d$ is not in general a standard ps.d.o. symbol in the (ξ,μ)-variables when $p^0(x,\xi)$ is a standard ps.d.o. symbol in the ξ-variable. For, in this case, the ξ-derivatives satisfy for $|\alpha|\neq 0$

$$|D_\xi^\alpha(p^0(x,\xi)+\omega\mu^d)|=|D_\xi^\alpha p^0(x,\xi)|\le c(x)\langle\xi\rangle^{d-|\alpha|}, \tag{1.5.4}$$

where

$$\langle\xi\rangle^{d-|\alpha|} \text{ is } O(\langle(\xi,\mu)\rangle^{d-|\alpha|}) \text{ on } \overline{\mathbb{R}}_+^{n+1} \quad \textit{if and only if } |\alpha|\le d. \tag{1.5.5}$$

(As a matrix norm, we usually take the operator norm in euclidean space. We use from now on the abbreviations $\langle\xi,\mu\rangle=(1+|\xi|^2+|\mu|^2)^{\frac12}$ and $|\xi,\mu|=(|\xi|^2+|\mu|^2)^{\frac12}$, as defined in (A.1).)

1.5.1 Remark. Let P be a pseudodifferential operator on $\mathbb{R}^n$ of order $d>0$; add an extra variable t and consider the "heat operator" ∂_t+P operating on functions of $x\in\mathbb{R}^n$ and $t\in\mathbb{R}$. Then in the differential operator case, ∂_t+P can be considered as a pseudodifferential operator with *anisotropic symbol* $i\tau+p(x,\xi)\in C^\infty(\mathbb{R}^{2n+1})$, allowing studies of parametrices (in $S_{\varrho,\delta}$ classes of ps.d.o.s) under suitable hypotheses on P. In the case where P is a genuine ps.d.o., with a principal symbol that is *not* polynomial in ξ, the symbol $i\tau+p(x,\xi)$ does *not* fall in the usual anisotropic classes, as considered in Fabes and Rivière F-R67], Piriou [Pi70,71], Lascar [Ls77], Shubin [Sh78], de Gosson [Go81], Rempel and Schulze [R-S82, Section 4.3.6] and elsewhere, which means that *the heat equation for ps.d.o.s escapes treatment in the standard anisotropic theory.* Therefore a different kind of symbol calculus is necessary, and even moreso in the case where boundary conditions are included.

The calculus we shall describe seems to have some relation to the calculus of Eskin and Čan Zui Ho (the announcement [C-E71] is all that has been available to us), except that they take ps.d.o.s in a larger sense — with very few estimates on ξ-derivatives, and no asymptotic expansions.

Other related calculi have been studied in Rempel and Schulze [R-S83,84] (for ps.d.o.s not necessarily satisfying the transmission condition) and Frank and Wendt [F-W82,84] (for ps.d.o.s with rational symbols). See also Vishik and Eskin [V-E67].

In Chapter 2, we study systematically the class $S_{1,0}^{d,\nu}$ of symbols $p(x,\xi,\mu)$ of order d depending on a parameter $\mu\ge 0$ (or $\mu\in\mathbb{R}$) and satisfying

$$|D_x^\beta D_\xi^\alpha D_\mu^j p(x,\xi,\mu)|\le c(x)(\langle\xi\rangle^{\nu-|\alpha|}+\langle\xi,\mu\rangle^{\nu-|\alpha|})\langle\xi,\mu\rangle^{d-\nu-j}. \tag{1.5.6}$$

Here the usual estimates in (ξ,μ) are satisfied for $|\alpha|\le\nu$, but the control in terms of $\langle\mu\rangle$ does not improve for larger $|\alpha|$. ν is called the *regularity number* and can have any real value; in some sense it measures how many "good" derivatives p has. When $p(x,\xi)$ is a usual symbol of order $d\ge 0$, the symbol $p(x,\xi)+\omega\mu^d$ (as well as $p(x,\xi)$ itself) is precisely in this class, of regularity $\nu=d$. (Differential operators are assigned the regularity $\nu=+\infty$.)

The subclass $S^{d,\nu}$ of *polyhomogeneous* symbols of order d and regularity ν consists of the symbols that have an asymptotic expansion

$$p(x,\xi,\mu)\sim\sum_{l\in\mathbb{N}}p_{d-l}(x,\xi,\mu) \tag{1.5.7}$$

where each $p_{d-l}(x,\xi,\mu)$ has the homogeneity property

$$p_{d-l}(x,t\xi,t\mu)=t^{d-l}p_{d-l}(x,\xi,\mu) \text{ for } t\ge 1, |\xi|\ge 1, \tag{1.5.8}$$

and (1.5.7) holds in the sense that $p-\sum_{l<M}p_{d-l}\in S^{d-M,\nu-M}_{1,0}$ for all M. When $p(x,\xi)$ is a usual polyhomogeneous symbol of order $d\ge 0$, then $p(x,\xi)+\omega\mu^d$ (as well as $p(x,\xi)$ itself) is in $S^{d,d}$. As usual, we also denote $p_d(x,\xi,\mu)$ by $p^0(x,\xi,\mu)$, the principal symbol, which is uniquely determined for $|\xi|\ge 1$.

Note that the homogeneity, although valid with respect to the variables (ξ,μ), is only assumed to hold for $|\xi|\ge 1$, which means that in the *non-compact* region

$$V_1=\{(\xi,\mu)\mid \mu\ge 0, |\xi|\le 1\} \tag{1.5.9}$$

the only control over p_{d-l} is furnished by estimates such as (1.5.6). This is precisely the problematic aspect of the symbols $p(x,\xi)+\omega\mu^d$, when p is truly pseudodifferential. Consider $p^0(x,\xi)$. When it is a homogeneous function of ξ that is not a polynomial, it has an irregularity at $\xi=0$, which can be handled in two ways: *Either* we modify $p^0(x,\xi)$ for small ξ to be a C^∞-function, loosing the homogeneity there, *or* we keep the strict homogeneity, but then we must deal with the fact that the function has a singularity at $\xi=0$. Denote the strictly homogeneous version of p^0 by p^h. For $d=0$, p^h is bounded near $\xi=0$, and for $d>0$ it is continuous; more precisely, the derivatives of p^h up to order l, where

$$l=\text{ largest integer } <d, \tag{1.5.10}$$

satisfy a Hölder condition of order $d-l$ (as shown in detail in Lemmas 2.1.9–10 later). Note that the regularity $\nu=d$ of p^0 indicates precisely the degree of Hölder continuity.

Both points of view will be used in the present text. The first point of view is the most commonly used in studies of ps.d.o.s; the ambiguity of p^0 near $\xi = 0$ is reflected in negligible error terms that occur everywhere in the theory anyway; it is only because of the parameter μ that the harmless compact region $|\xi| \le 1$ leads to the noncompact region V_1. The second point of view, the consideration of p^h, enters when we need certain precise informations for $\mu \to \infty$ (observe that when ξ is fixed and $\mu \to \infty$, the ray $\{(t\xi, t\mu) \mid t \ge 0\}$ approaches the μ-axis $\{\xi = 0\}$).

1.5.2 Definition. *A parameter-dependent, polyhomogeneous $N \times N$-matrix-formed symbol $p(x, \xi, \mu) \in S^{d,\nu}$ is said to be* **parameter-elliptic** *when its principal symbol p^0 can be defined such that there are positive continuous functions $c(x)$ and $c_0(x)$ for which $p^0(x, \xi, \mu)$ is invertible, satisfying*

$$|p^0(x, \xi, \mu)^{-1}| \le c(x) \langle \xi, \mu \rangle^{-d}, \tag{1.5.11}$$

for all $(\xi, \mu) \in \overline{\mathbb{R}}^{n+1}_+$ with $|\xi, \mu| \ge c_0(x)$.

For the original pseudodifferential operators that do not depend on μ, we can now formulate the definitions of parameter-elliptic and parabolic symbols.

1.5.3 Definition. *Let $p(x, \xi) \in S^d$ with $d > 0$ and let $P = \mathrm{OP}(p)$.*

1° *Let $\theta \in \mathbb{R}$. $P - \lambda$ and $p(x, \xi) - \lambda$ are said to be* **parameter-elliptic on the ray** $\lambda = -e^{i\theta} r$ *if the symbol*

$$\overline{p}(x, \xi, \mu) = p(x, \xi) + e^{i\theta} \mu^d, \tag{1.5.12}$$

which lies in $S^{d,d}$, is elliptic in the sense of Definition 1.5.2*, at each x.*

2° *$P + \partial_t$, $P - \lambda$ and $p(x, \xi) - \lambda$ are said to be* **parabolic** *if $p(x, \xi) - \lambda$ is parameter-elliptic on the rays $\lambda = -e^{i\theta} r$ for all $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.*

The ellipticity, parameter-ellipticity, resp. parabolicity, is said to hold *uniformly* when $c(x)$ and $c_0(x)$ in the estimates can be replaced by positive constants; this is automatically satisfied when x runs in a compact set.

When $d > 0$, the ellipticity definition may also be formulated in terms of the associated strictly homogeneous symbol, which is determined by

$$p^h(x, \xi) = |\xi|^d p^0(x, \xi/|\xi|) \text{ for } \xi \ne 0; \tag{1.5.13}$$

note that $p^h(x, \xi)$ extends to a continuous function taking the value 0 at $\xi = 0$, when $d > 0$. We then have

1.5.4 Lemma. *Let $p(x,\xi) \in S^d$ with $d > 0$, and let $|\omega| = 1$. The following statements* (a), (b) *and* (c) *are equivalent:*

(a) $p(x,\xi)+\omega\mu^d$ *is elliptic in the sense of Definition* 1.5.2 *(i.e., $p(x,\xi)-\lambda$ is parameter-elliptic on the ray $\lambda = -\omega r, r \ge 0$).*

(b) *For $|\xi| = 1, p^0(x,\xi)$ has no eigenvalues on the ray $\lambda = -\omega r$, $r \ge 0$, for each x.*

(c) *$p^h(x,\xi) + \omega\mu^d$ is bijective for all $(\xi,\mu) \in \overline{\mathbb{R}}_+^{n+1} \setminus 0$, for each x.*

Proof. (a) obviously implies (b). (b) implies that $p^h(x,\xi) + \omega\mu^d$ is bijective for all $\xi \neq 0$, all $\mu > 0$, and here the values $\xi = 0, \mu > 0$, are included simply because $p^h(x,0) = 0$, so (b) implies (c). Now let (c) hold; then since $p^h(x,\xi) + \omega\mu^d$ both is a continuous function of x and of $(\xi,\mu) \in \overline{\mathbb{R}}_+^{n+1} \setminus 0$ and is homogeneous in (ξ,μ) of degree d, there is a continuous function $c_1(x)$ so that

$$|(p^h(x,\xi) + \omega\mu^d)^{-1}| \le c_1(x)|\xi,\mu|^{-d} \text{ for } (\xi,\mu) \in \overline{\mathbb{R}}_+^{n+1} \setminus 0. \tag{1.5.14}$$

We can choose the definition of the smooth function $p^0(x,\xi)$ for $|\xi| \le 1$ such that $|p^h(x,\xi) - p^0(x,\xi)| \le 1$ for all (x,ξ). Then

$$(p^h + \omega\mu^d)^{-1}(p^0 + \omega\mu^d) = I - (p^h + \omega\mu^d)^{-1}(p^h - p^0), \tag{1.5.15}$$

where

$$|(p^h + \omega\mu^d)^{-1}(p^h - p^0)| \le c_1(x)|\xi,\mu|^{-d}.$$

Let $c_0(x) = (2c_1(x))^{1/d}$, then for $|\xi,\mu| \ge c_0(x)$,

$$|(p^h + \omega\mu^d)^{-1}(p^h - p^0)| \le \tfrac{1}{2},$$

so that (1.5.15) is invertible, by a Neumann series argument. Moreover, the inverse satisfies, for $|\xi,\mu| \ge c_0(x)$

$$\begin{aligned} |(p^0 + \omega\mu^d)^{-1}| &= |[I - (p^h + \omega\mu^d)^{-1}(p^h - p^0)]^{-1}(p^h + \omega\mu^d)^{-1}| \\ &\le 2c_1(x)|\xi,\mu|^{-d} \le c'(x)\langle\xi,\mu\rangle^{-d}, \end{aligned}$$

where we have used that $|\xi,\mu|^{-1} \le c_2(x)\langle\xi,\mu\rangle^{-1}$ for $|\xi,\mu| \ge c_0(x)$. This shows that (c) implies (a). $\square$

It is wellknown already for differential operators that *strong ellipticity* of P (positive definiteness of $p^0(x,\xi)+p^0(x,\xi)^*$ for $|\xi| \ge 1$) *implies parabolicity* of $P + \partial_t$, but that the converse holds only in the scalar case ($N = 1$). Parameter-ellipticity on a ray is of course weaker than strong ellipticity for any $N \ge 1$.

Note that when P satisfies the transmission condition, *parabolicity implies that P is of even order,* because for odd d, σ is an eigenvalue of $p^0(x',0,0,\xi_n)$ if and only if $-\sigma$ is an eigenvalue of $p^0(x',0,0,-\xi_n)$ (for $|\xi_n|\geq 1$).

We shall now consider the same questions for boundary value problems. Here we recall that the ellipticity definition in Section 1.2 required invertibility of the boundary symbol operator for all $|\xi'|\geq c>0$, so it is natural to expect that parameter-ellipticity amounts to the invertibility of a parameter-dependent boundary symbol operator for $|\xi',\mu|\geq c>0$. This is also correct in the differential operator case, but in the pseudodifferential case the mere invertibility will not suffice; some extra control is needed at $\xi'=0$ because of the lack of homogeneity of the C^∞ symbols on the region

$$V_1'=\{(\xi',\mu)\mid |\xi'|\leq 1,\mu\geq 0\}. \tag{1.5.16}$$

Let us consider an elliptic Green operator

$$\mathcal{A}=\begin{pmatrix}P_+ + G\\ T\end{pmatrix},\ \text{with } a^0(x',\xi',D_n)=\begin{pmatrix}p_+^0+g^0\\ t^0\end{pmatrix} \tag{1.5.17}$$

as in (1.2.53), and the associated realization B, defined as in Definition 1.4.1 when the hypotheses there are satisfied. The resolvent of B

$$R_\lambda=(B-\lambda)^{-1} \tag{1.5.18}$$

is the solution operator for the problem

$$\begin{aligned}(P_+ + G-\lambda)u&=f,\\ Tu&=0,\end{aligned}$$

and it will be studied along with the inhomogeneous problem

$$\mathcal{A}_\lambda u\equiv\begin{pmatrix}P_+ + G-\lambda I\\ T\end{pmatrix}u=\begin{pmatrix}f\\ \varphi\end{pmatrix}. \tag{1.5.19}$$

In fact, when $\mathcal{A}_\lambda$ is invertible, its inverse equals

$$\begin{pmatrix}P_+ + G-\lambda I\\ T\end{pmatrix}^{-1}=(R_\lambda\quad K_\lambda), \tag{1.5.20}$$

where R_λ is the resolvent and K_λ is a (parameter-dependent) Poisson operator. Together with $a^0(x',\xi',D_n)$ we consider $a^h(x',\xi',D_n)$

$$a^h(x',\xi',D_n)=\begin{pmatrix}p_+^h+g^h\\ t^h\end{pmatrix}, \tag{1.5.21}$$

the operator defined for $\xi'\neq 0$ by the *strictly homogeneous* symbols $p^h(x',0,\xi',\xi_n)$, $g^h(x',\xi',\xi_n,\eta_n)$ and $t^h(x',\xi',\xi_n)$, coinciding with p^0, g^0 and t^0 (respectively) for $|\xi'|\geq 1$. We use the notation $\overline{a}^0$ and $\overline{a}^h$ for the symbols associated with $\mathcal{A}_\lambda$ (where $-\lambda$, below in the form $\omega\mu^d$, is added to the first line in the matrix).

1.5.5 Definition. *Let $\mathcal{A}$ be a Green operator* (1.5.17)*, with P of order $d \in \mathbb{N}$, satisfying the transmission condition, with G of order d, and with $T = \{T_0, T_1, \dots, T_{d-1}\}$, T_k of order k (all symbols being polyhomogeneous).*

1° *Let $\theta \in \mathbb{R}$, and let $\omega = e^{i\theta}$. We say that $\mathcal{A}_\lambda$* (1.5.19) *is* **parameter-elliptic on the ray** $\lambda = -e^{i\theta} r$ *($r \geq 0$), when* (I)–(III) *hold:*

(I) *$p^0(x, \xi) + \omega\mu^d$ is bijective for $|\xi| = 1$, $\mu \geq 0$ and all x (i.e., $P - \lambda$ is parameter-elliptic on the ray $\lambda = -e^{i\theta} r$).*

(II) *The parameter-dependent boundary symbol operator*

$$\overline{a}^0(x', \xi', \mu, D_n) = \begin{pmatrix} p^0(x', 0, \xi', D_n)_+ + g^0(x', \xi', D_n) + \omega\mu^d \\ t^0(x', \xi', D_n) \end{pmatrix} \tag{1.5.22}$$

is bijective from $\mathcal{S}(\overline{\mathbb{R}}_+)^N$ to $\mathcal{S}(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^M$ for all $\mu \geq 0$, $|\xi'| \geq 1$ and all x'.

(III) *For all x', the strictly homogeneous boundary symbol operators $p^h(x', 0, \xi', D_n)$, $g^h(x', \xi', D_n)$ and $t^h(x', \xi', D_n)$ have limits for $\xi' \to 0$ in the respective principal symbol norms (to be explained further below), and the limiting parameter-dependent boundary symbol operator*

$$\overline{a}^h(x', 0, \mu, D_n) = \begin{pmatrix} p^h(x', 0, 0, D_n)_+ + g^h(x', 0, D_n) + \omega\mu^d \\ t^h(x', 0, D_n) \end{pmatrix}$$

is bijective from $\mathcal{S}(\overline{\mathbb{R}}_+)^N$ to $\mathcal{S}(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^M$ for $\mu > 0$ and all x'.

When (I)–(III) *hold, we also say that the μ-dependent system $\{P_+ + G + \omega\mu^d, T\}$ and its set of symbols are* **parameter-elliptic.**

2° *The system $\{\partial_t + P_+ + G, T\}$, the system $\mathcal{A}_\lambda$, and its symbol are said to be* **parabolic**, *when $\mathcal{A}_\lambda$ is parameter-elliptic on each of the rays $\lambda = -e^{i\theta} r$ with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.*

For differential boundary problems, part 1° of the definition gives precisely the parameter-elliptic systems considered in Agmon [Ag62,65], Agranovich and Vishik [A-V63] and Seeley [Se69′,69″,71]; part 2° gives the standard parabolic systems (references to classical works on these are given in Section 4.1).

Hypotheses (II) and (III) together could also be formulated as a single hypothesis (since $\overline{a}^0 = \overline{a}^h$ for $|\xi'| \geq 1$):

(II+III) *For all x', the strictly homogeneous parameter-dependent boundary symbol operator*

$$\overline{a}^h(x', \xi', \mu, D_n) = \begin{pmatrix} p^h(x', 0, \xi', D_n)_+ + g^h(x', \xi', D_n) + \omega\mu^d \\ t^h(x', \xi', D_n) \end{pmatrix} \tag{1.5.23}$$

is continuous (with respect to principal symbol norms) in $(\xi', \mu) \in \overline{\mathbb{R}}^n_+ \setminus 0$, and is bijective from $\mathcal{S}(\overline{\mathbb{R}}_+)^N$ to $\mathcal{S}(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^M$ for all $(\xi', \mu) \in \overline{\mathbb{R}}^n_+ \setminus 0$.

We prefer to keep (II) and (III) apart because there exist boundary value problems where (II) but not (III) is satisfied. Moreover, let us consider the following strengthened version of (II):

(II′) *The parameter-dependent boundary symbol operator* $\overline{a}^0(x', \xi', \mu, D_n)$ (1.5.22) *is bijective from* $\mathcal{S}(\overline{\mathbb{R}}_+)^N$ *to* $\mathcal{S}(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^M$ *for all* x' *and all* $(\xi', \mu) \in \overline{\mathbb{R}}^n_+$ *with* $|\xi', \mu| \geq c_0(x')$, *for some continuous function* $c_0(x') \geq 0$.

Even here, there exist pseudodifferential boundary problems satisfying (II′) but not (III); see Example 1.5.13 below. This is quite different from the situation for differential operator problems, where $\overline{a}^h$ itself is C^∞ in $(\xi', \mu) \in \overline{\mathbb{R}}^n_+$ (because of the polynomial structure of the symbols), so that one naturally takes $\overline{a}^0 \equiv \overline{a}^h$. Then (II′) implies (II+III).

For ps.d.o. problems, the invertibility of a given smooth principal boundary symbol operator a^0 as in (II′), does *not* imply (III). But it can be shown that (II′+III) is equivalent with (II+III); see Proposition 3.1.3 and Theorem 3.2.2 later.

Let us now explain the meaning of the convergence requirement in (III). (The symbol properties are taken up again in a systematic fashion in Section 2.3, in the framework of parameter-dependent symbol classes.)

Let T be a system of trace operators T_k of orders $k = 0, 1, \dots, d-1$, going from E to bundles F_k over Γ. (The orders of a given system can be modified to lie between 0 and $d-1$, by left composition with invertible ps.d.o.s over the boundary, so all trace operators are in principle included; however, such a modification may interfere with the validity of parameter-ellipticity.)

Let r be such that the entries in T are of class $\leq r$. Then each T_k is of the form

$$T_k = \textstyle\sum_{0 \leq j < r} S_{kj}\gamma_j + T'_k, \tag{1.5.24}$$

where T'_k is of class 0. In particular,

$$\begin{aligned} t^0_k(x', \xi) &= \textstyle\sum_{0 \leq j < r} s^0_{kj}(x', \xi')\xi^j_n + {t'}^0_k(x', \xi) \\ t^h_k(x', \xi) &= \textstyle\sum_{0 \leq j < r} s^h_{kj}(x', \xi')\xi^j_n + {t'}^h_k(x', \xi). \end{aligned} \tag{1.5.25}$$

The convergence for $\xi' \to 0$ is taken in the sense that

$$s^h_{kj}(x', \xi') \to s^h_{kj}(x', 0) \text{ for each } x', \tag{1.5.26}$$

$${t'}^h_k(x', \xi', \xi_n) \to {t'}^h_k(x', 0, \xi_n) \text{ in } L_{2,\xi_n} \text{ for each } x'; \tag{1.5.27}$$

the latter is a convergence with respect to the principal seminorm in (1.2.19) (the one with all indices zero). Now observe that each $s^h_{kj}(x', \xi')$ is homogeneous in ξ' of degree $k - j$. Then (1.5.26) *cannot hold unless* $k - j \geq 0$

and, in case $k-j=0$, $s^h_{kk}(x',\xi')$ *is constant in* ξ'. In other words, the T_k *must be of the form (of class* $\leq k+1$)

$$T_k = s_{kk}(x')\gamma_k + \textstyle\sum_{0\leq j<k} S_{kj}\gamma_j + T'_k. \tag{1.5.28}$$

For the ${t'}^h_k(x',\xi',\xi_n)$ we observe that by the homogeneity,

$$\|{t'}^h_k(x',\xi',\xi_n)\|_{L_{2,\xi_n}} \leq c_k(x')|\xi'|^{k+\frac12}, \tag{1.5.29}$$

so they *converge to zero* for $\xi'\to 0$, since $k\geq 0$. Note also that the $s^h_{kj}(x',\xi')$ with $k>j$ go to zero for $\xi'\to 0$, since they are homogeneous of degree $k-j$. Thus the *limit symbol* is simply

$$t^h_k(x',0,\xi_n) = s_{kk}(x')\xi_n^k. \tag{1.5.30}$$

The G term is similarly analyzed. It will in general be assumed to be of order d and has the form

$$G = \textstyle\sum_{0\leq j<r} K_j\gamma_j + G',$$

where G' is of class 0 and the K_j are Poisson operators of order $d-j$. Now the principal homogeneous symbols satisfy

$$\|k^h_j(x',\xi',\xi_n)\|_{L_{2,\xi_n}} \leq c_j(x')|\xi'|^{d-j-\frac12}, \tag{1.5.31}$$

since k^h_j is homogeneous in (ξ',ξ_n) of degree $d-j-1$, and

$$\|{g'}^h(x',\xi',\xi_n,\eta_n)\|_{L_{2,\xi_n,\eta_n}} \leq c(x')|\xi'|^d, \tag{1.5.32}$$

since ${g'}^h$ is homogeneous in (ξ',ξ_n,η_n) of degree $d-1$. The convergence for $\xi'\to 0$ is precisely taken in the sense that the k^h_j converge in L_{2,ξ_n}-norm and ${g'}^h$ converges in L_{2,ξ_n,η_n} norm (in accordance with (1.2.19) and (1.2.38)). We see that ${g'}^h$ is unproblematic, since $d>0$ so that

$${g'}^h(x',\xi',\xi_n,\eta_n)\to 0 \text{ for } \xi'\to 0,$$

and that the k^h_j with $j<d$ behave similarly, whereas the k^h_j with $j\geq d$ blow up for $\xi'\to 0$ (unless they are zero). It follows that G must necessarily be of the form

$$G = \textstyle\sum_{0\leq j<d} K_j\gamma_j + G',$$

i.e., *of class* $\leq d$. All together we have found

1.5.6 Lemma. *Let* $\begin{pmatrix} P_+ + G \\ T \end{pmatrix}$ *be a Green operator with* P *and* G *of order* $d > 0$, *and* $T = \{T_0, T_1, \dots, T_{d-1}\}$ *with* T_k *of order* k. *The associated strictly homogeneous symbols* g^h *and* t_k^h *have limits for* $\xi' \to 0$ *if and only if* G *and* T_k *are respectively of the form*

$$G = \sum_{0 \le j < d} K_j \gamma_j + G', \tag{1.5.33}$$
$$T_k = s_{kk}(x')\gamma_k + \sum_{0 \le j < k} S_{kj}\gamma_j + T_k',$$

with G' *and the* T_k' *of class* 0. *In the affirmative case, the limits of the strictly homogeneous symbols are as follows:*

$$g^h(x', \xi', \xi_n, \eta_n) \to 0, \quad t_k^h(x', \xi', \xi_n) \to s_{kk}(x')\xi_n^k, \text{ for } \xi' \to 0. \tag{1.5.34}$$

Also p_+^h has a limit for $\xi' \to 0$ in a related sense: Write

$$p^h(x', 0, \xi) = \sum_{0 \le l \le d} s_l^h(x', \xi')\xi_n^l + {p'}^h(x', 0, \xi),$$

where ${p'}^h$ is $O(|\xi'|^{d+1}|\xi', \xi_n|^{-1})$ (cf. (1.2.8,12)), then

$$s_d^h(x', \xi') = s_d(x') \text{ for all } \xi,$$
$$|s_l^h(x', \xi')| \le c_l(x')|\xi'|^{d-l} \to 0 \text{ for } \xi' \to 0, \text{ when } l < d, \tag{1.5.35}$$
$$\|{p'}^h(x', 0, \xi', \xi_n)\|_{L_{2,\xi_n}} \le c'(x')|\xi'|^{d+\frac{1}{2}} \to 0 \text{ for } \xi' \to 0,$$

so we have that

$$p^h(x', 0, \xi) \to s_d(x')\xi_n^d \text{ for } \xi' \to 0, \tag{1.5.36}$$

in this sense. When all the symbols converge in the described way, the operator family $\overline{a}^h(x', \xi', \mu, D_n) \colon H^d(\overline{\mathbb{R}}_+)^N \to L_2(\mathbb{R}_+)^N \times \mathbb{C}^M$ depends continuously on (ξ', μ), and we see that the limit for $\xi' \to 0$ is the *differential operator*

(1.5.37)
$$\overline{a}^h(x', 0, \mu, D_n) = \begin{pmatrix} s_d(x')D_n^d + \omega\mu^d \\ s_{00}(x')\gamma_0 \\ \vdots \\ s_{d-1,d-1}(x')\gamma_{d-1} \end{pmatrix} : H^d(\overline{\mathbb{R}}_+)^N \to \begin{matrix} L_2(\mathbb{R}_+)^N \\ \times \\ \mathbb{C}^{M_0} \\ \times \\ \vdots \\ \times \\ \mathbb{C}^{M_{d-1}} \end{matrix} .$$

It is also goes from $\mathcal{S}(\overline{\mathbb{R}}_+)^N$ to $\mathcal{S}(\overline{\mathbb{R}}_+)^N \times \prod_{0 \le k < d} \mathbb{C}^{M_k}$.

Condition (III) in Definition 1.5.5 requires bijectiveness of this operator, so we see immediately that a necessary condition for this is *surjectiveness* of each of the matrices $s_{kk}(x')$, i.e., *normality of* T. We have shown

1.5.7 Lemma. *In order for the system* $\mathcal{A}_\lambda = \begin{pmatrix} P_+ + G - \lambda \\ T \end{pmatrix}$ *to be parameter-elliptic on a ray, it is necessary that* T *be normal.*

1.5.8 Remark. The normality requirement is caused partly by our assumption that the strictly homogeneous symbols have a limit for $\xi' \to 0$. For $n = 2$, one can actually make do with less, namely the requirement that $\overline{a}^h(x_1, \xi_1, \mu, D_n)$ has invertible limits for $\xi_1 \to 0+$ and $\xi_1 \to 0-$ separately. We shall allow this for the general parameter-dependent operators considered in Section 3.2 but leave it out for systems (1.5.19) to keep the concepts simple. Also for $n > 2$, one could replace the limit hypothesis with some uniformity assumption on $s_k(x', \xi')$ for $\xi' \to 0$, allowing a ps.d.o. of order 0 as coefficient of γ_k in T_k. When $M_k = N$, parameter-ellipticity will then force S_{kk} to be an elliptic ps.d.o. of order 0, and it can practically be eliminated by composition with a parametrix, which makes the trace operator normal. When $M_k < N$, one must have a surjectively elliptic coefficient S_{kk}, and the theory becomes difficult because S_{kk} is only of "regularity 0" in the sense we discussed for p above. To restrict complications, we leave this possibility out of the general study here. (A special case where it occurs is studied in [G92], [G-Se95,95′].)

In the scalar case, the structure of (1.5.37) is fairly simple because s_d and the nontrivial s_{jj} are then just invertible functions. In this case we can show that when (I) and (II) hold, the normality is also *sufficient* for the validity of (III). For then (as shown below), *any* normal boundary problem for $s_d(x')D_n^d + \omega\mu^d$ with the correct number of nontrivial boundary conditions (the number of roots of $s_d(x')\xi_n^d + \omega\mu^d$ in $\mathbb{C}_+$) is uniquely solvable. When $N > 1$, the cases where $s_d(x')$ is diagonalizable and where the boundary space dimensions M_k are either 0 or N can be reduced to the preceding case. When general dimensions M_k are allowed, not all normal boundary operators (1.5.37) with the correct boundary space dimension $\sum M_k$ are wellposed, so the mentioned arguments do not work in general.

1.5.9 Proposition. *Let* $d \in \mathbb{N}_+$.

$1°$ *Let* $\lambda' \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$ *if* d *is even and* $\lambda' \in \mathbb{C} \setminus \mathbb{R}$ *if* d *is odd, and denote by* M *the number of roots of* $\tau^d - \lambda'$ *in* $\mathbb{C}_+$. *Let* J *be a subset of* $\{0, 1, \dots, d-1\}$ *with* M *elements. The operator*

$$a'(D_t) = \begin{pmatrix} D_t^d - \lambda' \\ \\ \{\gamma_j\}_{j\in J} \end{pmatrix} : H^d(\overline{\mathbb{R}}_+) \to \begin{matrix} L_2(\mathbb{R}_+) \\ \times \\ \mathbb{C}^M \end{matrix} \tag{1.5.38}$$

is bijective.

$2°$ *Let* $\omega \in \mathbb{C}$ *with* $|\omega| = 1$, *and consider* $\mathcal{A}_\mu = \{P_+ + G + \omega\mu^d, T\}$ *(for* $N \geq 1$*), with* P *and* G *of order* d *and class* $\leq d$, *and* $T = \{T_j\}_{j\in J}$ *having* *entries of order j and class $\le j+1$, going from E_Γ to F_j with* $\dim F_j = N$ *for each $j \in J$; here J is a subset of $\{0, 1, \dots, d-1\}$ with M elements as in* 1°. *Assume that $\mathcal{A}_\mu$ satisfies* (I) *and* (II) *of Definition* 1.5.5*, and that T is normal. If $N = 1$, or if $N > 1$ and $s_d(x')$ $(= p^0(x', 0, 0, 1))$ is smoothly diagonalizable, then $\mathcal{A}_\mu$ satisfies* (III).

PROOF. 1° The considerations here are related to the study of completely elliptic boundary problems in Hörmander [H58]. The d roots of $\tau^d - \lambda'$ have the form

$$\tau_l = e^{i(\sigma+2\pi l/d)}r, \qquad l = 0, 1, \dots, d-1, \tag{1.5.39}$$
$$\text{with } r = |\lambda'|^{1/d} \text{ and } \sigma = (\arg \lambda')/d$$

$(\arg \lambda' \in]0, 2\pi[\,)$. When d is even, the first $d/2$ roots lie in $\mathbb{C}_+$, and when d is odd, the first $(d+1)/2$ or $(d-1)/2$ of the roots lie in $\mathbb{C}_+$ (so $M = d/2$, resp. $(d \pm 1)/2$). Let us write $J = \{m_0, m_1, \dots, m_{M-1}\}$. Since the differential operator $D_t^d - \lambda'$ is surjective from $H^d(\mathbb{R})$ to $L_2(\mathbb{R})$, it suffices to show that the problem

$$\begin{aligned} (D_t^d - \lambda')u &= 0 \quad \text{on } \mathbb{R}_+, \\ \gamma_{m_k} u &= \varphi_k \text{ at } t = 0, \text{ for } k = 0, \dots, M-1, \end{aligned} \tag{1.5.40}$$

has a unique solution in $H^d(\overline{\mathbb{R}}_+)$ for each vector $\{\varphi_0, \dots, \varphi_{M-1}\} \in \mathbb{C}^M$. The full set of functions satisfying $(D_t^d - \lambda')u = 0$ on $\mathbb{R}$ is the span of the exponential solutions $\{\exp(i\tau_l t)\}_{0 \le l < d}$, and the subset of these lying in H^d on $\overline{\mathbb{R}}_+$ is spanned by $\{\exp(i\tau_l t)\}_{0 \le l < M}$. Inserting $u = \sum_{0 \le l < M} c_l \exp(i\tau_l t)$ in (1.5.40), we reduce the problem to solving the system of equations

$$\textstyle\sum_{0 \le l < M} c_l\, \tau_l^{m_k} = \varphi_k, \ 0 \le k < M, \tag{1.5.41}$$

and this is uniquely solvable when the matrix

$$X = \begin{pmatrix} \tau_0^{m_0} & \tau_0^{m_1} & \dots & \tau_0^{m_{M-1}} \\ \tau_1^{m_0} & \tau_1^{m_1} & \dots & \tau_1^{m_{M-1}} \\ \vdots & \vdots & & \vdots \\ \tau_{M-1}^{m_0} & \tau_{M-1}^{m_1} & \dots & \tau_{M-1}^{m_{M-1}} \end{pmatrix}$$

is invertible. In view of (1.5.39),

$$\tau_l^{m_k} = r^{m_k} \exp(i\sigma m_k + 2\pi m_k l/d),$$

so the entries in the kth column have the common factor $r^{m_k} \exp(i\sigma m_k)$, and

(1.5.43)

$$\det X = \Big[\textstyle\prod_{0 \le j < M} r^{m_j} \exp(i\sigma m_j)\Big] \det\big([\exp(i2\pi m_k/d)]^l\big)_{0 \le l, k < M}.$$

The last determinant is a Vandermonde determinant $\det((\lambda_k)^l)_{l,k}$, where the complex numbers $\lambda_k = \exp(i2\pi m_k/d)$ are nonzero and mutually distinct, since the m_k are distinct integers between 0 and $d-1$. Thus $\det X \neq 0$, and the unique solvability is proved.

2° Consider first the scalar case $N = 1$, and fix x'. Let $\mu > 0$. When (I) holds, the polynomial $s_d(x')\xi_n^d + \omega\mu^d$ can for $\mu > 0$ be written as $s_d(\xi_n^d - \lambda')$, with λ' as under 1°. When (II) holds, we have for each $\xi' \neq 0$ that the boundary space dimension $M_0 + \cdots + M_{d-1}$ (where the M_j are either 1 or 0) must equal the index of $p^h(x', 0, \xi', D_n)_+ + \omega\mu^d$ (since the singular Green operator $g^0(x', \xi', D_n)$ is compact from $H^d(\overline{\mathbb{R}}_+)$ to $L_2(\mathbb{R}_+)$). Since the operator family $p^h + \omega\mu^d$ depends continuously on $(\xi', \mu) \in \overline{\mathbb{R}}_+^n \setminus 0$, the index equals the index of $p^h(x', 0, 0, D_n)_+ + \omega\mu^d = s_d(D_n^d - \lambda')$; this index is precisely M as described above. The limit boundary symbol operator (1.5.37) reduces to the form (1.5.38) when we divide out the invertible coefficients and remove the zero-dimensional (trivial) trace operators, so 1° shows that the limit boundary symbol operator is bijective.

When $N > 1$ and $s_d(x')$ is smoothly diagonalizable (i.e., there is, in each local coordinate system at $\partial\Omega$, a diagonal matrix $c(x')$ and a regular matrix $b(x')$ depending C^∞ on x' so that $s_d(x') = b(x')\, c(x') b(x')^{-1}$), we reduce the original problem by replacing $P_+ + G + \omega\mu^d$ locally by $b(x')(P_+ + G + \omega\mu^d)b(x')^{-1}$ to a case where the limit operator on $\mathbb{R}_+$ is on diagonal form. Since the M_k equal N or 0, the limit trace operator can be reduced to a system of standard normal derivatives $\{\gamma_j\}_{j\in J}$. In this way, the limit boundary symbol operator decomposes into a set of operators belonging to the scalar case, where 1° can be applied. $\square$

1.5.10 Remark. Also for general normal problems, the dimension of the boundary bundle $M = \sum_{0\le k<d} M_k$ is determined (for parameter-elliptic systems) by the requirement that (1.5.37) be bijective, so the standard theory for such systems implies that M *equals the number of roots in* $\mathbb{C}_+$ *of* $\det(s_d(x')\xi_n^d + \omega\mu^d)$. (We take Ω connected (cf. Section A.5) so that this number is independent of x'.)

In the parabolic case, d must be even, as observed after Lemma 1.5.4. Then the ξ_n-roots of $\det(s_d(x')\xi_n^d + \omega\mu^d)$ come in pairs $\pm\tau$, so there are exactly $Nd/2$ roots in $\mathbb{C}_+$ and $Nd/2$ roots in $\mathbb{C}_-$, and $M = Nd/2$. We have all together:

$$\text{parabolicity implies } d \text{ even and } M \equiv \textstyle\sum_{k=0}^{d-1} M_k = Nd/2. \tag{1.5.44}$$

Contrary to this, parameter-ellipticity on a single ray $\lambda = -\omega\mu^d$ does not require that d or Nd should be even. For example, if $\psi(t)$ is a real C^∞ function on $\mathbb{R}$ with $\psi(t) = 1$ for $|t| \ge 1$ and $\psi(t) = 0$ for $|t| \le \frac{1}{2}$, then $\widetilde{\lambda}_+^1(\xi) + \mu = (|\xi'|\psi(\frac{|\xi'|}{\varepsilon|\xi|}) + i\xi_n)\psi(|\xi|) + \mu$ is for small ε a parameter-elliptic symbol, for which the Green system $\{\widetilde{\lambda}^1_{+,\mathbb{R}^n_+}+\mu,\gamma_0\}$ satisfies (I)–(III) in Definition 1.5.5. The symbol $\widetilde{\lambda}^1_-(\xi)+\mu=(|\xi'|\psi(\frac{|\xi'|}{\varepsilon|\xi|})-i\xi_n)\psi(|\xi|)+\mu$ defines an operator that by itself (with empty boundary condition) satisfies (I)–(III). They are also parameter-elliptic when μ is replaced by $e^{i\theta}\mu$, with $\theta\in\,]-\frac{\pi}{2},\frac{\pi}{2}[$, but not parabolic. (Such symbols were considered in [BM71], [R-S82], [G86]; related versions are considered below in Section 2.5.)

Here are some further examples of parameter-elliptic systems:

1.5.11 Example. The system

$$\begin{pmatrix}-\Delta-\lambda\\ \gamma_0\end{pmatrix} \tag{1.5.45}$$

associated with the Dirichlet problem for the Laplace operator is a very simple example of a differential operator system that is parameter-elliptic on all the rays $\lambda=-e^{i\theta}r$ with $\theta\in\,]-\pi,\pi[$; in particular, it is parabolic. Here (I)–(III) are easily verified by computation (it suffices to show injectiveness, for dimensional reasons), or one can appeal to the 1-coerciveness of $(-\Delta)_{\gamma_0}$. To get a pseudodifferential example, we can perturb (1.5.45) a little, replacing the parameter-independent operator by, say,

$$\mathcal{S}=\begin{pmatrix}-\Delta+K_0\gamma_0+K_1\gamma_1+G'\\ \gamma_0+T_0'\end{pmatrix}, \tag{1.5.46}$$

where K_0 and K_1 are Poisson operators of orders 2 and 1, respectively, G' is a singular Green operator of order 2 and class 0, and T_0' is a trace operator of order 0 and class 0. If the principal symbol norms of K_0,K_1,G' and T_0' are sufficiently small, $\mathcal{A}_\lambda$ will again be parameter-elliptic on all the rays $\lambda=-e^{i\theta}r$ with $\theta\in\,]-\pi,\pi[$. In fact, the condition (II+III) for (1.5.46) can also be characterized as the bijectiveness of $\overline{a}^h(x',\xi',\mu,D_n)\colon H^2(\overline{\mathbb{R}}_+)\to L_2(\mathbb{R}_+)\times\mathbb{C}$ for all $(\xi',\mu)\in\overline{\mathbb{R}}^n_+$ with $|\xi',\mu|=1$. Since $\overline{\mathbb{R}}^n_+\cap S^{n-1}$ is compact, there is an upper bound on the norm of the inverse $(\overline{a}^h)^{-1}$ for $(\xi',\mu)\in\overline{\mathbb{R}}_+\cap S^{n-1}$. A perturbation $\overline{a}^h(x',\xi',\mu,D_n)+s^h(x',\xi',D_n)=\overline{a}^h(I+(\overline{a}^h)^{-1}s^h)$ will then be bijective, e.g., if the operator $(\overline{a}^h)^{-1}s^h$ in $H^2(\overline{\mathbb{R}}_+)$ has norm $\le\delta<1$, which is assured by taking s^h small enough for $|\xi'|\le 1$.

1.5.12 Example. One can also consider perturbations of the system defining the Neumann problem,

$$\mathcal{A}'=\begin{pmatrix}-\Delta+K_0\gamma_0+K_1\gamma_1+G'\\ \gamma_1+S_0\gamma_0+T_1'\end{pmatrix}, \tag{1.5.47}$$

with K_0, K_1, G' as described above, S_0 a ps.d.o. of order 1 on the boundary, and T_1' a trace operator of order 1 and class 0. Again the unperturbed operator (with K_0, K_1, G', S_0 and T_1' being zero) has the parameter-ellipticity property on all rays $\lambda = -e^{i\theta}r$, $\theta \in]-\pi, \pi[$, so $\mathcal{A}'_\lambda$ is parameter-elliptic when the perturbing operators have a sufficiently small principal symbol norm.

In these examples, one can replace $-\Delta$ by more general strongly elliptic ps.d.o.s, as accounted for in Section 1.7.

Of course, many more examples can be given on the basis of the m-coercive cases in general (the discussion of these in Section 1.7 is included for that purpose), and here one finds in some cases (e.g., the Dirichlet case) that the m-coerciveness can be violated under small perturbations that preserve the parameter-ellipticity.

The examples given at the end of Remark 1.5.10 are parameter-elliptic but not coercive (in the sense of satisfying a Gårding inequality).

1.5.13 Example. We shall now give an example where (I), (II) and even (II′) are valid, but (III) (as well as the desired resolvent estimates) does not hold. Consider

$$\mathcal{A}_\lambda = \begin{pmatrix} -\Delta - \lambda \\ T \end{pmatrix}, \text{ with } T = (1 - \Delta_\Gamma)^{\frac12} \gamma_1 R_D, \tag{1.5.48}$$

where R_D is the solution operator for the Dirichlet problem

$$R_D \colon u \mapsto v, \text{ where } -\Delta v = u \text{ in } \Omega,\ \gamma_0 v = 0 \text{ at } \Gamma; \tag{1.5.49}$$

the bijective operator $(1 - \Delta_\Gamma)^{\frac12} = \mathrm{OP}'(\langle \xi' \rangle)$ has been included in order to give the trace operator nonnegative order. When $\lambda \in \overline{\mathbb{R}}_-$, the problem

$$\begin{aligned} (-\Delta - \lambda)u &= f \text{ in } \Omega, \\ Tu &= \varphi \text{ at } \Gamma \end{aligned} \tag{1.5.50}$$

can by insertion of $u = -\Delta v$ ($v = R_D u$ as in (1.5.49)) be transformed to the problem

$$\begin{aligned} (\Delta^2 + \lambda\Delta)v &= f && \text{in } \Omega, \\ \gamma_0 v &= 0 && \text{at } \Gamma, \\ \gamma_1 v &= (1 - \Delta_\Gamma)^{-\frac12}\varphi && \text{at } \Gamma, \end{aligned} \tag{1.5.51}$$

which is simply a Dirichlet problem for the strongly elliptic fourth-order operator $\Delta^2 + \lambda\Delta$. Since $\Delta^2 + \lambda\Delta \geq \Delta^2$ for $\lambda \in \overline{\mathbb{R}}_-$, there is a unique solution to (1.5.51) for all smooth data.

On the symbol level, we give $-\Delta$ the principal boundary symbol $\sigma(\xi')^2 + D_n^2$, where $\sigma(\xi')$ is a positive C^∞ function on $\mathbb{R}^{n-1}$ coinciding with $|\xi'|$ for $|\xi'| \geq 1$. Then the boundary symbol operator $r_D^0(x', \xi', D_n)$ for R_D depends smoothly on $\xi' \in \mathbb{R}^{n-1}$ and the boundary symbolic problem corresponding to (1.5.51),

$$\begin{aligned} (\sigma(\xi')^2 + D_n^2)^2 v + \mu^2(\sigma(\xi')^2 + D_n^2)v &= f && \text{on } \mathbb{R}_+, \\ \gamma_0 v &= 0 && \text{at } x_n = 0, \\ \gamma_1 v &= \sigma(\xi')^{-1}\varphi && \text{at } x_n = 0, \end{aligned} \tag{1.5.52}$$

is uniquely solvable for all $(\xi', \mu) \in \mathbb{R}^n$. This gives us a smooth boundary symbol operator $\overline{a}^0(x', \xi', \mu, D_n)$ associated with (1.5.48), *which is bijective for all* $(\xi', \mu) \in \mathbb{R}^n$. Thus (I) and (II$'$) (and hence (II)) are satisfied.

However, the trace operator T is of order 0 and class 0, so the symbol norm $\|t^h\|$ is $O(|\xi'|^{\frac{1}{2}})$ (cf. (1.5.29)); it goes to 0 for $\xi' \to 0$, and the limit operator

$$\overline{a}^h(x', 0, \mu, D_n) = \begin{pmatrix} D_n^2 + \mu^2 \\ \\ 0 \end{pmatrix} : \mathcal{S}(\overline{\mathbb{R}}_+) \to \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \times \\ \mathbb{C} \end{matrix} \tag{1.5.53}$$

is not bijective. Thus (III) is not satisfied.

Moreover, a closer analysis of the resolvent R_λ for (1.5.48) will show that it does not satisfy the estimates we can deduce when (I)–(III) hold, namely, its norm as an operator in $L_2(\Omega)$ is merely $O(\langle\lambda\rangle^{-\frac{3}{4}})$, where the operators satisfying (I)–(III) will have resolvents that are $O(\langle\lambda\rangle^{-1})$ for $|\lambda| \to \infty$. (More comments are given in Remark 3.2.19 and Section 4.7.)

The example was pointed out to us by Denise Huet (in 1981), who also told us about the relation to *singular perturbation theory*: Set $\mu^{-1} = \varepsilon$ (i.e., $-\lambda = \varepsilon^{-2}$) and multiply the first line in (1.5.51) by ε^2; then one arrives at a type of problem

$$\begin{aligned} \varepsilon^2\Delta^2 v - \Delta v &= g, \\ \gamma_0 v &= \varphi_0, \\ \gamma_1 v &= \varphi_1, \end{aligned} \tag{1.5.54}$$

which has been widely studied in singular perturbation theory, and where the point is to analyze the relation between v and the solution w of the following limit problem for $\varepsilon \to 0$:

$$\begin{aligned} -\Delta w &= g, \\ \gamma_0 w &= \varphi_0. \end{aligned} \tag{1.5.55}$$

At first sight, our resolvent analysis did not seem to be of any help in the study of (1.5.54–55), because (1.5.48) fails to satisfy (III) in the definition of parameter-ellipticity. However, we found out that with some further reductions, (1.5.54) can indeed be transformed to a parameter-elliptic resolvent problem; see the complete discussion in Section 4.7.

There do exist nonnormal realizations for which the resolvent is $O(\langle\lambda\rangle^{-1})$; see, e.g., Remark 1.7.17 later.

We shall now introduce a further notion that plays a great role in the precise calculus.

Our hypothesis (III) requires a certain continuity of the strictly homogeneous boundary symbol operators for $\xi' \to 0$. For the fine analysis we shall make, it will be necessary to actually measure the *amount* of continuity (and it will be possible to distinguish between various types of discontinuity) by introduction of the so-called *regularity number* ν for the boundary symbol operators. The concept is analogous to the concept we mentioned for $p(x,\xi)$ earlier, where the regularity number ν indicated the number of "good" estimates in terms of a calculus where (ξ,μ) is considered as the cotangent variable; another description of ν was the amount of Hölder continuity of the strictly homogeneous symbol for $\xi \to 0$ (when $\mu > 0$). All this generalizes to boundary symbols.

The full machinery will be introduced in Chapter 2 (where also negative regularity numbers will be included), and we here just list some practical definitions.

1.5.14 Definition. *Let d be a positive integer, and let $k \geq 0$.*

1° *A trace operator T of order k and class 0 is of regularity $\nu(T) = k+\frac{1}{2}$.*

2° *A trace operator of order k and class $k+1$ containing only standard terms*

$$T = \textstyle\sum_{j\leq k} S_{kj}\gamma_j$$

is of regularity $\nu(T)$, where $\nu(T)$ is the lowest value $k-j$ for which S_{kj} is not a differential operator ($\nu = +\infty$ if all S_{kj} are differential operators).

3° *A general trace operator of order k and class $k+1$,*

$$T = \textstyle\sum_{j\leq k} S_{kj}\gamma_j + T'$$

is of regularity $\nu(T) = \min\{\nu(\sum_{j\leq k} S_{kj}\gamma_j), \nu(T')\}$.

4° *A Poisson operator K of order k is of regularity $\nu(K) = k - \frac{1}{2}$.*

5° *A singular Green operator G of order d and class 0 is of regularity $\nu(G) = d$.*

6° *A singular Green operator G of order d and class $r \in [1,d]$ is of regularity $\nu(G) = d - r + \frac{1}{2}$.*

The motivation for 1° is (1.5.29), 2° is motivated by the degree of homogeneity of the $s^h_{kj}(x', \xi')$, and 3° then takes the weakest regularity number from the two preceding cases. 4° is motivated by (1.5.30) (for $k = d - j$). The motivation for 5° is (1.5.32), and 6° is motivated by (1.5.31), where the lowest value of $d - j - \frac{1}{2}$ for $j \le r - 1$ is $d - r + \frac{1}{2}$.

In particular, we have for G and T considered in Definition 1.5.5 that

$$\nu(G) = \min\{d, d - r + \tfrac{1}{2}\} \tag{1.5.56}$$

when G is of order d and class $r \le d$, and that

$$\nu(T_k) = \min\{1, k + \tfrac{1}{2}\}, \tag{1.5.57}$$

or possibly better, if more than the first of the coefficients S_{kj} are differential operators. The smallest occurring regularity is here $\frac{1}{2}$, and the largest is d, unless there are only differential operators involved ($G = 0$ in particular), so that the regularity is $+\infty$.

In Chapter 2, the regularity numbers will be defined more generally for parameter-dependent symbols, and we shall see how Definition 1.5.14 fits together with that (see in particular Proposition 2.3.14 ff).

1.5.15 Example. In the Dirichlet-type problem (1.5.46), $-\Delta$ has regularity $+\infty$ (and if it is replaced by a second-order ps.d.o. P, one gets regularity 2), G' has regularity 2, γ_0 and γ_1 have regularity $+\infty$, K_0 has regularity $\frac{3}{2}$, and K_1 and T'_0 contribute with regularity $\frac{1}{2}$. As shown in Section 1.6 later (see Example 1.6.13), selfadjointness of the corresponding realization (with K_0 eliminated) requires $K_1 = iT_0'^*$ and $G' = G'^*$, here the system is of the form

$$\mathcal{A} = \begin{pmatrix} -\Delta + iT_0'^*\gamma_1 + G' \\ \gamma_0 + T'_0 \end{pmatrix}. \tag{1.5.58}$$

So when $T'_0 \ne 0$, we are in a (relatively difficult) case where the regularity of the system is only $\frac{1}{2}$.

For the Neumann-type problem (1.5.47) one gets selfadjointness precisely when the system, with K_1 eliminated, is of the form (cf. Example 1.6.15):

$$\mathcal{A}' = \begin{pmatrix} -\Delta + iT_1'^*\gamma_0 + G' \\ \gamma_1 + S_0\gamma_0 + T'_1 \end{pmatrix} \tag{1.5.59}$$

where $G' = G'^*$ and $S_0 = -S_0^*$. Here G' again contributes with regularity 2, T'_1 and $T_1'^*$ are of regularity $\frac{3}{2}$ and S_0 is of regularity 1, so the regularity of the full system is 1, at worst.

Observe that the selfadjoint Neumann-type cases have better regularity than the Dirichlet-type cases.

Let us also mention the fact, encountered in Section 1.7, that the m-coerciveness of Neumann-type problems is stable under small perturbations, whereas it is not so for Dirichlet-type problems (cf. (1.7.61) ff).

The Dirichlet-type problems (and other problems with regularity $\frac{1}{2}$) come up with full importance in the applications, as in Section 4.7 and in Grubb and Solonnikov [G-So87–91], [G91,95′].

1.5.16 REMARK. The study of boundary problems for $-\Delta - \lambda$ as in Example 1.5.13 is carried further in the articles of Rempel and Schulze on complex powers of ps.d.o.s without the transmission condition [R-S83,84]. On one hand, they analyze the λ-dependence of the resolvent in certain cases where R_λ exists on rays but does not have minimal growth; on the other hand they exhibit examples where R_λ does have rays of minimal growth but where the boundary condition is not *normal*. The example presented in Remark 1.7.17 is one of theirs.

The papers of Rempel and Schulze are of interest because they include ps.d.o.s that do not satisfy the transmission condition, which gives complications in the form of so-called Mellin terms. However, for the case of ps.d.o.s satisfying the transmission condition (the Boutet de Monvel calculus), the unusual examples are strongly linked with the case of constant coefficient operators [R-S83]. For variable coefficient operators, [R-S84] introduces symbol classes with a hypothesis on first ξ'-derivatives that is principally very similar to what we call *regularity* 1 (as defined in [G79,81,82] and in this book). When this is combined with boundedness hypotheses for inverse principal symbols, one arrives at a symbol class where the strictly homogeneous symbols have invertible limits for $\xi' \to 0$ (the condition on ξ'-derivatives implies a Lipschitz continuity at $\xi' = 0$); in particular this *restricts* the setup to normal boundary conditions when applied to differential boundary problems or Boutet de Monvel-type problems (for $n > 2$). This is contrary to the impression given in [R-S83,84], where they make a point of being more general in this respect.

One may also observe that the general hypotheses of [R-S84] do not allow *regularity* $\frac{1}{2}$ (thereby excluding some Dirichlet-type boundary conditions), and that they have a requirement on $D_{\xi_j} D_{\xi_n}$-derivatives of the interior pseudodifferential operator P, which is part of what we call *regularity* 2, and which excludes the important example $P = (-\Delta)^{\frac{1}{2}}$ of a ps.d.o. not satisfying the transmission condition.

1.6 Adjoints

We here continue the analysis of normal boundary problems. The reader who is interested mainly in the general calculus of pseudodifferential boundary problems should bypass Sections 1.6 and 1.7 in a first reading.

We shall show that the full trace operator is surjective and that the realization is densely defined; the adjoint is seen to be another normal realization, and the selfadjoint realizations are characterized. Basic examples are discussed. This study is not necessary for the resolvent construction in itself but is important for applications; in particular it prepares for the variational theory in Section 1.7. The formulas inevitably look complicated as soon as one deals with matrices or operators of order > 2; we here build on the methods developed in [G73]. Since the calculus for uniformly estimated symbols has not yet been introduced (it is done in Chapter 2), the statements are formulated for compact manifolds; but they extend straightforwardly to operators on admissible manifolds (cf. Sections A.5 and 2.4).

1.6.1 Lemma. *Let A be a differential operator in $\widetilde{E}$ of order d. Let $T = S\varrho$ be a normal trace operator with S going from $E_\Gamma^d = \bigoplus_{0\le k<d} E_\Gamma$ to $F = \bigoplus_{0\le k<d} F_k$, and let $Z = \bigoplus_{0\le k<d} Z_k$ be the vector bundle formed of the kernels Z_k of the morphisms $S_{kk}\colon E_\Gamma \to F_k$ (so Z is the kernel of $S_{\mathrm{diag}}\colon E_\Gamma^d \to F$, where $S_{\mathrm{diag}} = (\delta_{kj} S_{kj})_{k,j<d}$; the diagonal part of S). For each k, let S'_{kk} be a morphism in E_Γ projecting E_Γ onto Z_k, and let S' be the $d\times d$ diagonal matrix with diagonal elements $S'_{00}, \dots, S'_{d-1,d-1}$. Let $\widetilde{S}$ be the $d\times d$ matrix of ps.d.o. blocks $\widetilde{S}_{kj} = \begin{pmatrix} S_{kj} \\ S'_{kj}\end{pmatrix}$ going from E_Γ^d to $\bigoplus_{0\le k<d}(F_k \oplus Z_k)$ (here k,j run in $\{0,1,\dots,d-1\}$); $\widetilde{S}$ is also written as $\begin{pmatrix} S \\ S'\end{pmatrix}$ for short.*

Then $\widetilde{S} = \begin{pmatrix} S \\ S'\end{pmatrix}$ is a bijection,

$$\widetilde{S}\colon \textstyle\prod_{0\le j<d} H^{s-j}(E_\Gamma) \xrightarrow{\sim} \prod_{0\le k<d}[H^{s-k}(F_k)\times H^{s-k}(Z_k)]$$

for any $s\in\mathbb{R}$, and the inverse $\widetilde{C}$ is again a triangular matrix of ps.d.o.s, $\widetilde{C} = (\widetilde{C}_{jk})_{j,k<d}$, where the $\widetilde{C}_{jk}$ go from $H^{s-k}(F_k)\times H^{s-k}(Z_k)$ to $H^{s-j}(E_\Gamma)$, are equal to 0 for $j<k$ and of order $j-k$ for $j\ge k$, the $\widetilde{C}_{kk}$ being homeomorphisms from $F_k\oplus Z_k$ to E_Γ. We here write $\widetilde{C}_{jk} = (\,C_{jk}\quad C'_{jk}\,)$ where

$$C_{jk}\colon H^{s-k}(F_k)\to H^{s-j}(E_\Gamma),\quad C'_{jk}\colon H^{s-k}(Z_k)\to H^{s-j}(E_\Gamma),$$

and define the matrices $C = (C_{jk})_{j,k<d}$ and $C' = (C'_{jk})_{j,k<d}$, writing $\widetilde{C} = (\,C\quad C'\,)$ for short, so that

$$(1.6.1)\qquad (\,C\quad C'\,)\begin{pmatrix} S \\ S'\end{pmatrix} = I \text{ in } \textstyle\prod_{0\le k<d} H^{s-k}(E_\Gamma),$$

$$\begin{pmatrix} S \\ S'\end{pmatrix}(\,C\quad C'\,) = \begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} \text{ in } \textstyle\prod_{0\le k<d} H^{s-k}(F_k)\times\prod_{0\le k<d} H^{s-k}(Z_k).$$

Then finally, Green's formula may be written (cf. (1.3.13)*)*

$$
\begin{aligned}
(1.6.2)\quad (Au,v)_\Omega-(u,A'v)_\Omega &= (\mathfrak{A}\varrho u,\varrho v)_\Gamma\\
&=(S\varrho u, I^\times S''\varrho v)_\Gamma+(S'\varrho u, I^\times S'''\varrho v)_\Gamma,
\end{aligned}
$$

where the operators S'' and S''' (continuous for any $s\in\mathbb{R}$),

$$
(1.6.3)\quad
\begin{aligned}
S''&=I^\times C^*\mathfrak{A}^*\varrho\colon\ \textstyle\prod_{0\le k<d}H^{s-k}(E_\Gamma)\to\prod_{0\le k<d}H^{s-k}(F_{d-k-1})\\
S'''&=I^\times C'^*\mathfrak{A}^*\varrho\colon\ \textstyle\prod_{0\le k<d}H^{s-k}(E_\Gamma)\to\prod_{0\le k<d}H^{s-k}(Z_{d-k-1})
\end{aligned}
$$

are again triangular systems of ps.d.o.s with morphisms in the diagonal; here C^* and ${C'}^*$ are surjective.

When Γ is noncharacteristic for A, $S''\varrho$ and $S'''\varrho$ are normal trace operators.

PROOF. Write $\widetilde{S}=\widetilde{S}_{\text{diag}}+\widetilde{S}_{\text{sub}}$; here $\widetilde{S}_{\text{diag}}$ is a diagonal matrix with homeomorphisms

$$
\widetilde{S}_{kk}=\begin{pmatrix}S_{kk}\\ \\ S'_{kk}\end{pmatrix}:E_\Gamma\to\begin{matrix}F_k\\ \oplus\\ Z_k\end{matrix}
$$

in the diagonal, and $\widetilde{S}_{\text{sub}}$ is lower triangular, with zeroes in and above the diagonal. It is then easy to invert $\widetilde{S}$: Observe, for example, that

$$
U=(\widetilde{S}_{\text{diag}})^{-1}\widetilde{S}
$$

is a triangular matrix of ps.d.o.s in E_Γ^d, with identities in the diagonal, so that

$$
U=I+V
$$

with V lower triangular and nilpotent. Then

$$
(1+V)^{-1}=1-V+V^2-\cdots+(-V)^{d-1},
$$

so that

$$
\begin{aligned}
(1.6.4)\quad \widetilde{C}=\widetilde{S}^{-1}&=U^{-1}(\widetilde{S}_{\text{diag}})^{-1}\\
&=(1-V+V^2-\cdots+(-V)^{d-1})(\widetilde{S}_{\text{diag}})^{-1}.
\end{aligned}
$$

Since the iterates of V are lower triangular, with zeroes in the diagonal, $\widetilde{C}$ is lower triangular and has $\widetilde{C}_{\text{diag}}=(\widetilde{S}_{\text{diag}})^{-1}$. $\widetilde{C}$ is now split into two matrices C and C' as explained in the lemma, and we insert the first identity of (1.6.1) in Green's formula for A, which gives

$$\begin{aligned}(\mathfrak{A}\varrho u, \varrho v)_\Gamma &= (\mathfrak{A}(CS + C'S')\varrho u, \varrho v)_\Gamma \\ &= (S\varrho u, C^*\mathfrak{A}^*\varrho v)_\Gamma + (S'\varrho u, C'^*\mathfrak{A}^*\varrho v)_\Gamma \\ &= (S\varrho u, I^\times S''\varrho v)_\Gamma + (S'\varrho u, I^\times S'''\varrho v)_\Gamma\end{aligned}$$

(since $I^\times I^\times = I$). Here S'' and S''' are lower triangular, since, for example (cf. (1.3.12)),

$$S'' = I^\times C^*\mathfrak{A}^{\times *} I^\times,$$

where $C^*\mathfrak{A}^{\times *} = (\mathfrak{A}^\times C)^*$ is upper triangular. That

$$\begin{aligned}C^*&\colon \textstyle\prod_j H^{-s+j}(E_\Gamma) \to \prod_k H^{-s+k}(F_k) \text{ and} \\ C'^*&\colon \textstyle\prod_j H^{-s+j}(E_\Gamma) \to \prod_k H^{-s+k}(Z_k)\end{aligned} \tag{1.6.5}$$

are both surjective follows from the fact that $\widetilde{C}^* = \begin{pmatrix} C^* \\ C'^* \end{pmatrix}$ is surjective.

It remains to show that $S''\varrho$ and $S'''\varrho$ are normal in the noncharacteristic case, but this follows readily from the fact that the diagonal elements in $\mathfrak{A}^\times$ are then homeomorphisms in E_Γ: Since

$$\begin{pmatrix} S_{kk} \\ S'_{kk} \end{pmatrix} \begin{pmatrix} C_{kk} & C'_{kk} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \text{ in } F_k \oplus Z_k, \tag{1.6.6}$$

the C_{kk} and C'_{kk} are injective morphisms from F_k (resp. Z_k) to E_Γ; then so are the diagonal elements in $\mathfrak{A}^\times C$ (resp. $\mathfrak{A}^\times C'$); and the diagonal elements in $(\mathfrak{A}^\times C)^*$ (resp. $(\mathfrak{A}^\times C')^*$) are then surjective from E_Γ to F_k (resp. Z_k), so that finally the diagonal elements in S'' resp. S''' are surjective from E_Γ to F_{d-k-1} (resp. Z_{d-k-1}). $\square$

In view of Lemma 1.3.1 and Proposition 1.3.2, one has the corollary:

1.6.2 Corollary. *Let P be a ps.d.o. in $\widetilde{E}$ of order d, satisfying the transmission condition at Γ, and let $T = S\varrho$ be normal as in Lemma* 1.6.1. *Then the conclusions of Lemma* 1.6.1 *are valid with A and A^* replaced by P_+ and P_+^*, taking as $\mathfrak{A}$ the Green's matrix satisfying Proposition* 1.3.2. *In particular, one has the formula with entries described in Lemma* 1.6.1:

$$(P_+u, v)_\Omega - (u, P_+^*v)_\Omega = (S\varrho u, I^\times S''\varrho v)_\Gamma + (S'\varrho u, I^\times S'''\varrho v)_\Gamma; \tag{1.6.7}$$

here $S'\varrho$ is normal, and the trace operators $S''\varrho$ and $S'''\varrho$ are normal when Γ is noncharacteristic for P.

Observe that Lemma 1.6.1 in particular shows that normality of $T = S\varrho$ (which requires surjectiveness of the *diagonal* elements) *implies surjectiveness of the full matrix* S, since $SC = I$ by (1.6.1). Since the Cauchy boundary operator ϱ is surjective (from $C^\infty(E)$ to $C^\infty(E_\Gamma)^d$ or from $H^d(E)$ to $\prod_{0\le k<d} H^{d-k-\frac12}(E_\Gamma)$; cf. Lemma 1.6.4 below), it follows that *the trace operator $T = S\varrho$ is surjective from $H^d(E)$ to $\prod_{0\le k<d} H^{d-k-\frac12}(F_k)$.*

1.6.3 Remark. The lemma and its corollary look a little simpler in the scalar case and the case (1.4.23), where they show the following, with notation as in Remark 1.4.4: When T is a normal trace operator $T = \{T_k\}_{k\in J}$, with $J \subset J_\varrho = \{0, 1, \dots, d-1\}$ and $T_k = \gamma_k + \sum_{j<k} S_{kj}\gamma_j$ for $k \in J$, then T is surjective from $H^d(E)$ to $\prod_{k\in J} H^{d-k-\frac12}(E_\Gamma)$. Here we write $T = S\varrho$ where $S = (S_{kj})_{k\in J, j\in J_\varrho}$ (with $S_{kk} = I$, $S_{kj} = 0$ for $j > k$). Now set

$$\begin{aligned} J' &= J_\varrho \setminus J \\ J'' &= \{k \in J_\varrho \mid d-k-1 \in J\} \\ J''' &= \{k \in J_\varrho \mid d-k-1 \notin J\} \end{aligned}$$

and let $S'\varrho = \{\gamma_k\}_{k\in J'}$. When Γ is noncharacteristic for P, there exist auxiliary normal (and surjective) trace operators $S''\varrho$ and $S'''\varrho$ from $H^d(E)$ to $\prod_{k\in J''} H^{d-k-\frac12}(E_\Gamma)$, resp. $\prod_{k\in J'''} H^{d-k-\frac12}(E_\Gamma)$, such that

$$(P_+u, v)_\Omega - (u, P_+^* v)_\Omega = (S\varrho u, \dot I^\times S''\varrho u)_\Gamma + (S'\varrho u, \dot I^\times S'''\varrho u)_\Gamma, \tag{1.6.8}$$

where $\dot I^\times$ indicates a reflection of the index set (replacing $\{k\}_{k\in J}$ by $\{d-k-1\}_{k\in J}$).

In particular, if T is of the form $T = \{\gamma_k\}_{k\in J}$, then S and S' are the projections

$$S = \mathrm{pr}_J \text{ and } S' = \mathrm{pr}_{J'} \tag{1.6.9}$$

mapping a vector $\{\varphi_0, \varphi_1, \dots, \varphi_{d-1}\}$ into $\{\varphi_k\}_{k\in J}$, resp. $\{\varphi_k\}_{k\in J'}$; the operators C and C' (satisfying (1.6.1)) are the injections

$$C = \mathrm{in}_J \text{ and } C' = \mathrm{in}_{J'}, \tag{1.6.10}$$

supplying $\{\varphi_k\}_{k\in J}$ with zeroes in the places indexed by $k \in J'$ (resp. supplying $\{\varphi_k\}_{k\in J'}$ with zeroes in the places indexed by $k \in J$). Their adjoints are again projections

$$C^* = \mathrm{pr}_J \text{ and } C'^* = \mathrm{pr}_{J'},$$

so all together

$$
\begin{aligned}
S'' &= \dot{I}^{\times}\,\mathrm{pr}_J\,\mathfrak{A}^*\varrho = \mathrm{pr}_{J''}\,I^{\times}\mathfrak{A}^{\times *}I^{\times}\varrho,\\
S''' &= \dot{I}^{\times}\,\mathrm{pr}_{J'}\,\mathfrak{A}^*\varrho = \mathrm{pr}_{J'''}\,I^{\times}\mathfrak{A}^{\times *}I^{\times}\varrho,
\end{aligned} \tag{1.6.11}
$$

in this case. (When T is of a more general form $S\varrho$, S'' and S''' will also contain terms derived from S.)

We now turn to the more general normal trace operators $T = S\varrho + T'$ considered in connection with pseudodifferential operators, and we shall show that also these are surjective. This is obtained by use of the following lemma, where $H^{(0,d)}(E_{\Sigma'_+})$ stand for the space of L_2 functions on Σ'_+ (a neighborhood of Γ where $x = (x', x_n)$; see (A.64)ff.), whose x' derivatives up to order d are in L_2 (cf. also (A.23), (A.69)ff).

1.6.4 Lemma. *The Cauchy trace operator $\varrho = \{\gamma_k\}_{0\le k<d}\colon H^d(E) \to \prod_{0\le k<d} H^{d-k-\frac12}(E_\Gamma)$ has a family of continuous right inverses $\mathcal{K}_\delta$ for $\delta > 0$ (Poisson operators) that map into functions supported in Σ'_+ and whose norms, as operators from $\prod_{0\le k<d} H^{s-k-\frac12}(E_\Gamma)$ to $H^{(0,s)}(E_{\Sigma'_+})$ are $O(\delta)$ for $\delta \to 0$, any s.*

Proof. We first consider the situation for $\Omega = \mathbb{R}^n_+$, $N = 1$. As is wellknown, ϱ defines a homeomorphism between $H^d(\overline{\mathbb{R}}^n_+) \ominus H^d_0(\overline{\mathbb{R}}^n_+)$ and $\prod_k H^{d-k-\frac12}(\mathbb{R}^{n-1})$, so one cannot expect to obtain a small norm with respect to the metric of $H^d(\overline{\mathbb{R}}^n_+)$. But the situation is different with $H^{(0,d)}(\overline{\mathbb{R}}^n_+)$. We construct the right inverse as in Hörmander [H63, Theorem 2.5.7]. Let $\zeta \in C_0^\infty(\mathbb{R})$, with $\zeta = 1$ on a neighborhood of 0 and $\operatorname{supp}\zeta \subset\,]-1,1[$. For any $\delta > 0$, the system of operators

$$\mathcal{K}^{(\delta)} = \{\mathcal{K}_0^{(\delta)}, \dots, \mathcal{K}_{d-1}^{(\delta)}\}$$

defined by

$$(\mathcal{K}_j^{(\delta)}\varphi)(x', x_n) = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}\zeta(\delta^{-2}\langle\xi'\rangle x_n)\tfrac{1}{j!}x_n^j\hat{\varphi}(\xi')\,d\!\!{}^-\xi' \tag{1.6.12}$$

furnishes a right inverse of ϱ; they are Poisson operators (cf. (1.2.29), (1.2.19) or Section 2.3). Here we have for $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})$ and any $s \in \mathbb{R}$

$$
\begin{aligned}
\|\mathcal{K}_j^{(\delta)}\varphi\|^2_{H^{(0,s)}(\overline{\mathbb{R}}^n_+)} &= c_j\int_{\mathbb{R}^{n-1}}\int_0^\infty \langle\xi'\rangle^{2s}|\zeta(\delta^{-2}\langle\xi'\rangle x_n)x_n^j\hat{\varphi}(\xi')|^2\,dx_n d\!\!{}^-\xi'\\
&= c_j\delta^{4j+2}\int_{\mathbb{R}^{n-1}}\langle\xi'\rangle^{2s-2j-1}|\hat{\varphi}(\xi)|^2\int_0^\infty|\zeta(t)t^j|^2\,dt d\!\!{}^-\xi'\\
&= c_j'\delta^{4j+2}\|\varphi\|^2_{H^{s-j-\frac12}(\mathbb{R}^{n-1})}.
\end{aligned} \tag{1.6.13}
$$

(One can similarly show that

$$\|D_{x_n}^k \mathcal{K}_j^{(\delta)}\varphi\|_{H^{(0,s)}}^2 = c_{j,k}\delta^{4j-4k+2}\|\varphi\|_{s+k-j-\frac{1}{2}}^2 .) \tag{1.6.14}$$

The desired result is obtained by taking $\mathcal{K}_\delta = \zeta\mathcal{K}^{(\delta)}$. The construction is carried over to the manifold situation by use of local coordinates. $\square$

1.6.5 Proposition. *Let $T = S\varrho + T'$ be a normal trace operator as defined in Definition 1.4.3. Then T is surjective from $H^d(E)$ to $\prod_{0\le k<d} H^{d-k-\frac{1}{2}}(F_k)$ (and from $C^\infty(E)$ to $\prod_k C^\infty(F_k)$), with a continuous right inverse K (a Poisson operator). For any $\varepsilon > 0$ and any $s \in \mathbb{R}$, K can be chosen such that it maps into functions supported in Σ'_+ and such that its norm as an operator from $\prod_{0\le k<d} H^{s-k-\frac{1}{2}}(F_k)$ to $H^{(0,s)}(E_{\Sigma'_+})$ is smaller than ε.*

Proof. Consider first the case where $S = I$. We search for a solution u of the equation

$$\varrho u + T'u = \varphi. \tag{1.6.15}$$

Since T' is of class 0, it is continuous from $H^{(0,d)}(E_{\Sigma'_+})$ to $\prod_k H^{d-k-\frac{1}{2}}(E_\Gamma)$ (cf. (1.2.27)), so by Lemma 1.6.4, the norm of $T'\mathcal{K}_\delta$ as an operator in $\prod_k H^{d-k-\frac{1}{2}}(E_\Gamma)$ goes to 0 for $\delta \to 0$. Let δ_0 be such that this norm is $\le \frac{1}{3}$ for $\delta \le \delta_0$ and the same holds on the boundary symbol level. Then $S_\delta = I + T'\mathcal{K}_\delta$ is a ps.d.o. in E_Γ^d and is elliptic (as a system $(S_{jk})_{j,k<d}$ where S_{jk} is of order $j-k$). Moreover, S_δ has an inverse, the convergent Neumann series

$$S_\delta^{-1} = (I + T'\mathcal{K}_\delta)^{-1} = \textstyle\sum_{m=0}^\infty (-T'\mathcal{K}_\delta)^m \tag{1.6.16}$$

with the norm of $\sum_{m=1}^\infty (-T'\mathcal{K}_\delta)^m \le \frac{1}{2}$. It is seen as in the proof of Proposition 1.4.2 (in a simpler version) that S_δ^{-1} differs from any parametrix of S_δ by an integral operator with C^∞ kernel, so S_δ^{-1} is a ps.d.o. in the calculus; it is elliptic. (The ellipticity is here taken in the extended sense mentioned after (1.2.54).) Finally, (1.6.15) has the solution

$$u = \mathcal{K}_\delta S_\delta^{-1}\varphi, \tag{1.6.17}$$

where $K = \mathcal{K}_\delta S_\delta^{-1}$ is a Poisson operator; we can take δ so small that the norm from $\prod_k H^{s-k-\frac{1}{2}}$ to $H^{(0,s)}(E_{\Sigma'_+})$ is $\le \varepsilon$.

Now let S be general, so we have to solve the equation

$$S\varrho u + T'u = \varphi. \tag{1.6.18}$$

Here we use Lemma 1.6.1 and consider the larger system

$$\begin{aligned} S\varrho u + T'u &= \varphi \\ S'\varrho u &= 0, \end{aligned} \tag{1.6.19}$$

which by composition with $\begin{pmatrix} C & C' \end{pmatrix}$ leads to the equation

$$\varrho u + C\,T'u = C\varphi,$$

CT' being again of class 0. By the first part of the proof, this has a solution

$$u = \mathcal{K}_\delta S_\delta^{-1} C\varphi = K\varphi,$$

where K can be estimated as asserted. $\square$

1.6.5′ Remark. The result of Proposition 1.6.5 extends to the situation of noncompact manifolds introduced later in this book. Here the proof that S_δ^{-1} belongs to the calculus is carried out as in the proof of Theorem 3.2.11, showing that Q_μ belongs to the calculus.

Also Lemma 1.6.8 extends; the operator $(I + CT'\mathcal{K}_\delta)^{-1}$ there is shown in a similar way to belong to the calculus.

Before applying these results, let us register their variants on the one-dimensional level.

1.6.6 Lemma.

1° *The Cauchy trace operator* $\varrho\colon H^d(\overline{\mathbb{R}}_+)^N \to \prod_{0\le j<d} \mathbb{C}^N$ *has a family of continuous right inverses* k_δ *for* $\delta > 0$ *(Poisson operators), mapping into functions supported in* $[0,1[$, *and with the property that the norm of* k_δ *as an operator from* $\prod_{0\le k<d} \mathbb{C}^N$ *to* $L_2(\mathbb{R}_+)^N$ *goes to* 0 *for* $\delta \to 0$.

2° *When* $t(x', \xi', D_n)\colon H^d(\overline{\mathbb{R}}_+)^N \to \prod_{0\le j<d} \mathbb{C}^{M_j}$ *is the boundary symbol operator associated with a normal trace operator* T, *it has a continuous right inverse* $k(x', \xi', D_n)$ *(a Poisson operator); here* k *can for any* $\varepsilon > 0$ *be taken with norm* $\le \varepsilon$ *as an operator from* $\prod_{0\le j<d} \mathbb{C}^{M_j}$ *to* $L_2(\mathbb{R}_+)^N$, *for* x' *in a compact set and* $|\xi'| = 1$.

Proof. One can take $k_\delta = \zeta(x_n) k^{(\delta)}$, where $k^{(\delta)}\varphi = \Sigma_{0\le j<d} k_j^{(\delta)} \varphi_j$ with $k_j^{(\delta)}\varphi_j = \zeta(\delta^{-2}x_n)\frac{1}{j!}x_n^j \varphi_j$ for $\varphi_j \in \mathbb{C}^N$; the proof then follows the same lines as above. $\square$

We recall that the Poisson operators on the one-dimensional level always map into $\mathcal{S}(\overline{\mathbb{R}}_+)^N$.

1.6.7 Corollary. *Consider polyhomogeneous symbols. Let P be elliptic. When $B = (P+G)_T$ is defined by a normal trace operator T, then $\begin{pmatrix} P_+ + G \\ T \end{pmatrix}$ is elliptic if and only if the principal boundary realization $b^0(x', \xi', D_n)$ (cf. (1.4.11′)) defines a bijection of $D(b^0(x', \xi', D_n))$ onto $L_2(\mathbb{R}_+)^N$ for all x', all $|\xi'| \geq 1$.*

Proof. As noted in Section 1.2, ellipticity holds precisely when

$$\begin{pmatrix} p^0(x', 0, \xi', D_n)_+ + g^0(x', \xi', D_n) \\ \\ t^0(x', \xi', D_n) \end{pmatrix} : H^d(\overline{\mathbb{R}}_+)^N \to \begin{matrix} L_2(\mathbb{R}_+)^N \\ \times \\ \mathbb{C}^M \end{matrix} \tag{1.6.20}$$

is bijective (for $|\xi'| \geq 1$). By the surjectiveness of t^0 (Lemma 1.6.6), the solvability question for the problem

$$\begin{aligned} (p^0_+ + g^0)u &= f, \quad f \in L_2(\mathbb{R}_+)^N, \\ t^0 u &= \varphi, \quad \varphi \in \mathbb{C}^M = \textstyle\prod_{0 \leq k < d} \mathbb{C}^{M_k}, \end{aligned} \tag{1.6.21}$$

can be reduced to the solvability question for

$$\begin{aligned} (p^0_+ + g^0)u &= f, \quad f \in L_2(\mathbb{R}_+)^N, \\ t^0 u &= 0, \end{aligned}$$

so that unique solvability of (1.6.21) with $u \in H^d(\overline{\mathbb{R}}_+)^N$ holds if and only if b^0 is bijective. $\square$

We can also use Lemma 1.6.4 to show that normal realizations have dense domains. In fact, we can show how the domains are homeomorphic to domains defined by normal boundary conditions *without* the nonlocal term T'. Since $C_0^\infty(\Omega)$ is a dense subset here, the denseness follows immediately. The elimination of T' can be convenient for special purposes (and is used in Sections 1.7 and 4.4), but in the functional calculus itself where we define functions of a realization B, one cannot simply replace B by another realization.

1.6.8 Lemma. *Let $T = S\varrho + T'$ (going from E to $\bigoplus_{0 \leq k < d} F_k$) be a normal trace operator associated with the order d.*

1° *Let C be a right inverse of S according to Lemma* 1.6.1. *There is a $\delta_0 > 0$ such that for $\delta \in]0, \delta_0]$, the operator*

$$\Lambda = I + \mathcal{K}_\delta C T' \tag{1.6.22}$$

(cf. Lemma 1.6.4*) is a homeomorphism in $H^s(E)$ for any $s \geq 0$, with inverse of the form*

$$\Lambda^{-1} = I + \mathcal{K}_\delta Q_\delta C T', \tag{1.6.22′}$$

for a δ-dependent ps.d.o. Q_δ on $\bigoplus_{0\le k<d} E_\Gamma$ that is bijective and elliptic in $\prod_{0\le k<d} H^{s-k}(E_\Gamma)$. Here one has the identity

$$T = S\varrho\Lambda. \tag{1.6.23}$$

In particular, Λ defines a bijection

$$\Lambda\colon \{u \in H^d(E) \mid Tu = 0\} \xrightarrow{\sim} \{v \in H^d(E) \mid S\varrho v = 0\}. \tag{1.6.23$'$}$$

2° *The set $\{u \in H^d(E) \mid Tu = 0\}$ is dense in $L_2(E)$.*

Proof. Consider Λ defined by (1.6.22). By Lemma 1.6.4, $\mathcal{K}_\delta$ maps $\prod_{0\le k<d} H^{-k-\frac12}(E_\Gamma)$ into $L_2(E)$ with a norm that is $O(\delta)$, so the norm of $\mathcal{K}_\delta CT'$ in $L_2(E)$ is $O(\delta)$. Similarly, the norm of the operator $T'\mathcal{K}_\delta C$ in $\prod_k H^{-k-\frac12}(E_\Gamma)$ is $O(\delta)$. We now take δ_0 so small that these norms are $\le \frac13$ for $\delta \in \,]0,\delta_0]$ and the same holds on the boundary symbol level; then we can invert Λ by a Neumann series

$$\Lambda^{-1} = I + \textstyle\sum_{m=1}^\infty (-\mathcal{K}_\delta CT')^m,$$

converging in operator norm in $L_2(E)$. In particular,

$$\begin{aligned}\textstyle\sum_{m=1}^\infty (-\mathcal{K}_\delta CT')^m &= \mathcal{K}_\delta CT' - \mathcal{K}_\delta \textstyle\sum_{m=1}^\infty (-CT'\mathcal{K}_\delta)^m CT' \\ &= -\mathcal{K}_\delta (1 + CT'\mathcal{K}_\delta)^{-1} CT'.\end{aligned}$$

It is seen as in the proof of Proposition 1.6.5 that the inverse $(I+CT'\mathcal{K}_\delta)^{-1}$ of the ps.d.o. $I + CT'\mathcal{K}_\delta$ in $\prod_k H^{-\frac12-k}(E_\Gamma)$ belongs to the calculus (and is elliptic). Thus Λ^{-1} has the form stated in (1.6.22$'$)ff. Now we see that when $v = \Lambda u$ for $u \in H^d(E)$, then since $\varrho\mathcal{K}_\delta = I$ and $SC = I$ (cf. (1.6.1)),

$$S\varrho v = S(\varrho u + \varrho\mathcal{K}_\delta CT'u) = (S\varrho + T')u,$$

which shows (1.6.23). In particular, $S\varrho v = 0$ if and only if $S\varrho u + T'u = 0$, which completes the proof of 1°.

Now 2° is an immediate consequence, for $C_0^\infty(\Omega, E)$ is contained in $\{v \in H^d(E) \mid S\varrho v = 0\}$, and is dense in $L_2(E)$, which by Λ^{-1} carries over to the denseness of $\Lambda^{-1}C_0^\infty(\Omega, E) \subset \{u \in H^d(E) \mid Tu = 0\}$ in $L_2(E)$. $\square$

The lemma shows that normal realizations are densely defined. When B is a normal realization, we can then analyze its *adjoint* B^* (as $L_2(E)$-operators). The *formal adjoint* B' will be defined as the operator acting like B^* and with domain

$$D(B') = D(B^*) \cap H^d(E). \tag{1.6.24}$$

1.6.9 Theorem. *Let P be a ps.d.o. of order $d \in \mathbb{N}$ in $\widetilde{E}$, satisfying the transmission condition at Γ, and assume that Γ is noncharacteristic for P. Let G be an s.g.o. in E of order d and class $\le d$, let T be a normal trace operator (going from E to $\bigoplus_{0\le k<d} F_k$), and let $B = (P+G)_T$ be the realization acting like $P_+ + G$ and with domain*

$$D(B) = \{u \in H^d(E) \mid Tu = 0\}.$$

Write

$$T = S\varrho + T', \quad G = K\varrho + G',$$

where T' and G' are of class 0, and define S', C, C', S'' and S''' as in Lemma 1.6.1, Corollary 1.6.2. Then the formal adjoint B' of B is the realization $(P^* + \widetilde{G})_{\widetilde{T}}$, where $\widetilde{G}$ is the s.g.o. of order d and class $\le d$,

$$\begin{aligned}\widetilde{G} &= -{T'}^* I^\times S''\varrho + {G'}^* - {T'}^* C^* K^* \\ &= -{T'}^* C^* \mathfrak{A}^* \varrho + {G'}^* - {T'}^* C^* K^*,\end{aligned} \tag{1.6.25}$$

and $\widetilde{T}$ is the normal trace operator going from E to $\bigoplus_{0\le k<d} Z_{d-k-1}$, defined by

$$\widetilde{T} = S'''\varrho + I^\times {C'}^* K^* = I^\times {C'}^* \mathfrak{A}^* \varrho + I^\times {C'}^* K^*. \tag{1.6.26}$$

In the polyhomogeneous case, when B is elliptic, B^* equals B' and is likewise elliptic.

Proof. Let $v \in H^d(E)$; we shall determine when it belongs to $D(B^*)$. Using the Green's formula (1.6.7) adapted specially to $S\varrho$, we have for $u \in D(B)$,

$$\begin{aligned}((P_+{+}G)u, v)_\Omega &= (P_+u, v)_\Omega + (K\varrho u, v)_\Omega + (G'u, v)_\Omega \\ &= (u, P_+^* v)_\Omega + (S\varrho u, I^\times S''\varrho v)_\Gamma + (S'\varrho u, I^\times S'''\varrho v)_\Gamma \\ &\quad + ((CS + C'S')\varrho u, K^* v)_\Gamma + (u, {G'}^* v)_\Omega \\ &= (u, P_+^* v + {G'}^* v)_\Omega - (T'u, I^\times S''\varrho v + C^* K^* v)_\Gamma \\ &\quad + (S'\varrho u, I^\times S'''\varrho v + {C'}^* K^* v)_\Gamma \\ &= (u, (P_+^* + {G'}^* - {T'}^* I^\times S''\varrho - {T'}^* C^* K^*)v)_\Omega \\ &\quad + (S'\varrho u, (I^\times S'''\varrho + {C'}^* K^*)v)_\Gamma \\ &= (u, (P_+^* + \widetilde{G})v)_\Omega + (S'\varrho u, I^\times \widetilde{T} v)_\Gamma;\end{aligned} \tag{1.6.27}$$

cf. (1.6.1). It is clear that when $\widetilde{T}v = 0$, then $v \in D(B^*)$ and $B^*v = (P_+^* + \widetilde{G})v$, so $B^* \supset (P^* + \widetilde{G})_{\widetilde{T}}$. Conversely, we must show that when $v \in D(B^*) \cap H^d(E)$, then $\widetilde{T}v = 0$, which is seen as follows:

Observe first that when $v \in D(B^*) \cap H^d(E)$, the last line in (1.6.27) is a continuous functional on u in the $L_2(E)$-norm, so there is a constant $c(v)$ for which

$$|(S'\varrho u, I^\times \widetilde{T}v)_\Gamma| \le c(v)\|u\|_0, \text{ for } u \in D(B). \tag{1.6.28}$$

We only have to consider the case where $\widetilde{T}$ is not void, i.e., $\sum_k \dim Z_k > 0$. Assume that there is a $v \in D(B^*) \cap H^d(E)$ such that $\widetilde{T}v = \psi \ne 0$; we shall show that this leads to a contradiction. Let $\varphi \in \prod_{0 \le k < d} C^\infty(Z_k)$, with

$$(\varphi, I^\times \psi) \ne 0. \tag{1.6.29}$$

Applying Proposition 1.6.5 to the normal trace operator $\begin{pmatrix} T \\ S'\varrho \end{pmatrix}$, we can for each $\varepsilon > 0$ find $u_\varepsilon \in H^d(E)$, supported in Σ'_+ and with

$$Tu_\varepsilon = 0, \quad S'\varrho u_\varepsilon = \varphi,$$

and $\|u_\varepsilon\|_{(0,d)} \le \varepsilon$. Now $u_\varepsilon \in D(B)$, so by (1.6.28)

$$|(S'\varrho u_\varepsilon, I^\times \widetilde{T}v)| = |(\varphi, I^\times \psi)| \le c(v)\|u_\varepsilon\|_0 \le c(v)\|u_\varepsilon\|_{(0,d)} \to 0 \text{ for } \varepsilon \to 0;$$

this contradicts (1.6.29). Then $\widetilde{T}v$ must equal 0, and we have shown that $B' = (P^* + \widetilde{G})_{\widetilde{T}}$. Clearly, $\widetilde{T}$ is normal.

Consider finally the polyhomogeneous and elliptic case. We first show the ellipticity of the adjoint, working on the boundary symbol level. Recall that in the elliptic case, the associated boundary symbol operator $\begin{pmatrix} p^0_+ + g^0 \\ t^0 \end{pmatrix}$ is bijective from $H^d(\overline{\mathbb{R}}_+)^N$ to $L_2(\mathbb{R}_+)^N \times \mathbb{C}^M$ with inverse $(r^0 \quad k^0)$, and the boundary symbol realization $b^0 = (p^0_+ + g^0)_{t^0}$ is bijective from $D(b^0)$ to $L_2(\mathbb{R}_+)^N$ with inverse r^0 (at each (x', ξ') with $|\xi'| \ge 1$). The preceding arguments give that b^0 is densely defined and that b^{0*} is an extension of the boundary symbol realization b'^0 corresponding to the system

$$\begin{pmatrix} (p^{0*})_+ + \widetilde{g}^0 \\ \widetilde{t}^0 \end{pmatrix} \tag{1.6.30}$$

associated with (1.6.25)–(1.6.26), in such a way that

$$D(b^{0*}) \cap H^d(\overline{\mathbb{R}}_+)^N = D(b'^0). \tag{1.6.31}$$

Now r^0 is a bounded operator in $L_2(\mathbb{R}_+)^N$ of the form $(p^0)^{-1}_+ + g^0_1$, where $(p^0)^{-1}$ is a ps.d.o. of order $-d$ and where g^0_1 is an s.g.o. of order $-d$ and class 0. Then

$$(r^0)^* = ((p^0)^{-1}_+ + \widetilde{g}^0)^* = (p^{0*})^{-1}_+ + g^{0*}_1,$$

which has a similar structure as r^0; so in particular it maps $L_2(\mathbb{R}_+)^N$ into $H^d(\overline{\mathbb{R}}_+)^N$. Then since b^0 is a bijection of the dense domain $D(b^0)$ onto $L_2(\mathbb{R}_+)^N$, it follows from an elementary result on Hilbert space operators that the adjoint b^{0*} is the inverse of the adjoint r^{0*}. In particular, the domain of b^{0*} is contained in $H^d(\overline{\mathbb{R}}_+)^N$, so in fact

$$b^{0*} = b'^0 = ((p^{0*})_+ + \widetilde{g}^0)_{\widetilde{t}^0}.$$

This implies that the problem

$$b'^0 u = f, \text{ that is, } (p^{0*}_+ + \widetilde{g}^0)u = f, \ \widetilde{t}^0 u = 0,$$

has a unique solution $u \in H^d(\overline{\mathbb{R}}_+)^N$ for any $f \in L_2(\mathbb{R}_+)^N$, and hence $\{P^* + \widetilde{G}, \widetilde{T}\}$ and B' are *elliptic*, in view of Corollary 1.6.7.

It remains to show that $B' = B^*$ in the elliptic case. We here use that B has a "good" parametrix R_0 as stated in Proposition 1.4.2. Since R_0 is a bounded operator in $L_2(E)$, one has by standard arguments

$$R_0^* B^* \subset (BR_0)^*, \text{ which equals } I + \mathcal{S}^*.$$

Then if $f \in D(B^*)$, one finds that

$$f = R_0^* B^* f - \mathcal{S}^* f \in H^d(E),$$

since $R_0^* = (P^{-1})^*_+ + G_0^*$ maps $L_2(E)$ into $H^d(E)$ and $\mathcal{S}^*$ is smoothing. Thus $B^* = B'$, in view of (1.6.24). $\square$

1.6.9′ Example. As a special exercise, let us find the formal adjoint of the "minimal operator" B_0 associated with $P_+ + G$, i.e., the realization defined from the system $\{P_+ + G, \varrho\}$. Here when $G = K\varrho + G'$, we have in fact that $B_0 = (P + G')_\varrho$, since $\varrho u = 0$ on the domain. The trace operator is ϱ (in particular $T' = 0$), so that $S = I = C$ in the terminology of Lemma 1.6.1, and S' and C' are zero (or void), for the range of S' is zero-dimensional. The formulas (1.6.25) and (1.6.26) give that $\widetilde{G} = G'^*$, and $\widetilde{T}$ is void. Thus the formal adjoint $(B_0)'$ equals the realization acting like $P^*_+ + G'^*$ and with domain $H^d(E)$, i.e., the "maximal operator" for $P^*_+ + G'^*$.

It follows in particular that when $B = (P + G)_T$ is a normal realization for which $D(B) \supset H_0^d(E)$ (i.e., $B \supset B_0$), then the formal adjoint B' acts like $P^* + G'^*$ (since $B' \subset (B_0)'$), where G'^* is *of class* 0. Note that as a special consequence, if B is selfadjoint with $D(B) \supset H_0^d(E)$, it acts like $P_+ + G_1$ with G_1 of class 0.

An obvious corollary of the theorem is that it suffices for formal selfadjointness of B, that

$$\widetilde{G} = G \text{ and } \widetilde{T} = \Psi T \tag{1.6.32}$$

for some homeomorphism Ψ, where $\widetilde{G}$ and $\widetilde{T}$ are defined by (1.6.25) and (1.6.26).

However, the full condition (1.6.32) is not *necessary* for selfadjointness, since G is in general not completely determined from a realization $B = (P+G)_T$. In fact, since $Tu = 0$ for $u \in D(B)$, one has that

$$(P+G)_T = (P+G+K_1T)_T$$

for any Poisson operator K_1. So in order to characterize the selfadjoint realizations, we first investigate the extent to which a realization $B = (P+G)_T$ determines G and T. Here, when $T = S\varrho + T'$ and $G = K\varrho + G'$, an insertion of $Tu = 0$ and $\varrho u = CS\varrho u + C'S'\varrho u$ gives that Gu can be replaced by

$$G_1u = KC'S'\varrho u + (G' - KCT')u = K_1S'\varrho u + G_1'u,$$
$$\text{with } K_1 = KC' \text{ and } G_1' = G' - KCT',$$

so we can always assume that G is given in the form $G = KS'\varrho + G'$, to make the analysis more precise.

1.6.10 Theorem. *Let $B = (P+G)_T$ and $\overline{B} = (P+\overline{G})_T$ be normal realizations of a pseudodifferential operator P of order $d > 0$, the trace operators $T = S\varrho + T'$ (resp. $\overline{T} = \overline{S}\varrho + \overline{T}'$) mapping into bundles $F = \bigoplus_{0\le k<d} F_k$ (resp. $\overline{F} = \bigoplus_{0\le k<d} \overline{F}_k$). Then we have, with the notation of Lemma* 1.6.1 *and Corollary* 1.6.2 *applied to the two operator systems:*

1° *$D(B) = D(\overline{B})$ holds if and only if $Z_k = \overline{Z}_k$ for $k = 0, \dots, d-1$, and there exists a triangular homeomorphism $\Psi = (\Psi_{jk})_{0\le j,k<d}$,*

$$\Psi\colon \textstyle\prod_{0\le k<d} H^{s-k}(F_k) \to \prod_{0\le k<d} H^{s-k}(\overline{F}_k), \tag{1.6.33}$$

with bijective morphisms $\Psi_{kk}\colon F_k \to \overline{F}_k$ in the diagonal and ps.d.o.s Ψ_{jk} of order $j-k$ below it, such that

$$\overline{T} = \Psi T, \text{ i.e., } \overline{S} = \Psi S \text{ and } \overline{T}' = \Psi T'. \tag{1.6.34}$$

2° *Let $D(B) = D(\overline{B})$. Since $Z_k = \overline{Z}_k$ for all k according to 1°, the operator S' defined in Lemma* 1.6.1 *supplies S as well as $\overline{S}$ to homeomorphisms in $\prod_{0\le k<d} H^{s-k}(E_\Gamma)$. Assume (as we may) that the singular Green operators G and $\overline{G}$ have been reduced to the form*

$$G = KS'\varrho + G', \text{ resp. } \overline{G} = \overline{K}S'\varrho + \overline{G}', \tag{1.6.34'}$$

by insertion of the homogeneous boundary conditions; K and $\overline{K}$ go from $\prod_{0\le k<d} H^{d-k-\frac12}(Z_k)$ to $H^d(E)$. Then $B=\overline{B}$ if and only if

$$K=\overline{K},\quad G'=\overline{G}'. \tag{1.6.35}$$

Proof. In 1°, the mentioned properties of course imply $D(B)=D(\overline{B})$. Conversely, assume that $D(B)=D(\overline{B})$, i.e.,

$$Tu=0 \iff \overline{T}u=0,\ \text{ for } u\in H^d(E). \tag{1.6.36}$$

Let Λ be a bijection in $H^s(E)$ for $s\ge 0$, as defined in Lemma 1.6.8, such that $T=S\varrho\Lambda$; note that

$$\Lambda - I=\mathcal{K}_\delta CT' \text{ and } \Lambda^{-1}-I=\mathcal{K}_\delta Q_\delta CT'$$

are singular Green operators of order and class 0. For $v=\Lambda u\in H^d(E)$, we then have

$$\begin{aligned} S\varrho v=0 &\iff T\Lambda^{-1}v=0 \iff \overline{T}\Lambda^{-1}v=0\\ &\iff (\overline{S}\varrho+\overline{T}')\Lambda^{-1}v=0 \iff \overline{S}\varrho v+T''v=0,\\ &\text{where } T''=\overline{S}\varrho(\Lambda^{-1}-1)+\overline{T}'\Lambda^{-1} \text{ is of class 0.}\end{aligned}$$

Since $S\varrho v=0$ is satisfied by all $v\in C_0^\infty(\Omega,E)$, and such v also have $\overline{S}\varrho v=0$, it follows that $T''v=0$ for $v\in C_0^\infty(\Omega,E)$, and hence $T''\equiv 0$ since it is of class 0. This proves the identity

$$\overline{S}\varrho=\overline{S}\varrho\Lambda^{-1}+\overline{T}'\Lambda^{-1},$$

which implies

$$\overline{S}\varrho\Lambda=\overline{S}\varrho+\overline{T}'=\overline{T}.$$

So we now get from (1.6.36), taking $v=\Lambda u\in H^d(E)$

$$S\varrho v=0 \iff \overline{S}\varrho v=0,\ \text{ for } v\in H^d(E). \tag{1.6.37}$$

Since these trace operators are without "interior term", the techniques of [G73] can be applied. Let us denote

$$\begin{aligned} Z^t(S)&=\{\,\varphi\in \textstyle\prod_{0\le k<d} H^{t-k-\frac12}(E_\Gamma)\mid S\varphi=0\,\},\\ R^t(S)&=S\textstyle\prod_{0\le k<d} H^{t-k-\frac12}(E_\Gamma),\end{aligned}$$

with analogous definitions for other operators. Recall that S is triangular and is the sum of a diagonal part and a (nilpotent) subtriangular part

$$S = S_{\mathrm{diag}} + S_{\mathrm{sub}},$$

where S_{diag} has morphisms S_{kk} in the diagonal. The same holds for S', C and C' defined in Lemma 1.6.1 as well as for $\overline{S}$ and its associated operators $\overline{S}'$, $\overline{C}$ and $\overline{C}'$. Since ϱ is surjective from $H^d(E)$ to $\prod_{0\le k<d} H^{d-k-\frac12}(E_\Gamma)$, (1.6.37) is equivalent with the statement

$$Z^d(S) = Z^d(\overline{S}). \tag{1.6.38}$$

Since $SC = I$, one has that $Z^d(S) = R^d(I - CS)$, so (1.6.38) implies

$$\overline{S}(I - CS) = 0. \tag{1.6.39}$$

All the matrices here are triangular, and we find, taking the diagonal part, that

$$\overline{S}_{\mathrm{diag}}(I - C_{\mathrm{diag}}S_{\mathrm{diag}}) = 0.$$

Then since $S_{\mathrm{diag}}C_{\mathrm{diag}} = I$ (the diagonal part of the equation $SC = I$), we find, using the above steps backwards, that

$$Z^d(\overline{S}_{\mathrm{diag}}) \supset R^d(I - C_{\mathrm{diag}}S_{\mathrm{diag}}) = Z^d(S_{\mathrm{diag}}).$$

The argument also works with S and $\overline{S}$ interchanged, so all together

$$Z^d(S_{\mathrm{diag}}) = Z^d(\overline{S}_{\mathrm{diag}}).$$

Since S_{diag} is a morphism, we may here identify

$$\begin{aligned} Z^d(S_{\mathrm{diag}}) &= \{\, \varphi \in \textstyle\prod_{0\le k<d} H^{d-k-\frac12}(E_\Gamma) \mid S_{\mathrm{diag}}\varphi = 0 \,\} \\ &= \textstyle\prod_{0\le k<d} H^{d-k-\frac12}(Z_k), \end{aligned}$$

with a similar statement for $\overline{S}$, so it follows that $Z_k = \overline{Z}_k$ for $k = 0, 1, \dots, d-1$. Then also $F_k \simeq \overline{F}_k$ for each k.

Finally, (1.6.39) shows that

$$\overline{S} = \overline{S}CS = \Psi S,$$

where $\Psi = \overline{S}C$ is a surjective triangular ps.d.o. from $\prod H^{t-k-\frac12}(F_k)$ to $\prod H^{t-k-\frac12}(\overline{F}_k)$ with morphisms in the diagonal. Since $F_k \simeq \overline{F}_k$, it follows that Ψ is bijective, with bijective morphisms in the diagonal. (This is shown in [G73, Lemma 1.13] by induction in p for the submatrices $(\Psi_{jk})_{j,k\le p}$.) We now also have

$$\overline{T} = \overline{S}\varrho\Lambda = \Psi S\varrho\Lambda = \Psi T,$$

so the proof of 1° is complete.

Now consider 2°. Since S' defined in Lemma 1.6.1 is a supplement of S as well as $\overline{S}$, both G and $\overline{G}$ can be replaced by the reduced forms (1.6.34′).

Clearly, (1.6.35) implies $B = \overline{B}$. For the converse direction, we again use Λ to transform the domain of B. Let $B = \overline{B}$, then

$$(P_+ + G)\Lambda v = (P_+ + \overline{G})\Lambda v \text{ for all } v \text{ with } S\varrho v = 0, \tag{1.6.40}$$

and hence, for these v,

(1.6.41)
$$0 = (G - \overline{G})\Lambda v = [(K - \overline{K})S'\varrho + G' - \overline{G}']\Lambda v = (K - \overline{K})S'\varrho v + G''v,$$
$$\text{where } G'' = (K - \overline{K})S'\varrho(\Lambda - I) + (G' - \overline{G}')\Lambda \text{ is of class } 0.$$

Since $\varrho v = 0$ when $v \in C_0^\infty(\Omega, E)$, the identity (1.6.41) is satisfied for all $v \in C_0^\infty(\Omega, E)$, so we can conclude that $G'' \equiv 0$. Hence also

$$(K - \overline{K})S'\varrho v = 0 \text{ for all } v \in H^d(E) \text{ with } S\varrho v = 0. \tag{1.6.42}$$

Here $S'\varrho v$ runs through the full space $\prod_{0\le k<d} H^{d-k-\frac{1}{2}}(Z_k)$, so it follows that $K = \overline{K}$. Now $G'' = 0$ implies $(G' - \overline{G}')\Lambda = 0$, and hence $G' = \overline{G}'$. $\square$

As an application of Theorems 1.6.9 and 1.6.10, we can characterize the formally selfadjoint realizations $B = (P + G)_T$. We here assume that G is given in reduced form as in (1.6.34′)

$$G = KS'\varrho + G'.$$

Then K must be replaced by KS' in the formulas (1.6.25) and (1.6.26) for the terms in the adjoint $B' = (P^* + \widetilde{G})_{\widetilde{T}}$, which leads to the simpler formulas

$$\begin{aligned} \widetilde{T} &= S'''\varrho + I^\times C'^* S'^* K^* = S'''\varrho + I^\times K^* \equiv I^\times C'^* \mathfrak{A}^* \varrho + I^\times K^*, \\ \widetilde{G} &= -T'^* I^\times S''\varrho + G'^* - T'^* C^* S'^* K^* = -T'^* I^\times S''\varrho + G'^* \\ &\equiv -T'^* C^* \mathfrak{A}^* \varrho + G'^*, \end{aligned} \tag{1.6.43}$$

where we have used that $S'C' = I$ and $S'C = 0$; cf. (1.6.1).

We assume from now on that Γ is noncharacteristic for P, so that $\mathfrak{A}$ is invertible. If $D(B) = D(B')$, then the above theorem tells us that Z_k equals the kernel $\overline{Z}_k$ of

$$(S''')_{kk} = (I^\times {C'}^*\mathfrak{A}^*)_{kk} = i{C'}^*_{d-k-1,d-k-1}s_d(x') \text{ for each } k$$

(recall (1.3.11) ff.), which is a vector bundle morphism of E_Γ onto Z_{d-k-1}; hence $\overline{Z}_k \simeq F_{d-k-1}$, and consequently $Z_k \simeq F_{d-k-1}$ for $k = 0, \dots, d-1$. Moreover, there is a triangular pseudodifferential homeomorphism Ψ (with vector bundle morphisms in the diagonal) so that

$$\widetilde{T} = \Psi T; \text{ in particular } I^\times {C'}^*\mathfrak{A}^* = \Psi S \text{ and } I^\times K^* = \Psi T'. \tag{1.6.44}$$

By use of the identity $SC = I$ we find the explicit formula for Ψ:

$$\Psi = I^\times {C'}^*\mathfrak{A}^* C; \tag{1.6.45}$$

it follows in particular that the right-hand side is a homeomorphism.

In particular, the diagonal part provides an explicit vector bundle homeomorphism of F_k onto Z_{d-k-1}:

$$\Psi_{kk} \equiv {C'}^*_{d-k-1,d-k-1}s_d^* C_{kk} \colon F_k \xrightarrow{\sim} Z_{f-k-1}, \text{ for } k = 0, \dots, d-1. \tag{1.6.46}$$

In order to apply part 2° of the above theorem, we reduce $\widetilde{G}$ to the form (1.6.34′) (by insertion of $I = CS + C'S'$ and the equation $S\varrho u = -T'u$, which holds on $D(B')$ when it equals $D(B)$)

$$\begin{aligned} \widetilde{G}u &= -{T'}^* C^*\mathfrak{A}^*(CS + C'S')\varrho u + {G'}^* u \\ &= -{T'}^* C^*\mathfrak{A}^* C'S'\varrho u + {G'}^* u + {T'}^* C^*\mathfrak{A}^* CT'u. \end{aligned}$$

Then part 2° of the above theorem tells us that when $P = P^*$, then $B = B'$ implies

$$\begin{aligned} K &= -{T'}^* C^*\mathfrak{A}^* C' \\ G' &= {G'}^* + {T'}^* C^*\mathfrak{A}^* CT', \end{aligned} \tag{1.6.48}$$

and conversely, when (1.6.44) and (1.6.48) hold, $B = B'$. Noting in particular (cf. (1.3.7)) that

$$P = P^* \text{ implies } \mathfrak{A}^* = -\mathfrak{A}, \tag{1.6.48′}$$

we see that the first line in (1.6.48) can also be written

$$K = {T'}^* C^*\mathfrak{A} C',$$

which is equivalent with the last statement in (1.6.44), in view of (1.6.45). We can also insert $\mathfrak{A}^* = -\mathfrak{A}$ in the formula for G'. All together, we have found:

1.6.11 Theorem. *Let P be a ps.d.o. of order $d \geq 0$, with Γ non-characteristic for P, and let $B = (P+G)_T$ be a normal realization, with $T = S\varrho + T'$, and with G given in the reduced form*

$$G = KS'\varrho + G'. \tag{1.6.49}$$

Then the formal adjoint B' equals $(P^* + \widetilde{G})_{\widetilde{T}}$, where

$$\begin{aligned} \widetilde{T} &= S'''\varrho + I^\times K^* = I^\times C'^* \mathfrak{A}^* \varrho + I^\times K^* \\ \widetilde{G} &= -T'^* I^\times S''\varrho + G'^* = -T'^* C^* \mathfrak{A}^* \varrho + G'^*. \end{aligned} \tag{1.6.50}$$

Moreover, one has:

1° *$D(B) = D(B')$ holds if and only if*

$$\widetilde{T} = \Psi T, \text{ in particular } S''' = \Psi S \text{ and } I^\times K^* = \Psi T', \tag{1.6.51}$$

with a homeomorphism Ψ of $\prod_k H^{s-k}(F_k)$ onto $\prod_k H^{s-k}(Z_{d-k-1})$. In the affirmative case, Ψ is determined by the formula

$$\Psi = I^\times C'^* \mathfrak{A}^* C, \tag{1.6.52}$$

and the diagonal elements in Ψ define vector bundle homeomorphisms of F_k onto Z_{d-k-1}; cf. (1.6.46).

2° *Let $P = P^*$, so in particular, $\mathfrak{A}^* = -\mathfrak{A}$. Then $B = B'$ holds if and only if the conditions in 1° are satisfied along with*

$$G' = G'^* - T'^* C^* \mathfrak{A} C T'. \tag{1.6.53}$$

As a consequence, one then also has

$$K = T'^* \Psi^* I^\times = T'^* C^* \mathfrak{A} C'. \tag{1.6.54}$$

When $\{P_+ + G, T\}$ is elliptic, these conditions imply that B is selfadjoint.

Let us see what the results look like in the case where the F_k equal E_Γ or 0 (in particular the scalar case), as in Remark 1.6.3. Here

$$\begin{aligned} F_k &= E_\Gamma \text{ for } k \in J, \quad F_k = 0 \quad \text{for } k \in J', \\ Z_k &= 0 \quad \text{for } k \in J, \quad Z_k = E_\Gamma \text{ for } k \in J', \end{aligned}$$

so the property $F_k \simeq Z_{d-k-1}$ implies that J' must be the "reflected" set of J, that is,

$$J' = \{\, 0 \leq k < d \mid d-k-1 \in J \,\} = \dot{I}^\times J. \tag{1.6.55}$$

In particular, d must be even, $d = 2m$.

The special cases where J equals $J_0 = \{0, 1, \ldots, m-1\}$, resp. $J_1 = \{m, m+1, \ldots, 2m-1\}$, represent generalizations of the Dirichlet, resp. Neumann, condition; we shall analyze these in detail in the examples below. We denote

(1.6.56)
$$\begin{aligned} \gamma u &= \{\gamma_0 u, \ldots, \gamma_{m-1} u\} = \{\gamma_j u\}_{j\in J_0}, \quad \text{ranging in } E^0 = \textstyle\bigoplus_{j\in J_0} E_\Gamma, \\ \nu u &= \{\gamma_m u, \ldots, \gamma_{2m-1} u\} = \{\gamma_j u\}_{j\in J_1}, \text{ ranging in } E^1 = \textstyle\bigoplus_{j\in J_1} E_\Gamma, \end{aligned}$$

and we split $\mathfrak{A}$ in blocks corresponding to the two index sets J_0 and J_1

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{A}^{00} & \mathfrak{A}^{01} \\ \mathfrak{A}^{10} & 0 \end{pmatrix}, \tag{1.6.57}$$

so that Green's formula takes the form

$$(P_+ u, v)_\Omega - (u, P_+^* v)_\Omega = (\mathfrak{A}^{00}\gamma u + \mathfrak{A}^{01}\nu u, \gamma v) + (\mathfrak{A}^{10}\gamma u, \nu v)_\Gamma; \tag{1.6.58}$$

here $\mathfrak{A}^{01}$ and $\mathfrak{A}^{10}$ are invertible. γ and ν are called the Dirichlet, resp. the Neumann, trace operator.

1.6.12 Example. Let $d = 2m$. *Dirichlet-type boundary conditions* are conditions where $J = J_0$; here $T = S\gamma + T'$ where S is a bijective operator in $\prod_{j\in J_0} H^{s-j}(E_\Gamma)$ (in view of the triangular form and the surjectiveness of the diagonal part), so by composition with S^{-1} one can reduce the condition $Tu = 0$ to the form

$$T^0 u \equiv \gamma u + T'^0 u = 0, \tag{1.6.59}$$

where T'^0 is of class 0. We denote (as in (1.4.5))

(1.6.60)
$$H^s_{E^0} = \textstyle\prod_{0\le k<m} H^{s-k-\frac12}(E_\Gamma) \text{ and } H^s_{E^1} = \prod_{m\le k<2m} H^{s-k-\frac12}(E_\Gamma).$$

Note that in this case (recall (1.6.9–10))

$$\begin{aligned} S &= \mathrm{pr}_{J_0}, \quad S' = \mathrm{pr}_{J_1}, \\ C &= \mathrm{in}_{J_0} \quad C' = \mathrm{in}_{J_1}, \end{aligned} \tag{1.6.61}$$

and hence

$$\begin{aligned} S'' &= I^\times \,\mathrm{pr}_{J_0} \mathfrak{A}^* \varrho = \dot I^\times \mathfrak{A}^{00*}\gamma + \dot I^\times \mathfrak{A}^{10*}\nu \\ S''' &= I^\times \,\mathrm{pr}_{J_1} \mathfrak{A}^* \varrho = \dot I^\times \mathfrak{A}^{01*}\gamma, \end{aligned} \tag{1.6.62}$$

where $\dot{I}^\times$ indicates a reflection of the index set as in Remark 1.6.3. When $B = (P+G)_{T^0}$, G is in reduced form when γ is eliminated from it, i.e.,

$$G = K^1\nu + G'. \tag{1.6.63}$$

Then the formal adjoint B' equals $(P^* + \widetilde{G})_{\widetilde{T}}$, with

$$\begin{aligned}\widetilde{T} &= \dot{I}^\times \mathfrak{A}^{01^*}\gamma + \dot{I}^\times K^1 = \dot{I}^\times \mathfrak{A}^{01^*}(\gamma + (\mathfrak{A}^{01^*})^{-1}K^{1^*})\\ \widetilde{G} &= -T'^{0^*}(\mathfrak{A}^{00^*}\gamma + \mathfrak{A}^{10^*}\nu) + G'^*.\end{aligned}$$

In particular, $D(B) = D(B')$ holds if and only if

$$T'^0 = (\mathfrak{A}^{01^*})^{-1}K^{1^*}, \tag{1.6.64}$$

and in this case, $\widetilde{G}$ satisfies, for $u \in D(B)$,

$$\widetilde{G}u = -T'^{0^*}\mathfrak{A}^{10^*}\nu u + G'^* u + T'^{0^*}\mathfrak{A}^{00^*}T'^0 u, \tag{1.6.64$'$}$$

which shows how the reduced form looks.

Finally, when $P = P^*$, one has that $B = B'$ if and only if (1.6.64) holds and

$$G' = G'^* + T'^{0^*}\mathfrak{A}^{00^*}T'^0; \tag{1.6.65}$$

here one can use that

$$P = P^* \text{ implies } \mathfrak{A}^{01} = -\mathfrak{A}^{10^*} \text{ and } \mathfrak{A}^{00^*} = -\mathfrak{A}^{00}. \tag{1.6.65$'$}$$

Note that *when* $T'^0 = 0$ in the selfadjoint case, the singular Green operator (when in reduced form) is *necessarily of class* 0. Note also that we have for the particular case where G and T'^0 are zero:

$$(P_\gamma)' = (P^*)_\gamma. \tag{1.6.66}$$

1.6.13 Example. In the special case where $P = -\Delta$, the terms in Green's formula are particularly simple;

$$\begin{gathered}\mathfrak{A} = i\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}\\ (-\Delta u, v)_\Omega - (u, -\Delta v)_\Omega = i(\gamma_1 u, \gamma_0 v) + i(\gamma_0 u, \gamma_1 v)_\Gamma,\end{gathered} \tag{1.6.67}$$

(recall that $\gamma_1 u = -i\partial_{x_n} u\big|_\Gamma$). Consider a Dirichlet-type realization $B = (-\Delta + G)_{T_0}$ with G in reduced form,

$$T_0 = \gamma_0 + T_0', \quad G = K_1\gamma_1 + G'. \tag{1.6.68}$$

In this case, selfadjointness holds if and only if

$$K_1 = iT_0'^*, \quad G' = G'^*. \tag{1.6.69}$$

Note in particular that a realization $(-\Delta)_{T_0}$ without singular Green term can be selfadjoint *only when* $T_0' = 0$. And, a realization $(-\Delta + G)_{\gamma_0}$ can only be selfadjoint when G is *of class* 0 (G in reduced form).

1.6.14 Example. Let $d = 2m$. *Neumann-type boundary conditions* are conditions where (1.4.23) holds with $J = \{m, \dots, 2m-1\} = J_1$. Here $Tu = 0$ may be reduced to the form

$$T^1 u \equiv \nu u + S^{10}\gamma u + T'^1 u = 0, \tag{1.6.70}$$

and T^1 is continuous from $H^{2m}(E)$ to $H^{2m}_{E^1}$ (cf. (1.6.60)). Now the general notation of Lemma 1.6.1 specializes to the present case as follows:

$$S\varrho u = \nu u + S^{10}\gamma u, \quad S'\varrho u = \gamma u,$$

where S and S' are written together in block notation as

$$\widetilde{S} = \begin{pmatrix} I & 0 \\ S^{10} & I \end{pmatrix} = \begin{pmatrix} S' \\ S \end{pmatrix}, \tag{1.6.71}$$

which has the inverse
(1.6.72)

$$\widetilde{C} = \begin{pmatrix} I & 0 \\ -S^{10} & I \end{pmatrix} = (\, C' \quad C \,), \text{ with } C = \begin{pmatrix} 0 \\ I \end{pmatrix}, C' = \begin{pmatrix} I \\ -S^{10} \end{pmatrix}.$$

Then

$$C^* = (\, 0 \quad I \,) \text{ and } C'^* = (\, I \quad -S^{10^*} \,) \tag{1.6.73}$$

so that

$$\begin{aligned} C^*\mathfrak{A}^*\varrho u &= (\, 0 \quad I \,) \begin{pmatrix} \mathfrak{A}^{00^*} & \mathfrak{A}^{10^*} \\ \mathfrak{A}^{01^*} & 0 \end{pmatrix} \begin{pmatrix} \gamma u \\ \nu u \end{pmatrix} = \mathfrak{A}^{01^*}\gamma u, \\ C'^*\mathfrak{A}^*\varrho u &= (\, I \quad -S^{10^*} \,) \begin{pmatrix} \mathfrak{A}^{00^*}\gamma u + \mathfrak{A}^{10^*}\nu u \\ \mathfrak{A}^{01^*}\gamma u \end{pmatrix} \\ &= \mathfrak{A}^{10^*}\nu u + (\mathfrak{A}^{00^*} - S^{10^*}\mathfrak{A}^{01^*})\gamma u. \end{aligned} \tag{1.6.74}$$

Consider a realization $B = (P+G)_{T^1}$ with T^1 of the form (1.6.70) and G in reduced form

$$G = K^0\gamma + G'. \tag{1.6.75}$$

Then the formal adjoint equals $(P^* + \widetilde{G})_{\widetilde{T}}$ with

$$\begin{aligned} \widetilde{T} &= \dot{I}^\times[\mathfrak{A}^{10^*}\nu u + (\mathfrak{A}^{00^*} - S^{10^*}\mathfrak{A}^{01^*})\gamma u] + \dot{I}^\times K^{0^*} \\ \widetilde{G} &= -T'^{1^*}\mathfrak{A}^{01^*}\gamma u + G'^*. \end{aligned} \tag{1.6.76}$$

We see that $D(B) = D(B')$ holds if and only if

$$\widetilde{T} = \dot{I}^{\times}\mathfrak{A}^{10*}T^1,$$

which can be specified into the following conditions

$$\begin{aligned} T'^1 &= (\mathfrak{A}^{10*})^{-1}K^{0*} \\ S^{10} &= (\mathfrak{A}^{10*})^{-1}(\mathfrak{A}^{00*} - S^{10*}\mathfrak{A}^{01*}). \end{aligned} \tag{1.6.77}$$

Furthermore, if $P = P^*$, then $B = B'$ if and only if (1.6.77) holds and

$$G' = G'^*. \tag{1.6.78}$$

Again, the term of class 0 in T^1 is directly linked with the boundary term $K^0\gamma$ in G; in formally selfadjoint cases one of them cannot be nonzero without the other one being so.

1.6.15 Example. In the special case where $P = -\Delta$, and $B = (-\Delta + G)_T$ is a Neumann-type realization with G in reduced form:

$$T = \gamma_1 + S_0\gamma_0 + T_1', \quad G = K_0\gamma_0 + G', \tag{1.6.79}$$

the criteria for formal selfadjointness are:

$$\begin{aligned} K_0 &= iT_1'^*, \\ S_0 &= -S_0^*, \\ G' &= G'^*. \end{aligned} \tag{1.6.80}$$

1.6.16 Example. Let us also include a simple example of a system acting on vector valued functions, where the terms in the trace operator maps into lower-dimensional bundles. Let $E = \overline{\Omega} \times \mathbb{C}^N$, $N > 1$, and let Π be an orthogonal projection of $\mathbb{C}^N$ onto a subspace V of dimension $k \geq 1$, with $k < N$. Let $\Pi^\perp$ be the projection onto $V^\perp$. Then

$$\mathcal{A} = \begin{pmatrix} -\Delta \\ \\ \Pi\gamma_0 \\ \\ \Pi^\perp\gamma_1 \end{pmatrix} : C^\infty(E) \to \begin{array}{c} C^\infty(E) \\ \times \\ C^\infty(\Pi E_\Gamma) \\ \times \\ C^\infty(\Pi^\perp E_\Gamma) \end{array} \tag{1.6.81}$$

defines a selfadjoint elliptic realization (in fact it breaks up into a Dirichlet problem for the sections in $\Pi E = \overline{\Omega} \times V$ and a Neumann problem for the sections in $\Pi^\perp E = \overline{\Omega} \times V^\perp$). The ingredients in Lemma 1.6.1 are here:

$$
\begin{aligned}
F &= \Pi E_\Gamma \oplus \Pi^\perp E_\Gamma = (\Gamma \times V) \oplus (\Gamma \times V^\perp),\\
Z &= \Pi^\perp E_\Gamma \oplus \Pi E_\Gamma = (\Gamma \times V^\perp) \oplus (\Gamma \times V),\\
S &= \begin{pmatrix} \Pi & 0 \\ 0 & \Pi^\perp \end{pmatrix} : E_\Gamma \oplus E_\Gamma \to F,\\
S' &= \begin{pmatrix} \Pi^\perp & 0 \\ 0 & \Pi \end{pmatrix} : E_\Gamma \oplus E_\Gamma \to Z,\\
C = S^* &= \begin{pmatrix} \mathrm{in}_V & 0 \\ 0 & \mathrm{in}_{V^\perp} \end{pmatrix} : F \to E_\Gamma \oplus E_\Gamma,\\
C^* = {S'}^* &= \begin{pmatrix} \mathrm{in}_{V^\perp} & 0 \\ 0 & \mathrm{in}_V \end{pmatrix} : Z \to E_\Gamma \oplus E_\Gamma;
\end{aligned}
\tag{1.6.82}
$$

S, S' and C are morphisms. In view of (1.6.67), Green's formula (1.6.2) takes the form

$$
\begin{aligned}
&(-\Delta u, v)_{L_2(\Omega)^N} - (u, -\Delta v)_{L_2(\Omega)^N} = \\
&\quad i\Big(\begin{pmatrix} \Pi\gamma_0 u \\ \Pi^\perp \gamma_1 u \end{pmatrix}, \begin{pmatrix} \Pi\gamma_1 v \\ \Pi^\perp \gamma_0 \end{pmatrix}\Big)_\Gamma + i\Big(\begin{pmatrix} \Pi^\perp\gamma_0 u \\ \Pi \gamma_1 u \end{pmatrix}, \begin{pmatrix} \Pi^\perp\gamma_1 v \\ \Pi \gamma_0 v \end{pmatrix}\Big)_\Gamma.
\end{aligned}
\tag{1.6.83}
$$

Let us add a term of class 0 to the trace operator in (1.6.81), replacing it by

$$T = \begin{pmatrix} \Pi(\gamma_0 + T_0') \\ \Pi^\perp(\gamma_1 + T_1') \end{pmatrix}$$

where T_0' and T_1' are of class 0. (The second line could also include a term $\Pi^\perp S_0 \gamma_0$, which, of course, would complicate the calculations.) Let us also add a singular Green term to $-\Delta$, taken in reduced form (cf. (1.6.49) and (1.6.82))

$$G = K_0 \Pi^\perp \gamma_0 + K_1 \Pi \gamma_1 + G',$$

with G' of class 0. So we study the operator

$$
\mathcal{A} = \begin{pmatrix} -\Delta + K_0\Pi^\perp\gamma_0 + K_1\Pi\gamma_1 + G' \\ \Pi(\gamma_0 + T_0') \\ \Pi^\perp(\gamma_1 + T_1') \end{pmatrix}
\tag{1.6.84}
$$

where K_0 and K_1 are Poisson operators of orders 2 (resp. 1) from $\Gamma \times V$ (resp. $\Gamma \times V^\perp$) to E; G' is a s.g.o. of class 0 and order 2; and T_0' and T_1' are trace operators of class 0 and orders 0 (resp. 1). Let B be the corresponding realization. Then $B' = (-\Delta + \widetilde{G})_{\widetilde{T}}$, where

$$
\begin{aligned}
\widetilde{T} &= I^\times \begin{pmatrix} \Pi^\perp & 0 \\ 0 & \Pi \end{pmatrix} (-i) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \varrho + I^\times \begin{pmatrix} \Pi^\perp K_0^* \\ \Pi K_1^* \end{pmatrix} \\
&= -i \begin{pmatrix} \Pi\gamma_0 + iK_1^* \\ \Pi^\perp\gamma_1 + iK_0^* \end{pmatrix}, \\
\widetilde{G} &= -\left(T_0'^*\Pi \quad T_1'^*\Pi^\perp \right) \begin{pmatrix} \Pi & 0 \\ 0 & \Pi^\perp \end{pmatrix} (-i) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \varrho + G'^* \\
&= iT_1'^*\Pi^\perp\gamma_0 + iT_0'^*\Pi\gamma_1 + G'^*.
\end{aligned} \tag{1.6.85}
$$

So, $D(B) = D(B')$ if and only if

$$K_0 = iT_1'^*\Pi^\perp, \quad K_1 = iT_0'^*\Pi; \tag{1.6.86}$$

since $\Delta = \Delta^*$, one has $B = B'$ precisely when (1.6.86) holds together with

$$G' = G'^*. \tag{1.6.87}$$

Similar calculations, carried out for more general elliptic operators, will give formulas of the same kind but with modifications stemming from $\mathfrak{A}$ and S^{10} as in (1.6.65) and (1.6.77).

For other boundary conditions, the calculation of specific examples seems hardly more enlightening than the general formulas.

1.7 Semiboundedness and coerciveness

A special class of boundary problems satisfying the parabolicity hypothesis introduced in Section 1.5 is the class of systems $\{P_+ + G, T\}$ defining lower-bounded, "variational" realizations (with the property of *m-coerciveness* described in detail below). These are of interest, on one hand, because there are wellestablished results in functional calculus for them (e.g. the theory of contraction semigroups generated by such operators, with applications to the heat equation), on the other hand because they provide a good source of *examples* of boundary value problems that enter into our theory. A reader who is interested mainly in the general calculus should bypass this section in a first reading.

We consider polyhomogeneous operators. Recall that the ps.d.o. P of order d is called *strongly elliptic* on M (where M is a set in Σ), when the principal symbol $p^0(x,\xi)$ satisfies the (matrix) inequality, with $c(x) > 0$,

$$\operatorname{Re} p^0(x,\xi) \equiv \tfrac{1}{2}[p^0(x,\xi) + p^0(x,\xi)^*] \geq c(x)|\xi|^d I \text{ for } x \in M, |\xi| \geq 1. \tag{1.7.1}$$

(One can also include operators for which the inequality (1.7.1) is obtained after multiplication by a (matrix) function of x.) Recall also that when P satisfies (1.7.1) on a compact manifold Σ without boundary, it satisfies the Gårding inequality (for any $\varepsilon > 0$)

$$\text{(1.7.2)}\quad \operatorname{Re}(Pu, u) \geq c\|u\|_{d/2}^2 - c_1\|u\|_{d/2-\varepsilon}^2 \text{ for } u \in H^{d/2}(\Sigma) \text{ (or } H^{d/2}(\widetilde{E})),$$

with $c > 0$.

Note that strong ellipticity of P requires that $(p^0(x,\xi)v, v)_{\mathbb{C}^N}$, for each x and each $v \in \mathbb{C}^N$, takes its values in a proper cone when $\xi \in \mathbb{R}^n$, $|\xi| \geq 1$, since

$$\text{(1.7.3)}\qquad |\operatorname{Im}(p^0(x,\xi)v, v)| \leq c'(x)|\xi|^d|v|^2 \leq c''(x)\operatorname{Re}(p^0(x,\xi)v, v),$$

when (1.7.1) holds. In particular, when P furthermore satisfies the transmission condition at $x_n = 0$, d must be *even*, since

$$p^0(x', 0, 0, \xi_n) = (-1)^d p^0(x, 0, 0, -\xi_n) \neq 0$$

for $|\xi_n| \geq 1$, so (1.7.3) cannot hold for d odd.

Now let P be strongly elliptic of even order $d = 2m$ on Σ. We assume throughout that P satisfies the transmission condition at Γ. It is of interest to discuss which *realizations* B of P in $L_2(E)$ that satisfy lower-boundedness inequalities like (1.7.2). We say that B is *m-coercive*, when there exist constants $C > 0, c > 0$ and $c_1 \in \mathbb{R}$ so that

$$\text{(1.7.4)}\qquad C\|u\|_m^2 \geq \operatorname{Re}(Bu, u) \geq c\|u\|_m^2 - c_1\|u\|_0^2 \text{ for } u \in D(B).$$

(Also here one can include the realizations satisfying (1.7.4) after composition with a morphism, but we restrict the attention to systems satisfying (1.7.4) to fix the ideas.) One can call the systems $\begin{pmatrix} P_+ + G \\ T \end{pmatrix}$, for which $(P+G)_T$ is *m-coercive*, the *strongly elliptic* systems.

It is wellknown in the differential operator case that the Dirichlet realization of a strongly elliptic operator of order $2m$ is m-coercive; see Gårding [Gå53]. Larger classes of m-coercive elliptic realizations were introduced by Agmon and others in the Fifties, and the complete set of m-coercive realizations defined by normal boundary conditions was thoroughly discussed in [G70–74].

We here give some basic steps in the corresponding analysis for pseudodifferential boundary problems.

Realizations satisfying (1.7.4) can be expected to be linked with variational theory, and a standard ingredient here is the so-called Lax-Milgram Lemma (that has its roots in the Friedrichs construction and enters in early work by Vishik; cf. Lions and Magenes [L-M68, Ch. 2.10]). We formulate a version of the lemma with an outline of its proof that will be used in the following.

1.7.1 Lemma. *Let V and H be Hilbert spaces, such that V is identified with a dense subspace of H, the respective norms satisfying (with $c > 0$)*

$$\|u\|_V \geq c\|u\|_H \text{ for } u \in V. \tag{1.7.5}$$

Let $s(u,v)$ be a sesquilinear form on V that is continuous on V, and V-coercive:

$$\operatorname{Re} s(u,u) \geq c_0\|u\|_V^2 - k\|u\|_H^2 \text{ for } u \in V, \tag{1.7.6}$$

with $c_0 > 0$. Let S be the operator in H defined from s and V as follows

$$\begin{aligned} D(S) &= \{u \in V \mid \exists\, f \in H \text{ so that } s(u,v) = (f,v)_H \text{ for all } v \in V\} \\ Su &= f, \end{aligned} \tag{1.7.7}$$

and let $s^*(u,v)$ be the adjoint form on V

$$s^*(u,v) = \overline{s(v,u)} \text{ for } u, v \in V. \tag{1.7.8}$$

Then S is a closed, densely defined operator in H, its adjoint S^* is precisely the analogous operator defined from s^* and V, and both S and S^* have their numerical range and spectrum lying in a sector C of the complex plane (with α and $\gamma \in \mathbb{R}$, $\beta \geq 0$)

$$C = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \alpha,\ |\operatorname{Im} z| \leq \beta(\operatorname{Re} z + \gamma)\}. \tag{1.7.9}$$

Proof. The operator S is welldefined, since V is dense in H. Assume first $k = 0$. The dense injection $V \subset H$ and the identification of H with H^* induce an injection $H \subset V^*$ (the space of continuous, conjugate linear functionals on V), such that

$$\langle w, \overline{v}\rangle_{V^*,V} = (w,v)_H \text{ when } w \in H,\ v \in V,$$

here $\langle w, \overline{v}\rangle$ denotes the value of the functional $w \in V^*$ on $v \in V$; cf. (A.12).

The sesquilinear form $s(u,v)$ induces a continuous operator $\mathfrak{S}$ from V to V^* by the requirement

$$s(u,v) = \langle \mathfrak{S}u, \overline{v}\rangle_{V^*,V} \text{ for all } u \text{ and } v \in V,$$

and the adjoint form $s^*(u,v)$ induces in this way precisely the adjoint $\mathfrak{S}^*$ (going from V to V^*). It follows from the inequality (1.7.6) for s and s^* (with $k = 0$) that $\mathfrak{S}$ and $\mathfrak{S}^*$ are homeomorphisms of V onto V^*. With the injection $H \subset V^*$ defined as above, S is now simply the restriction of $\mathfrak{S}$ to the space

$$D(S) = \{u \in V \mid \mathfrak{S}u \in H\}, \tag{1.7.10}$$

and it follows that S is a bijection of $D(S)$ onto H. Define analogously S' by restriction of $\mathfrak{S}^*$; then it follows by use of (1.7.5) that S^{-1} and $(S')^{-1}$ are bounded operators in H, and by (1.7.8) that they are adjoints of one another. Since $(S')^{-1}$ is injective, $R(S^{-1}) = D(S)$ is dense in H, so S^* is welldefined, and it is seen that S^* equals S'. Note that S satisfies

$$\operatorname{Re}(Su, u) \geq c_0 \|u\|_V^2 \geq c_0 c^2 \|u\|_H^2,$$

and that 0 is in the resolvent set of S; then the spectrum is contained in the halfplane $\{z \in \mathbb{C} \mid \operatorname{Re} z \geq c_0 c^2\}$. The same holds for S^*.

When $k \neq 0$, we replace s by

$$s_k(u, v) = s(u, v) + (ku, v)_H,$$

defining an operator S_k by the above procedure, and it is seen that $S_k = S + kI$; in this case the spectrum and the numerical range of S and S^* lie in $\{z \in \mathbb{C} \mid \operatorname{Re} z \geq c_0 c^2 - k\}$. To show that they lie in a sector C, one uses that $|\operatorname{Im} s_k(u, u)| \leq \beta \operatorname{Re} s_k(u, u)$ for some β in view of (1.7.6) and the continuity of s on $V \times V$; then the above arguments can be applied to rotations $e^{\pm i\theta} S_k$ for a suitable $\theta > 0$. $\square$

When the situation of the lemma holds, we say for short that S is the *variational operator* associated with s on V in H, and operators generated in this way are called *variational*. One can show that $D(S)$ is dense even in V, so that a given variational operator S in H *determines* the form s and the space V it is associated with, by closure of the form $(Su, v)_H$ (the V-norm of $u \in D(S)$ being equivalent with $\operatorname{Re}(Su, u) + k\|u\|_H^2$ for a suitably large fixed k).

In the following theorem, the lemma will be applied with V equal to $H_0^m(E)$, which, as we recall, is the closure of $C_0^\infty(\Omega, E)$ and identifies with the space of functions $f \in H^m(E)$ for which $\gamma_0 f = \cdots = \gamma_{m-1} f = 0$. The theorem extends the classical result of Gårding [Gå53] for differential operators.

1.7.2 Theorem. *Let P be strongly elliptic in $\widetilde{E}$ of even order $2m > 0$.*

1° *The Dirichlet realization P_γ (hence also the system $\{P_+, \gamma\}$) is elliptic.*

2° *The Dirichlet boundary symbol realization $b^0(x', \xi, D_n) = p^0(x', 0, \xi', D_n)_\gamma$ is m-coercive and positive, satisfying*

$$\operatorname{Re}(p_+^0 u, u) \geq c(x', \xi') \|u\|_m^2 \text{ for all } u \in H_0^m(\overline{\mathbb{R}}_+)^N, \tag{1.7.11}$$

with $c(x',\xi')>0$*, for any* x'*, any* $|\xi'|\geq 1$*. Moreover,* b^0 *is the variational operator associated with the sesquilinear form*

$$s(u,v)=\langle p^0_+u,\overline{v}\rangle_{H^{-m},H^m_0} \tag{1.7.12}$$

on $H^m_0(\overline{\mathbb{R}}_+)^N$ *in* $L_2(\mathbb{R}_+)^N$*, the adjoint being the analogous operator for* p^{0*}*.*

3° *The realization* P_γ *(with domain* $H^{2m}(E)\cap H^m_0(E)$*) is* m*-coercive:*

$$C\|u\|^2_m\geq \operatorname{Re}(P_+u,u)\geq c\|u\|^2_m-c_1\|u\|^2_0 \text{ for } u\in D(P_\gamma), \tag{1.7.13}$$

and it is the variational operator associated with the sesquilinear form

$$b(u,v)=\langle P_+u,\overline{v}\rangle_{H^{-m}(E),H^m_0(E)} \tag{1.7.14}$$

on $H^m_0(E)$ *in* $L_2(E)$*, the adjoint being the analogous operator for* P^**.* P_γ *and* P^*_γ *have their numerical range and spectrum in a sector* C *as in* (1.7.9).

Proof. We begin the proof on the boundary symbol level. Here we note first that the strong ellipticity of P implies by Fourier transformation, since P is of order $2m$,

$$C\|u\|^2_m\geq \operatorname{Re}(p^0u,u)\geq c\|u\|^2_m \text{ for } u\in\mathcal{S}(\mathbb{R})^N,$$

with $C\geq c>0$ (we can modify p^0 to be positive for $|\xi|\leq 1$), and this holds in particular for $u\in C^\infty_0(\mathbb{R}_+)^N$. Thus $\operatorname{Re}(p^0u,u)$ is a norm equivalent with the H^m-norm, and in particular the completion $H^m_0(\overline{\mathbb{R}}_+)^N$ of $C^\infty_0(\mathbb{R}_+)^N$ in the H^m-norm is identical with the completion V of $C^\infty_0(\mathbb{R}_+)^N$ in the norm $\operatorname{Re}(p^0u,u)$. Hence the sesquilinear form

$$s(u,v)=(p^0_+u,v) \text{ for } u,v\in C^\infty_0(\mathbb{R}_+)^N \tag{1.7.15}$$

extends by continuity to a sesquilinear form $s(u,v)$ on $V=H^m_0(\overline{\mathbb{R}}_+)^N$, which is continuous and satisfies

$$\operatorname{Re} s(u,u)\geq c\|u\|^2_m \text{ for } u\in V. \tag{1.7.16}$$

We now apply Lemma 1.7.1 and the concepts introduced there. In the present case, where $V=H^m_0(\overline{\mathbb{R}}_+)^N$ and $H=L_2(\mathbb{R}_+)^N$, V^* is customarily identified with $H^{-m}(\overline{\mathbb{R}}_+)^N(\subset\mathcal{D}'(\mathbb{R}_+)^N)$, whereby the duality $\langle w,\overline{v}\rangle_{V^*,V}$ coincides with the distribution duality when $v\in C^\infty_0(\mathbb{R}_+)^N$. Moreover, $p^0_+=r^+p^0e^+$ maps $H^m_0(\overline{\mathbb{R}}_+)^N$ continuously into $H^{-m}(\overline{\mathbb{R}}_+)^N$ with an adjoint $(p^0_+)^*$ (from $H^m_0(\overline{\mathbb{R}}_+)^N$ to $H^{-m}(\overline{\mathbb{R}}_+)^N$) satisfying

$$(p^0_+)^*=(p^{0^*})_+\equiv r^+p^{0^*}e^+ \text{ on } H^m_0(\overline{\mathbb{R}}_+)^N.$$

This implies that the extended sesquilinear form s on $V = H_0^m$ is in fact described by

$$s(u,v) = \langle p_+^0 u, \overline{v} \rangle_{H^{-m}, H_0^m}, \tag{1.7.17}$$

which shows that $\mathfrak{S}$ *equals* p_+^0 (as an operator from H_0^m to H^{-m}). It follows in particular that

$$Su = p_+^0 u \text{ for } u \in D(S). \tag{1.7.18}$$

Now recall the hypoellipticity property (Lemma 1.3.4), which shows that $D(S) \subset H^{2m}(\overline{\mathbb{R}}_+)^N$, so all together $D(S) \subset H^{2m}(\overline{\mathbb{R}}_+)^N \cap H_0^m(\overline{\mathbb{R}}_+)^N$. On the other hand, if $u \in H^{2m}(\overline{\mathbb{R}}_+)^N \cap H_0^m(\overline{\mathbb{R}}_+)^N$, then $f = p_+^0 u \in L_2(\mathbb{R}_+)^N$, and

$$s(u,v) = \langle p_+^0 u, \overline{v} \rangle_{H^{-m}, H_0^m} = (p_+^0 u, v)_{L_2} = (f, v)_{L_2}$$

for all $v \in V$, so $u \in D(S)$. We have then shown that

$$D(S) = H^{2m}(\overline{\mathbb{R}}_+)^N \cap H_0^m(\overline{\mathbb{R}}_+)^N. \tag{1.7.19}$$

(1.7.18) and (1.7.19) together show that S is the realization of p^0 on $\mathbb{R}_+$ defined by the (clearly normal) boundary condition $\gamma u = 0$, i.e.,

$$S = (p^0)_\gamma.$$

Since S is a bijection of $D(S)$ onto $L_2(\mathbb{R}_+)^N$, the realization is elliptic, in view of Corollary 1.6.7. All together, this shows 1° and 2°.

Once the ellipticity is established, the rest is easy in view of Theorem 1.6.9. Like in the preceding analysis, we can extend the sesquilinear form

$$b(u,v) = (Pu, v) \text{ for } u, v \in C_0^\infty(\Omega, E)^N$$

by continuity to a sesquilinear form $b(u,v)$ on $H_0^m(E)^N$, and in view of (1.7.2), b is $H_0^m(E)^N$-coercive. Then b on $H_0^m(E)^N$ induces a variational operator B. This operator is an extension of the realization P_γ, since $D(P_\gamma) \subset V$ and

$$(P_\gamma u, v)_{L_2} = (P_+ u, v)_{L_2} = \langle P_+ u, \overline{v} \rangle_{H^{-m}, H_0^m} = b(u,v)$$

for $u \in D(P_\gamma)$, $v \in V$. Since $(P_+ u, v) = (u, (P^*)_+ v)$ for u and $v \in C_0^\infty(\Omega, E)^N$, it follows by extension by continuity that the adjoint form b^* equals

$$b^*(u,v) = \langle (P^*)_+ u, \overline{v} \rangle_{H^{-m}, H_0^m}.$$

The adjoint B^* of B is the variational operator defined from b^* on $H_0^m(E)^N$ in $L_2(E)^N$, so it satisfies analogously with B,

$$B^* \supset (P^*)_\gamma. \tag{1.7.20}$$

By 1° and 2° above, P_γ is elliptic, and (1.6.66), jointly with Theorem 1.6.9, shows that

$$(P_\gamma)^* = (P^*)_\gamma.$$

Then (1.7.20) implies

$$B \subset P_\gamma$$

so we have all together that $B = P_\gamma$. The general properties of P_γ and $(P^*)_\gamma$ now follow from Lemma 1.7.1. $\square$

In particular, if P is formally selfadjoint and strongly elliptic, then P_γ is *selfadjoint lower-bounded*, and it equals *the Friedrichs extension* of $P\big|_{C_0^\infty(\Omega,E)}$ (as the variational construction shows).

Note also that part 2° of the theorem implies:

1.7.3 Corollary. *When P is strongly elliptic in $\widetilde{E}$ of order $2m$, then p_+^0 is surjective from $H^{2m}(\overline{\mathbb{R}}_+)^N$ to $L_2(\mathbb{R}_+)^N$ (and from $\mathcal{S}(\overline{\mathbb{R}}_+)^N$ to $\mathcal{S}(\overline{\mathbb{R}}_+)^N$), with nullspace dimension equal to mN; in particular (cf.* (A.107)*)*

$$\operatorname{index} p_+^0 = mN. \tag{1.7.21}$$

With the above results as basis, one can analyze many more realizations of P defined by normal boundary conditions. First of all, the realizations P_T without a G and without "interior" terms in T can be treated by a slight extension of the theory of [G70–74]. We shall go through this rapidly, giving a variant that uses the right inverse of S according to Lemma 1.6.1 (rather than [G73, Section 1.3]). More general cases are treated later.

Let P be of even order $2m > 0$, and write Green's formula in the form

$$(P_+u, v)_\Omega - (u, P_+^*v)_\Omega = (\mathfrak{A}^{00}\gamma u + \mathfrak{A}^{01}\nu u, \gamma v)_\Gamma + (\mathfrak{A}^{10}\gamma u, \nu v)_\Gamma, \tag{1.7.22}$$

which was explained in (1.6.56)–(1.6.58). Write $P = A + P'$ as in Lemma 1.3.1; then there exists an integro-differential form $a(u,v)$ on $H^m(E) \times H^m(E)$ with smooth coefficients such that one has the "halfways" Green's formula for A,

$$(Au, v)_\Omega = a(u,v) + (\mathfrak{A}^{01}\nu u, \gamma v);$$

cf. [G71,73]. Then we also have

$$(P_+u, v) = a(u,v) + p'(u,v) + (\mathfrak{A}^{01}\nu u, \gamma v) \tag{1.7.23}$$

where

$$p'(u,v) = (P'_+u,v) = (u,(P'^*)_+v)$$

satisfies

$$|p'(u,v)| \le c\|u\|_m\|v\|_m \text{ for } u,v \in H^{2m}(E);$$

this follows by interpolation from the inequalities

$$|(P'_+u,v)| \le c_1\|u\|_{2m}\|v\|_0, \quad |(u,P'^*_+v)| \le c_2\|u\|_0\|v\|_{2m},$$

or one can show it directly from the structure of P'. All together

$$(P_+u,v) = p(u,v) + (\mathfrak{A}^{01}\nu u,\gamma v) \text{ for } u,v \in H^{2m}(E), \tag{1.7.24}$$

where $p(u,v) = a(u,v) + p'(u,v)$ satisfies

$$|p(u,v)| \le c\|u\|_m\|v\|_m. \tag{1.7.25}$$

Consider a normal trace operator without "interior" term

$$T = S\varrho\colon H^{2m}(E) \to \prod_{0\le k<2m} H^{2m-k-\frac{1}{2}}(F_k). \tag{1.7.26}$$

It is advantageous to split S in blocks indexed by $J_0 = \{0,1,\dots,m-1\}$ and $J_1 = \{m,\dots,2m-1\}$:

$$S = \begin{pmatrix} S^{00} & 0 \\ S^{10} & S^{11} \end{pmatrix} \tag{1.7.27}$$

where $S^{\delta\varepsilon}$ goes from $\prod_{j\in J_\varepsilon} H^{2m-j-\frac{1}{2}}(E_\Gamma)$ to $\prod_{k\in J_\delta} H^{2m-k-\frac{1}{2}}(F_k)$. The other matrices occurring in Lemma 1.6.1 are split similarly:

$$S' = \begin{pmatrix} S'^{00} & 0 \\ 0 & S'^{11} \end{pmatrix}, \quad C = \begin{pmatrix} C^{00} & 0 \\ C^{10} & C^{11} \end{pmatrix}, \quad C' = \begin{pmatrix} C'^{00} & 0 \\ C'^{10} & C'^{11} \end{pmatrix}, \tag{1.7.28}$$

which corresponds to a splitting of $\widetilde{S}$ and $\widetilde{C}$:

$$\widetilde{S} = \begin{pmatrix} \widetilde{S}^{00} & 0 \\ \widetilde{S}^{10} & \widetilde{S}^{11} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} S^{00} \\ S'^{00} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} S^{10} \\ 0 \end{pmatrix} & \begin{pmatrix} S^{11} \\ S'^{11} \end{pmatrix} \end{pmatrix} \tag{1.7.29}$$

etc. Because of the triangular nature of the operators, the inversion formulas (1.6.1) imply in particular

$$\begin{aligned} &C^{\delta\delta}S^{\delta\delta} + C'^{\delta\delta}S'^{\delta\delta} = 1 && \text{on } \textstyle\bigoplus_{J_\delta} E_\Gamma, \\ &\begin{pmatrix} S^{\delta\delta}C^{\delta\delta} & S^{\delta\delta}C'^{\delta\delta} \\ S'^{\delta\delta}C^{\delta\delta} & S'^{\delta\delta}C'^{\delta\delta} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} && \text{on } \begin{matrix} \bigoplus_{J_\delta} F_k \\ \oplus \\ \bigoplus_{J_\delta} Z_k \end{matrix}, \end{aligned} \tag{1.7.30}$$

for $\delta = 0$ and 1, the S-operators being surjective and the C-operators injective. Now the boundary condition may be written

$$\begin{aligned} S^{00}\gamma u \qquad\qquad &= 0, \\ S^{10}\gamma u + S^{11}\nu u &= 0. \end{aligned} \tag{1.7.31}$$

In view of (1.7.30), the condition (1.7.31) may be written equivalently as:

$$\begin{aligned} &\text{(i)} \qquad \gamma u = C'^{00}\varphi_0 \text{ and} \\ &\text{(ii)} \qquad \nu u = -C^{11}S^{10}\gamma u + C'^{11}\varphi_1, \\ &\text{for some } \{\varphi_0, \varphi_1\} \in \textstyle\prod_{J_0} H^{2m-k-\frac12}(Z_k) \times \prod_{J_1} H^{2m-k-\frac12}(Z_k), \end{aligned} \tag{1.7.32}$$

for one can take $\varphi_0 = S'^{00}\gamma u$ and $\varphi_1 = S'^{11}\nu u$; (1.7.32) gives a parametrization of the space of boundary values. We first study *weak semiboundedness* (see (1.7.33) below), which is *necessary* for m-coerciveness, or for P_T being variational or even just semibounded in L_2-norm.

1.7.4 Theorem. *Let P be a ps.d.o. on $\widetilde{E}$ of even order $2m > 0$, with Γ noncharacteristic for P. Let P_T be the realization defined by a normal trace operator of the form $T = S\varrho$. Then P_T is weakly semibounded, i.e., there exists a $\theta \in \mathbb{R}$ and a constant c such that*

$$\operatorname{Re} e^{i\theta}(P_+u, u) \le c\|u\|_m^2 \textit{ for all } u \in D(P_T) \tag{1.7.33}$$

if and only if

$$C'^{00*}\mathfrak{A}^{01}C'^{11} = 0. \tag{1.7.34}$$

In the affirmative case, one has in fact

$$\begin{aligned} (\mathfrak{A}^{01}\nu u, \gamma v) &= (\mathfrak{A}^{01}C^{11}S^{10}\gamma u, \gamma v) \textit{ and} \\ |(P_+u, v)| &\le c'\|u\|_m\|v\|_m \textit{ for all } u, v \in D(P_T). \end{aligned} \tag{1.7.35}$$

Proof. We say that $f(u,v)$ is m-bounded when $|f(u,v)| \le c\|u\|_m\|v\|_m$.

The only term in the right-hand side of (1.7.23) with $u = v$ that is not always m-bounded is $(\mathfrak{A}^{01}\nu u, \gamma u)$. Inserting (1.7.32), we may write it as

$$\begin{aligned} (\mathfrak{A}^{01}\nu u, \gamma u)_\Gamma &= (\mathfrak{A}^{01}(-C'^{11}S^{10}\gamma u + C'^{11}\varphi_1), \gamma u)_\Gamma \\ &= (\mathfrak{A}^{01}C'^{11}\varphi_1, C'^{00}\varphi_0)_\Gamma - (\mathfrak{A}^{01}C'^{11}S^{10}\gamma u, \gamma u)_\Gamma. \end{aligned} \tag{1.7.36}$$

The last term here is again m-bounded, so (1.7.33) holds if and only if

$$\operatorname{Re} e^{i\theta}(C'^{00*}\mathfrak{A}^{01}C'^{11}\varphi_1, \varphi_0) \le c_1\|u\|_m^2 \text{ for } u \in D(P_T) \text{ with (1.7.32).} \tag{1.7.37}$$

Addition of $w \in C_0^\infty(\Omega, E)$ to u does not change the left-hand side. Recall that

$$\inf\{\|u+w\|_m \mid w \in C_0^\infty(\Omega, E)\} \simeq \|\gamma u\|_{\prod_{J_0} H^{m-k-\frac12}(E_\Gamma)},$$

since $C_0^\infty(\Omega, E)$ is dense in the nullspace $H_0^m(E)$ of the surjective continuous mapping $\gamma\colon H^m(E) \to \prod_{J_0} H^{m-k-\frac12}(E_\Gamma)$. Then (1.7.37) holds if and only if, for some constant c_2,

$$\operatorname{Re} e^{i\theta}(C'^{00*}\mathfrak{A}^{01}C'^{11}\varphi_1, \varphi_0) \le c_2\|\gamma u\|^2_{\prod_{J_0} H^{m-k-\frac12}(E_\Gamma)} \quad \text{for } u \in D(P_T) \text{ with (1.7.32)}. \tag{1.7.38}$$

This holds, of course, if $C'^{00*}\mathfrak{A}^{01}C'^{11} = 0$, and then (1.7.35) is also valid. Conversely, since $\gamma u = S^{00}\varphi_0$, (1.7.38) implies

$$\operatorname{Re} e^{i\theta}(C'^{00*}\mathfrak{A}^{01}C'^{11}\varphi_1, \varphi_0) \le c_3\|\varphi_0\|^2_{\prod_{J_0} H^{m-k-\frac12}(Z_k)}, \tag{1.7.39}$$

where φ_1 and φ_0 run freely in $\prod_{J_1} H^{2m-k-\frac12}(Z_k)$, resp. $\prod_{J_0} H^{2m-k-\frac12}(Z_k)$. This implies (1.7.34). $\square$

Throughout the proof it is used that the various matrix operators over Γ have the appropriate continuity properties, which we do not always recall explicitly.

Note in particular that the Dirichlet condition corresponds to the case where C'^{00} is zero-dimensional, and the Neumann-type conditions correspond to the case where C'^{11} is zero-dimensional, so in these cases (1.7.34) is trivially satisfied, and the realizations are weakly semibounded (as we know more or less already).

As observed for the differential operator case in earlier works, the condition (1.7.34) is *not always stable* under lower-order perturbations of the problem. It holds stably, when F satisfies (1.4.23) and J is one of the sets $\{0, 1, \dots, p-1, m, m+1, m+p-1\}$ (cf., e.g., Fujiwara and Shimakura [F-S70]), but otherwise it requires exact identity between certain operators, which can be violated by perturbation of low-order terms.

The link between weakly semibounded boundary problems and sesquilinear forms can now be elaborated as in [G73]. The discussion of m-coerciveness will be based on a decomposition that was derived, in the differential operator case, from an abstract theorem in [G74]; the more direct proof that we give here was explained in [G73′]. We now assume that P is strongly elliptic, and we set

$$P^r = \tfrac12(P + P^*);$$

it is formally selfadjoint and strongly elliptic. By application of Theorem 1.7.2 to P^r, we have (after adding a constant to P if necessary) that P^r_γ is selfadjoint positive, satisfying

$$(P^r_+ u, u) \geq c\|u\|^2_m \text{ for } u \in D(P^r_\gamma), \tag{1.7.40}$$

with $c > 0$. Now we decompose $H^{2m}(E)$ as

$$H^{2m}(E) = D(P^r_\gamma) \dot{+} Z^{2m}(P^r_+),$$

where $Z^{2m}(P^r_+)$ is the space of $z \in H^{2m}(E)$ with $P^r_+ z = 0$; the decomposition is described by

$$\begin{gathered} u = u^r_\gamma + u^r_\zeta, \\ u^r_\gamma = (P^r_\gamma)^{-1} P^r_+ u, \quad u^r_\zeta = u - u^r_\gamma. \end{gathered} \tag{1.7.41}$$

Observe that since

$$P^r_+ u^r_\zeta = 0, \quad \text{ and } \gamma u^r_\gamma = 0 \text{ so that } \gamma u^r_\zeta = \gamma u, \tag{1.7.42}$$

u^r_ζ is likewise determined by

$$u^r_\zeta = K^r \gamma u, \tag{1.7.43}$$

where K^r is the Poisson operator $K^r \colon \varphi \mapsto z$ solving

$$\begin{aligned} P^r_+ z &= 0 \text{ in } \Omega, \\ \gamma z &= \varphi \text{ at } \Gamma. \end{aligned} \tag{1.7.44}$$

Let us now also define the ps.d.o. Q^r over Γ by

$$Q^r = \nu K^r; \tag{1.7.45}$$

then Q^r maps γz into νz when $z \in Z^{2m}(P^r_+)$. (Q^r is an *elliptic* ps.d.o. if and only if the Neumann problem for P^r is elliptic.)

1.7.5 Lemma. *For any $u \in H^{2m}(E)$ one has*

$$\operatorname{Re}(P_+ u, u)_\Omega = (P^r_+ u^r_\gamma, u^r_\gamma)_\Omega + \operatorname{Re}(\mathfrak{A}^{01} \nu u, \gamma u)_\Gamma + (\mathfrak{B} \gamma u, \gamma u) \tag{1.7.46}$$

where $\mathfrak{B}$ is the ps.d.o. system on Γ

$$\mathfrak{B} = \tfrac{1}{2}[\mathfrak{A}^{00*} + (\mathfrak{A}^{10*} - \mathfrak{A}^{01}) Q^r]. \tag{1.7.47}$$

When P is selfadjoint, $\mathfrak{B} = -\frac{1}{2}\mathfrak{A}^{00} - \mathfrak{A}^{01} Q^r$; cf. (1.6.65$'$).

PROOF. Let $u \in H^{2m}(E)$ and set $v = u^r_\gamma$, $z = u^r_\zeta$. By Green's formula (1.7.22), one has that

$$\begin{aligned}
(v, P_+ v)_\Omega &= (P_+^* v, v)_\Omega, \\
(z, P_+ v)_\Omega &= (P_+^* z, v)_\Omega + (\gamma z, \mathfrak{A}^{01} \nu v)_\Gamma, \\
(z, P_+ z)_\Omega &= (P_+^* z, z)_\Omega + (\gamma z, \mathfrak{A}^{00} \gamma z + \mathfrak{A}^{01} \nu z) + (\nu z, \mathfrak{A}^{10} \gamma z) \\
&= (P_+^* z, z)_\Omega + ([\mathfrak{A}^{00^*} + \mathfrak{A}^{10*} Q^r] \gamma z, \gamma z) + (\gamma z, \mathfrak{A}^{01} Q^r \gamma z).
\end{aligned}$$

Then

$$\begin{aligned}
\operatorname{Re}(P_+ u, u) &= \tfrac{1}{2}(P_+ u, u) + \tfrac{1}{2}(u, P_+ u) \\
&= \tfrac{1}{2}[(P_+ v, v) + (P_+ v, z) + (P_+ z, v) + (P_+ z, z)] \\
&\quad + \tfrac{1}{2}[(v, P_+ v) + (v, P_+ z) + (z, P_+ v) + (z, P_+ z)] \\
&= (P_+^r v, v) + (v, P_+^r z) + (P_+^r z, v) + (P_+^r z, z) + \tfrac{1}{2}(\mathfrak{A}^{01} \nu v, \gamma z) \\
&\quad + \tfrac{1}{2}(\gamma z, \mathfrak{A}^{01} \nu v) + \tfrac{1}{2}([\mathfrak{A}^{00^*} + \mathfrak{A}^{10^*} Q^r] \gamma z, \gamma z) + \tfrac{1}{2}(\gamma z, \mathfrak{A}^{01} Q^r \gamma z) \\
&= (P_+^r v, v) + \operatorname{Re}(\mathfrak{A}^{01}(\nu u - Q^r \gamma u), \gamma z) \\
&\quad + \tfrac{1}{2}([\mathfrak{A}^{00^*} + \mathfrak{A}^{10^*} Q^r] \gamma z, \gamma z) + \tfrac{1}{2}(\gamma z, \mathfrak{A}^{01} Q^r \gamma z),
\end{aligned}$$

since $P_+^r z = 0$, $\nu v = \nu(u - u^r_\zeta) = \nu u - Q^r \gamma u$ and $\gamma z = \gamma u$. This gives (1.7.46). $\square$

1.7.6 Example. In the case $P = -\Delta$, P^r equals P, and the formula (1.7.46) becomes

$$\operatorname{Re}(P_+ u, u) = (P_+ u^r_\gamma, u^r_\gamma) + \operatorname{Re}(i\gamma_1 u, \gamma_0 u) + (-iQ^r \gamma_0 u, \gamma_0 u)$$

where we recall that $i\gamma_1 u = \partial u / \partial n$, and iQ^r is the ps.d.o. sending γz into $\partial z / \partial n$ when $\Delta z = 0$. In this case, $-iQ^r$ is a positive ps.d.o. on Γ with principal symbol $|\xi'|$.

Since P^r_γ satisfies (1.7.40), it depends on the boundary term in (1.7.46) whether a given realization P_T is m-coercive. When P_T is weakly semibounded, we find by insertion of (1.7.32) in (1.7.46), in view of (1.7.35),

$$\begin{aligned}
\text{(1.7.49)} \quad \operatorname{Re}(P_+ u, u) &= (P_+^r u^r_\gamma, u^r_\gamma)_\Omega + ((\mathfrak{B} - \mathfrak{A}^{01} C^{11} S^{10}) \gamma u, \gamma u)_\Gamma \\
&= (P_+^r u^r_\gamma, u^r_\gamma)_\Omega + (C'^{00^*} (\mathfrak{B} - \mathfrak{A}^{01} C^{11} S^{10}) C'^{00} \varphi_0, \varphi_0),
\end{aligned}$$

when $u \in D(P_T)$ and $\gamma u = C'^{00} \varphi_0$.

1.7.7 Lemma. *For any $\varepsilon \in [0, \frac{1}{2}[$, there are positive constants c_ε and C_ε so that for $u \in H^{m-\varepsilon}(E)$ (with the notation (1.6.60)),*

$$c_\varepsilon \|u\|^2_{m-\varepsilon} \le \|u^r_\gamma\|^2_{m-\varepsilon} + \|\gamma u\|^2_{H^{m-\varepsilon}_{E^0}} \le C_\varepsilon \|u\|^2_{m-\varepsilon}. \tag{1.7.50}$$

PROOF. Recall from (1.7.43) that

$$u^r_\gamma = u - K^r \gamma u$$

for $u \in H^{2m}(E)$; this definition extends to $H^m(E)$, as does the formula $u^r_\gamma = (P^r_\gamma)^{-1} P_+$ in view of the proof of Theorem 1.7.2. Here

$$\|K^r \gamma u\|_s \le c_s \|\gamma u\|_{H^s_{E^0}}$$

for any s. This implies the left-hand inequality in (1.7.50):

$$\begin{aligned} \|u\|_{m-\varepsilon} &= \|u^r_\gamma + K^r \gamma u\|_{m-\varepsilon} \le \|u^r_\gamma\|_{m-\varepsilon} + \|K^r \gamma u\|_{m-\varepsilon} \\ &\le \|u^r_\gamma\|_{m-\varepsilon} + c_{m-\varepsilon} \|\gamma u\|_{H^{m-\varepsilon}_{E^0}}. \end{aligned}$$

The right-hand inequality now follows by using that γ is continuous from $H^s(E)$ to $H^s_{E^0}$ for $s > m - \frac{1}{2}$,

$$\begin{aligned} \|u^r_\gamma\|_{m-\varepsilon} + \|\gamma u\|_{H^{m-\varepsilon}_{E^0}} &= \|u - K^r \gamma u\|_{m-\varepsilon} + \|\gamma u\|_{H^{m-\varepsilon}_{E^0}} \\ &\le \|u\|_{m-\varepsilon} + c_1 \|\gamma u\|_{H^{m-\varepsilon}_{E^0}} \le c_2 \|u\|_{m-\varepsilon}. \qquad \square \end{aligned}$$

1.7.8 Theorem. *Let P be a strongly elliptic ps.d.o. of order $2m > 0$, and let P_T be the realization defined by a normal trace operator T of the form $T = S\rho$. It is necessary and sufficient for m-coerciveness of P_T that the following two conditions hold:*

(1) *P_T is weakly semibounded, i.e.,*

$$C'^{00*} \mathfrak{A}^{01} C'^{11} = 0. \tag{1.7.51}$$

(ii) *The ps.d.o. over Γ (cf. (1.7.47))*

$$\mathcal{L} = C'^{00*} (\mathfrak{B} - \mathfrak{A}^{01} C^{11} S^{10}) C'^{00}, \tag{1.7.52}$$

acting in $\prod_{k \in J_0} H^{m-k-\frac{1}{2}}(Z_k)$, is strongly elliptic, i.e.,

$$\sigma^0(\mathcal{L}) + \sigma^0(\mathcal{L})^* \text{ is positive definite}$$

at each (x', ξ') *with* $|\xi'| > 1$.

When furthermore $M \equiv \dim \bigoplus_{0 \le k < 2m} F_k$ *equals* mN, *then* (i) *and* (ii) *imply that* P_T *is elliptic.*

PROOF. Clearly, weak semiboundedness is necessary for m-coerciveness, so we may assume (1.7.51) in view of Theorem 1.7.4. Then, as shown in (1.7.49), one has for $u \in D(P_T)$,

$$\operatorname{Re}(P_+ u, u) = (P_+^r u_\gamma^r, P_+^r u_\gamma^r) + \operatorname{Re}((\mathfrak{B} - \mathfrak{A}^{01} C^{11} S^{10})\gamma u, \gamma u). \tag{1.7.53}$$

Since $\gamma u = C'^{00}\varphi_0$ and $\varphi_0 = S'^{00}\gamma u$, where φ_0 runs through the full space $\prod_{J_0} H^{2m-k-\frac{1}{2}}(Z_k)$ when u runs through $D(P_T)$, one has in view of the continuity of the operators C'^{00} and S'^{00}: There exist $\varepsilon \in \,]0, \frac{1}{2}[$, $c > 0$ and $c' \in \mathbb{R}$, so that

$$\operatorname{Re}((\mathfrak{B} - \mathfrak{A}^{01} C^{11} S^{10})\gamma u, \gamma u) \ge c\|\gamma u\|^2_{H^m_{E^0}} - c'\|\gamma u\|^2_{H^{m-\varepsilon}_{E^0}} \tag{1.7.54}$$

holds for $u \in D(P_T)$, *if and only if*

$$\operatorname{Re}(\mathcal{L}\varphi_0, \varphi_0) \ge c_1\|\varphi_0\|^2_{\{m-k-\frac{1}{2}\}} - c_1'\|\varphi_0\|^2_{\{m-\varepsilon-k-\frac{1}{2}\}} \tag{1.7.55}$$

holds for $\varphi_0 \in \prod_{J_0} H^{m-k-\frac{1}{2}}(Z_k)$, with $c_1 > 0$ and $c_1' \in \mathbb{R}$. Here $\|\varphi_0\|_{\{m-\varepsilon-k-\frac{1}{2}\}}$ is the norm in $\prod_{k \in J_0} H^{m-\varepsilon-k-\frac{1}{2}}(Z_k)$. (1.7.55) holds precisely when $\mathcal{L}$ is strongly elliptic. In view of (1.7.40), we find by use of Lemma 1.7.7 that when (1.7.54) holds,

$$\operatorname{Re}(P_+ u, u) \ge c''\|u\|^2_m - c_1''\|u\|^2_{m-\varepsilon} \text{ for } u \in D(P_T); \tag{1.7.56}$$

i.e., P_T is strongly elliptic. This shows that (ii) implies m-coerciveness.

Conversely, assume (1.7.56). We can substitute u by $u - w_j \quad (j \in \mathbb{N})$, where $w_j \in C_0^\infty(\Omega, E)$ and $w_j \to u_\gamma^r$ in $H_0^m(E)$; then since $(u - w_j)_\gamma^r = u_\gamma^r - w_j$ and

$$\|u_\gamma^r - w_j\|_m \to 0, \quad \|u_\gamma^r - w_j\|_{m-\varepsilon} \to 0,$$
$$(P_+^r(u_\gamma^r - w_j), u_\gamma^r - w_j) \to 0$$

for $j \to \infty$, and $\gamma(u - w_j) = \gamma u$, it follows that (1.7.54) holds. Then also (1.7.55) holds, and the necessity of (ii) is proved.

Finally, consider the case where $M = mN$. This means that $\bigoplus_{k<2m} F_k$ and $\bigoplus_{k<2m} Z_k$ have the same dimension. The above calculations, carried out on the boundary symbol level, show that $b^0 = (p^0)_{t^0}$ has positive lower bound and hence

$$\begin{pmatrix} p_+^0 \\ t^0 \end{pmatrix} : H^{2m}(\overline{\mathbb{R}}_+)^N \to \begin{array}{c} L_2(\mathbb{R}_+)^N \\ \times \\ \mathbb{C}^M \end{array}$$

is injective; then in view of Corollary 1.7.3 it must be *bijective*, and P_T is elliptic (cf. Corollary 1.6.7). □

1.7.9 REMARK. One can furthermore show, exactly as in [G71,73], that when $M = mN$, weak semiboundedness of P_T implies weak semiboundedness of the formal adjoint $(P^*)_{\widetilde{T}}$ (which likewise has no "interior" terms); in fact, weak semiboundedness here is equivalent with the property

$$\gamma D(P_T) = \gamma D((P^*)_{\widetilde{T}}) \tag{1.7.57}$$

and with the equation

$$S^{00}(\mathfrak{A}^{01^*})^{-1}S^{11^*} = 0. \tag{1.7.58}$$

Moreover, in this case a consideration of b^0 and b^{0^*} shows that m-coerciveness of P_T will imply m-coerciveness of $(P^*)_{\widetilde{T}}$, which is then elliptic and equal to the true adjoint of P_T. In particular, both operators are *variational*. The associated sesquilinear forms have domain equal to the closed subspace of $H^m(E)$ defined by

$$V_{S^{00}} = \{u \in H^m(E) \mid S^{00}\gamma u = 0\}. \tag{1.7.59}$$

(When S is differential, the sesquilinear forms can be realized as expressions $a_T(u,v) + p'(u,v)$, where p' is as in (1.7.23)ff. and a_T is an integro-differential form, specially adapted to the boundary condition as in [G73].)

The proof of Theorem 1.7.8 for differential operators was originally based on a more abstract analysis of the realizations $\widetilde{P}$ of P_+, acting like P_+ and with

$$D(P_{\min}) \subset D(\widetilde{P}) \subset D(P_{\max}); \tag{1.7.60}$$

cf. [G68,74]. This analysis is a further development of theories of Krein [Kr47], Birman [Bi56] and Vishik [Vi52].

We now turn to general realizations $B = (P+G)_T$ where "interior" terms are allowed. Here B need no longer act like P_+, and its domain need not satisfy inclusions as in (1.7.60) (recall, e.g., Example 1.4.5), so the abstract theory of [G74] cannot be used. However, it is possible to discuss m-coerciveness and semiboundedness by direct use of the boundary operator calculus.

Let us first consider weak semiboundedness, i.e., the property that for some θ and $c \in \mathbb{R}$,

$$\operatorname{Re} e^{i\theta}(Bu,u)_\Omega \le c\|u\|_m^2 \text{ when } u \in D(B), \tag{1.7.61}$$

necessary for the other semiboundedness or variationality properties we are interested in. It is not hard to show that Neumann-type realizations are weakly semibounded. But we now meet the surprising fact that Dirichlet-type realizations need *not* be weakly semibounded; there is a necessary and sufficient condition on the interior terms for this to hold, and the condition is *exact*, i.e., not stable under small perturbations. Let us first consider the Neumann- and Dirichlet-type conditions and afterwards give the more technical discussion of the general case. We write G and T as in Examples 1.6.12–15:

$$G = K^1\nu + K^0\gamma + G', \tag{1.7.62}$$

with a Dirichlet-type trace operator

$$T^0 = \gamma + T'^0 \tag{1.7.63}$$

or a Neumann-type trace operator

$$T^1 = \nu + S^0\gamma + T'^1. \tag{1.7.64}$$

By (1.7.24), one has in general

$$\begin{aligned}(Bu,v)_\Omega &= p(u,v) + (Gu,v)_\Omega + (\mathfrak{A}^{01}\nu u,\gamma v)_\Gamma \\ &= (K^1\nu u,v)_\Omega + (\mathfrak{A}^{01}\nu u,\gamma v)_\Gamma + m\text{-bounded terms.}\end{aligned} \tag{1.7.65}$$

Now in the Neumann-type problems where $T^1u = 0$,

$$\begin{aligned}(K^1\nu u,v)_\Omega + (\mathfrak{A}^{01}\nu u,\gamma v)_\Gamma &= -(K^1(S^0\gamma u + T'^1u),v)_\Omega \\ - (\mathfrak{A}^{01}(S^0\gamma u + T'^1u),\gamma v)_\Gamma &= -(S^0\gamma u + T'^1u, K^{1^*}v + \mathfrak{A}^{01^*}\gamma v)_\Gamma,\end{aligned}$$

which is m-bounded, since all the operators are defined on $H^m(E)$, and the orders fit together. (For example, $T'^1\colon H^m(E) \to \prod_{k\in J_1} H^{m-k-\frac12}(E_\Gamma)$ and $K^{1^*}\colon H^m(E) \to \prod_{k\in J_1} H^{-m+k+\frac12}(E_\Gamma)$ are continuous.) So in this case, weak semiboundedness holds without restrictions, and one moreover has m-boundedness:

$$|(Bu,v)| \le c\|u\|_m\|v\|_m \text{ for } u,v\in D(B). \tag{1.7.66}$$

Now consider a Dirichlet-type condition. Here, when $u\in D(B)$ (so that $\gamma u = -T'^0u$),

$$(K^1\nu u,u) + (\mathfrak{A}^{01}\nu u,\gamma u) = (\nu u, T''u) \tag{1.7.67}$$

where T'' is the trace operator of class 0

$$T'' = K^{1^*} - \mathfrak{A}^{01^*} T'^0. \tag{1.7.68}$$

Since ν is not defined as a continuous operator on $H^m(E)$, there is something wrong when $T''u \neq 0$. Of course, $T'' = 0$ *implies* weak semiboundedness (in fact, (1.7.66) will hold). We shall now show that $T'' = 0$ is also *necessary* for weak semiboundedness. This requires a deeper analysis of the surjectiveness properties of our trace operators. For the present case, we need the construction of a sequence u_ε in $D(B)$ where $(\nu u_\varepsilon, T''u_\varepsilon)$ is not dominated by $\|u_\varepsilon\|_m^2$ if $T'' \neq 0$. A tool for constructing this is furnished by the following lemma, which is formulated such that it is also useful in the general case.

1.7.10 Lemma. *Let T be a normal trace operator associated with the order $2m$, and write it in blocks corresponding to the index sets $J_0 = \{0, 1, \dots, m-1\}$ and $J_1 = \{m, m+1, \dots, 2m-1\}$:*

$$T = \begin{pmatrix} S^{00} & 0 \\ S^{10} & S^{11} \end{pmatrix} \begin{pmatrix} \gamma \\ \nu \end{pmatrix} + \begin{pmatrix} T'^0 \\ T'^1 \end{pmatrix}. \tag{1.7.69}$$

Let S' be a morphism supplementing S as in Lemma 1.6.1, *written in blocks*

$$S' = \begin{pmatrix} S'^{00} & 0 \\ 0 & S'^{11} \end{pmatrix}, \tag{1.7.70}$$

so that the full system $\{T, S'\rho\}$ is normal. Let $\varphi \in \prod_{k\in J_1} H^{2m-k-\frac12}(Z_k)$, with $\varphi \neq 0$. There is a family of functions $u_\varepsilon \in H^{2m}(E)$ for $\varepsilon \in]0, \varepsilon_0]$ such that for each ε,

$$Tu_\varepsilon = 0, \quad S'^{00}\gamma u_\varepsilon = 0, \quad S'^{11}\nu u_\varepsilon = \varphi, \text{ and} \tag{1.7.71}$$

$$\|u_\varepsilon\|_m \le c(\varphi)\varepsilon, \tag{1.7.72}$$

where $c(\varphi)$ depends only on φ. In fact, $c(\varphi) \le$ const.$\|\varphi\|_{\prod_{k\in J_1} H^{m-k-\frac12}(Z_k)}$.

Proof. The right inverse $\mathcal{K}_\delta$ of ρ constructed in Lemma 1.6.4 can be written

$$\begin{gathered} \mathcal{K}_\delta = \begin{pmatrix} \mathcal{K}_\delta^0 & \mathcal{K}_\delta^1 \end{pmatrix}, \text{ where} \\ \gamma\mathcal{K}_\delta^0 = I, \quad \gamma\mathcal{K}_\delta^1 = 0, \quad \nu\mathcal{K}_\delta^0 = 0, \quad \nu\mathcal{K}_\delta^0 = I; \end{gathered} \tag{1.7.73}$$

it is seen from the estimates (1.6.14), applied with $k = 0, 1, \ldots, m$ and $s = m - k$, that

$$\|\mathcal{K}_\delta^1 \psi_1\|_{H^m(E)} \le \delta \text{ const. } \|\psi_1\|_{\prod_{j \in J_1} H^{m-j-\frac{1}{2}}(E_\Gamma)}, \tag{1.7.74}$$

for all $\psi_1 \in \prod_{J_1} H^{m-j-\frac{1}{2}}(E_\Gamma)$. Now we search for the solution of (1.7.71) in the form $u_\varepsilon = \begin{pmatrix} \mathcal{K}_\delta^0 & \mathcal{K}_\varepsilon^1 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}$, which gives

$$\begin{pmatrix} \begin{pmatrix} S^{00}\gamma + T'^0 \\ S'^{00}\gamma \end{pmatrix} \\ \begin{pmatrix} S^{11}\nu + S^{10}\gamma + T'^1 \\ S'^{11}\nu \end{pmatrix} \end{pmatrix} (\mathcal{K}_\delta^0 \psi_0 + \mathcal{K}_\varepsilon^1 \psi_1) = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \end{pmatrix} \tag{1.7.75}$$

and hence, in view of (1.7.73) (recall also (1.7.29))

$$\begin{aligned} &\widetilde{S}^{00}\psi_0 + \begin{pmatrix} T'^0 \\ 0 \end{pmatrix} \mathcal{K}_\delta^0 \psi_0 = -\begin{pmatrix} T'^0 \\ 0 \end{pmatrix} \mathcal{K}_\varepsilon^1 \psi_1, \\ &\widetilde{S}^{11}\psi_1 + \begin{pmatrix} S^{10}\psi_0 \\ 0 \end{pmatrix} + \begin{pmatrix} T'^1 \\ 0 \end{pmatrix} (\mathcal{K}_\delta^0 \psi_0 + \mathcal{K}_\varepsilon^1 \psi_1) = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}. \end{aligned} \tag{1.7.76}$$

The first line can be solved for ψ_0 (as in Proposition 1.6.5) by taking δ so small that the norm of

$$\widetilde{\widetilde{S}} = (\widetilde{S}^{00})^{-1} \begin{pmatrix} T'^0 \\ 0 \end{pmatrix} \mathcal{K}_\delta^0$$

in $H^s_{E^0}$ (cf. (1.6.60)) is $\le \frac{1}{3}$ for $s \in [0, 2m]$; it is seen from (1.6.13) that the tangential order s does not really interfere with the estimates. Then

$$\psi_0 = -[1 + \widetilde{\widetilde{S}}]^{-1} (\widetilde{S}^{00})^{-1} \begin{pmatrix} T'^0 \\ 0 \end{pmatrix} \mathcal{K}_\varepsilon^1 \psi_1 = \widetilde{T} \mathcal{K}_\varepsilon^1 \psi_1, \tag{1.7.77}$$

with a trace operator $\widetilde{T}$ of class 0. δ is kept fixed in the following, and we insert (1.7.77) in the second line of (1.7.76). This leads to an equation

$$\widetilde{S}^{11}\psi_1 + \widetilde{\widetilde{T}} \mathcal{K}_\varepsilon^1 \psi_1 = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}.$$

Now we can choose ε_0 so small that the norm of $(\widetilde{S}^{11})^{-1} \widetilde{\widetilde{T}} \mathcal{K}_\varepsilon^1$ is $\le \frac{1}{3}$ for $\varepsilon \le \varepsilon_0$; then we invert $I + (\widetilde{S}^{11})^{-1} \widetilde{\widetilde{T}} \mathcal{K}_\varepsilon^1$ by a Neumann series. This gives

$$\psi_1 = [I + (\widetilde{S}^{11})^{-1} \widetilde{\widetilde{T}} \mathcal{K}_\varepsilon^1]^{-1} (\widetilde{S}^{11})^{-1} \begin{pmatrix} 0 \\ \varphi \end{pmatrix}. \tag{1.7.78}$$

With ψ_1 and ψ_0 determined by (1.7.78) and (1.7.77), we have a solution of (1.7.71)

$$u_\varepsilon = \mathcal{K}_\delta^0\psi_0 + \mathcal{K}_\varepsilon^1\psi_1 = (I + \mathcal{K}_\delta^0\widetilde{T})\mathcal{K}_\varepsilon^1\psi_1,$$

and since $\mathcal{K}_\delta^0\widetilde{T}$ is a singular Green operator of order and class 0 (hence bounded in $H^m(E)$), it follows in view of (1.7.74) that (1.7.72) is also satisfied. The last statement is seen from an inspection of the proof. $\square$

In the special case (1.7.63), the lemma means that for any $\varphi \in H^m_{E^1}$ (1.6.60), there is a family of functions $u_\varepsilon \in H^{2m}(E)$ such that

$$T^0 u_\varepsilon = 0, \nu u_\varepsilon = \varphi, \text{ and } \|u_\varepsilon\|_m \le c\varepsilon \text{ for } \varepsilon \in]0, \varepsilon_0]. \tag{1.7.79}$$

1.7.11 Proposition. *Let the ps.d.o. P be of order $2m$ on $\widetilde{E}$. Let $B = (P+G)_T$ be the realization of P determined by a Dirichlet-type boundary condition $T^0 \equiv \gamma u + T'^0 u = 0$, with G written as in* (1.7.62)*. Then B is weakly semibounded if and only if*

$$K^{1^*} = \mathfrak{A}^{01^*} T'^0, \tag{1.7.80}$$

and in the affirmative case, B is m-bounded (satisfies (1.7.66)*).*

Proof. Let $T'' = K^{1^*} - \mathfrak{A}^{01^*}T'^0$. That $T'' = 0$ implies weak semiboundedness and (1.7.66) was accounted for above. Now we shall show that the inequality

$$\operatorname{Re} e^{i\theta}(Bu, u) \le c\|u\|_m^2 \text{ for } u \in D(B),$$

for some θ and c, implies $T'' = 0$. To fix the ideas, we can take $\theta = 0$. Assume that $T'' \ne 0$, then for some $v \in L_2(E)$, $T''v \ne 0$. Since the set of $u \in H^{2m}(E)$, for which $T^0 u = 0$ and $\nu u = 0$, is dense in $L_2(E)$ (Lemma 1.6.8), we can find such a u for which $T''u \ne 0$; we denote $T''u = \varphi$. Now take a sequence u_ε satisfying (1.7.79) and let $v_\varepsilon = \varepsilon u + u_{a\varepsilon}$, where $a \in]0,1]$ will be fixed later. In view of (1.7.65) and (1.7.67), we have (for $\varepsilon \le \varepsilon_0$)

$$\operatorname{Re}(Bv_\varepsilon, v_\varepsilon) = \operatorname{Re}(\nu v_\varepsilon, T''v_\varepsilon) + O(\|v_\varepsilon\|_m^2),$$

where

$$\operatorname{Re}(\nu v_\varepsilon, T''v_\varepsilon) = \operatorname{Re}(\varphi, \varepsilon\varphi + T''u_{a\varepsilon}) = \varepsilon\|\varphi\|_{L_2}^2 + \operatorname{Re}(\varphi, T''u_{a\varepsilon}).$$

Since T'' is continuous on $H^m(E)$, *we can choose a so small* that

$$\begin{aligned} |\operatorname{Re}(\varphi, T''u_{a\varepsilon})| &\le \|\varphi\|_{H^m_{E^1}} \|T''u_{a\varepsilon}\|_{(H^m_{E^1})^*} \le c_1\|\varphi\|_{H^m_{E^1}}\|u_{a\varepsilon}\|_m \\ &\le c_2 a\varepsilon c(\varphi)\|\varphi\|_{H^m_{E^1}} \le \tfrac{\varepsilon}{2}\|\varphi\|_{L_2}^2, \end{aligned}$$

whereby

$$\operatorname{Re}(\nu u_\varepsilon, T'' v_\varepsilon) \ge \tfrac{\varepsilon}{2}\|\varphi\|^2_{L_2}.$$

On the other hand,

$$\|v_\varepsilon\|^2_m = \|u_{a\varepsilon} + \varepsilon u\|^2_m \le \varepsilon^2 (c^2 c(\varphi)^2 + \|u\|_m)^2 \le c_3 \varepsilon^2.$$

The inequality

$$\varepsilon \|\varphi\|^2_{L_2} \le \varepsilon^2 c_1$$

cannot hold for $\varepsilon \to 0$ with $\varphi \ne 0$, so the hypothesis $T'' \ne 0$ is contradicted. $\square$

Note that (1.7.80) is part of the condition for selfadjointness; cf. (1.6.64).

Now consider the general case. The result has the same flavor as in the preceding cases but is more technical to state and to prove. With the formulation (1.7.69) we write the condition $Tu = 0$ in the form

$$\begin{aligned} &S^{00}\gamma u + T'^0 u = 0,\\ &S^{11}\nu u + S^{10}\gamma u + T'^1 u = 0. \end{aligned} \tag{1.7.81}$$

We define S'^{00}, C^{00}, etc., by Lemma 1.6.1 and (1.7.28) ff.

1.7.12 Lemma. *The function $u \in H^{2m}(E)$ satisfies* (1.7.81) *if and only if it satisfies*

$$\begin{aligned} \gamma u &= C'^{00}\varphi_0 - C^{00} T'^0 u\\ \nu u &= C'^{11}\varphi_1 - C^{11} S^{10}\gamma u - C^{11} T'^1 u \end{aligned} \tag{1.7.82}$$

for some $\varphi_0 \in \prod_{k\in J_0} H^{2m-k-\frac12}(Z_k)$ and $\varphi_1 \in \prod_{k\in J_1} H^{2m-k-\frac12}(Z_k)$; in fact, one then has

$$\varphi_0 = S'^{00}\gamma u \text{ and } \varphi_1 = S'^{11}\nu u. \tag{1.7.83}$$

For any given pair $\{\varphi_0, \varphi_1\}$, there exists u satisfying (1.7.82).

Proof. If u satisfies (1.7.81) and we define φ_0 and φ_1 by (1.7.83), then (1.7.82) is obtained by multiplication on the left by C^{00} (resp. C^{11}), using that $C^{\delta\delta}S^{\delta\delta} = I - C'^{\delta\delta}S'^{\delta\delta}$ for $\delta = 0, 1$. If u satisfies (1.7.82) then (1.7.83) follows by composition with S'^{00} (resp. S'^{11}), and (1.7.81) follows by composition with S^{00} (resp. S^{11}); cf. (1.7.30). The last statement follows from the fact that the trace operator $\{T, S'^{00}\gamma, S'^{11}\nu\}$ is normal and hence surjective. $\square$

1.7.13 Theorem. *Let $B = (P + G)_T$ be a normal realization on E of a pseudodifferential operator P of order $2m$ on $\widetilde{E}$, and write G and T as in* (1.7.62) *and* (1.7.69). *Then B is weakly semibounded (satisfies* (1.7.61) *for some c and θ) if and only if the following equations hold:*

$$C'^{00*}\mathfrak{A}^{01}C'^{11} = 0, \tag{1.7.84}$$

$$C'^{11*}(K^{1*} - \mathfrak{A}^{01*}C^{00}T'^{0}) = 0. \tag{1.7.85}$$

When they hold, B is in fact m-bounded (satisfies (1.7.66)*).*

In particular, symmetric realizations have these properties.

Proof. In view of (1.7.65) and Lemma 1.7.12, we have

$$\begin{aligned}(Bu, v) &= (K^1\nu u, v) + \mathfrak{A}^{01}\nu u, \gamma v) + m\text{-bd. terms}\\ &= (C'^{11}S'^{11}\nu u - C^{11}S^{10}\gamma u - C^{11}T'^{1}u, K^{1*}v + \mathfrak{A}^{01*}\gamma v) + m\text{-bd. terms}\\ &= (C'^{11}S'^{11}\nu u, K^{1*}v + \mathfrak{A}^{01*}\gamma v) + m\text{-bd. terms.}\end{aligned}$$

Consider the inequality (1.7.61), with $\theta = 0$ to fix the ideas. It clearly holds if and only if for some c_1,

$$\operatorname{Re} f(u, u) \le c_1\|u\|_m^2 \text{ for } u \in D(B), \tag{1.7.86}$$

where

$$f(u, v) = (C'^{11}S'^{11}\nu u, K^{1*}v + \mathfrak{A}^{01*}\gamma v). \tag{1.7.87}$$

Now when $u, u' \in D(B)$, we have from Lemma 1.7.12, that

$$\begin{aligned}f(u, u') &= (C'^{11}\varphi_1, K^{1*}u' + \mathfrak{A}^{01*}(C'^{00}\varphi_0' - C^{00}T'^{0}u'))\\ &= (\varphi_1, C'^{11*}\mathfrak{A}^{01*}C'^{00}\varphi_0' + T''u'),\end{aligned}$$

where

$$T'' = C'^{11*}(K^{1*} - \mathfrak{A}^{01*}C^{00}T'^{0}); \text{ it is of class 0.}$$

If (1.7.84) and (1.7.85) hold, then $f(u, u') = 0$ so that B is weakly semibounded and satisfies (1.7.66). Conversely assume (1.7.86). To see that T'' must be the zero operator, suppose that $T''v \neq 0$ for some $v \in L_2(E)$. Since the set of $u \in H^{2m}(E)$, for which Tu, $S'^{00}\gamma u$ and $S'^{11}\gamma u$ are zero, is dense in $L_2(E)$ (Lemma 1.6.8), there is a u with these properties such that $T''u = \varphi \neq 0$. Now take a sequence u_ε according to Lemma 1.7.10, and set $v_\varepsilon = \varepsilon u + u_{a\varepsilon}$, with $a \le 1$ to be chosen later. The proof now proceeds just as in Proposition 1.7.11: On one hand,

$$\operatorname{Re} f(v_\varepsilon, v_\varepsilon) = \operatorname{Re}(\varphi, T''v_\varepsilon) = \operatorname{Re}(\varphi, \varepsilon\varphi + T''u_{a\varepsilon}) \ge \varepsilon\|\varphi\|^2 - |(\varphi, T''u_{a\varepsilon})|,$$

where we can take a so small that $|(\varphi, T'' u_{a\varepsilon})| \leq \frac{\varepsilon}{2}\|\varphi\|^2$, so that

$$\operatorname{Re} f(v_\varepsilon, v_\varepsilon) \geq \tfrac{\varepsilon}{2}\|\varphi\|^2.$$

On the other hand,

$$\|v_\varepsilon\|_m^2 = \|\varepsilon u + u_{a\varepsilon}\|_m^2 \leq \varepsilon^2 \cdot c,$$

and then

$$\operatorname{Re} f(v_\varepsilon, v_\varepsilon) \leq c_1 \|v_\varepsilon\|_m^2 \leq c_2 \varepsilon^2.$$

This cannot hold for $\varepsilon \to 0$, and it follows that T'' must equal 0.

Thus, in the case where (1.7.86) holds,

$$f(u,u) = (\varphi_1, C'^{11*} \mathfrak{A}^{01*} C'^{00} \varphi_0) \text{ for } u \in D(B),$$

where $\varphi_1 = S'^{11} \nu u$ and $\varphi_0 = S'^{00} \gamma u$. We shall now show that (1.7.86) also implies that

$$\mathfrak{C} \equiv C'^{11*} \mathfrak{A}^{01*} C'^{00} = 0$$

(or, equivalently, $\mathfrak{C}^* = 0$; this is (1.7.84)). If $\mathfrak{C}$ is not zero, there exists a smooth φ_0 with $\mathfrak{C}\varphi_0 \neq 0$, and there exists a $u \in D(B)$ with

$$Tu = 0, \quad C'^{11} \nu u = 0 \text{ and } S'^{00} \gamma u = \varphi_0,$$

since this system of trace operators is normal. Construct u_ε according to Lemma 1.7.10 with $\varphi = \mathfrak{C}\varphi_0$. Now take $v_\varepsilon = \varepsilon u + u_\varepsilon$. Again we have that

$$\|v_\varepsilon\|_m^2 \leq \text{ const. } \varepsilon^2,$$

whereas

$$\begin{aligned} \operatorname{Re} f(v_\varepsilon, v_\varepsilon) &= \operatorname{Re}(S'^{11} \nu v_\varepsilon, \mathfrak{C} S'^{00} \gamma v_\varepsilon) = \operatorname{Re}(\mathfrak{C}\varphi_0, \varepsilon \mathfrak{C}\varphi_0) \\ &= \varepsilon \|\mathfrak{C}\varphi_0\|^2. \end{aligned}$$

For $\varepsilon \to 0$ we get a contradiction to (1.7.86), which shows that $\mathfrak{C}$ must equal zero, and the proof is complete. □

Finally, let us consider the question of m-coerciveness for realizations $B = (P+G)_T$ satisfying the properties listed in Theorem 1.7.13. By use of Lemma 1.7.5 one finds for $u \in D(B)$, in view of Lemma 1.7.12 and Theorem 1.7.13:

$$\begin{aligned} &\operatorname{Re}(Bu,u) = \operatorname{Re}(P_+ u, u) + \operatorname{Re}(Gu,u) \\ &= (P_+^r u_\gamma^r, u_\gamma^r) + \operatorname{Re}(\mathfrak{A}^{01} \nu u, \gamma u) + (\mathfrak{B}\gamma u, \gamma u) + \operatorname{Re}(Gu,u) \\ &= (P_+^r u_\gamma^r, u_\gamma^r) + \operatorname{Re}(\nu u, \mathfrak{A}^{01*} \gamma u + K^{1*} u) \\ &\qquad + (\mathfrak{B}\gamma u, \gamma u) + \operatorname{Re}(K^0 \gamma u, u) + \operatorname{Re}(G' u, u) \\ &= (P_+^r u_\gamma^r, u_\gamma^r) + \operatorname{Re}(-C^{11} S^{10} \gamma u - C^{11} T'^1 u, \mathfrak{A}^{01*} \gamma u + K^{1*} u) \\ &\qquad + (\mathfrak{B}\gamma u, \gamma u) + \operatorname{Re}(K^0 \gamma u, u) + \operatorname{Re}(G' u, u) \\ &\equiv s(u,u), \end{aligned} \tag{1.7.88}$$

where we used that $f(u,u)$ defined by (1.7.87) is zero. Of course, there is no hope to write this as an integro-differential form like in the differential operator case; let us note that $s(u,u)$, or rather the associated sesquilinear form

$$s(u,v) \equiv \tfrac{1}{4} \textstyle\sum_{k=0,1,2,3} i^k s(u+i^k v, u+i^k v), \tag{1.7.89}$$

has the general structure

$$s(u,v) = (P_+^r u_\gamma^r, v_\gamma^r) + (G'' u, v) + (K' \gamma u, v) + (\mathfrak{C} \gamma u, \gamma v), \tag{1.7.90}$$

where G'' is an s.g.o. of order $2m$ and class 0, and K' is a Poisson operator and $\mathfrak{C}$ a ps.d.o. on Γ, of the appropriate orders, so that $s(u,v)$ is continuous on $H^m(E)$. In view of (1.7.41–43) and (1.7.82), $s(u,u)$ may for $u \in D(B)$ be further reduced to the form

$$s(u,u) = (P_+^r u_\gamma^r, u_\gamma^r) + (G'' u, u) + (K_0' \varphi_0, u) + (\mathfrak{C}_0 \varphi_0, \varphi_0), \quad \text{where } \varphi_0 = S'^{00} \gamma u. \tag{1.7.91}$$

In this way, the m-coerciveness problem for $(P+G)_T$ is reduced to an m-coerciveness problem for a continuous sesquilinear form on a subset of $H^m(E)$. In the differential operator case (as considered in [G71,74]), the s.g.o. term with G'' and the "mixed" term with K_0' do not appear, and m-coerciveness holds if and only if $\mathfrak{C}_0$ is strongly elliptic. For the present case one finds at least that it *suffices* for m-coerciveness that $G'' \geq 0$, K_0 is suitably small and $\mathfrak{C}_0$ is strongly elliptic. We shall not give a complete analysis here but instead restrict ourselves to some indications of what more can be done.

The first thing we observe is that for the question of m-coerciveness of $s(u,v)$ on $D(B)$ it is only the "Dirichlet" part of the boundary condition, namely the condition

$$S^{00} \gamma u + T_0' u = 0, \tag{1.7.92}$$

that matters. For one has:

1.7.14 Lemma. *The closure of $D(B)$ in $H^m(E)$ equals the set*

$$V = \{ u \in H^m(E) \mid S^{00} \gamma u + T_0' u = 0 \}. \tag{1.7.93}$$

In particular, $s(u,v)$ satisfies the inequality

$$s(u,u) \geq c \|u\|_m^2 - c_0 \|u\|_0^2 \tag{1.7.94}$$

for $u \in D(B)$ if and only if it does so for $u \in V$.

PROOF. Clearly, $D(B) \subset V$, so we have to prove that $D(B)$ is dense in V in the $H^m(E)$-topology.

Let $v \in V$. First take a sequence of functions $w_j \in C^\infty(E)$ such that $w_j \to v$ in $H^m(E)$; then by continuity, $S^{00}\gamma w_j + T_0' w_j \to 0$ in $\prod_k H^{m-k-\frac{1}{2}}(F_k)$. Next, we construct $x_j \in C^\infty(E)$ such that $x_j \to 0$ in $H^m(E)$ and

$$S^{00}\gamma x_j + T_0' x_j = S^{00}\gamma w_j + T_0' w_j \quad \text{for each } j.$$

This is found by setting $x_j = K_\delta^0 \varphi_j$ (cf. (1.7.73)) and using the existence of C^{00} so that $S^{00}C^{00} = I$; here φ_j must solve

$$S^{00}(\varphi_j + C^{00}T_0' K_\delta^0 \varphi_j) = S^{00}(\gamma w_j + C^{00} T_0' w_j),$$

which is obtained by taking

$$\varphi_j = (1 + C^{00}T_0' K_\delta^0)^{-1}(\gamma w_j + C^{00}T_0' w_j);$$

this is welldefined for δ sufficiently small. By continuity, φ_j goes to 0 in $\prod H^{m-k-\frac{1}{2}}(E_\Gamma)$ and hence $x_j \to 0$ in $H^m(E)$. So now we have a sequence of smooth functions $y_j = w_j - x_j \to v$ in $H^m(E)$, with $S^{00}\gamma y_j + T_0' y_j = 0$. Here

$$S^{11}\nu y_j + S^{10}\gamma y_j + T_1' y_j = \rho_j$$

exists for each j. By a variant of Lemma 1.7.10, we can construct $z_j \in H^m(E)$ such that

$$\begin{aligned} S^{10}\gamma z_j + T_0' z_j &= 0 \\ S^{11}\nu z_j + S^{10}\gamma z_j + T_1' z_j &= \rho_j \end{aligned}$$

for each j, and $\|z_j\|_m \to 0$ for $j \to \infty$. Then finally $z_j - y_j$ lies in $D(B)$ and approximates v in $H^m(E)$.

The last statement in the lemma is an easy consequence. $\square$

The next observation is that we can simplify the sesquilinear form problem by transformation to a more local condition by use of Lemma 1.6.8.

Let C^{00} be the right inverse of S^{00} defined above and set

$$\Lambda = I + \mathcal{K}_\delta^0 C^{00} T_0'; \tag{1.7.95}$$

then by Lemma 1.6.8, there is a $\delta_0 > 0$ so that for $\delta \in]0, \delta_0]$, Λ is a bijection in $H^s(E)$ *for all* $s \geq 0$, with an inverse of the form $\Lambda^{-1} = I + \mathcal{K}_\delta^0 Q_\delta C^{00} T_0'$ (Q_δ being a ps.d.o. in $\prod_{0 \leq k < m} H^{-k}(E_\Gamma)$) such that

$$T_0 \equiv S^{00}\gamma + T_0' = S^{00}\gamma\Lambda. \tag{1.7.96}$$

Λ defines in particular a bijection of V (1.7.93) onto

$$V_1 = \{v \in H^m(E) \mid S^{00}\gamma v = 0\}. \tag{1.7.97}$$

In this way, the validity of (1.7.94) for $u \in V$ is equivalent with the validity of an inequality

$$s_1(v,v) \geq c_1\|v\|_m^2 - c_2\|v\|_0^2 \text{ for } v \in V_1, \tag{1.7.98}$$

where s_1 is the sesquilinear form

$$s_1(u,v) \equiv s(\Lambda u, \Lambda v) \text{ for } u,v \in H^m(E); \tag{1.7.99}$$

note that it is again of the kind (1.7.90).

The advantage of reducing the problem to one where V is replaced by V_1 in (1.7.97) is that now

$$H_0^m(E) \subset V_1 \subset H^m(E),$$

so that we can write

$$V_1 = H_0^m(E) \dot{+} W,$$

where $W \simeq \gamma V_1$ represents the boundary values (in fact K^r (1.7.43) defines a bijection of γV_1 onto W). Here m-coerciveness of s_1 requires m-coerciveness of its restrictions to $H_0^m(E)$ and to W; the former represents m-coerciveness of the Dirichlet problem for an associated operator $P_+ + G'''$, and the latter can be reduced to strong ellipticity of a certain pseudodifferential system over Γ. Conversely, we expect that m-coerciveness of these two restrictions will imply m-coerciveness of the full form on V_1 by methods like those in [G70,74], under reasonable hypotheses on s_1.

The detailed analysis will be left to a treatment elsewhere, and we here just end with two particular examples.

1.7.15 Example. Consider a formally selfadjoint Dirichlet-type realization, as described in Example 1.6.12. (Note that $P^r = P$ here; to avoid changes, we keep the notation of (1.7.41)ff. anyway.) For $u \in D(B)$ we have, using Lemma 1.7.5, formulas (1.6.64)–(1.6.65′) and the fact that $\gamma u = -T'^0 u$,

$$\begin{aligned}
\operatorname{Re}(Bu,u) &= \operatorname{Re}(P_+u,u) + \operatorname{Re}(Gu,u) \\
&= (P_+u_\gamma^r, u_\gamma^r) - \operatorname{Re}(\mathfrak{A}^{01}\nu u, T'^0 u) + (\mathfrak{B}T'^0 u, T'^0 u) \\
&\qquad + \operatorname{Re}(K^1\nu u, u) + \operatorname{Re}(G'u,u) \\
&= (P_+u_\gamma^r, u_\gamma^r) + (G''u,u),
\end{aligned} \tag{1.7.100}$$

where

$$G'' = T'^{0*}(-\tfrac{1}{2}\mathfrak{A}^{00} - \mathfrak{A}^{01}Q^r)T'^0 + \tfrac{1}{2}(G' + G'^*)$$
$$= -T'^{0*}(\mathfrak{A}^{00} + \mathfrak{A}^{01}Q^r)T'^0 + G'.$$

When $G'' \geq 0$,

$$\operatorname{Re}(Bu, u) \geq (P_+ u^r_\gamma, u^r_\gamma) \geq c\|u^r_\gamma\|^2_m, \tag{1.7.101}$$

this gives at least positivity of the realizations. As for m-coerciveness, we observe that for $\varepsilon \in [0, \frac{1}{2}[$, one has by (1.7.43), when $u \in D(B)$,

$$\|u^r_\gamma\|_{m-\varepsilon} = \|u - K^r\gamma u\|_{m-\varepsilon} = \|u + K^r T'^0 u\|_{m-\varepsilon} \simeq \|u\|_{m-\varepsilon}$$
$$\text{when } I + K^r T'^0 \text{ is invertible;}$$

here the invertibility holds at least when T'^0 has a sufficiently small norm, or when $K^r T'^0$ has a lower bound > -1. In those cases, the inequality (1.7.101) implies m-coerciveness. Then one can furthermore allow a small negative lower bound for G''.

In the case $P = -\Delta$, $\mathfrak{A}^{00} = 0$ and $-\mathfrak{A}^{01}Q^r = -iQ^r$ is a positive selfadjoint ps.d.o. with principal symbol $|\xi'|$, so the first term in G'' is ≥ 0. Then it suffices that $\operatorname{Re} G' \geq 0$ or has a suitably small negative lower bound.

By use of (1.7.95–97), one gets another criterion. Here the space

$$V = \{u \in H^m(E) \mid \gamma u + T'_0 u = 0\}$$

can be replaced by $V_1 = H^m_0(E)$ by insertion of $u = \Lambda^{-1}v = (I - \mathcal{K}^0_\delta T_1)v$ for suitable δ and T_1. Noting that $u^r_\gamma = (1 - K^r\gamma)u$ we then find for the form $\operatorname{Re}(Bu, u)$, when $\gamma v = 0$,

(1.7.102)
$$\begin{aligned}
\operatorname{Re}(Bu, u) &= \operatorname{Re}(B\Lambda^{-1}v, \Lambda^{-1}v) \\
&= (P_+(I - K^r\gamma)\Lambda^{-1}v, (I - K^r\gamma)\Lambda^{-1}v) + (G''\Lambda^{-1}v, \Lambda^{-1}v) \\
&= (P_+(v + (K^r - \mathcal{K}^0_\delta)T_1 v), v + (K^r - \mathcal{K}^0_\delta)T_1 v) + (G''\Lambda^{-1}v, \Lambda^{-1}v) \\
&= (P_+ v, v) + (G''' v, v) \equiv s_1(v, v)
\end{aligned}$$

where G''' is the s.g.o. of class 0 and order $2m$:

$$G''' = (\Lambda^{-1})^* G'' \Lambda^{-1} + T_1^*(K^r - \mathcal{K}^0_\delta)^* P_+ (K^r - \mathcal{K}^0_\delta) T_1 + 2\operatorname{Re}[P_+(K^r - \mathcal{K}^0_\delta)T_1].$$

Here m-coerciveness on $H^m_0(E)$ is obtained, e.g., if $G''' \geq 0$, but also a moderate negativity can be allowed, since P_+ itself is m-coercive on $H^m_0(E)$. (We expect that the m-coerciveness of $P_+ + G'''$ on $H^m_0(E)$ can be characterized by a positivity estimate of a model sesquilinear form at the boundary.)

1.7.16 Example. Let $P = -\Delta$ and consider a formally selfadjoint Neumann-type realization, as described in Example 1.6.15. Here we have for $u \in D(B)$, since $\gamma_1 u = -S_0\gamma_0 u - T'^1 u$,

$$\begin{aligned}\operatorname{Re}(Bu,u) &= (P_+u_\gamma^r, u_\gamma^r) - \operatorname{Re}(i(S_0\gamma_0 u + T_1'u), \gamma_0 u) \\ &\quad + (\mathfrak{B}\gamma_0 u, \gamma_0 u) + \operatorname{Re}(K_0\gamma_0 u, u) + \operatorname{Re}(G'u,u) \\ &= (P_+u_\gamma^r, u_\gamma^r) + ((iS_0 - iQ^r)\gamma_0 u, \gamma_0 u) - 2\operatorname{Re}(iT_1'u, \gamma_0 u) + (G'u,u),\end{aligned} \tag{1.7.103}$$

by use of (1.6.80) and Lemma 1.7.5, as in the preceding example. Note that $iS_0 - iQ^r$ is selfadjoint with iQ^r positive (with symbol $|\xi'|$). The form is to be considered on $H^1(E)$ (in view of Lemma 1.7.14), so we do not need a transformation as in Example 1.7.15. Now 1-coerciveness is assured, for example, if $G' \geq 0$, iS_0 has principal symbol $> -|\xi'|$ and T_1' is small, for we can then obtain that

$$\operatorname{Re}(Bu,u) \geq c_1\|u_\gamma^r\|_1^2 - c_\varepsilon\|u_\gamma^r\|_{1-\varepsilon}^2 + c_1'\|\gamma_0 u\|_{\frac12}^2 - c_\varepsilon'\|\gamma_0 u\|_{\frac12-\varepsilon}^2$$

with positive constants c_1 and c_1'; this gives 1-coerciveness by Lemma 1.7.7. Also a small negative lower bound on G' can be allowed. Sharper criteria can probably be obtained if $u = u_\gamma^r + K^r\gamma_0 u$ is inserted in the last terms in (1.7.103).

1.7.17 Remark. It is seen from the preceding discussion how the normality of the boundary condition is of great help in the systematic discussion of coerciveness problems. Let us also remark, however, that there do exist elliptic boundary problems that are not normal but define m-coercive realizations anyway. An example is the following:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \mathbb{R}_+^n, \\ D_{x_1}\gamma_1 u + (1-\Delta_{x'})\gamma_0 u &= \varphi \quad \text{at } x_n = 0, \end{aligned} \tag{1.7.104}$$

which was considered in [R-S83,84] in connection with resolvent studies. Since the coefficient of γ_1 is not invertible, the boundary condition is not normal, and it remains nonnormal if multiplied by $(1-\Delta_{x'})^{-\frac12}$. However, the realization B of $-\Delta$ with domain

$$D(B) = \{u \in H^2(\overline{\mathbb{R}}_+^n) \mid D_{x_1}\gamma_1 u + (1-\Delta_{x'})\gamma_0 u = 0\} \tag{1.7.105}$$

is 1-coercive, for one finds by writing the boundary condition in the form

$$\gamma_0 u = -(1-\Delta_{x'})^{-1}D_{x_1}\gamma_1 u\big[\equiv -S\gamma_1 u\big] \tag{1.7.106}$$

and applying Green's formula that

$$
\begin{aligned}
\operatorname{Re}(Bu,u) &= \operatorname{Re}\Big[\sum_{j=1}^{n} \|D_j u\|_0^2 + i\int_{\mathbb{R}^{n-1}} \gamma_1 u \overline{S\gamma_1 u}\, dx'\Big] \\
&= \sum_{j=1}^{n} \|D_j u\|_0^2 = \|u\|_1^2 - \|u\|_0^2 \text{ for } u \in D(B),
\end{aligned}
\tag{1.7.107}
$$

since S is a constant coefficient ps.d.o. on $\mathbb{R}^{n-1}$ with real symbol. It is not hard to check that the problem is elliptic; an argument as in Theorem 1.6.9 shows that the adjoint B^* is the analogous operator defined by the boundary condition

$$
\gamma_0 u = +S\gamma_1 u. \tag{1.7.108}
$$

Here B^* is likewise elliptic and 1-coercive.

In particular, the inequalities for B and B^* imply that $(B-\lambda)^{-1}$ exists for $\operatorname{Re}\lambda \le -1$ and satisfies

$$
\|(B-\lambda)^{-1} f\|_1 + (|\operatorname{Re}\lambda| - 1)\|(B-\lambda)^{-1} f\|_0 \le \|f\|_0 \tag{1.7.109}
$$

there, so the norm of the resolvent in $L_2(\mathbb{R}^n_+)$ is $O(|\operatorname{Re}\lambda|^{-1})$ for $\operatorname{Re}\lambda \to -\infty$. In particular, the norm is $O(\langle\lambda\rangle^{-1})$ on each ray $\{\lambda = -re^{i\theta} \mid r \in [r_0, +\infty[\}$ with argument $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, as shown in [R-S83,84]. (The considerations extend to various other nonnormal boundary conditions of the form $S_1\gamma_1 u + S_0\gamma_0 u = 0$.)

In the systematic calculus of variable coefficient operators, this type of example has too low "regularity" to enter in our general theory, and only partial results are obtained; see the discussion in Remark 3.2.19. Actually, the systematic variable coefficient calculus of [R-S84] contains requirements that essentially imply regularity 1 (in a less apparent way) and which are not verified by this example, so it is not covered by their general theory either, as one might think; it is linked with the constant coefficient case. (See also our Remark 1.5.16.)