Introductory workshop

Analysis of Singular Spaces

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Index theory introduction

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I. Summary on pseudodifferential operators.

The *symbol* $p(x,\xi)$ is in $S^d_{1,0}(\Omega,\mathbb{R}^n)$, when

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le c(x)\langle\xi\rangle^{d-|\alpha|},\ \forall\alpha,\beta;$$

here $\langle\xi\rangle = (1+|\xi|^2)^{\frac12}$. It is *classical*, $\in S^d(\Omega,\mathbb{R}^n)$, when there exist $p_{d-l}(x,\xi)$ for $l\in\mathbb{N}_0$ such that

(i) $p_{d-l}(x,t\xi) = t^{d-l}p_{d-l}(x,\xi)$ for $|\xi|\ge 1,\ t\ge 1$,

(ii) $p(x,\xi) - \sum_{0\le l<M} p_{d-l}(x,\xi) \in S^{d-M}_{1,0}(\Omega,\mathbb{R}^n)$, $\forall M$;

in short, $p \sim \sum_{l\in\mathbb{N}_0} p_{d-l}$ in $S^d_{1,0}(\Omega,\mathbb{R}^n)$.

Elliptic, when the principal symbol $p_d(x,\xi)\ne 0$, $\forall x\in\Omega$, $|\xi|\ge 1$.

The associated pseudodifferential operator — ψdo — is

$$Pu(x) \equiv \operatorname{Op}\big(p(x,\xi)\big)u(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}p(x,\xi)\hat u(\xi)\,d\xi.$$

Operators "in (x,y)-form" are defined via *oscillatory integrals*:

$$\operatorname{Op}\big(p(x,y,\xi)\big)u(x) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}p(x,y,\xi)u(y)\,d\xi dy;$$

here $p(x,y,\xi)\in S^d_{1,0}(\Omega\times\Omega,\mathbb{R}^n)$. We say that $p\sim p_1$, $P\sim P_1$, if $p-p_1$ resp. $P-P_1$ is of order $-\infty$.

Theorem 1. $\operatorname{Op}(p)\operatorname{Op}(p') \sim \operatorname{Op}(p'')$, *where*

$$p'' = p \# p' \sim \sum_{\alpha \in \mathbb{N}_0^n} \tfrac{1}{\alpha!} D_\xi^\alpha p \, \partial_x^\alpha p' \quad \text{in } S_{1,0}^{d+d'}(\Omega, \mathbb{R}^n), \tag{1}$$

the Leibniz product.

Theorem 2. *When* p *is elliptic* $\in S^d(\Omega, \mathbb{R}^n)$, *then* $\exists q \in S^{-d}(\Omega, \mathbb{R}^n)$ *such that* $p \# q \sim 1$. *Then* $P = \operatorname{Op}(p)$ *and* $Q = \operatorname{Op}(q)$ *satisfy*

$$PQ = I - \mathcal{R}_1, \quad QP = I - \mathcal{R}_2, \quad \mathcal{R}_1 \text{ and } \mathcal{R}_2 \text{ of order } -\infty. \tag{2}$$

Q is called a *parametrix* of P, q a parametrix symbol.

Theorem 3. *When* $p \in S_{1,0}^d(\Omega, \mathbb{R}^n)$, *then (for Sobolev spaces)*

$$\operatorname{Op}(p) : H^s_{\mathrm{comp}}(\Omega) \to H^{s-d}_{\mathrm{loc}}(\Omega), \forall s \in \mathbb{R}. \tag{3}$$

Theorem 4. *The space of classical* ψ*do's is invariant under* C^∞ *coordinate changes. The principal symbol has a meaning independent of the choice of coordinates. In particular, ellipticity is defined invariantly.*

Allows the definition of ψdo's on smooth manifolds. Let X be a compact manifold without boundary, P elliptic on X. Then (2) and (3) hold on X (we can drop "comp" and "loc"). By Rellich's theorem, $\mathcal{R}_1$ and $\mathcal{R}_2$ are compact operators in any $H^s(X)$. Then

$$P : H^s(X) \to H^{s-d}(X)$$

is a Fredholm operator, with finite dimensional kernel (the nullspace) and cokernel $(H^{s-d}(X)/\operatorname{ran} P)$, independently of s. In this way P has an *index:*

$$\operatorname{index} P = \dim\ker P - \dim\operatorname{coker} P.$$

Moreover, P^* is likewise elliptic of order d, and

$$\dim\operatorname{coker} P = \dim\ker P^*.$$

The index depends only on p_d. The Atiyah-Singer index theorem (1964) shows that this integer is equal to a certain constant defined by algebraic topology from P and X.

The theory likewise works for matrix-formed operators (square matrices in elliptic considerations), and their generalizations to operators in vector bundles over manifolds.

Details of the ψdo theory can be found e.g. in Chapters 7–8 of a coming book: http://www.math.ku.dk/~grubb/distribution.htm

II. An index formula.

We now assume $d > 0$. The operators P^*P and PP^* are elliptic of order $2d$; they define selfadjoint nonnegative realizations in $L_2(X)$ with discrete spectrum going to ∞.

Lemma 5. $\ker P^*P = \ker P$ *and* $\ker PP^* = \ker P^*$. *For* $\lambda > 0$,

$$\dim\ker(P^*P - \lambda) = \dim\ker(PP^* - \lambda).$$

There is a nice trick from the early days of index theory: Let $\varphi(\lambda)$ be a function on $\overline{\mathbb{R}}_+$ with $\varphi(0) = 1$, and let M be a discrete subset of $\overline{\mathbb{R}}_+$ containing 0. Then

$$\operatorname{index} P = \dim\ker P - \dim\ker P^*$$

$$= \textstyle\sum_{\lambda\in M} \varphi(\lambda)\big(\dim\ker(P^*P - \lambda) - \dim\ker(PP^* - \lambda)\big).$$

Applying this with $\varphi(\lambda) = e^{-t\lambda}$, $t > 0$, and M containing the eigenvalues of P^*P and PP^*, we get:

$$\begin{aligned}
&\operatorname{index} P = \\
&= \textstyle\sum_M e^{-t\lambda}\big(\dim\ker(P^*P-\lambda) - \dim\ker(PP^*-\lambda)\big) \\
&= \textstyle\sum_{j\in\mathbb{N}} e^{-t\lambda_j(P^*P)} - \sum_{j\in\mathbb{N}} e^{-t\lambda_j(PP^*)} \\
&= \operatorname{Tr} e^{-tP^*P} - \operatorname{Tr} e^{-tPP^*};
\end{aligned}$$

here the $\lambda_j(P^*P)$, $j\in\mathbb{N}$, denote the eigenvalues of P^*P repeated according to multiplicity, with a similar definition of $\lambda_j(PP^*)$. It is known that

$$\operatorname{Tr} e^{-tP^*P} = c_{-n}t^{-\frac{n}{2d}} + \dots + c_{-1}t^{-\frac{1}{2d}} + c_0 + o(t) \text{ for } t\to 0$$

(a heat trace expansion), and similarly for PP^*, so we get by subtraction a formula

$$\operatorname{index} P = h(t) + c_0(P^*P) - c_0(PP^*) + g(t),$$

where $g(t)\to 0$ for $t\to 0$, and $h(t)$ blows up for $t\to 0$ unless it is 0. Since the left-hand side is independent of t, we conclude first that $h(t)=0$ and next that $g(t)=0$, obtaining

Theorem 6. $\operatorname{index} P = c_0(P^*P) - c_0(PP^*)$.

The constants $c_0(P^*P)$ and $c_0(PP^*)$ can (in principle) be calculated from the first n terms of the symbol of P in local coordinates. (We then say they are "local". Constants that depend on the full structure are called "global".)

There is in fact a form $a(x)$ on X so that

$$\operatorname{index} P = c_0(P^*P) - c_0(PP^*) = \int_X a(x).$$

III. Other index formulas.

One can approach the index by other operator families associated with P^*P and PP^*. Let B be an elliptic ψdo of order m such that the resolvent $(B-\lambda)^{-1}$ exists and is $O(\lambda^{-1})$ in $L_2(X)$ for $\lambda \to \infty$ on rays in $V = \{\lambda \in \mathbb{C} \mid \operatorname{ang} \lambda \in [\frac{\pi}{2} - \varepsilon, \frac{3\pi}{2} + \varepsilon]\}$.

Three operator families:

Resolvent $(B-\lambda)^{-1}$,

Heat operator e^{-tB},

Power operator B^{-s} (defined as 0 on $\ker B$).

Can be obtained from one another:

$$\text{Resolvent } (B-\lambda)^{-1} \quad \underset{\text{Laplace transf.}}{\overset{\text{Cauchy int.}}{\rightleftarrows}} \quad e^{-tB} \text{ Heat operator}$$

$$\text{Cauchy int.} \searrow\!\sim \qquad\qquad \sim\!\swarrow \text{Mellin transf.}$$

$$\Gamma(s)B^{-s}$$

Power operator

Cauchy integrals:

$$B^{-s} = \tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s}(B-\lambda)^{-1}\,d\lambda, \quad e^{-tB} = \tfrac{i}{2\pi}\int_{\mathcal{C}'} e^{-tB}(B-\lambda)^{-1}\,d\lambda$$

where $\mathcal{C}$ is a curve in $(V \cup \{|\lambda| \le \delta\}) \setminus \overline{\mathbb{R}}_-$ around the nonzero eigenvalues; $\mathcal{C}'$ runs in V around all eigenvalues.

Three equivalent asymptotic trace expansions:

Along with B elliptic of order $m > 0$ we consider A of order $\nu \in \mathbb{R}$.

The resolvent trace expansion:

$$\operatorname{Tr}(A(B-\lambda)^{-N}) \sim \sum_{j\ge 0} \tilde{c}_j (-\lambda)^{-\frac{\nu+n-j}{m}-N} + \sum_{k\ge 0} \big(\tilde{c}'_k \log(-\lambda) + \tilde{c}''_k \big) (-\lambda)^{-k-N}, \tag{5}$$

for $\lambda \to \infty$ in V. ($N > (\nu+n)/m$.)

The heat trace expansion:

$$\operatorname{Tr}(Ae^{-tB}) \sim \sum_{j\ge 0} c_j t^{\frac{j-\nu-n}{m}} + \sum_{k\ge 0} (-c'_k \log t + c''_k) t^k, \tag{6}$$

for $t \to 0+$.

The complex power trace expansion:

$$\Gamma(s) \operatorname{Tr}(AB^{-s}) \sim \sum_{j\ge 0} \frac{c_j}{s + \frac{j-\nu-n}{m}} - \frac{1}{s} \operatorname{Tr}(A\Pi_0(B)) + \sum_{k\ge 0} \Big(\frac{c'_k}{(s+k)^2} + \frac{c''_k}{s+k} \Big), \tag{7}$$

where the right-hand side gives the pole structure of the meromorphic extension. *Same constants* as in heat trace expansion; related to tilde-constants by universal formulas. Division by $\Gamma(s)$ makes double poles simple. The c_j and c'_k are “local”, the c''_k “global”.

We denote

$$C_0(A,B) = \tilde c_{n+\nu} + \tilde c_0'' = c_{n+\nu} + c_0'', \quad C_{-1}(A,B) = \tilde c_0' = c_0',$$

where we set $\tilde c_{n+\nu} = c_{n+\nu} = 0$ if $n+\nu \notin \mathbb{N}_0$. When $A = I$, $C_{-1}(I,B) = 0$, and

$$\operatorname{index} P = C_0(I, P^*P) - C_0(I, PP^*).$$

The constants $C_{-1}(A,B)$ and $C_0(A,B)$ play a role as generalized trace-functionals on A. In fact, $mC_{-1}(A,B)$ is independent of B; it is the *noncommutative residue*

$$\operatorname{res}(A) = mC_{-1}(A,B) \tag{8}$$

introduced by Wodzicki in 1984 (note that $C_{-1}(A,B)$ is the residue of $\operatorname{Tr}(AB^{-s})$ at the pole 0). It is *tracial*, i.e., vanishes on commutators, and satisfies:

$$\operatorname{res}(A) = \tfrac{1}{(2\pi)^n} \int_{\widetilde X} \int_{|\xi|=1} \operatorname{tr} a_{-n}(x,\xi)\, dS(\xi) dx$$

(one point of the formula is that this has an invariant meaning on X). It vanishes when A is a differential operator, in particular if $A = I$.

The constant $C_0(A, B)$ is by some authors called the *zeta-regularized trace*. For A of noninteger order, and for integer-order cases with suitable symbol parity properties in relation to the dimension, this equals the *canonical trace*, as introduced by Kontsevich and Vishik in 1994; then it also has tracial properties, and can be determined from $a(x, \xi)$ by a Hadamard type finite-part integral.

One can play around much more with these formulas and expansions. For example, when A is an elliptic operator and $\widetilde{A}$ is a parametrix, then

$$\operatorname{index} A = C_0([A, \widetilde{A}], B); \text{ here } [A, \widetilde{A}] = A\widetilde{A} - \widetilde{A}A$$

(found in Melrose-Nistor).

Remark. To give an impression of how the trace expansions are shown, consider first a case where B is an elliptic differential operator of order $m > n$ so that the resolvent is trace-class, and let $\lambda \in \mathbb{R}_-$. Agmon observed that if we write $-\lambda = \mu^m$, $\mu \in \mathbb{R}$ (m is even in this case), and consider μ as an extra cotangent variable, the symbol

$$\bar{b}(x,\xi,\mu) = b(x,\xi) + \mu^m$$

is elliptic as a function of $(\xi,\mu) \in \mathbb{R}^{n+1}$. Construct the parametrix symbol

$$\bar{q}(x,\xi,\mu) = \bar{q}_{-m} + \bar{q}_{-m-1} + \cdots, \quad \bar{q}_{-m} = \tfrac{1}{b(x,\xi)+\mu^m}.$$

For each term, the kernel $K_{-m-j}(x,y,\mu)$ of the operator $\operatorname{Op}(\bar{q}_{-m-j})$ satisfies, by homogeneity,

$$K_{-m-j}(x,x,\mu) = \tfrac{1}{(2\pi)^n} \int \bar{q}_{-m-j}(x,\xi,\mu)\, d\xi$$
$$= \mu^{-m-j+n} \tfrac{1}{(2\pi)^n} \int \bar{q}_{-m-j}(x,\eta,1)\, d\eta,$$

which by integration in x produces the j-th term in (5). When B is a pseudodifferential operator, the singularities at $\xi = 0$ in the strictly homogeneous symbols give trouble, but a finer analysis can sort this out, showing how logarithmic terms arise.

IV. The zeta and eta functions.

The *zeta function* of B is the function

$$\zeta(B,s) = \operatorname{Tr} B^{-s};$$

it is well-defined as a holomorphic function for $\operatorname{Re} s > n/m$, since

B^{-s} is trace-class then. As already mentioned in connection with power functions, it has a meromorphic extension to $\mathbb{C}$. We get as a special case of (7) by division by $\Gamma(s)$ and the information that $c_0' = 0$ when $A = I$:

$$\zeta(B, s) \sim \sum_{j \geq 0, \frac{j-n}{m} \notin \mathbb{N}_0} \frac{b_j}{s + \frac{j-n}{m}} + \sum_{k>0} \frac{b_k'}{s+k}.$$

When B is a differential operator, some poles vanish due to parity of symbol terms.

When B is selfadjoint nonnegative, $\zeta(B, s)$ can also be read as

$$\zeta(B, s) = \sum_{\text{eigenvalues} \neq 0} \lambda_j(B)^{-s},$$

the eigenvalues repeated according to multiplicity.

When B is selfadjoint not lower bounded, one is interested in

$$\zeta(B^2, s/2) = \sum_{\text{eigenvalues} \neq 0} |\lambda_j(B)|^{-s} = \operatorname{Tr}(|B|^{-s}),$$

with behavior as above, and also

$$\eta(B, s) = \sum_{\text{eigenvalues} \neq 0} \operatorname{sign} \lambda_j(B) |\lambda_j(B)|^{-s} = \operatorname{Tr}(B|B|^{-s-1}),$$

the *eta function*. An interesting case is when B is a differential operator system of order 1; let $m = 1$ in the following. The pole structure of the eta function is covered by (7):

$$\Gamma(s)\,\mathrm{Tr}(\tfrac{B}{|B|}|B|^{-s}) \sim \textstyle\sum_{j\ge 0} \frac{c_j}{s+j-n} + \sum_{k\ge 0}\Big(\frac{c'_k}{(s+k)^2} + \frac{c''_k}{s+k}\Big), \quad (9)$$

which by division by $\Gamma(s)$ gives the pole structure

$$\eta(B,s) = \mathrm{Tr}(\tfrac{B}{|B|}|B|^{-s}) \sim \textstyle\sum_{k\ge -n} \frac{b_k}{s+k}. \quad (10)$$

The coefficient b_0 equals $\mathrm{res}(\frac{B}{|B|})$ (cf. (8)); it is "local".

There is a deep result here, shown partially by Atiyah, Patodi and Singer in 1976 with the proof completed by Gilkey 1981, that in fact $b_0 = 0$, so that the eta function has a *value* at zero, $\eta(B,0)$ (equal to c''_0 from (9)). This value is "global".

Note that also the noncommutative residue of the positive eigenprojection $\Pi_>(B) = \frac{1}{2}(I + \frac{B}{|B|})$ is zero then. Wodzicki 1984 observed this and generalized it to a result on the stability of the zeta value at 0 under different choices of the ray where λ^{-s} has its jump, also in nonselfadjoint cases.

We can view the eta-functions as

$$\eta(B,s) = \textstyle\sum_{\text{eigenvalues}>0} \lambda_j(B)^{-s} - \sum_{\text{eigenvalues}<0} |\lambda_j(B)|^{-s},$$

which measures the "spectral asymmetry" of B.

Another function of interest in this connection is the derivative $\zeta'(B,s) = \frac{d}{ds}\zeta(B,s)$, extended meromorphically to $\mathbb{C}$, in particular its value at $s=0$. One can show that $-\zeta'(B,0)$ can be viewed as the logarithm of a determinant of B (by generalization from finite dimensional cases); it is often called the *zeta-determinant* of B. It is nonlocal, and its evaluation is on the same level of difficulty as the eta value. In view of the formula

$$\operatorname{Tr}(\tfrac{d}{ds}B^{-s}) = -\operatorname{Tr}((\log B)\,B^{-s}),$$

extended meromorphically from large $\operatorname{Re} s$, we have

$$\log\det B \equiv -\zeta'(B,0) = C_0(\log B, B),$$

extending the notation introduced for $\operatorname{Tr} AB^{-s}$ to the non-classical case where A is replaced by $\log B$. This goes rather naturally for ψdo's on closed manifolds, but is somewhat problematic for generalizations to manifolds with boundary.

V. The Atiyah-Patodi-Singer problem.

The Atiyah-Singer index theorem can with some extra efforts be generalized to elliptic differential operators on manifolds with boundary, provided with differential boundary conditions satisfying the Shapiro-Lopatinskii condition (also called "elliptic boundary problems"), the index is then local as in the boundaryless case.

But while striving to break up the study of general higher-order problems into first-order pieces, the authors got the idea of using Dirac operators as building blocks, justified by geometric considerations. The difficulty here is that a Dirac operator D is first-order and does not always have a Shapiro-Lopatinskii boundary condition. They took recourse to certain *pseudodifferential* boundary conditions. This makes the study technically harder and introduces global constants along with the local ones.

Let X be a smooth compact n-dimensional manifold with boundary $\partial X = X'$, and let D be a first-order elliptic differential operator on X, going from a vector bundle E_1 to E_2, both of dimension

N. On a collar neighborhood $X_c = X' \times [0,c[$ of X' with points $x = (x', x_n)$, let D have the form

$$D = \sigma(\tfrac{\partial}{\partial x_n} + A),$$

where A is a selfadjoint elliptic first-order operator in the bundle $E_1' = E_1|_{X'}$ and σ is a unitary morphism from E_1' to E_2' (lifted to X_c). Let $\Pi_\geq$ be the nonnegative eigenprojection for A. The APS problem is the boundary value problem

$$Du = f \text{ on } X, \quad \Pi_\geq \gamma_0 u = 0; \tag{11}$$

here $\gamma_0 u = u|_{X'}$. This problem is well-posed in the sense that the L_2-realization $D_\geq$ of D with domain

$$D(D_\geq) = \{u \in H^1(X, E_1) \mid \Pi_\geq \gamma_0 u = 0\}$$

is a Fredholm operator. Atiyah, Patodi and Singer (1975) showed the index formula

$$\operatorname{index} D_\geq = \int_X a(x) - \tfrac{1}{2}(\eta(A,0) + \nu_0(A)); \tag{12}$$

here $a(x)$ is the usual form entering in the index formula for D on the doubled manifold (but integrated only over X), and $\nu_0(A) = \dim\ker A$.

The proof by A-P-S is a tricky combination of functional analysis (using the spectral theory for A) with the study of D on the doubled manifold. Although they used that

$$\operatorname{index} D_{\geq} = \operatorname{Tr} e^{-tD_{\geq}^* D_{\geq}} - \operatorname{Tr} e^{-tD_{\geq} D_{\geq}^*},$$

they avoided the need to actually calculate trace expansions for these two "heat traces" individually.

That was done much later, in G 1992 down to and including the crucial constant term, showing that each of $\operatorname{Tr} e^{-tD_{\geq}^* D_{\geq}}$ and $\operatorname{Tr} e^{-tD_{\geq} D_{\geq}^*}$ contributes half of the piece $\frac{1}{2}(\eta(A,0) + \nu_0(A))$. Full expansions were established by G and Seeley 1996 with much information on all the coefficients in terms of A and D. The "non-product case" (where x_n-dependence is allowed) was left open by A-P-S; it was treated in G 1992, and full expansions were obtained in another G and Seeley 1995 paper using ψdo methods.

VI. Other boundary conditions.

In the product case, $D^* = D$ means on X_c that

$$\sigma^2 = I, \quad \sigma A = -A\sigma. \tag{13}$$

In this case, $D_{\geq}$ is selfadjoint iff $\ker A = \{0\}$. We can replace $\Pi_{\geq}$ by other projections Π, studying expansion coefficients.

(i) When $\ker A \neq \{0\}$, there is a decomposition (Palais 1965, Douglas-Wojciechowski 1991, Müller 1994, Dai-Freed 1994)

$$\ker A = V \oplus V^{\perp}, \quad V^{\perp} = \sigma V.$$

Take $\Pi = \Pi_{>} + \Pi_V$, then D_Π is selfadjoint (with index 0). Lesch-Wojciechowski 1996 classified the selfadjoint cases with $V \subset \ker A$. Other finite rank perturbations have also been studied.

(ii) Take Π with

$\Pi - \Pi_{\geq}$ of order $-\infty$ (e.g. Π equal to the true Calderón projector, Booss-Wojciechowski book 1993),

$\Pi - \Pi_{\geq}$ of order $-n$ (several authors),

$\Pi - \Pi_{\geq}$ of order -1 (several authors).

(iii) Allow projections differing *principally* from $\Pi_\geq$. Brüning and Lesch 1999 introduced a family $\Pi(\theta)$ giving selfadjoint realizations; G 1999 studied *all well-posed* conditions (in the sense of Seeley 1969), getting full trace expansions, also in non-product cases.

We list the power operator versions expressing the pole structure of meromorphic extensions. The zeta function expansion is

$$\zeta({D_\Pi}^* D_\Pi, s) = \operatorname{Tr}(({D_\Pi}^* D_\Pi)^{-s}) \sim$$
$$\frac{1}{\Gamma(s)}\Big[\sum_{-n\leq k<0} \frac{a_k}{s+\frac{k}{2}} + \frac{\nu_0(D_\Pi)}{s} + \sum_{k=0}^{\infty}\Big(\frac{a'_k}{(s+\frac{k}{2})^2} + \frac{a''_k}{s+\frac{k}{2}}\Big)\Big]. \quad (14)$$

The eta function expansion is

$$\eta({D_\Pi}^* D_\Pi, s) = \operatorname{Tr}(D({D_\Pi}^* D_\Pi)^{-\frac{s+1}{2}}) \sim$$
$$\frac{1}{\Gamma(\frac{s+1}{2})}\Big[\sum_{-n<k<0} \frac{b_k}{s+k} + \sum_{k=0}^{\infty}\Big(\frac{b'_k}{(s+k)^2} + \frac{b''_k}{s+k}\Big)\Big]. \quad (15)$$

Pole at $s=0$? (14) implies

$$\zeta({D_\Pi}^* D_\Pi, s) \sim \sum_{-n\leq k<0} \frac{\bar{a}_k}{s+\frac{k}{2}} + \sum_{k=0}^{\infty} \frac{\bar{a}'_k}{s+\frac{k}{2}}, \quad (16)$$

with $\bar{a}_0' = a_0'$. This coefficient is local, and it has been a challenge to show its vanishing; the most general result is (to my knowledge) from G 2003:

Theorem 7. *If the principal symbol of* Π *commutes with the principal symbol of* A^2*, then* $a_0' = 0$*.*

Moreover, the value $a_0'' = \zeta({D_\Pi}^* D_\Pi, 0)$ *is then explicitly derived from* Π *and* $\ker D_\Pi$*, modulo local contributions.*

For the eta function expansion in (15), division by $\Gamma(\frac{s+1}{2})$ does not remove the double pole at 0. However, consider selfadjoint cases; they require

$$\Pi = -\sigma \Pi^\perp \sigma$$

in addition to (13). We find here that $b_0' = 0$ under the hypotheses of Theorem 7, so there is at most a simple pole. Moreover, if $b_0''(D_\Pi) = 0$ for a selfadjoint D_Π, it will also be 0 for a selfadjoint perturbation $D_{\overline{\Pi}}$ when $\Pi - \overline{\Pi}$ has order $\le -n$. (Results from G 2003, Lei 2003, proved earlier for particular cases.)

Final remark. The lectures moreover included: 1) An introduction to Boutet de Monvel's theory of pseudodifferential boundary operators, furnishing a complete calculus where elliptic boundary value problems and their solution operators are incorporated. 2) An explanation of the Calderón projectors $C^\pm$ associated with a Dirac-type operator, and the solvability of the problem (11) with $\Pi_\geq$ replaced by C^+. (More on these topics e.g. in Ch. 10–11 of the book referred to on page 5 and below.) 3) The definition of well-posed boundary conditions for D — they are not elliptic in the usual sense, only injectively elliptic. (More details in G 1999.)

The works listed below refer to further contributions to the theories.

References

[ABP] M. F. Atiyah and R. Bott and V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 279–330.

[APS] M. F. Atiyah and V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemanninan geometry I*, Math. Proc. Camb. Phil. Soc. **77** (1975), 43–69.

[AS] M. F. Atiyah and I. M. Singer, *The index of elliptic operators*, Ann. Math. **87** (1968), 484–530.

[BW] B. Booss-Bavnbek and K. Wojciechowski, *Elliptic boundary problems for Dirac operators*, Birkhäuser, Boston, 1993.

[BL] J. Brüning and M. Lesch, *On the eta-invariant of certain non-local boundary value problems*, Duke Math. J. **96** (1999), 425–468.

[DF] X. Dai and D. Freed, *η-invariants and determinant lines*, J. Math. Phys. **35** (1994), 5155–5194.

[DW] R. G. Douglas and K. P. Wojciechowski, *Adiabatic limits of the η-invariant, the odd-dimensional Atiyah-Patodi-Singer problem*, Comm. Math. Phys. **142** (1991), 139–168.

[Gi] P. B. Gilkey, *The residue of the global eta function at the origin*, Adv. Math. **40** (1981), 290–307.

[G1] G. Grubb, *Heat operator trace expansions and index for general Atiyah-Patodi-Singer boundary problems*, Comm. Part. Diff. Equ. **17** (1992), 2031–2077.

[G2] ———, *Trace expansions for pseudodifferential boundary problems for Dirac-type operators and more general systems*, Arkiv f. Mat. **37** (1999), 45–86.

[G3] ———, *Poles of zeta and eta functions for perturbations of the Atiyah-Patodi-Singer problem*, Comm. Math. Phys. **215** (2001), 583–589.

[G4] ———, *Spectral boundary conditions for generalizations of Laplace and Dirac operators*, Comm. Math. Phys. **240** (2003), 243–280.

[G5] ———, *Distributions and Operators*, GTM, Springer-Verlag, New York, 2008, forthcoming book.

[GS1] G. Grubb and R. Seeley, *Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems*, Invent. Math. **121** (1995), 481–529.

[GS2] ———, *Zeta and eta functions for Atiyah-Patodi-Singer operators*, J. Geom. An. **6** (1996), 31–77.

[KV] M. Kontsevich and S. Vishik, *Geometry of determinants of elliptic operators*, Functional Analysis on the Eve of the 21'st Century (Rutgers Conference in honor of I. M. Gelfand 1993), Vol. I (S. Gindikin et al., eds.), Progr. Math. 131, Birkhäuser, Boston, 1995, pp. 173–197.

[L] Y. Lei, *The regularity of the eta function for perturbations of order −(dim X) of the Atiyah-Patodi-Singer boundary problem*, Comm. Part. Diff. Equ. **28** (2003), 1587–1596.

[LW] M. Lesch and K. P. Wojciechowski, *On the eta-invariant of generalized Atiyah-Patodi-Singer boundary value problems*, Illinois J. Math. **40** (1996), 30–46.

[MN] R. Melrose and V. Nistor, *Homology of pseudodifferential operators I. Manifolds with boundary*, manuscript, arXiv: funct-an/9606005.

[M] W. Müller, *Eta invariants and manifolds with boundary*, J. Diff. Geom. (1994), 311–377.

[P] R. S. Palais, *Seminar on the Atiyah-Patodi-Singer index theorem*, Ann. Math. Studies 57, Princeton University Press, Princeton, N. J., 1965, 366 pp.

[S] R. T. Seeley, *Topics in pseudo-differential operators*, CIME Conf. on Pseudo-Differential Operators 1968, Edizioni Cremonese, Roma, 1969, pp. 169–305.

[W] M. Wodzicki, *Local invariants of spectral asymmetry*, Invent. Math. **75** (1984), 143-178.