Boundary problems for fractional Laplacians

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Introduction

The fractional Laplacian $(-\Delta)^a$, $0 < a < 1$, has attracted recent interest from researchers both in probability, finance, mathematical physics and geometry. For a bounded smooth open set $\Omega \subset \mathbb{R}^n$, it is known that there is unique solvability in $L_\infty(\Omega)$ of the problem

$$(-\Delta)^a u = f \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega},$$

called the homogeneous Dirichlet problem. But the results on the regularity of the solutions have been somewhat sparse.
• Vishik, Eskin, Shamir 1960's: $u \in \dot{H}^{\frac{1}{2}+a-\varepsilon}(\overline{\Omega})$.
• Some analysis of the behavior of solutions at $\partial\Omega$ when data are C^∞, Eskin '81, Bennish '93, Chkadua and Duduchava '01.

Recent activity:
• Ros-Oton and Serra (arXiv 2012) show by potential theoretic and integral operator methods, when Ω is $C^{1,1}$, that

$$f \in L_\infty(\Omega) \implies u \in d^a C^\alpha(\overline{\Omega}),$$

for some $\alpha > 0$. Moreover, $u \in C^a(\overline{\Omega}) \cap C^{2a}(\Omega)$. Lifted to at most $\alpha \leq 1$ when f is more smooth. They state that they do not know of other regularity results for $(-\Delta)^a$ in the literature.

• G (arXiv13) presents a new systematic theory of pseudodifferential boundary problems covering $(-\Delta)^a$, showing

$$f \in L_\infty(\Omega) \implies u \in d^a C^{a-\varepsilon}(\overline{\Omega}), \tag{1}$$

$$f \in C^t(\overline{\Omega}) \implies u \in d^a C^{a+t-\varepsilon}(\overline{\Omega}), \text{ all } t > 0. \tag{2}$$

This theory will be the subject of the talk.

Latest news:

• Ros-Oton, Serra can now also show (1) (assuming only Ω that is $C^{1,1}$). By singular integral operator and potential theoretic methods (arXiv14).

• We can remove ε in (1) (if $a \neq \frac{1}{2}$) and in (2) (except when $a + t$ or $2a + t$ is integer); **optimal** (arXiv14).

1. Pseudodifferential operators

Pseudodifferential operators (ψdo's) were introduced in the 1960's as a generalization of singular integral operators (Calderon, Zygmund, Seeley, Kohn, Nirenberg, Hörmander, Giraud, Mikhlin, ...) They systematize the use of the Fourier transform $\mathcal{F}u = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,dx$:

$$Pu = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u = \mathcal{F}^{-1}\big(p(x,\xi)\hat{u}(\xi)\big) = \mathrm{OP}(p)u,$$

$$\text{where } p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha, \text{ the symbol.}$$

This extends to more general functions $p(x,\xi)$ as symbols. In the classical theory, symbols are taken *polyhomogeneous*:

$$p(x,\xi) \sim \sum_{j\in\mathbb{N}_0} p_j(x,\xi), \text{ where } p_j(x,t\xi) = t^{m-j} p_j(x,\xi)$$

for $|\xi| \ge 1, t \ge 1$ (here the order $m \in \mathbb{C}$).
The elliptic case is when the principal symbol p_0 is invertible; then $Q = \mathrm{OP}(p_0^{-1})$ is a good approximation to an inverse of P. The theory extends to manifolds by use of local coordinates.

An example of a ψdo is $(-\Delta)^a = \mathrm{OP}(|\xi|^{2a})$, of order $2a$, but also variable coefficients are allowed. E.g., for a strongly elliptic differential operator A of order $2k$, A^a is a ψdo of order $2ka$.

Let Ω be a smooth open subset of $\mathbb{R}^n$. There is a need to consider boundary value problems

$$P_+ u = f \text{ on } \Omega, \quad Tu = \varphi \text{ on } \partial\Omega;$$

here $P_+ = r^+ P e^+$ is the truncation of P to Ω (r^+ restricts to Ω, e^+ extends by zero on $\complement\Omega$), and T is a trace operator.

Boutet de Monvel in 1971 introduced a calculus treating this when P is of *integer order* and has the *transmission property*:

P_+ maps $C^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$.

Solution operators for the problem are typically of the form

$$\begin{pmatrix} Q_+ + G & K \end{pmatrix},$$

where $Q \sim P^{-1}$ and G is an auxiliary operator called a singular Green operator, and K is a Poisson operator (going from $\partial\Omega$ to Ω).

But there are many interesting ψdo's not having the transmission property, e.g. $(-\Delta)^{\frac{1}{2}}$ does not have it, although of order 1.

2. The μ-transmission property

Definition 1. *For* $\operatorname{Re}\mu > -1$, $\mathcal{E}_\mu(\overline{\Omega})$ *consists of the functions* u *of the form*

$$u(x) = \begin{cases} d(x)^\mu v(x) \text{ for } x \in \Omega, \text{ with } v \in C^\infty(\overline{\Omega}), \\ 0 \text{ for } x \in \complement\Omega; \end{cases}$$

where $d(x)$ *is* > 0 *on* Ω, *belongs to* $C^\infty(\overline{\Omega})$, *and is proportional to* $\operatorname{dist}(x, \partial\Omega)$ *near* $\partial\Omega$. *More generally for* $j \in \mathbb{N}$, $\mathcal{E}_{\mu-j}$ *is spanned by the distribution derivatives up to order* j *of functions in* $\mathcal{E}_\mu$.

In Hörmander's book '85 Th. 18.2.18, for a classical ψdo P of order m:

Theorem 2. r^+P *maps* $\mathcal{E}_\mu(\overline{\Omega})$ *into* $C^\infty(\overline{\Omega})$ **if and only if** *the symbol has the* μ*-transmission property for* $x \in \partial\Omega$, *with* N *denoting the interior normal:*

$$\partial_x^\beta \partial_\xi^\alpha p_j(x, -N) = e^{\pi i(m-2\mu-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x, N),$$

for all indices.

It is a, possibly twisted, parity along the normal to $\partial\Omega$. Simple parity is the case $m = 2\mu$; it holds for $|\xi|^{2a}$ with $m = 2a$, $\mu = a$. Boutet de Monvel's transmission property is the case $m \in \mathbb{Z}$, $\mu = 0$.
The operators are for short said to be of *type* μ.

3. Solvability with homogeneous boundary conditions

The μ-transmission property was actually introduced far earlier by Hörmander in a lecture note from IAS 1965-66, distributed by photocopying. I received it in 1980, and have only last year studied it in depth. It contains much more, namely a *solvability theory* in L_2 Sobolev spaces for operators of type μ, which in addition have a certain factorization property of the principal symbol.

Definition 3. *P (of order m) has the factorization index μ_0 when, in local coordinates where Ω is replaced by $\mathbb{R}^n_+$ with coordinates (x', x_n),*

$$p_0(x', 0, \xi', \xi_n) = p_-(x', \xi', \xi_n) p_+(x', \xi', \xi_n),$$

with $p_\pm$ homogeneous in ξ of degrees μ_0 resp. $m - \mu_0$, and $p_\pm$ extending to $\{\operatorname{Im} \xi_n \lessgtr 0\}$ analytically in ξ_n.

Here $OP(p_\pm(x', \xi))$ on $\mathbb{R}^n$ preserve support in $\overline{\mathbb{R}}^n_+$ resp. $\overline{\mathbb{R}}^n_-$.

Example: For $(-\Delta)^a$ on $\mathbb{R}^n$ we have

$$|\xi|^{2a} = (|\xi'|^2 + \xi_n^2)^a = (|\xi'| - i\xi_n)^a (|\xi'| + i\xi_n)^a,$$

so that $p_\pm = (|\xi'| \pm i\xi_n)^a$, and the factorization index is a.

We denote $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. The operators $\Xi_\pm^\mu = \mathrm{OP}((\langle\xi'\rangle \pm i\xi_n)^\mu)$ play a great role in the theory.

Based on the factorization, Vishik and Eskin showed in '64 (extension to L_p by Shargorodsky '95, $1 < p < \infty$, $1/p' = 1 - 1/p$):

Theorem 4. *When P is elliptic of order m and has the factorization index μ_0, then*

$$r^+P\colon \dot{H}_p^s(\overline{\Omega}) \to \overline{H}_p^{s-\operatorname{Re} m}(\Omega)$$

is a Fredholm operator for $\operatorname{Re}\mu_0 - 1/p' < s < \operatorname{Re}\mu_0 + 1/p$.

Here we denote

$$H_p^s(\mathbb{R}^n) = \{u \in \mathcal{S}' \mid \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L_p(\mathbb{R}^n)\},$$
$$\dot{H}_p^s(\overline{\Omega}) = \{u \in H_p^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}, \quad \overline{H}_p^s(\Omega) = r^+H_p^s(\mathbb{R}^n).$$

The notation with $\dot{H}$ and $\overline{H}$ stems from Hörmander '65 and '85.

Note that s runs in a small interval $\,]\operatorname{Re}\mu_0 - 1/p', \operatorname{Re}\mu_0 + 1/p[\,$. The problem is now to find the solution space for higher s.

For this, Hörmander introduced for $p = 2$ a particular space combining the $\dot{H}$ and the $\overline{H}$ definitions:

Definition 5. *For $\mu \in \mathbb{C}$ and $s > \operatorname{Re}\mu - 1/p'$, the space $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ is defined by*

$$H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) = \Xi_+^{-\mu} e^+ \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+).$$

Here $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \subset \mathcal{S}'(\mathbb{R}^n)$, supported in $\overline{\mathbb{R}}^n_+$, and there holds:

Proposition 6. *Let $s > \operatorname{Re}\mu - 1/p'$. Then*

$$\Xi_+^{-\mu} e^+ \colon \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+) \to H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \text{ has the inverse}$$
$$r^+ \Xi_+^{\mu} \colon H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \to \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+),$$

and $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ is a Banach space with the norm

$$\|u\|_{\mu(s)} = \|r^+ \Xi_+^{\mu} u\|_{\overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+)}.$$

Note the jump at $x_n = 0$ in $e^+ \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+)$.

One has that $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \supset \dot{H}_p^{s}(\overline{\mathbb{R}}^n_+)$, and elements of $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ are locally in H_p^s on $\mathbb{R}^n_+$, but they are not in general H_p^s up to the boundary. The definition generalizes to $\Omega \subset \mathbb{R}^n$ by use of local coordinates.

These are Hörmander's μ-spaces, very important since they turn out to be the correct solution spaces.

The spaces $H_p^{\mu(s)}$ replace the $\mathcal{E}_\mu$ in a Sobolev space context, in fact one has:

Proposition 7. *Let $\overline{\Omega}$ be compact, and let $s > \operatorname{Re}\mu - 1/p'$. Then*

$$\mathcal{E}_\mu(\overline{\Omega}) \subset H_p^{\mu(s)}(\overline{\Omega}) \textit{ densely, and } \bigcap\nolimits_s H_p^{\mu(s)}(\overline{\Omega}) = \mathcal{E}_\mu(\overline{\Omega}).$$

We can now state the basic theorems:

Theorem 8. *When P is of order m and type μ, r^+P maps $H_p^{\mu(s)}(\overline{\Omega})$ continuously into $\overline{H}_p^{s-\operatorname{Re} m}(\Omega)$ for all $s > \operatorname{Re}\mu - 1/p'$.*

Theorem 9. *Let P be elliptic of order m, with factorization index μ_0, and of type $\mu_0 \pmod 1$. Let $s > \operatorname{Re}\mu_0 - 1/p'$. The solutions u in $\dot{H}_p^{\operatorname{Re}\mu_0 - 1/p' + \varepsilon}(\overline{\Omega})$ of the equation*

$$r^+Pu = f, \quad f \textit{ given in } \overline{H}_p^{s-\operatorname{Re} m}(\Omega),$$

belong to $H_p^{\mu_0(s)}(\overline{\Omega})$. Moreover, the mapping

$$r^+P \colon H_p^{\mu_0(s)}(\overline{\Omega}) \to \overline{H}_p^{s-\operatorname{Re} m}(\Omega) \quad (4)$$

is Fredholm.

This represents a *homogeneous Dirichlet problem.*

The proofs in the old 1965 notes (for $p = 2$) are long and difficult. One of the difficulties is that the $\Xi_\pm^\mu$ are not truly ψdo's in n variables, the derivatives of the symbols $(\langle\xi'\rangle \pm i\xi_n)^\mu$ do not decrease for $|\xi| \to \infty$ in the required way.

More recently we have found (G '90) a modified choice of symbol that gives true ψdo's $\Lambda_\pm^{(\mu)}$ with the same holomorphic extension properties for $\operatorname{Im}\xi_n \lessgtr 0$; they can be used instead of $\Xi_\pm^\mu$, also for $p \neq 2$.

This allows a reduction of some of the considerations to cases where the Boutet de Monvel calculus (extended to H_p^s in G '90) can be applied.

In fact, when we for Theorem 9 introduce

$$Q = \Lambda_-^{(\mu_0 - m)} P \Lambda_+^{(-\mu_0)},$$

we get a ψdo of order 0 and type 0, with factorization index 0; then

$$r^+ Pu = f, \text{ with } \operatorname{supp} u \subset \overline{\Omega},$$

can be transformed to the equation

$$r^+ Qv = g, \text{ where } v = \Lambda_+^{(\mu_0)} u,\ g = r^+ \Lambda_-^{(\mu_0 - m)} e^+ f.$$

Here the Boutet de Monvel calculus applies to Q and provides good solvability properties for all $s > \operatorname{Re}\mu_0 - 1/p'$.

Since $\bigcap_s H_p^{\mu(s)}(\overline{\Omega}) = \mathcal{E}_\mu(\overline{\Omega})$, and $\bigcap_s \overline{H}_p^{s-\operatorname{Re} m}(\Omega) = C^\infty(\overline{\Omega})$, one finds as a corollary when $s \to \infty$:

Corollary 10. *Let P be as in Theorem 9 and let u be a function supported in $\overline{\Omega}$. If $r^+Pu \in C^\infty(\overline{\Omega})$, then $u \in \mathcal{E}_{\mu_0}(\overline{\Omega})$. Moreover, the mapping*

$$r^+P \colon \mathcal{E}_{\mu_0}(\overline{\Omega}) \to C^\infty(\overline{\Omega})$$

is Fredholm.

One can furthermore show that the finite dimensional kernel and cokernel (a complement of the range) of the mapping in Corollary 10 serve as kernel and cokernel also in the mappings for finite s in Theorem 9.

Note the sharpness: The functions in $\mathcal{E}_{\mu_0}$ have the behavior $u(x) = d(x)^{\mu_0} v(x)$ at the boundary with $v \in C^\infty(\overline{\Omega})$; they are *not in* C^∞ *themselves*, when $\mu_0 \notin \mathbb{N}_0$!

We find using Poisson operators from the Boutet de Monvel calculus that also the $H_p^{\mu(s)}$-spaces give rise to a factor $d(x)^\mu$, namely

$$H_p^{\mu(s)}(\overline{\Omega}) \subset e^+ d(x)^\mu \overline{H}_p^{s-\operatorname{Re}\mu}(\Omega) + \dot{H}_p^s(\overline{\Omega}), \text{ if } s > \operatorname{Re}\mu + 1/p$$

(with $\dot{H}_p^s(\overline{\Omega})$ replaced by $\dot{H}_p^{s-\varepsilon}(\overline{\Omega})$ if $s - \operatorname{Re}\mu \in \mathbb{N}$).

4. Hölder estimates

There is currently much interest for Hölder estimates of solutions. One can apply Sobolev embeddings to the H_p^s-spaces and let $p \to \infty$. Another even more efficient method is to extend our results to general function spaces, including Hölder-Zygmund spaces. By Johnsen '96, the Boutet de Monvel theory extends to $F^s_{p,q}$ (Triebel-Lizorkin) spaces and $B^s_{p,q}$ (Besov) spaces for $s \in \mathbb{R}$, $0 < p, q \leq \infty$ ($p < \infty$ in the F-case). This includes $B^s_{\infty,\infty}$ (Hölder-Zygmund spaces) that equal ordinary Hölder spaces when $s \in \mathbb{R}_+ \setminus \mathbb{N}$. We shall write $B^s_{\infty,\infty} = C^s_*$ for all $s \in \mathbb{R}$. Also the mapping properties of $\Lambda^\mu_\pm$ and the description of the μ-spaces $C_*^{\mu(s)}$ extend. Theorem 9 is generalized to:

Theorem 11. *Let P be elliptic of order m, with factorization index μ_0, and of type $\mu_0 \pmod 1$. Let $s > \operatorname{Re}\mu_0 - 1$. The solutions u in $\dot{C}_*^{\operatorname{Re}\mu_0 - 1 + \varepsilon}(\overline{\Omega})$ of the equation*

$$r^+ Pu = f, \quad f \text{ given in } \overline{C}_*^{s - \operatorname{Re} m}(\Omega),$$

belong to $C_*^{\mu_0(s)}(\overline{\Omega})$. Moreover, the following mapping is Fredholm:

$$r^+ P \colon C_*^{\mu_0(s)}(\overline{\Omega}) \to \overline{C}_*^{s - \operatorname{Re} m}(\Omega). \quad (5)$$

Here the μ-spaces satisfy:

$$C_*^{\mu(s)}(\overline{\Omega}) \subset e^+ d(x)^\mu \overline{C}_*^{s-\operatorname{Re}\mu}(\Omega) + \dot{C}_*^s(\overline{\Omega}), \text{ if } s > \operatorname{Re}\mu$$

(with $\dot{C}_*^s(\overline{\Omega})$ replaced by $\dot{C}_*^{s-\varepsilon}(\overline{\Omega})$ if $s - \operatorname{Re}\mu \in \mathbb{N}$).

This has the consequence for $(-\Delta)^a$, $0 < a < 1$:

Corollary 12. *Assume $u \in e^+ L_\infty(\Omega)$. If u solves*

$$r^+(-\Delta)^a u = f, \qquad (6)$$

then

$$f \in L_\infty(\Omega) \implies u \in e^+ d(x)^a C^a(\overline{\Omega}) \ (\textit{when } a \neq \tfrac{1}{2}).$$
$$f \in C^t(\overline{\Omega}) \implies u \in e^+ d(x)^a C^{t+a}(\overline{\Omega}) \ (\textit{when } t + a, t + 2a \notin \mathbb{N}).$$

The problem (6) *is uniquely solvable in these spaces.*

These results are **optimal**. In the exceptional cases we subtract ε from the Hölder exponent. The first estimate, with $C^{a-\varepsilon}$, has very recently been obtained by Ros-Oton and Serra for $C^{1,1}$-domains (with nonlinear implications).

Let $\operatorname{Re}\mu_0 > 0$. Since $u(x) = x_n^{\mu_0} v(x)$ with $v \in C^\infty(\overline{\mathbb{R}}^n_+)$ implies $u(x) = x_n^{\mu_0 - 1} x_n v(x)$ with $x_n v \in C^\infty(\overline{\mathbb{R}}^n_+)$, $\mathcal{E}_{\mu_0}(\overline{\mathbb{R}}^n_+) \subset \mathcal{E}_{\mu_0 - 1}(\overline{\mathbb{R}}^n_+)$.
Similar statement for $\overline{\Omega}$ with x_n replaced by $d(x)$.
When $u \in \mathcal{E}_{\mu_0 - 1}(\overline{\mathbb{R}}^n_+)$, it equals $x_n^{\mu_0 - 1} w$ for a $w \in C^\infty(\overline{\mathbb{R}}^n_+)$, so

$$\gamma_{\mu_0 - 1, 0} u \equiv \gamma_0(x_n^{-\mu_0 + 1} u) = \gamma_0 w \in C^\infty(\partial\mathbb{R}^n_+)$$

is well-defined, takes all values. We see that

$$\mathcal{E}_{\mu_0}(\overline{\mathbb{R}}^n_+) = \{u \in \mathcal{E}_{\mu_0 - 1}(\overline{\mathbb{R}}^n_+) \mid \gamma_{\mu_0 - 1, 0} u = 0\}.$$

This extends to the Sobolev and Hölder-Zygmund spaces, where

$$H_p^{\mu_0(s)}(\overline{\Omega}) \subset H_p^{(\mu_0 - 1)(s)}(\overline{\Omega}), \quad C_*^{\mu_0(s)}(\overline{\Omega}) \subset C_*^{(\mu_0 - 1)(s)}(\overline{\Omega}).$$

Theorem 13. *When $\operatorname{Re}\mu_0 > 0$, the mapping $\gamma_{\mu_0 - 1, 0}$ extends to continuous and surjective mappings*

$$\gamma_{\mu_0 - 1, 0} \colon H_p^{(\mu_0 - 1)(s)}(\overline{\Omega}) \to B_p^{s - \operatorname{Re}\mu_0 + 1/p'}(\partial\Omega), \text{ when } s > \operatorname{Re}\mu_0 - 1/p',$$
$$\gamma_{\mu_0 - 1, 0} \colon C_*^{(\mu_0 - 1)(s)}(\overline{\Omega}) \to C_*^{s - \operatorname{Re}\mu_0 + 1}(\partial\Omega), \text{ when } s > \operatorname{Re}\mu_0 - 1,$$

with kernel $H_p^{\mu_0(s)}(\overline{\Omega})$ resp. $C_*^{\mu_0(s)}(\overline{\Omega})$.

Now Theorem 13 can be combined with Theorems 9 and 11 to show the solvability of *nonhomogeneous Dirichlet problems*:

Theorem 14. *Let P be as in Theorem 9. The mappings*

$$\{r^+P,\gamma_{\mu_0-1,0}\}\colon H_p^{(\mu_0-1)(s)}(\overline{\Omega})\to \overline{H}_p^{s-\operatorname{Re} m}(\Omega)\times B_p^{s-\operatorname{Re}\mu_0+1/p'}(\partial\Omega)$$
$$\{r^+P,\gamma_{\mu_0-1,0}\}\colon C_*^{(\mu_0-1)(s)}(\overline{\Omega})\to \overline{C}_*^{s-\operatorname{Re} m}(\Omega)\times C_*^{s-\operatorname{Re}\mu_0+1}(\partial\Omega)$$

are Fredholm for $s>\operatorname{Re}\mu_0-1/p'$, resp. $s>\operatorname{Re}\mu_0-1$.

For $(-\Delta)^a$, $0<a<1$, this gives:

Corollary 15. *There is a unique solution $u\in C_*^{(a-1)(2a)}(\overline{\Omega})$ of*

$$r^+(-\Delta)^a u=f\in L_\infty(\Omega),\quad \gamma_{a-1,0}u=\varphi\in C^{a+1}(\partial\Omega);$$

it satisfies (with modifications for $a=\frac12$),

$$u\in e^+d(x)^{a-1}C^{a+1}(\overline{\Omega})+\dot{C}^{2a}(\overline{\Omega}).$$

Moreover,

$$f\in C^t(\overline{\Omega}),\ \varphi\in C^{t+a+1}(\partial\Omega)\implies u\in e^+d(x)^{a-1}C^{t+a+1}(\overline{\Omega})+\dot{C}^{2a+t}(\overline{\Omega}),$$

(with modifications if $t+a$ or $t+2a$ is integer).

Note that since $a-1<0$, the term in $d(x)^{a-1}C^{t+a+1}(\overline{\Omega})$ can blow up at the boundary; it does so nontrivially when $\varphi \neq 0$.

In the studies by potential- and integral operator-methods, this phenomenon is called "large solutions" (blowing up at $\partial\Omega$). We see that such solutions appear very naturally, with a precise singularity, when the boundary condition is nonhomogeneous.

Further remarks. Our study moreover allows treatments of other boundary conditions (also vector valued). For example, $r^+(-\Delta)^a u = f$ with a *Neumann condition* $\partial_n(d(x)^{1-a}u)|_{\partial\Omega} = \psi$ can be shown to be Fredholm solvable.

The current efforts for problems involving the fractional Laplacian are often concerned with nonlinear equations where it enters, and there is an interest also in generalizations with low regularity of the domain or the coefficients.

For problems where Δ itself enters, one has a old and well-known background theory of boundary value problems in the smooth case. This has been absent in the case of $(-\Delta)^a$, and we can say that the present results provide that missing link.

The methods used in the current literature on $(-\Delta)^a$ are often integral operator methods and potential theory. Here is one of the strange formulations used there:

Because of the nonlocal nature of $(-\Delta)^a$, when we consider a subset $\Omega \subset \mathbb{R}^n$, auxiliary conditions may be given as *exterior conditions*, where the value of the unknown function is prescribed on the complement of Ω.

Then the homogeneous Dirichlet problem is formulated as:

$$\begin{cases} r^+(-\Delta)^a U &= f \text{ on } \Omega, \\ U &= g \text{ on } \complement\Omega. \end{cases} \tag{8}$$

The nonhomogeneous Dirichlet problem is then formulated (for more general U) as:

$$\begin{cases} r^+(-\Delta)^a U &= f \text{ on } \Omega, \\ U &= g \text{ on } \complement\Omega, \\ d(x)^{1-a} U &= \varphi \text{ on } \partial\Omega. \end{cases} \tag{9}$$

(Abatangelo arXiv November '13.) We can show, when Ω is smooth: *Within the framework of H_p^s and C_*^s spaces,* (8) *and* (9) *can be reduced to problems with the unknown u supported in $\overline{\Omega}$ (i.e., $g = 0$), uniquely solved by our preceding theorems.*