Boundary problems for fractional-order pseudodifferential operators

Gerd Grubb
Copenhagen University

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1. Introduction

The fractional Laplacian $(-\Delta)^a$ on $\mathbb{R}^n$, $0 < a < 1$, is currently of great interest mathematical physics and differential geometry, probability and finance. It enters in linear as well as nonlinear equations.
It is a pseudodifferential operator (ψdo) of order $2a$, as described by use of the Fourier transform $\mathcal{F}\colon u \mapsto \mathcal{F}u = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,dx$;

$$(-\Delta)^a u = \mathrm{Op}(|\xi|^{2a})u = \mathcal{F}^{-1}(|\xi|^{2a}\hat{u}(\xi)).$$

In recent nonlinear and probability studies, it is viewed as a singular integral operator,

$$(-\Delta)^a u(x) = c_{n,a} PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|y|^{n+2a}}\, dy.$$

One of the difficulties with the operator is that it is *nonlocal*, in contrast to differential operators. This is problematic when one wants to study it on a subset Ω of $\mathbb{R}^n$. We take Ω smooth.
The discussion we give in the following works for classical ψdo's P of order $2a$ with symbol $p \sim \sum_{j\in\mathbb{N}_0} p_j(x,\xi)$ being *even*:

$$p_j(x,-\xi) = (-1)^j p_j(x,\xi),\ \text{all } j.$$

E.g. $P = A(x,D)^a$, where $A(x,D)$ is a 2' order strongly elliptic diff. op.

Overview of methods to treat boundary problems for P:

(a) The variational construction: Define the sesquilinear form

$$p_0(u,v) = \int_\Omega Pu\,\bar v\,dx, \quad u,v \in C_0^\infty(\Omega).$$

It is completed to a form on $\dot H^a(\overline\Omega)$, coercive when P is strongly elliptic, and then inducing an operator P_D in $L_2(\Omega)$ acting like r^+P with domain

$$D(P_D) = \{u \in \dot H^a(\overline\Omega) \mid r^+Pu \in L_2(\Omega)\}.$$

Here $\dot H^a(\overline\Omega) = \{u \in H^a(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline\Omega\}$, and r^+ indicates restriction to Ω. P_D has been known for many years, but an exact description of $D(P_D)$ for $a \ge \frac12$ has not been available until recently.

(b) Vishik and Eskin considered in the 1960's the problem

$$r^+(-\Delta)^a u = f \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline\Omega,$$

called the homogeneous Dirichlet problem, in more general spaces, using ψdo methods. A result is that $D(P_D) = \dot H^{2a}(\overline\Omega)$ when $a < \frac12$, and $D(P_D) \subset \dot H^{a+\frac12-\varepsilon}(\overline\Omega)$ when $a \ge \frac12$. The techniques are mainly estimates for constant-coefficient operators, perturbation and localization.

(c) Recent real integral operator methods: A trick by Caffarelli and Silvestre '07 to view $(-\Delta)^a$ on $\mathbb{R}^n$ as the Dirichlet-to-Neumann operator for a degenerate elliptic differential boundary value problem on $\mathbb{R}^n \times \mathbb{R}_+$. (Not easy to use for subsets of $\mathbb{R}^n$.) Methods from potential theory, no use of Fourier transforms (no reference to Vishik-Eskin or ψdo's), results in Hölder spaces. This works in low smoothness.
For the question of regularity of solutions an interesting result was shown by Ros-Oton and Serra (JMPA '14) on the homogeneous Dirichlet problem for $(-\Delta)^a$: $f \in L_\infty$ implies $u \in d^a C^\alpha(\overline{\Omega})$ for small α; here $d(x) = \operatorname{dist}(x, \partial\Omega)$. Improved later to $\alpha < a$.
Extensions to more general translation-invariant singular integral operators with even kernels.

(d) Very recent ψdo methods: G'14,'15,'16, based on the a-transmission property introduced by Hörmander (book 1985, earlier unpublished notes). Allows x-dependent operatos.
The Boutet de Monvel theory is not directly applicable, since it deals with integer-order ψdo's having the 0-transmission property. However, some questions for our P can be reduced to questions within the BdM calculus.

I shall report on the development of **(d)**, focusing on the latest results.

2. The pseudodifferential strategy

Let Ω be an open subset of $\mathbb{R}^n$ that is either bounded smooth or equal to $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$. r^+ stands for restriction from $\mathbb{R}^n$ to Ω, e^+ stands for extension by zero from Ω to $\mathbb{R}^n$. r^- and e^- are similar for $\complement\overline{\Omega}$.

Recall some spaces:

• *The power-distance spaces:* $\mathcal{E}_\mu(\overline{\Omega})$ equals $e^+ d^\mu C^\infty(\overline{\Omega})$ for $\operatorname{Re}\mu > -1$, where $d(x)$ is a smooth positive extension into Ω of $\operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$. For general $\mu \in \mathbb{C}$, $\mathcal{E}_{\mu-k}(\overline{\Omega}) = \operatorname{span} D^{(k)}\mathcal{E}_\mu(\overline{\Omega})$, where $D^{(k)}$ runs through smooth differential operators of order k.

• *The Sobolev spaces* (Bessel-potential spaces): For $1 < p < \infty$, $s \in \mathbb{R}$, $\langle\xi\rangle = (|\xi|^2 + 1)^{\frac{1}{2}}$,

$$\begin{aligned} H^s_p(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L_p(\mathbb{R}^n)\}, \\ \overline{H}^s_p(\Omega) &= r^+ H^s_p(\mathbb{R}^n), \\ \dot{H}^s_p(\overline{\Omega}) &= \{u \in H^s_p(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}. \end{aligned}$$

When $p = 2$, we omit p.
The notation with $\overline{H}$ and $\dot{H}$ stems from Hörmander's works.

• *The Hölder spaces* $C^{k,\sigma}(\overline{\Omega})$ where $k \in \mathbb{N}_0$, $0 < \sigma \leq 1$, are also denoted $C^s(\overline{\Omega})$ with $s = k + \sigma$ when $\sigma < 1$. For $s \in \mathbb{N}_0$, $C^s(\overline{\Omega})$ is the usual space of continuously differentiable functions.
Hölder-Zygmund-spaces $C^s_*(\overline{\Omega})$ generalize Hölder spaces to all $s \in \mathbb{R}$ with good interpolation properties. (The spaces C^s_* are also known as the Besov spaces $B^s_{\infty,\infty}$.)
Also here the notation $\overline{C}$ and $\dot{C}$ can be used.

The μ-transmission property was introduced by Hörmander ('85 book Th. 18.2.15, and lecture notes IAS Princeton '65). Let us just recall it for real a:

Definition. *A classical ps.d.o. of order m is said to satisfy the a-transmission condition at $\partial\Omega$ (for short: to be of type a), when*

$$\partial_x^\beta \partial_\xi^\alpha p_j(x, -\nu) = e^{\pi i(m-2a-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x, \nu),$$

for all indices; here $x \in \partial\Omega$ and ν denotes the interior normal at x.

Boutet de Monvel's transmission condition is the case $a = 0$, found independently. Both authors inspired from Vishik and Eskin, Dokl.'64.

NB! Our operators P of order $2a$ and with *even* symbol satisfy the a-transmission condition for any Ω; in particular $(-\Delta)^a$ does so.

Hörmander showed (Th. 18.2.18 in book '85, and notes):

Theorem 1. *For a classical ψdo P, the a-transmission property at $\partial\Omega$ is necessary and sufficient in order that*

$$r^+P \text{ maps } \mathcal{E}_a(\overline{\Omega}) \text{ into } C^\infty(\overline{\Omega}).$$

In the affirmative case, when P is elliptic, there holds for $u \in \dot{H}^a(\overline{\Omega})$:

$$u \in \mathcal{E}_a(\overline{\Omega}) \iff r^+Pu \in C^\infty(\overline{\Omega}),$$

and the mapping from u to r^+Pu is Fredholm (for bounded Ω) .

This solves the homogeneus Dirichlet problem for P when $f \in C^\infty(\overline{\Omega})$,

$$r^+Pu = f \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega}. \qquad (1)$$

The proof takes place in Sobolev spaces, for simplicity let $p = 2$.

Order-reducing operators of plus/minus type are an important tool. The basic definition is:

$$\Xi^t_\pm = \mathrm{Op}(\chi^t_\pm) \text{ on } \mathbb{R}^n, \quad \chi^t_\pm = (\langle\xi'\rangle \pm i\xi_n)^t.$$

These symbols extend analytically in ξ_n to $\operatorname{Im}\xi_n \lessgtr 0$. Hence, by the Paley-Wiener theorem, $\Xi^t_\pm$ preserve support in $\overline{\mathbb{R}}^n_\pm$. Then for all $s \in \mathbb{R}$,

$$\Xi^t_+ \colon \dot{H}^s(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} \dot{H}^{s-t}(\overline{\mathbb{R}}^n_+), \quad r^+\Xi^t_- e^+ \colon \overline{H}^s(\mathbb{R}^n_+) \xrightarrow{\sim} \overline{H}^{s-t}(\mathbb{R}^n_+).$$

In fact, Ξ^t_+ and $r^+\Xi^t_- e^+$ are adjoints. The inverses are Ξ^{-t}_+, $r^+\Xi^{-t}_- e^+$.

The symbols $\chi^t_\pm$ are not standard pseudodifferential, the control over ξ'-derivatives is too weak. There is another family with full control, $\lambda^t_\pm$ defining ψdo's $\Lambda^t_\pm$ with properties like those for $\Xi^t_\pm$; in particular the support-preserving, adjoint and invertibility properties as above (G'90).

There is also a definition of similar operators $\Lambda^{(t)}_\pm$ for Ω, constructed by local coordinates.

The operators Ξ^a_+ and Λ^a_+ also map $\mathcal{E}_a(\overline{\mathbb{R}}^n_+) \cap \mathcal{E}'$ to $e^+ C^\infty(\overline{\mathbb{R}}^n_+)$. They have an additional interesting property: When $u \in \mathcal{E}_a(\overline{\mathbb{R}}^n_+) \cap \mathcal{E}'$, then

$$\gamma_0(\Xi^a_+ u) = \Gamma(a+1)\gamma_0(u/x_n^a).$$

This follows from formulas for Fourier transformation of homogeneous functions. It also holds with Ξ^a_+ replaced by Λ^a_+.

The spaces $\mathcal{E}_a$ were generalized by Hörmander to Sobolev space settings as the a-**transmission spaces.** For $\mathbb{R}^n_+$,

$$H^{a(s)}(\overline{\mathbb{R}}^n_+) = \Xi_+^{-a} e^+ \overline{H}^{s-a}(\mathbb{R}^n_+), \text{ for } s-a > -\tfrac{1}{2}.$$

Here $e^+\overline{H}^{s-a}(\mathbb{R}^n_+)$ generally has a jump at $x_n = 0$; it is mapped by Ξ_+^{-a} to a singularity of the type x_n^a. In fact, we can show:

$$H^{a(s)}(\overline{\mathbb{R}}^n_+) \begin{cases} = \dot{H}^s(\overline{\mathbb{R}}^n_+) \text{ if } -\frac{1}{2} < s-a < \frac{1}{2}, \\ \subset e^+ x_n^a \overline{H}^{s-a}(\mathbb{R}^n_+) + \dot{H}^s(\overline{\mathbb{R}}^n_+) \text{ if } s-a > \frac{1}{2}, \end{cases}$$

with $\dot{H}^s(\overline{\mathbb{R}}^n_+)$ replaced by $\dot{H}^{s-\varepsilon}(\overline{\mathbb{R}}^n_+)$ if $s-a-\frac{1}{2} \in \mathbb{N}$.
Replacement of $\Xi^t_\pm$ by $\Lambda^t_\pm$ gives the same spaces.

In the curved situation, we define the corresponding spaces by use of local coordinates, or directly by use of the families $\Lambda^{(t)}_\pm$.

Now consider P, elliptic of order $2a > 0$ with even symbol. We want to solve the Dirichlet problem (1) for a bounded open smooth set Ω. The main idea is to use that the a-transmission property implies that the ψdo

$$Q = \Lambda_-^{(-a)} P \Lambda_+^{(-a)}$$

is of order 0 and type 0, hence belongs to the Boutet de Monvel calculus. *Moreover*, the truncated operator $Q_+ = r^+ Q e^+$ is elliptic in that calculus *without additional trace or Poisson operators*. This can be seen in the scalar case by use of a factorization of the principal symbol q_0 in plus/minus factors q_0^+ and q_0^- of order 0 (it can also be shown for systems when P is strongly elliptic).

By the BdM calclulus, Q_+ defines a Fredholm operator

$$Q_+ \colon \overline{H}^s(\Omega) \xrightarrow{\sim} \overline{H}^s(\Omega), \text{ all } s > -\tfrac{1}{2}.$$

Going back to P, we can show that for u supported in $\overline{\Omega}$,

$$r^+ P u = r^+ \Lambda_-^{(a)} e^+ Q_+ \Lambda_+^{(a)} u.$$

Then, arguing carefully with the operators $\Lambda_\pm^{(a)}$, $r^\pm$ and $e^\pm$, it is possible to lift the mapping properties of Q_+ to r^+P and obtain:

Theorem 2. *Let P be a classical ψdo of order $2a$ with even symbol, elliptic avoiding a ray. Let $s > a - \frac{1}{2}$. The homogeneous Dirichlet problem (1), considered for $u \in \dot{H}^{a-\frac{1}{2}+\varepsilon}(\overline{\Omega})$, satisfies:*

$$f \in \overline{H}^{s-2a}(\Omega) \implies u \in H^{a(s)}(\overline{\Omega}), \ \textit{the a-transmission space.}$$

Moreover, the mapping from u to f is Fredholm:

$$r^+P \colon H^{a(s)}(\overline{\Omega}) \to \overline{H}^{s-2a}(\Omega).$$

So the a-transmission spaces are the **domain spaces** for homogeneous Dirichlet problems.

Question: Can one define a nontrivial Dirichlet boundary value? and get a well-posed nonhomogeneous Dirichlet problem?

3. Nonhomogeneous boundary conditions

Consider the case $\Omega = \mathbb{R}^n_+$. Here $u \in \mathcal{E}_a$ means that $u = e^+ x_n^a v$ with $v \in C^\infty(\overline{\mathbb{R}}^n_+)$. By a Taylor expansion of v,

$$u(x) = x_n^a v(x',0) + x_n^{a+1}\partial_n v(x',0) + \tfrac{1}{2}x_n^{a+2}\partial_n^2 v(x',0) + \dots \text{ for } x_n > 0. \qquad (2)$$

If $u \in \mathcal{E}_{a-1}$, $u = e^+ x_n^{a-1} w$ with $w \in C^\infty(\overline{\mathbb{R}}^n_+)$, we have analogously:

$$u(x) = x_n^{a-1} w(x',0) + x_n^{a}\partial_n w(x',0) + \tfrac{1}{2}x_n^{a+1}\partial_n^2 w(x',0) + \dots \text{ for } x_n > 0. \qquad (3)$$

The only structural difference between (2) and (3) is the first term in (3), $x_n^{a-1} w(x',0)$. We conclude:

$\mathcal{E}_a(\overline{\mathbb{R}}^n_+)$ is the subset of $\mathcal{E}_{a-1}(\overline{\mathbb{R}}^n_+)$ for which $\gamma_0(u/x_n^{a-1}) = 0$.

Moreover, there is clearly a bijection

$$\gamma_{a-1,0} \colon \mathcal{E}_{a-1}(\overline{\mathbb{R}}^n_+)/\mathcal{E}_a(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} C^\infty(\mathbb{R}^{n-1}),$$

represented by the mapping from u to $w(x',0) = \gamma_0(u/x_n^{a-1})$ in (3).

There is a similar result for $\overline{\Omega}$, replacing x_n by d.

In the Sobolev space setting, the map $\gamma_{a-1,0} \colon u \mapsto \Gamma(a)\gamma_0(u/d^{a-1})$ from $\mathcal{E}_{a-1}(\overline{\Omega})$ to $C^\infty(\partial\Omega)$ extends to $\gamma_{a-1,0} \colon H^{(a-1)(s)}(\overline{\Omega}) \to H^{s-a+\frac{1}{2}}(\partial\Omega)$ for $s > a - \frac{1}{2}$.

This gives a bijection

$$\gamma_{a-1,0}\colon H^{(a-1)(s)}(\overline{\Omega})/H^{a(s)}(\overline{\Omega}) \xrightarrow{\sim} H^{s-a+\frac{1}{2}}(\partial\Omega).$$

When we adjoin this mapping to the mapping in Theorem 2, we get a Fredholm solvable *nonhomogeneous Dirichlet problem*:

Theorem 3. *When P is as above, then for $s > a - \frac{1}{2}$,*

$$\{r^+P, \gamma_{a-1,0}\}\colon H^{(a-1)(s)}(\overline{\Omega}) \to \overline{H}^{s-2a}(\Omega) \times H^{s-a+\frac{1}{2}}(\partial\Omega)$$

is a Fredholm mapping.

Note that when $a < 1$, the solutions with $\gamma_{a-1,0}u \neq 0$ are "large" at $\partial\Omega$, since $u = d^{a-1}v$ for a nice v with nonzero boundary value.

References for nonhomogeneous Dirichlet problems: G'14–'16, Abatangelo'15.

The above results can also be obtained in H_p^s-spaces and in Triebel-Lizorkin scales $F_{p,q}^s$ and Besov scales $B_{p,q}^s$; in particular the Hölder-Zygmund scale C_*^s.

One can also define Neumann problems:

On the space $H^{(a-1)(s)}(\overline{\mathbb{R}}^n_+)$ we have in fact a longer expansion when s is large enough (by Taylor expansion of u/x_n^{a-1}, now normalized with Gamma coefficients):

$$u(x) = \tfrac{1}{\Gamma(a)} x_n^{a-1} u_0(x') + \tfrac{1}{\Gamma(a+1)} x_n^{a} u_1(x') + \tfrac{1}{\Gamma(a+2)} x_n^{a+1} u_2(x') + \dots,\; x_n > 0,$$

$$\gamma_{a-1,0} u = u_0 = \Gamma(a)\gamma_0(u/x_n^{a-1}),\text{ the Dirichlet value,}$$

$$\gamma_{a-1,1} u = u_1 = \Gamma(a+1)\gamma_0(\partial_{x_n}(u/x_n^{a-1})),\text{ the Neumann value,}$$

and generally $\gamma_{a-1,j} u = u_j$. Note that $u' = u - \frac{1}{\Gamma(a)} x_n^{a-1} u_0$ is in $H^{a(s)}(\overline{\mathbb{R}}^n_+)$, and that u_1 is the first coefficient in u',

$$\gamma_{a-1,1} u = \gamma_{a,0} u',\text{ when } u' = u - \tfrac{1}{\Gamma(a)} x_n^{a-1} u_0.$$

In particular, when $u_0 = 0$, then $\gamma_{a-1,1} u = \gamma_{a,0} u$. For $\Omega \subset \mathbb{R}^n$, such formulas hold with d instead of x_n. We have for the Neumann problem:

Theorem 4. *When P is principally like $(-\Delta)^a$, then for $s > a + \frac{1}{2}$,*

$$\{r^+ P, \gamma_{a-1,1}\} \colon H^{(a-1)(s)}(\overline{\Omega}) \to \overline{H}^{s-2a}(\Omega) \times H^{s-a-\frac{1}{2}}(\partial\Omega)$$

is a Fredholm mapping.

There is a corollary on the homogeneous Neumann problem.

4. Green's formula

One of the difficult questions for our operators is to establish integration-by-parts formulas. An important paper of Ros-Oton and Serra (ARMA '14) showed that for real functions u, v solving a homogeneous Dirichlet problem,

$$\int_\Omega ((-\Delta)^a u\,\partial_j v + \partial_j u\,(-\Delta)^a v)\,dx = \int_{\partial\Omega} \nu_j(x)\,\gamma_{a,0}u\,\gamma_{a,0}v\,d\sigma; \quad (6)$$

here ν_j is the j'th component of the normal vector ν.

This was surprising, because the boundary term is *local*, and the formula holds for a *curved* boundary, cleanly without messy extra integrals. It is equivalent with a so-called Pohozaev formula, leading to uniqueness results for nonlinear problems.

Ros-Oton and Serra have with Valdinoci extended the formula to other x-independent operators, and we have shown a generalization valid for x-dependent ψdo's, with $u, v \in H^{a(s)}(\overline{\Omega})$, $s > a + \frac{1}{2}$, JDE'16.

In (6), since the Dirichlet value $\gamma_{a-1,0}u$ is zero, $\gamma_{a,0}u$ equals the Neumann value $\gamma_{a-1,1}u$; the same holds for v.

We have recently been able to show a much more general formula, involving nontrivial Neumann as well as Dirichlet data:

Theorem 5. *Let P be a classical ψdo of order $2a > 0$ with even symbol. When $u, v \in H^{(a-1)(s)}(\overline{\Omega})$, then for $s > a + \frac{1}{2}$,*

$$\int_{\Omega} (Pu\, \bar{v} - u\, \overline{P^* v})\, dx = \int_{\partial\Omega} (s_0 u_1\, \bar{v}_0 - s_0 u_0\, \bar{v}_1 + Bu_0\, \bar{v}_0)\, dx', \quad (7)$$

with $u_0 = \gamma_{a-1,0} u,\ u_1 = \gamma_{a-1,1} u,\ v_0 = \gamma_{a-1,0} v,\ v_1 = \gamma_{a-1,1} v$;

here $s_0(x) = p_0(x, \nu(x))$, and B is a first-order ψdo on $\partial\Omega$.

Remarkably, no ellipticity is assumed for this theorem. The **proof** is based on reductions to applications of the finer details of BdM theory: Let $\Omega = \mathbb{R}^n_+$. Let K_0 be the Poisson operator with symbol $(\langle \xi' \rangle + i\xi_n)^{-1}$, it satifies $\gamma_0 K_0 = I$. Then $K_{a-1,0} = \Xi_+^{1-a} e^+ K_0$ is a right inverse of $\gamma_{a-1,0}$. Write $u = u' + K_{a-1,0} u_0$; then $\gamma_{a-1,0} u' = 0$, so $u' \in H^{a(s)}(\overline{\Omega})$. Do the same for v. Then

$$\begin{aligned} \langle r^+ Pu, v \rangle &= I_1 + I_2 + I_3 + I_4, \\ I_1 &= \langle r^+ Pu', v' \rangle,\ I_2 = \langle r^+ PK_{a-1,0} u_0, v' \rangle, \\ I_3 &= \langle r^+ Pu', K_{a-1,0} v_0 \rangle,\ I_4 = \langle r^+ PK_{a-1,0} u_0, K_{a-1,0} v_0 \rangle. \end{aligned}$$

Similarly, $\langle u, r^+ P^* u \rangle = I_1' + I_2' + I_3' + I_4'$.

It is not so hard to prove that $I_1 - I_1' = \langle r^+ Pu', v' \rangle - \langle u', r^+ P^* u' \rangle = 0$.
The main part of the proof consists of analyzing $I_1 + I_3 + I_4 - I_2' - I_3' - I_4'$.
This produces a lot of nonlocal terms, however, the nonlocal contributions from $I_2 + I_3 - I_2' - I_3'$ cancel out, giving only the local contribution

$$\langle s_0 u_1, v_0 \rangle - \langle s_0 u_0, v_1 \rangle.$$

The last terms $I_4 - I_4'$ give a generally nontrivial contribution $\langle Bu_0, v_0 \rangle$ (even when a is integer), with B a ψdo in general.

The proof for a curved boundary is deduced from this by localization.

For constant-coefficient operators on $\mathbb{R}^n_+$, $B = 0$ if the symbol of P is real and symmetric in ξ_n, e.g. for $(-\Delta + m^2)^a$.

Corollary 6. *When $u \in H^{(a-1)(s)}(\overline{\Omega})$, $v \in H^{a(s)}(\overline{\Omega})$, then for $s > a + \frac{1}{2}$,*

$$\int_\Omega (Pu\, \bar{v} - u\, \overline{P^* v})\, dx = -\int_{\partial\Omega} s_0 u_0\, \bar{v}_1\, dx'. \quad (8)$$

Follows from Theorem 5 since $v_0 = 0$.
Note that in this case $v_1 = \gamma_{a-1,1} v = \gamma_{a,0} v$.
In (8), the boundary contribution is completely local.

Formula (8) was otained in Abatangelo '15 for $P = (-\Delta)^a$, $0 < a < 1$.

The Pohozaev formulas (cf. (6)) can be deduced from (8).

General formulas as in Theorem 5 containing nontrivial Neumann data u_1, v_1 when u_0 and $v_0 \neq 0$ have to our knowledge not been shown earlier.

Many of the efforts around $(-\Delta)^a$ are directed towards showing results for linear or nonlinear equations where $-\Delta$ is replaced by $(-\Delta)^a$ (or a second-order elliptic PDO A is replaced by A^a). The new Green's formula is an addition to this picture. It allows to find the adjoint of a boundary problem, and will for example make it possible to study general operators defined by nonlocal boundary conditions; applications of the theory of extensions in Functional Analysis.

Let us also point out that the study of the Neumann condition $\gamma_{a-1,1}u = \psi$ for our operators is a new field, where not much has been done, neither for linear nor nonlinear questions.

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