Heat kernel estimates for elliptic pseudodifferential operators

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Abstract

Strongly elliptic classical pseudodifferential operators P include fractional Laplacians of order $a > 0$ and their perturbations, in particular the Dirichlet-to-Neumann operator of order 1, associated with the Laplacian on a smooth manifold with boundary. By pseudodifferential methods we show that the kernels of the heat semigroups $\exp(-tP)$ they generate, satisfy Poissonian estimates. In particular, when P is selfadjoint, uniform estimates for complex t with positive real part are obtained. Joint work with Heiko Gimperlein.

H. Gimperlein and G. Grubb: Heat kernel estimates for pseudodifferential operators, fractional Laplacians and Dirichlet-to-Neumann operators, arXiv:1302.6529, to appear in J. Evol. Eq.

1. Sectorially elliptic pseudodifferential operators

Consider P, classical sectorially elliptic ψdo of order $d \in \mathbb{R}_+$ (possibly a system) on a closed n-dimensional Riemanninan C^∞-manifold M.

The principal symbol $p^0(x,\xi)$ of P has spectrum in a sector $\{\lambda \mid |\arg \lambda| \le \varphi_0\}$ with $\varphi_0 \in [0, \frac{\pi}{2}[$. The heat semigroup $V(t) = e^{-tP}$ exists for $|\arg t| < \frac{\pi}{2} - \varphi_0$, defined from the resolvent $Q_\lambda = (P-\lambda)^{-1}$ by a Cauchy integral around the spectrum. Estimates of kernel $\mathcal{K}_V(x,y,t)$?

Examples:

1° $d = 1$, $M = \partial \widetilde{M}$ for an $n+1$-dimensional smooth Riemanninan manifold $\widetilde{M}$ with boundary. The Dirichlet-to-Neumann operator P_{DN} maps the boundary value u of a harmonic function $\tilde{u}$ on $\widetilde{M}$ into the normal derivative $\partial_\nu \tilde{u}$. P_{DN} is a selfadjoint nonnegative elliptic ψdo of order 1 on M (when signs are chosen conveniently).

2° $d \in \mathbb{R}_+$. The powers $(-\Delta)^a$ of the Laplace-Beltrami operator on M are nonnegative selfadjoint elliptic ψdo's of order $d = 2a$.

3° Generalizations of these cases, e.g. where in 1° the Laplacian on $\widetilde{M}$ is replaced by a general strongly elliptic 2' order operator, possibly a system, or e.g. where Δ on M is replaced by perturbations by lower-order terms, or more general sectorially elliptic systems.

It is known from works of Seeley, also shown in G. book '96, that Q_λ exists for λ in a sectorial region
$W_{r_0,\varepsilon} = \{\lambda \in \mathbb{C} \mid |\lambda| \geq r_0,\ \arg\lambda \in [\varphi_0 + \varepsilon, 2\pi - \varphi_0 - \varepsilon]\}$,
and that a parametrix Q'_λ consistent with Q_λ exists on a larger set
$V_{\delta,\varepsilon} = W_{r_0,\varepsilon} \cup \{|\lambda| \leq \delta\} \cup \{\operatorname{Re}\lambda < \inf_{|\xi|=1} \operatorname{Re} p^0(x,\xi)\}$.
The symbol $q(x,\xi,\lambda)$ of Q_λ is holomorphic in λ and satisfies in local coordinates, for λ on rays $\{\lambda = \mu^d e^{i\varphi} \mid \mu \in \mathbb{R}_+\}$ in $V_{\delta,\varepsilon}$, all M:

$$q(x,\xi,\lambda) = \sum_{0 \leq l < M} q_{-d-l} + q'_M, \text{ where } q_{-d} = (p^0(x,\xi) - \lambda)^{-1},$$

$$|D_x^\beta D_\xi^\alpha q_{-d-l}| \leq C\langle\xi\rangle^{d-l-|\alpha|}\langle\xi,\mu\rangle^{-2d}, \text{ for } l + |\alpha+\beta| > 0,$$

where $\langle\xi\rangle = (|\xi|^2+1)^{\frac{1}{2}}$, $\langle\xi,\mu\rangle = (|\xi|^2+\mu^2+1)^{\frac{1}{2}}$. There is a similar estimate of the remainder symbol q'_M. The heat semigroup $V(t) = e^{-tP}$ is defined from P by the Cauchy integral formula

$$V(t) = \tfrac{i}{2\pi}\int_{\mathcal{C}} e^{-t\lambda}(P-\lambda)^{-1}\,d\lambda,$$

where $\mathcal{C}$ is a contour around the spectrum of P, e.g. of the form $\partial W_{r_0,\varepsilon}$.
Here the symbol $v(x,\xi,t) = v_{-d} + \cdots + v_{-d-M+1} + v'_M$ satisfies e.g.

$$|D_x^\beta D_\xi^\alpha v_{-d-l}(x,\xi,t)| \leq C\langle\xi\rangle^{d-l-|\alpha|}te^{-c't}.$$

We define $V_{-d-l}(t)$ and $V'_M(t)$ to be operators with symbols v_{-d-l}, v'_M in local coordinates. G '96 showed $\sup_{x,y}$-estimates for the kernels $\mathcal{K}_V(x,y,t)$, $\mathcal{K}_{V_{-d-l}}$, $\mathcal{K}_{V'_M}$, obtaining the asymptotic expansion of $\operatorname{Tr} V(t)$ in powers of t with $\log t$ for $t \to 0$.

The first task is to generalize this to Poisson-like estimates for $t \in \mathbb{R}_+$.

Prop. A. (Taylor '81) *Let $a(x,\xi)$ satisfy $|D_\xi^\alpha a(x,\xi)| \le C\langle\xi\rangle^{r-|\alpha|}$ for some $r \in \mathbb{R}$, $N > n + r$, all $|\alpha| \le N$. Let $A = \operatorname{Op}(a)$.*
Then the kernel $\mathcal{K}_A(x,y) = \mathcal{F}^{-1}_{\xi\to z} a(x,\xi)|_{z=x-y}$ is $O(|x-y|^{-N})$ for $|x-y| \to \infty$, and satisfies for $|x-y| > 0$:

$$|\mathcal{K}_A(x,y)| \le C \begin{cases} |x-y|^{-r-n} \text{ if } r > -n, \\ |\log|x-y|| + 1 \text{ if } r = -n, \\ 1 \text{ if } r < -n. \end{cases}$$

It holds in particular when $a \in S^r_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$.

Theorem 1. 1° *The kernels $\mathcal{K}_{V_{-d-l}}(x,y,t)$ satisfy in local coordinates, for some $c' > 0$,*

$$|\mathcal{K}_{V_{-d-l}}(x,y,t)| \le Ce^{-c't} \begin{cases} t\,(|x-y| + t^{\frac{1}{d}})^{l-d-n} \text{ if } d-l > -n, \\ t\,(|\log(|x-y| + t^{\frac{1}{d}})| + 1) \text{ if } d-l = -n, \\ t \text{ if } d-l < -n, \end{cases}$$

2° *The kernel* $\mathcal{K}_V(x,y,t)$ *satisfies on* M

$$|\mathcal{K}_V(x,y,t)| \le Ce^{-c_1 t} t\,(d(x,y)+t^{\frac{1}{d}})^{-d-n},$$

for any $c_1 < \gamma(P) = \inf \operatorname{Re} \operatorname{spec} P$.

When the eigenvalues λ of P with real part equal to $\gamma(P)$ are semisimple (geometric multiplicity equals algebraic multiplicity), there is an estimate with $c_1 = \gamma(P)$.

NB! Nonselfadjoint P are allowed. E.g. there is in probability theory an interest for operators such as $P = (-\Delta)^{\frac{1}{2}} + b(x)\cdot\nabla + c(x)$, real b and c, smooth and bounded with bounded derivatives. Since

$$p^0(x,\xi) = |\xi| + ib\cdot\xi, \quad \operatorname{Re} p^0 = |\xi|,$$

the operator is strongly elliptic and nonselfadjoint, and the theorem applies to it. Treated with other methods e.g. by Xie and Zhang '12.

Recall that $p^0(x,\xi)$ has its spectrum in a sector $\{|\arg\lambda| \le \varphi_0\}$ with $\varphi_0 < \frac{\pi}{2}$. Let $\theta_0 = \frac{\pi}{2} - \varphi_0$. By rotating P, one can extend the estimates to t in sectors $\{|\arg t| \le \theta_0 - \varepsilon\}$, $\varepsilon > 0$.

Consider P selfadjoint ≥ 0. Then $V(t)$ exists for all complex t with $|\arg t| < \frac{\pi}{2}$. The above methods give uniform estimates in sectors $\{|\arg t| \leq \frac{\pi}{2} - \varepsilon\}$, but it a more difficult question to find out what happens when $\arg t \to \pm\frac{\pi}{2}$.
From the estimates $c_1|\xi|^d \leq p^0(x,\xi) \leq c_2|\xi|^d$ follow the resolvent estimates, when $\arg\lambda = \varphi$:

$$|q_{-d}(x,\xi,\lambda)| = |(p^0(x,\xi) - \lambda)^{-1}| \leq C|\sin\varphi|^{-1}\langle\xi,\mu\rangle^{-d}.$$

Note that $\partial_{x_j} q_{-d} = -q_{-d}(\partial_{x_j} p^0)q_{-d}$, so each time we take a derivative, an extra factor $|\sin\varphi|^{-1}$ comes in. We get for $l + |\alpha+\beta| > 0$:

$$|D_x^\beta D_\xi^\alpha q_{-d-l}(x,\xi,\lambda)| \leq C|\sin\varphi|^{-2l-1-|\alpha|-|\beta|}\langle\xi\rangle^{d-l-|\alpha|}\langle\xi,\mu\rangle^{-2d}.$$

Now it is important to economize with the use of derivatives in estimates. Still based on Prop. A, we find estimates in terms of $t = |t|e^{i\theta}$ such as

$$|\mathcal{K}_{V_{-d-l}}(x,y,t)| \leq$$
$$Ce^{-c'\operatorname{Re} t}\begin{cases}(\cos\theta)^{-N_l}|t|\,(|x-y| + |t|^{\frac{1}{d}})^{l-d-n} \text{ if } d-l > -n,\\ (\cos\theta)^{-N_l}|t|\,(|\log(|x-y| + |t|^{\frac{1}{d}})| + 1) \text{ if } d-l = -n,\\ (\cos\theta)^{-N_l}|t| \text{ if } d-l < -n,\end{cases}$$

$$N_l = \begin{cases} \max\{\frac{n}{d}, [d-1+n]+3\} \text{ if } l=0, \\ \max\{2l+1+\frac{n-l}{d}, 2l+2+[d-1+n]\} \text{ if } l>0, d-l>-n, \\ 2l+2 \text{ if } d-l=-n, \\ 2l+1 \text{ if } d-l<-n. \end{cases}$$

Roughly, $N_l \lesssim 2l+n+d$.
This is not so bad, but we are really after the full kernel, which includes the remainder $V'_M = V - \sum_{l<M} V_{-d-l}$, that must be estimated in an exact form. In G '96, exact remainders were found (by spectral invariance arguments), but estimated only for t on rays, not for $\theta \to \pm\frac{\pi}{2}$.
The problem is studied for the remainder $Q'_M = Q_\lambda - \sum_{l<M} Q_{-d-l}$, by a mix of functional analysis estimates and ψdo estimates. We shall use:

Prop. B. (Agmon '62) *Let T be a bounded linear operator in $L_2(\Omega)$ such that T and T^* map into $H^m(\Omega)$ for an $m>n$. Then T has a continuous and bounded kernel $\mathcal{K}_T(x,y)$ satisfying*

$$|\mathcal{K}_T(x,y)| \le C(\|T\|_{0,m} + \|T^*\|_{0,m})^{n/m} \|T\|_{0,0}^{1-n/m}.$$

Prop. C. (Marschall '87) *Let* $a(x,\xi)$ *satisfy* $|D_x^\beta D_\xi^\alpha a(x,\xi)| \le C_0 \langle\xi\rangle^{-|\alpha|}$ *for* $|\alpha| \le N$, $|\beta| \le 1$, *for some* $N > \frac{n}{2}$. *Then* $A = \operatorname{Op}(a)$ *is bounded on* $L_2(\mathbb{R}^n)$, *and* $\|A\|_{0,0} \le C\, C_0$.

It holds in particular for $a \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$.
The remainder is captured in the formula

$$Q'_M = Q'_M(P-\lambda)Q_\lambda = R_M Q_\lambda, \text{ where}$$
$$R_M = (Q_\lambda - \sum\nolimits_{l<M} Q_{-d-l})(P-\lambda) = 1 - \sum\nolimits_{l<M} Q_{-d-l}(P-\lambda)$$

is a ψdo of order $-M$ constructed from known symbols.
Also in the symbol compositions, we must deal with remainders. Here results are best when the right-hand factor is independent of λ:

Lemma 2. *When*

$$|D_x^\beta D_\xi^\alpha a(x,\xi,\lambda)| \le C|\sin\varphi|^{-N-|\alpha|-|\beta|}\langle\xi\rangle^{d_1-|\alpha|}\langle\xi,\mu\rangle^{-2d},$$
$$|D_x^\beta D_\xi^\alpha b(x,\xi)| \le C\langle\xi\rangle^{d_2-|\alpha|},$$

then $\operatorname{Op}(a)\operatorname{Op}(b) = \operatorname{Op}(c)$, *where*

$$c(x,\xi,\lambda) = \sum\nolimits_{|\alpha|<L} \tfrac{1}{\alpha!}\, D_\xi^\alpha a(x,\xi,\lambda)\partial_x^\alpha b(x,\xi) + c_L(a,b),$$
$$|D_x^\beta D_\xi^\alpha c_L(a,b)| \le C|\sin\varphi|^{-N-L-|\alpha|-|\beta|}\langle\xi\rangle^{d_1+d_2-L-|\alpha|}\langle\xi,\mu\rangle^{-2d}.$$

For the symbol r_M of R_M this gives (also other compositions are needed)

$$|D_x^\beta D_\xi^\alpha r_M| \le C|\sin\varphi|^{-2M-|\alpha|-|\beta|}\langle\xi\rangle^{d-M-|\alpha|}\langle\xi,\mu\rangle^{-2d},$$

For the exact operators, the formula

$$Q_\lambda = -\lambda^{-1} + \lambda^{-1} Q_\lambda P$$

gives a useful extra decay in λ when we define V'_M by a contour integral. Here we have some estimates by functional analysis:

$$\|Q_\lambda\|_{s,s} \le C|\sin\varphi|^{-1}|\lambda|^{-1}, \quad \|Q_\lambda P\|_{s,s} \le C|\sin\varphi|^{-1}.$$

Estimating ψdo norms by use of Prop. C and resulting kernels by Prop. B, and performing the contour integration, where $\cos\theta \sim |\sin\varphi|$, we arrive at

Theorem 2. The remainder kernel $\mathcal{K}_{V'_M}$ satisfies for $M > 2d + n + 2$, $t = |t|e^{i\theta}$:

$$|\mathcal{K}_{V'_M}(x,y,t)| \le C(\cos\theta)^{-2d-\frac{7}{2}n-7}|t|.$$

Combining this with the estimates for the homogeneous terms V_{-d-l}, we finally obtain:

Theorem 3. *In the case where P is selfadjoint ≥ 0,*

$$|\mathcal{K}_V(x,y,t)| \leq C(\cos\theta)^{-N} e^{-\gamma(P)\,\mathrm{Re}\,t} \frac{|t|}{(d(x,y)+|t|^{\frac{1}{d}})^d}((d(x,y)+|t|^{\frac{1}{d}})^{-n}+1),$$

$$N = \max\{\tfrac{n}{d}, \tfrac{7n}{2}+4d+7\}.$$

Ouhabaz and ter Elst '13 have a similar result just for P_{DN}, where $d=1$, with

$$N = 2n(n+1) \text{ compared to our } N = \tfrac{7n}{2}+11.$$

It is nonlinear in n and larger than ours when $n \geq 6$.

Methods: Multiple commutator estimates for semigroups defined from iterates of P_{DN}, refined $(L_p \to L_q)$-estimates for pseudodifferential operators (Coifman-Meyer and others), Riesz potentials, interpolation, and other tools.

For t real, they also treat P_{DN} for $-\Delta + v(x)$, $v \in L_\infty, v \geq 0$. (A corollary by domination methods.) Our study allows any smooth real v.

We can also show lower estimates for $t > 0$:

Theorem 4. *Let $0 < d < 2$ and let $P = (-\Delta)^{d/2} + P'$, P' of order $d-1$. Then there is an $r > 0$ such that*

$$|\mathcal{K}_V(x,y,t)| \geq ct\,(d(x,y) + t^{\frac{1}{d}})^{-d-n},\ \textit{for } d(x,y) + t^{\frac{1}{d}} \leq r.$$

This follows from precise estimates for $(-\Delta)^{d/2}$ combined with our estimates applied to the first remainder $V - V_{-d}$. It is valid in particular for P_{DN}.

If time permits, discuss operators on domains.

3. Fractional Laplacians on domains

Results on $(-\Delta)^a$ restricted to open subsets $\Omega \subset \mathbb{R}^n$ are generally obtained by functional analytic methods, positivity considerations etc. There seems to be a need for clarification of what regularity properties the solutions have of an equation

$$(-\Delta)^a u = f \text{ on } \Omega.$$

Take Ω smooth and bounded. The Boutet de Monvel theory of pseudodifferential boundary problems only works when a is integer.

Ros-Oton and Serra showed '12, when $0 < a < 1$, Ω is $C^{1,1}$:

For some $0 < \alpha < \min\{a, 1-a\}$ (x_n a normal coordinate),

$$f \in L_\infty(\Omega) \implies u \in x_n^a C^\alpha(\overline{\Omega}).$$

Lifted up to $u \in x_n^a C^\gamma(\overline{\Omega})$ with a $\gamma \le 1$ when f is better.

Ψdo methods? By Vishik and Eskin '60 (L_p extension by Shargorodsky '95), $(-\Delta)^a$ has a *factorization* with index a:

$$|\xi|^a = (|\xi'| - i\xi_n)^a (|\xi'| + i\xi_n)^a,$$

extending analytically to $\operatorname{Im} \xi_n > 0$ resp. $\operatorname{Im} \xi_n < 0$, which implies

$$\|u\|_{H_p^s} \le C \|f\|_{H_p^{s-2a}} \text{ for } s \in a +]-1/p', 1/p[\,.$$

For $s = 1/p + a - \varepsilon$, $p \to \infty$, this gives at best $u \in C^{a-\varepsilon}(\overline{\Omega})$.

But there are some other considerations by Hörmander, described in a typed lecture note from IAS Princeton 1965 (I received it 1980).

More generally than Boutet de Monvel's two-sided transmission condition, he defined the μ-transmission condition for any $\mu \in \mathbb{C}$, for any classical ψdo P of order $m \in \mathbb{C}$. In local coordinates, at points $x \in \partial\Omega$ with interior normal N, it requires:

$$\partial_x^\beta \partial_\xi^\alpha p_j(x, -N) = e^{\pi i(m-j-|\alpha|-2\mu)} \partial_x^\beta \partial_\xi^\alpha p_j(x, N).$$

It is satisfied by $(-\Delta)^a$ with $m = 2a$, $\mu = a$. Boutet's case is where $\mu = 0$ and $m \in \mathbb{Z}$.

Proposition. (H book '85, Th. 18.2.18.) *The μ-transmission condition is necessary and sufficient for P to map $\mathcal{E}_\mu = x_n^\mu C^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$.*

The notes develop a solvability theory in L_2-Sobolev spaces for μ-transmission operators, departing from Vishik and Eskin's estimates.

I have at present worked out an extension to L_p-Sobolev spaces, drawing on the understanding that has been developed since 1965, in particular:

- a joint work GH '90 on 0-transmission operators of any real order;
- the extension of Boutet's calculus to L_p-spaces G '90, with sharp order-reduction operators.

By combination with Sobolev embedding, this implies for any $a, t > 0$:

Application. *When* $r_\Omega(-\Delta)^a u = f$ *for some* u *supported in* Ω, *then*

$$f \in L_\infty(\Omega) \implies u \in x_n^a C^{a-\varepsilon}(\overline{\Omega}),$$
$$f \in C^t(\overline{\Omega}) \implies u \in x_n^a C^{t+a-\varepsilon}(\overline{\Omega}).$$

The first result sharpens the result by Ros-Oton and Serra '12, the second generalizes it to high t (when Ω is smooth).