Eigenvalue asymptotics for nonsmooth singular Green operators

Gerd Grubb
Copenhagen University

Graz, August 2012

Plan

1. Singular Green operators in the smooth case
2. Spectral estimates for nonsmooth ψdbo's
3. Nonsmooth resolvent differences

1. Singular Green operators in the smooth case

- Ω bounded open $\subset \mathbb{R}^n$ with C^∞-boundary $\partial\Omega = \Sigma$.
- A strongly elliptic on Ω, C^∞-coefficients,

$$Au = -\sum\nolimits_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \sum\nolimits_{j=1}^{n} a_j\partial_j u + a_0 u, \text{ with}$$
$$\operatorname{Re}\sum\nolimits_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2 \text{ for } x \in \overline{\Omega}, \xi \in \mathbb{R}^n; \quad c_0 > 0.$$

- $\partial_n^j u|_\Sigma = \gamma_j u$, $j \in \mathbb{N}_0$. $\nu u = \sum n_j \gamma_0(a_{jk}\partial_k u)$ $(= \gamma_1 u$ when $A = -\Delta)$, $\vec{n} = (n_1, \dots, n_n)$ the normal to Σ.
- The *Dirichlet realization* A_γ acts like A with $D(A_\gamma) = \{u \in H^2(\Omega) \mid \gamma_0 u = 0\}$,
- Define a *Neumann-type realization* $A_{\nu,C}$ with $D(A_{\nu,C}) = \{u \in H^2(\Omega) \mid \nu u = C\gamma_0 u\}$; C a first-order diff.op. on Σ.

If both are invertible, then $A_{\nu,C}^{-1} - A_\gamma^{-1}$ is a *singular Green operator*.

When B is compact in a Hilbert space H, set $s_j(B) = \mu_j(B^*B)^{\frac{1}{2}}$.

It is well-known (starting with Weyl 1912,...) that each of the operators A_γ and $A_{\nu,C}$ has a spectral asymptotic behavior

$$s_j(A_\gamma^{-1}) \text{ and } s_j(A_{\nu,C}^{-1}) \sim c_A j^{-2/n} \text{ for } j \to \infty, \quad (1)$$

with a constant c_A determined from A. Remainders improved to $O(j^{-3/n})$ or more exact formulas (Hörmander '69, Ivrii '80s,...).

It is also well-known (Birman and Solomyak, Grubb in '70s and '80s) that

$$s_j(A_{\nu,C}^{-1} - A_\gamma^{-1}) \sim c\, j^{-2/(n-1)} \text{ for } j \to \infty. \quad (2)$$

Again the remainder can be refined using Hörmander, Ivrii results.

The "weak Schatten class" $\mathfrak{S}_{p,\infty}(H)$ consists of those B such that $s_j(B)$ is $O(j^{-1/p})$ for $j \to \infty$; with quasi-norm $\mathbf{N}_p(B) \equiv \sup_j s_j(B) j^{1/p}$. These are just *upper estimates*.

Here A_γ^{-1} and $A_{\nu,C}^{-1}$ are in $\mathfrak{S}_{n/2,\infty}$, and $A_{\nu,C}^{-1} - A_\gamma^{-1} \in \mathfrak{S}_{(n-1)/2,\infty}$.

The dimension $n-1$ comes in because the resolvent difference has its essential effect in the neighborhood of the boundary $\partial\Omega$.
More generally, the *singular Green operators* defined by Boutet de Monvel '71 in a calulus of *pseudodifferential boundary operators* (ψdbo's) satisfy, by G '84:

When G is a singular Green operator on Ω of order $-t<0$ (and class zero), then

$$s_j(G) \sim c_G\, j^{-t/(n-1)} \text{ for } j \to \infty. \quad (3)$$

Question: Extend asymptotic estimates like (2) and (3) to operators with nonsmooth x-dependence.

Upper estimates for (2) are known from Birman '62, when the coefficients are in $C^0 \cap W^{1,\infty}$ and $\partial\Omega$ is C^2.

Reference: G '12, arXiv:1205.0094.

The ψdbo calculus deals with matrices:

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} H^{s+d}(\Omega)^N \\ \times \\ H^{s+d}(\Sigma)^M \end{matrix} \to \begin{matrix} H^{s}(\Omega)^{N'} \\ \times \\ H^{s}(\Sigma)^{M'} \end{matrix}, \text{ where}$$

- P is a pseudodifferential operator (ψdo) on $\mathbb{R}^n$ of order d, and $P_+ = r^+ P e^+$ is its truncation to Ω (r^+ restricts to Ω and e^+ extends by zero).
- T is a trace operator from Ω to Σ of order $d - \frac{1}{2}$, K is a Poisson operator from Σ to Ω of order $d + \frac{1}{2}$, S is a ψdo on Σ of order d.
- G is a singular Green operator of order d, e.g. of type KT.

P and G are defined in local coordinates by Fourier integrals

$$(Pu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, d\xi,$$

$$(Gu)(x) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} \int_0^\infty \tilde{g}(x', x_n, y_n, \xi')\acute{u}(\xi', y_n)\, dy_n d\xi',$$

(when G is of class 0). Here $\hat{u} = \mathcal{F}u$, $\acute{u}(\xi', y_n) = \mathcal{F}_{y' \to \xi'} u(y', y_n)$, $y' = (y_1, \dots, y_{n-1})$.

For the 2' order elliptic operator A we have the examples:

- $Q = A^{-1}$ is the ψdo inverse of A on $\mathbb{R}^n$, Q_+ its truncation to Ω,
- $A_\gamma^{-1} = Q_+ + G_\gamma$, where G_γ is the s.g.o. $-K_\gamma \gamma_0 Q_+$; here K_γ is the Poisson solution operator for the Dirichlet problem.
- $A_{\nu,C}^{-1} - A_\gamma^{-1} = K_\gamma L^{-1} (K'_\gamma)^*$, a *Krein resolvent formula*. Here $L = C - P_{\gamma,\nu}$, where $P_{\gamma,\nu}$ is the Dirichlet-to-Neumann operator νK_γ, a ψdo on Σ.

The ψdbo calculus was generalized to symbols with C^τ- Hölder continuity in x ($\tau > 0$) by Abels '05. The Krein resolvent formula was extended to nonsmooth cases by Abels-G-Wood '12, when the coefficients of A are in W^1_q (for some $q > n$) and the domain has a $B^{3/2}_{p,2}$-boundary; this contains $C^{3/2+\varepsilon}$-domains for all $\varepsilon > 0$.

Some tools for spectral estimates:

Lemma A. *If Ξ is bounded smooth m-dimensional, and $B \in \mathcal{L}(L_2(\Xi), H^t(\Xi))$ with $t > 0$, then $B \in \mathfrak{S}_{m/t,\infty}$; indeed,*

$$\mathbf{N}_{m/t}(B) \equiv \sup_j s_j(B) j^{t/m} \leq C \|B\|_{\mathcal{L}(L_2,H^t)}.$$

Lemma B. 1° *Let $B = B_0 + R$, then for $j \to \infty$,*

$$\lim s_j(B_0) j^{1/p} = C_0,\ \lim s_j(R) j^{1/p} = 0 \implies \lim s_j(B) j^{1/p} = C_0.$$

2° *Let $B = B_M + B'_M$ for $M \in \mathbb{N}_0$, then $\lim s_j(B_M) j^{1/p} = C_M$, $\lim_M C_M = C_0$ and $\lim_M \mathbf{N}_p(B'_M) = 0$ imply $\lim s_j(B) j^{1/p} = C_0$.*

Lemma C. *When P is a classical ψdo system of order $-t < 0$, cut down to Ξ, with principal symbol $p^0(x,\xi)$, then $\lim_j s_j(P) j^{t/m} = c(p^0)^{t/m}$, where*

$$c(p^0) = \tfrac{1}{m(2\pi)^m} \int_\Xi \int_{|\xi|=1} \operatorname{tr}((p^{0*} p^0)^{m/2t})\, d\omega dx. \quad (4)$$

2. Spectral estimates for nonsmooth ψdbo's

The pseudodifferential calculus for symbols $p(x,\xi)$ with full estimates in ξ but only C^τ-smoothness in x ($\tau > 0$) was developed by Kumano-go and Nagase '78, J. Marschall '87 and M. Taylor '91. Here when $P_i = \mathrm{OP}(p_i(x,\xi))$ of order d_i, we only have

$$P_i \colon H^{s+d_i}(\mathbb{R}^m) \to H^s(\mathbb{R}^m) \text{ for } |s| < \tau,$$

$$P_1 P_2 - \mathrm{OP}(p_1 p_2) \colon H^{s+d_1+d_2-\theta}(\mathbb{R}^m) \to H^s(\mathbb{R}^m) \text{ for } s, s+d_1 \in\,]-\tau+\theta, \tau[\,.$$

Marschall shows that operator norms depend on N symbol estimates (N linked with the dimension). Then Lemma C extends:

Theorem 1. *If P is a classical C^τ-smooth ψdo system of order $-t < 0$, defined on a compact m-dimensional C^∞-manifold Ξ without boundary, then*

$$s_j(P) j^{t/m} \to C(p^0), \text{ for } j \to \infty.$$

Proof: Approximate P by C^∞-smooth operators P_k obeying Lemma C. Now $\|P - P_k\|_{\mathcal{L}(H^{-t},H^0)} \to 0$, so $P_k \to P$ in $\mathfrak{S}_{m/t,\infty}$ by Lemma A. Apply Lemma B to the decompositions $P = P_k + (P - P_k)$.

We now address the question for nonsmooth singular Green operators on smooth bounded domains $\Omega \subset \mathbb{R}^n$, in particular resolvent differences, where the *boundary dimension* $n-1$ should come in.

The nonsmooth ψdbo calculus (Abels '05) has similar difficulties as the ψdo calculus: Sobolev mapping properties are valid only for s in a small interval, in particular this holds for remainders in composition rules $\mathcal{A}_1\mathcal{A}_2 - \mathrm{OP}(a_1 \circ a_2)$.

For spectral estimates the calculus must be sharpened to operator norms depending on specific *finite* sets of symbol seminorms, as in Marschall's ψdo treatment.

However, there is an additional difficulty in the application of spectral theory to nonsmooth singular Green operators:

If $G = G_0 + R$ on Ω, where G_0 of order $-t$ has the expected asymptotic behavior

$$s_j(G_0) \sim C(G_0) j^{-t/(n-1)},$$

and R is of lower order, bounded from $H^{-t-\theta}(\Omega)$ to $H^0(\Omega)$ for some $\theta > 0$, then Lemma A for operators on Ω only gives that

$$s_j(R) \leq C j^{-(t+\theta)/n}.$$

But $j^{-(t+\theta)/n}$ decreases faster than $j^{-t/(n-1)}$ only when

$$\theta > t/(n-1),$$

and we usually do not have such large values of θ available.

So the remainders arising in compositions and approximations are a major problem. One has to involve the boundary more directly.

Consider a C^τ-smooth s.g.o. of order $-t$ and class 0 on $\mathbb{R}^n_+$,

$$(Gu)(x) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} \int_0^\infty \tilde g(x', x_n, y_n, \xi')\acute{u}(\xi', y_n)\, dy_n d\xi'.$$

Theorem 2. *Let G be selfadjoint ≥ 0 on $\mathbb{R}^n_+$, and let $\psi(x) = \psi_0(x')\psi_n(x_n)$ with C_0^∞ functions equal to 1 near 0. Then*

$$\mu_j(\psi G\psi) j^{t/(n-1)} \to c(\psi_0^2 g^0)^{t/(n-1)} \text{ for } j \to \infty;$$

$$c(\psi_0^2 g^0) = \tfrac{1}{(n-1)(2\pi)^{(n-1)}} \int_\Sigma \int_{|\xi'|=1} \operatorname{tr}((\psi_0^2 g^0(x', \xi', D_n))^{(n-1)/t})\, d\omega dx'. \quad (5)$$

The proof uses that the symbol-kernel $\tilde g$ has a *rapidly convergent* double expansion in Laguerre functions $\varphi_m(x_n, \sigma)$,

$$\tilde g(x', x_n, y_n, \xi') = \sum\nolimits_{l,m\in\mathbb{N}_0} c_{lm}(x', \xi')\varphi_l(x_n, \langle\xi'\rangle)\varphi_m(y_n, \langle\xi'\rangle).$$

Here $\langle\xi'\rangle = (1 + |\xi'|^2)^{\frac{1}{2}}$, the c_{lm} are ψdo symbols of order $-t$, and the $\varphi_m(x_n, \sigma)$ are of the form $\mathrm{pol}_m(x_n)e^{-x_n\sigma}$, with polynomials of degree m in x_n with coefficients depending on σ.
The $\varphi_m(x_n, \sigma)$, $m \in \mathbb{N}_0$, are an orthonormal basis of $L_2(\mathbb{R}_+)$.

Let Φ_m denote the Poisson operator with symbol-kernel $\varphi_m(x_n, \langle\xi'\rangle)$, then we can write, with C^τ-smooth ψdo's C_{lm} of order $-t$ on $\mathbb{R}^{n-1}$,

$$G = \sum\nolimits_{l,m\in\mathbb{N}_0} \Phi_l C_{lm} \Phi_m^*.$$

The idea of proof is then (we leave out cut-off functions): Write

$$G = G_M + G_M^\dagger, \text{ where}$$

$$G_M = \sum\nolimits_{l,m<M} \Phi_l C_{lm} \Phi_m^*, \quad G_M^\dagger = \sum\nolimits_{l \text{ or } m\geq M} \Phi_l C_{lm} \Phi_m^*.$$

We can use the rapid decrease of the C_{lm}, combined with the control of $\mathfrak{S}_{p,\infty}$-quasinorms in terms of finite sets of symbol seminorms, to show that $G_M^\dagger \to 0$ in $\mathfrak{S}_{(n-1)/t,\infty}$ for $M \to \infty$. Next,

$$G_M = \begin{pmatrix} \Phi_0 & \cdots & \Phi_{M-1} \end{pmatrix} \begin{pmatrix} C_{00} & \cdots & C_{0,M-1} \\ \vdots & & \vdots \\ C_{M-1,0} & \cdots & C_{M-1,M-1} \end{pmatrix} \begin{pmatrix} \Phi_0^* \\ \vdots \\ \Phi_{M-1}^* \end{pmatrix}$$

$$= \mathcal{K}_M \mathcal{C}_M \mathcal{K}_M^*.$$

Hence the j-th eigenvalue satisfies

$$\mu_j(G_M) = \mu_j(\mathcal{K}_M \mathcal{C}_M \mathcal{K}_M^*) = \mu_j(\mathcal{C}_M \mathcal{K}_M^* \mathcal{K}_M) = \mu_j(\mathcal{C}_M),$$

since $\mathcal{K}_M^* \mathcal{K}_M = I_M$ in view of the orthonormality of the Laguerre system.

Here $\mathcal{C}_M$ is an $M \times M$-matrix-formed C^τ-smooth ψdo of order $-t$ on $\mathbb{R}^{n-1}$, to which Theorem 1 applies to give a spectral asymptotic estimate.

Now Lemma B is applied to the decomposition $G = G_M + G_M^\dagger$ for $M \to \infty$, to complete the proof.

There is an extension of the theorem to selfadjoint C^τ-smooth s.g.o.s on bounded open smooth sets $\Omega \subset \mathbb{R}^n$.

3. Nonsmooth resolvent differences

Finally consider the Krein resolvent formula

$$A_{\nu,C}^{-1} - A_\gamma^{-1} = K_\gamma L^{-1}(K'_\gamma)^* \equiv G_C,$$

for A with W^1_q-coefficients; take Ω smooth to begin with. We want to find spectral asymptotics of G_C; recall that K_γ is the Dirichlet Poisson operator, and $L = C - P_{\gamma,\nu}$.

In the selfadjoint case, G_C the sum of a s.g.o. of order -2 (as treated above) and a lower-order term. However, perturbation methods fail, since the lower-order term is linked with dimension n.

Instead we shall use that G_C is here already in a product form passing via the boundary, and we can even allow nonselfadjointness.

In the original boundary problems for $A = -\sum_{j,k=1}^n \partial_j a_{jk} \partial_k + \sum_{j=1}^n a_j \partial_j$ $+a_0$, we approximate the coefficients by C^∞-functions a^ε_{jk}, a^ε_j (by convolution with an approximate identity), and we likewise approximate C by smoothed out versions C^ε.

Following the construction of A_γ^{-1}, K_γ and L^{-1} in Abels-G-Wood '12, we can show that for $\varepsilon \to 0$,

$$\begin{aligned} \|K_\gamma^\varepsilon - K_\gamma\|_{\mathcal{L}(H^{s-\frac12}(\Sigma),H^s(\Omega))} &\to 0, \text{ each } s \in [0,2], \\ \|K_\gamma'^\varepsilon - K_\gamma'\|_{\mathcal{L}(H^{s-\frac12}(\Sigma),H^s(\Omega))} &\to 0, \text{ each } s \in [0,2], \qquad (6) \\ \|(L^\varepsilon)^{-1} - L^{-1}\|_{\mathcal{L}(H^{\frac12}(\Sigma),H^{\frac32}(\Sigma))} &\to 0. \end{aligned}$$

It follows by use of Lemma A that

$$G_C^\varepsilon - G_C = K_\gamma^\varepsilon (L^\varepsilon)^{-1}(K_\gamma'^\varepsilon)^* - K_\gamma L^{-1}(K_\gamma')^* \to 0 \text{ in } \mathfrak{S}_{(n-1)/2,\infty},$$

for $\varepsilon \to 0$.

Then, since the result is known in the smooth case, we conclude by use of Lemma B:

Theorem 3. *For the resolvent difference $G_C = K_\gamma L^{-1}(K'_\gamma)^*$ defined from the Dirichlet realization and a Neumann-type realization of a strongly elliptic operator A with W^1_q-smooth coefficients, $q > n$, on a bounded smooth set Ω,*

$$s_j(G_C) j^{2/(n-1)} \to c(g^0_C)^{2/(n-1)} \text{ for } j \to \infty,$$

where $c(g^0_C)$ is defined similarly to (5).

Earlier, Birman '62 had upper estimates when coefficients are in $C^0 \cap W^{1,\infty}$. The constant satisfies:

Theorem 4. *With $l^0(x',\xi')$ denoting the principal symbol of L and $\lambda^\pm(x',\xi')$ denoting the root in $\mathbb{C}_\pm$ of the principal symbol $a^0(x',0,\xi',\xi_n)$ of A (as a polynomial in ξ_n, in local coordinates)*

$$c(g^0_C) = \frac{1}{(n-1)(2\pi)^{(n-1)}} \int_\Sigma \int_{|\xi'|=1} |4(l^0)^2 \operatorname{Im}\lambda^+ \operatorname{Im}\lambda^-|^{-(n-1)/4}\, d\omega dx'. \quad (7)$$

There is a recent extension to nonsmooth sets Ω:

Assume that Ω has a $B^{3/2}_{p,2}$-boundary; here $p, q > 2$, $p < \infty$, with

$$1 - \frac{n}{q} \geq \frac{1}{2} - \frac{n-1}{p} \equiv \tau_0 > 0.$$

Note that $C^{3/2+\varepsilon} \subset B^{3/2}_{p,2} \subset C^{1+\tau_0}$ for all $\varepsilon > 0$. Birman assumed a C^2-boundary to get upper estimates (in $\mathfrak{S}_{(n-1)/2,\infty}$).

Theorem 5. *The asymptotic estimate for G_C extends to this case.*

Proof ingredients: One can construct a sequence of $C^{1+\tau_0}$-diffeomorphisms $\lambda_l : \overline{\Omega} \to \overline{\Omega}_l$ where the $\overline{\Omega}_l$ are C^∞-domains, such that $\lambda_l \to \mathrm{Id}$ on a neighborhood of $\overline{\Omega}$, for $l \to \infty$.

The boundary value problems carried over to $\overline{\Omega}_l$ define Krein terms G_{C_l}, to which the preceding considerations apply.

Moreover, with $\varrho_l(x)$ denoting the square root of the Jacobian,

$$\varrho_l = (|\det(\partial\lambda_{l,j}/\partial x_k)_{j,k=1,\dots,n})|)^{1/2},$$

G_C is *unitarily equivalent* with $\varrho_l^{-1} G_{C_l} \varrho_l$, for each l.

Since $\sup \varrho_l,\ \sup \varrho_l^{-1} \to 1$ for $l \to \infty$, we can use perturbation arguments to carry the spectral estimates for the G_{C_l} over to G_C.