Resolvent differences and their spectral estimates

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Plan

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1. Spectral upper estimates

A compact selfadjoint nonnegative operator T in a Hilbert space H has a decreasing sequence of eigenvalues $\mu_j(T) \to 0$.

(i) *Spectral upper estimate*, with $\alpha > 0$:

$$\mu_j(T) \text{ is } O(j^{-\alpha}).$$

(ii) *Spectral asymptotic estimate (Weyl-type estimate)*:

$$\mu_j(T)j^{\alpha} \to C(T), \text{ for } j \to \infty.$$

(iii) *Spectral asymptotic estimate with remainder*, with $\beta > \alpha$:

$$\mu_j(T) - C(T)j^{-\alpha} \text{ is } O(j^{-\beta}).$$

They correspond to estimates of the counting function $N'(t; T) = \#\{\mu_j > 1/t\}$, for $t \searrow 0$, e.g. (iii) is equivalent with

$$N'(t; T) - C(T)^{1/\alpha} t^{1/\alpha} \text{ is } O(t^{1/\alpha - (\beta-\alpha)/\alpha}).$$

For general compact operators T, this applies to $s_j(T) = \mu_j(T^*T)^{1/2}$, s-numbers. The weak Schatten class $\mathfrak{S}_{(p)}$ has (i) with $p = 1/\alpha$.

Let Ω be smooth open $\subset \mathbb{R}^n$, with *bounded boundary* $\partial\Omega = \Sigma$. Denote $\partial_n^j u|_\Sigma = \gamma_j u$, $j \in \mathbb{N}_0$. Consider a symmetric strongly elliptic second-order differential operator on Ω with real C^∞-coefficients and an associated sesquilinear form

$$Au = -\sum\nolimits_{j,k=1}^{n} \partial_j(a_{jk}(x)\partial_k u) + a_0(x)u,$$
$$a(u,v) = \sum\nolimits_{j,k=1}^{n} (a_{jk}\partial_k u, \partial_j v) + (a_0 u, v), \text{ with}$$
$$a(u,u) \geq c\|u\|^2_{H^1(\Omega)} - k\|u\|^2_{L_2(\Omega)} \text{ for } u \in H^1(\Omega);$$

$c > 0$, $k \geq 0$. Set $\nu u = \sum n_j \gamma_0(a_{jk}\partial_k u)$ ($= \gamma_1 u$ when $A = -\Delta$). The maximal operator $A_{\max}$ acts like A in $L_2(\Omega)$ with domain

$$D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\};$$

the minimal operator $A_{\min}$ equals $\overline{A|_{C_0^\infty}}$, with $D(A_{\min}) = H_0^2(\Omega)$; $A_{\max}$ and $A_{\min}$ are adjoints.

The set $\mathcal{M}$ of *realizations* $\widetilde{A}$ of A are the operators with $A_{\min} \subset \widetilde{A} \subset A_{\max}$; they are defined by *boundary conditions*.
Examples:

The Dirichlet realization A_γ with $D(A_\gamma) = \{u \in H^2(\Omega) \mid \gamma_0 u = 0\}$.

Neumann-type (Robin) realizations A_χ with $\chi u = \nu u - b\gamma_0 u$, for a real C^∞-function b; $D(A_\chi) = \{u \in H^2(\Omega) \mid \chi u = 0\}$.

Mixed realizations A_{χ,Σ_+}. Here Σ is a union of closed subsets $\Sigma_+ \cup \Sigma_-$ with $\Sigma_+^\circ \cap \Sigma_-^\circ = \emptyset$, and $D(A_{\chi,\Sigma_+})$ is the set of $u \in H^1(\Omega) \cap D(A_{\max})$ such that

$$\chi u = 0 \text{ on } \Sigma_+, \ \gamma_0 u = 0 \text{ on } \Sigma_-.$$

All can be defined variationally from sesquilinear forms $a_\gamma(u, v)$, $a_\chi(u, v)$ resp. $a_{\chi,\Sigma_+}(u, v)$, acting like $a(u, v) + (b\gamma_0 u, \gamma_0 v)$, considered on $H_0^1(\Omega)$, $H^1(\Omega)$, resp. $H^1_{\Sigma_+}(\Omega) = \{u \in H^1(\Omega) \mid \operatorname{supp} \gamma_0 u \subset \Sigma_+\}$.

They are selfadjoint lower bounded, we can assume them positive by adding a constant to A.

M.Sh. Birman showed in 1962 the upper estimate:

Theorem 1. *For* $\widetilde{A}$ *equal to* A_χ *or* A_{χ,Σ_+}, *the difference between* $\widetilde{A}^{-1}$ *and* A_γ^{-1} *is compact and belongs to* $\mathfrak{S}_{((n-1)/2)}$, *i.e.,*

$$\mu_j(\widetilde{A}^{-1} - A_\gamma^{-1}) \text{ is } O(j^{-2/(n-1)}). \qquad (1)$$

The proof used the min-max principle for Rayleigh quotients, combined with the fact that $\widetilde{A}^{-1} - A_\gamma^{-1}$ acts essentially in a subspace of $Z = \ker A_{\max} \subset L_2(\Omega)$.
Although Birman's 1962 paper was not officially translated to English until 2008 (by M.Z. Solomyak), unofficial translations existed from the mid-sixties in USA.
NB! Birman only assumed a piecewise C^2-boundary. According to Solomyak's translation, the proofs work even for Lipschitz domains.
The coefficients a_{jk} are in Birman's paper assumed to be continuous with locally bounded first derivatives.

2. Spectral asymptotic estimates in smooth cases

Improvements of the spectral upper estimates to asymptotic estimates were worked out by Solomyak and Birman together. Here the coefficients and the boundary are assumed to be smooth. The result for exterior domains was published in 1980:

Theorem 2. *For second-order operators on exterior domains there is an asymptotic estimate*

$$\mu_j(\widetilde{A}^{-1} - A_\gamma^{-1})j^{2/(n-1)} \to C_0 \text{ for } j \to \infty, \qquad (2)$$

where C_0 is a constant defined from the symbols.

In the paper it is mentioned that the result was obtained by Birman and Solomyak for bounded domains in '78 and '79, and that related results were shown by Koshevnikov '78 and H'melnitski '78.

The results for bounded domains were in fact preceded by a precise result of G'74:

Theorem 3. 1° *Let A be a strongly elliptic system of order $2m$, Ω bounded smooth. Let A_γ be the Dirichlet realization, A_B a realization defined by a normal elliptic boundary condition, both selfadjoint. Then there is a "Kreĭn resolvent formula"*

$$A_B^{-1} - A_\gamma^{-1} = \mathrm{i}_V J \mathcal{L}^{-1} J^* \operatorname{pr}_V,$$

where $\mathcal{L}^{-1}$ is an elliptic pseudodifferential operator of order $-2m$ in a vector bundle F over Σ, and J is an **isometry** *from $L_2(\Sigma, F)$ to a closed subspace $V \subset Z = \ker A_{\max}$.*
2° *It follows that $\mu_j^\pm(A_B^{-1} - A_\gamma^{-1}) = \mu_j^\pm(\mathcal{L}^{-1})$, and hence*

$$\mu_j^\pm(A_B^{-1} - A_\gamma^{-1}) - C^\pm j^{-2m/(n-1)} \text{ is } O(j^{-(2m+1)/(n-1)}),$$

with constants $C^\pm$ determined from the principal symbols.

The asymptotic estimate with remainder follows from Hörmander '68 for elliptic ψdo's, when the principal symbol eigenvalues of $\mathcal{L}$ are simple; this restriction is removed by results of Ivrii '82.
We have recently checked that the proof extends to exterior domains, for uniformly strongly elliptic systems.

The proof was based on a general extension theory by G'68, which was a further development of the abstract theories of Kreĭn'47, Vishik'52 and Birman'56, combined with a full interpretation for concrete realizations of elliptic operators.

In G'84 the calculus of *pseudodifferential boundary operators* initiated by Boutet de Monvel 1971 was brought into the picture, in particular *singular Green operators*.

Theorem 4. 1° *When G is a singular Green operator of class zero and negative order $-r$ on Ω, then*

$$s_j(G)j^{r/(n-1)} \to C(g^0) \text{ for } j \to \infty, \quad (3)$$

with a constant defined from the principal symbol g^0 of G.

2° *The difference $A_B^{-1} - A_\gamma^{-1}$ is a singular Green operator on Ω of class zero and order $-2m$. The asymptotic estimate*

$$s_j(A_B^{-1} - A_\gamma^{-1})j^{2m/(n-1)} \to C_0 \text{ for } j \to \infty$$

follows, also in nonselfadjoint cases.

Moreover, G'84 gave a general strategy of how to reduce the question for an exterior domain to a bounded set (showing that cutoffs at infinity give negligible errors). The method has been streamlined in a recent publication G'11 in Applic. Anal.

This clears up the question of Weyl-type spectral asymptotic estimates for all smooth, uniformly strongly elliptic operators with smooth normal elliptic boundary conditions at a compact boundary.

Remainder estimates for singular Green operators can be shown in "lucky" cases. For example, when $G = KK^*$ for a Poisson operator K, then

$$\mu_j(G) = \mu_j(KK^*) = \mu_j(K^*K);$$

where K^*K is a ψdo of order $-r$ on Σ. If K^*K is *elliptic*, one gets the estimate with optimal remainder

$$\mu_j(G) - C_0 j^{-r/(n-1)} = O(j^{-(r+1)/(n-1)}).$$

3. Recent studies

Besides the extension theories of Kreĭn'47, Vishik'52. Birman'56, G'68, there has been another development, mostly aimed towards ODE, formulated in terms of *boundary triples* and *Weyl-Titchmarsh m-functions*: Kochubei'75, Gorbachuk-Gorbachuk'84 (book translated '91), Derkach-Malamud'87, Malamud-Mogilevski'97, '02.... The theory involves not only *operators* but also *relations*. Applications to PDE are given e.g. in Amrein-Pearson'04, Behrndt-Langer'07, Ryshov'07, Brown-Marletta-Naboko-Wood'08,.... A connection between the two types of theories is explained in Brown-G-Wood'09.

A central theme in all cases is to establish *Kreĭn resolvent formulas*

$$(\widetilde{A}-\lambda)^{-1}-(A_\gamma-\lambda)^{-1}=\mathcal{G}(\lambda) \text{ “acting on” } \Sigma.$$

Example. For our A of second order, let K_γ denote the *Poisson operator* $K_\gamma : \varphi \mapsto u$ solving the semihomogeneous Dirichlet problem

$$Au = 0 \text{ on } \Omega,\ \gamma_0 u = \varphi \text{ on } \Sigma,$$

it maps $H^{s-\frac{1}{2}}(\Sigma) \xrightarrow{\sim} H^s(\Omega) \cap Z$ for $s \geq 0$.

In particular for $s = 0$, $K_\gamma : H^{-\frac{1}{2}}(\Sigma) \to L_2(\Omega)$ and $K_\gamma^* : L_2(\Omega) \to H^{\frac{1}{2}}(\Sigma)$. Let $P_{\gamma,\chi} = \chi K_\gamma$, the *Dirichlet-to-Neumann operator* (ψdo on Σ of order 1). Then

$$A_\chi^{-1} - A_\gamma^{-1} = K_\gamma L^{-1} K_\gamma^*, \qquad (4)$$

where L acts like $-P_{\gamma,\chi}$. There is also a λ-dependent version. (4) follows from G'68 and also from the boundary triple theories.

Recent papers by Behrndt-Langer'07, Malamud'10, Behrndt-Langer-Lotoreichik'11 use Krein resolvent formulas to show spectral upper estimates for general resolvent differences. They use s-number estimates of Sobolev space embeddings. Improvements to *Weyl-type* estimates in nonsmooth cases need further analytic tools.

Two new cases:

Case 1. *The resolvent difference between a Robin realization and the Neumann realization (with $b = 0$).*

Behrndt-Langer-Lobanov-Lotoreichik-Popov'10 showed essentially:

Theorem 5. *For $A = -\Delta$, $\chi = \nu - b\gamma_0$, on a bounded smooth domain,*

$$|\mu_j(A_\chi^{-1} - A_\nu^{-1})| \le Cj^{-3/(n-1)}. \qquad (5)$$

For smooth b, this is an immediate consequence of the G'84 theorem on singular Green operators, since $A_\chi^{-1} - A_\nu^{-1}$ is a singular Green operator of order -3. Then one even has

$$|\mu_j(A_\chi^{-1} - A_\nu^{-1})| j^{3/(n-1)} \to C_0 \text{ for } j \to \infty, \qquad (6)$$

with a specific C_0. But the BLLLP'10 proof works also when b is just in $L_\infty(\Sigma)$. Are there *Weyl-type formulas* for nonsmooth b?

We have recently shown (J. Spec. Th.'11):

Theorem 6. *In the second-order variable coefficient case, let the Robin coefficient b be piecewise C^ε with a jump at $\partial\Sigma_+$. Then the limit formula* (6) *holds.*

An important ingredient in the proof is the spectral estimate of Laptev'81 for a ψdo Q on Σ with opposite cutoffs $1_{\Sigma_+} Q 1_{\Sigma_-}$; here the dimension $n-1$ in spectral estimates is replaced by $n-2$.

Case 2. *The resolvent difference between a mixed realization and the Dirichlet realization.* This is much more nonsmooth than Case 1, since the boundary condition jumps from order 0 to order 1 at the interface between Σ_- and Σ_+. The realization A_{χ,Σ_+} is far from regular; its domain is at best in $H^{\frac{3}{2}-\varepsilon}(\Omega)$, $\varepsilon > 0$ (Shamir'68).

Let $X = H_0^{-\frac{1}{2}}(\Sigma_+)$, the subspace of distributions in $H^{-\frac{1}{2}}(\Sigma)$ supported in Σ_+. Its dual space is $X^* = H^{\frac{1}{2}}(\Sigma_+^\circ) = r^+ H^{\frac{1}{2}}(\Sigma)$, where r^+ denotes restriction to Σ_+°. Let e^+ denote extension from Σ_+° to Σ by 0 on Σ_-. For an operator Q over Σ, denote the truncation $r^+ Q e^+ = Q_+$. Let V be the subspace of $Z = \ker A_{\max}$ that is mapped to X by γ_0, and denote the restriction of γ_0 to V by γ_V, so $\gamma_V \colon V \xrightarrow{\sim} X$.

In J. Math. An. Appl.'11 we show:

Theorem 7. *For the mixed problem there is an operator L mapping $D(L) \subset X$ to X^* such that the Kreĭn resolvent formula holds:*

$$A_{\chi,\Sigma_+}^{-1} - A_\gamma^{-1} = \mathrm{i}_V \gamma_V^{-1} L^{-1} (\gamma_V^{-1})^* \operatorname{pr}_V, \quad (7)$$

Here L acts like $-P_{\gamma,\chi,+}$ and has $D(L) \subset H_0^{1-\varepsilon}(\Sigma_+)$.

NB! L^{-1} acts like $-(P_{\gamma,\chi,+})^{-1}$, which NOT the same as $-P_{\chi,\gamma,+}$.

Tools for the spectral analysis: Since the 70's, a Weyl-type spectral asymptotic formula is known to hold for a classical ψdo P of negative order $-t$ on a closed m-dimensional manifold Σ:

$$s_j(P)j^{t/m} \to c(P)^{t/m} \text{ for } j \to \infty, \text{ with}$$
$$c(P) = \frac{1}{m(2\pi)^m} \int_\Sigma \int_{|\xi|=1} |p^0(x,\xi)|^{m/t} \, d\omega dx. \quad (9)$$

Extensions:
Birman-Solomyak'77: (9) extends to symbols homogeneous of degree $-t$ in ξ and just continuous in ξ on $\{|\xi| = 1\}$, and somewhat better than continuous in x (depending on m).
G JST'11: (9) extends to products of truncated ψdo's $P_{1,+}P_{2,+}\dots P_{r,+}$ where the P_k are classical ψdo's of order $-t_k < 0$; for the constant, $|p_1^0 p_2^0 \dots p_r^0|$ is then integrated over Σ_+.

Perturbation: Let $P' = P + R$, where $s_j(R) = o(j^{-t/m})$, then also $s_j(P')j^{t/m} \to c(P)^{t/m}$ for $j \to \infty$.

We want to find the eigenvalue asymptotics of the operator

$$G = \mathrm{i}_V \gamma_V^{-1} L^{-1} (\gamma_V^{-1})^* \,\mathrm{pr}_V \,.$$

The operator is compact nonnegative so that the s-numbers $s_j(G)$ are eigenvalues $\mu_j(G)$. By the general rule $\mu_j(B_1 B_2) = \mu_j(B_2 B_1)$,

$$\mu_j(G) = \mu_j(L^{-1} (\gamma_V^{-1})^* \,\mathrm{pr}_V \,\mathrm{i}_V \gamma_V^{-1}) = \mu_j(L^{-1} (\gamma_V^{-1})^* \gamma_V^{-1}). \quad (8)$$

Here we can show that with $P_1 = K_\gamma^* K_\gamma$, elliptic ψdo of order -1,

$$(\gamma_V^{-1})^* \gamma_V^{-1} = r^+ K_\gamma^* K_\gamma e^+ = P_{1,+} \text{ on } L_2(\Sigma_+).$$

This reduces the problem to studying $\mu_j(L^{-1} P_{1,+})$.
To understand $L^{-1} = -(P_{\gamma,\chi,+})^{-1}$, restrict the attention to $A = -\Delta + a_0(x)$. In the half-space constant-coefficient case where $\Sigma = \mathbb{R}^{n-1}$, $\Sigma_\pm = \overline{\mathbb{R}}^{n-1}_\pm$, $A = -\Delta + \alpha^2$ with $\alpha > 0$, $\chi = \nu$, we have with $|\xi', \alpha| = (|\xi'|^2 + \alpha^2)^{1/2}$,

$$P_{\gamma,\nu} = -\,\mathrm{Op}(|\xi', \alpha|) = -\mathcal{F}^{-1} |\xi', \alpha| \mathcal{F}.$$

Now some detailed calculations from Eskin's book '81 show that

$$-(P_{\gamma,\nu,+})^{-1}v = (\Lambda_+)_+(\Lambda_-)_+v,$$

for $v \in L_2(\mathbb{R}^{n-1}_+)$, where $\Lambda_\pm = \mathrm{Op}\big((|\xi'',\alpha| \pm i\xi_{n-1})^{-1/2}\big)$, $\xi'' = (\xi_1, \dots, \xi_{n-2})$; nonstandard ψdo's of order $-\frac{1}{2}$.
A difficulty here is to handle the truncation between Λ_+ and Λ_-, which is not covered by Laptev's estimates. But Birman-Solomyak'77 is useful in a proof that $(\Lambda_+)_+(\Lambda_-)_+$ differs from $-P_{\nu,\gamma,+}$ by an operator with a better spectral behavior, over compact pieces of $\mathbb{R}^{n-1}$.
Coordinate changes are delicate (not quite covered by Eskin's book).
The end result is that the eigenvalues behave principally like the eigenvalues of a composition $P_{2,+}P_{\nu,\gamma,+}P_{2,+}$ of truncated ψdo's of negative order, $P_2 = P_1^{1/2}$. By the result for truncated ψdo's:
Theorem 8. *When $A = -\Delta + a_0(x)$,*

$$\mu_j(A^{-1}_{\chi,\Sigma_+} - A^{-1}_\gamma)j^{2/(n-1)} \to C_+^{2/(n-1)} \text{ for } j \to \infty,$$

$$C_+ = \tfrac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma_+} \int_{|\xi|=1} |p^0(x,\xi)p_1^0(x,\xi)|^{(n-1)/2}\, d\omega dx'.$$

Since p^0 and p_1^0 are independent of x' when $A = -\Delta + a_0(x)$, we have in fact, with a universal constant c_n,

$$C_+ = c_n \operatorname{area}(\Sigma_+).$$

NB! The spectral asymptotics detects the area of Σ_+!

E.B. Davies has remarked that since both C_+ and $A^{-1}_{\chi,\Sigma_+} - A^{-1}_\gamma$ increase with Σ_+, one can extend the result to nonsmooth interfaces $\partial\Sigma_+$, by approximation from inside and outside by smooth surfaces.

Asymptotic estimates with remainders are more delicate; for these we would need an extension of Laptev's results to the occurring nonstandard ψdo's.

Nonsmooth domains.

Kreĭn resolvent formulas have been established in the following cases:

• Posilicano-Raimondi'09 treat second-order selfadjoint cases on $C^{1,1}$-domains,

• G'08 treats second-order nonselfadjoint cases on $C^{1,1}$-domains,

• Gesztesy-Mitrea'08 treat Δ on $C^{3/2+\varepsilon}$-domains,

• Gesztesy-Mitrea'11 treat Δ on quasi-convex Lipschitz domains,

• Abels-G-Wood (preprint) treat second-order nonselfadjoint cases on $B^{3/2}_{p,2}$-domains by pseudodifferential methods (this includes $C^{3/2+\varepsilon}$-domains).

Upper spectral estimates are in selfadjoint cases covered by Birman'62 to a large extent.
Asymptotic spectral estimates do not seem to have been worked out yet. We expect that the results of Birman-Solomyak'77 will be useful also in these questions.