Boundary problems for fractional Laplacians and other fractional-order operators

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Introduction

The fractional Laplacian $(-\Delta)^a$ on $\mathbb{R}^n$, $0 < a < 1$, is currently of great interest in probability, finance, mathematical physics and differential geometry. In particular, it enters in nonlinear equations.
One way to describe it is as an integral operator with a convolution kernel:

$$(-\Delta)^a u(x) = c_{n,a} PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|y|^{n+2a}}\, dy.$$

Another way to describe it is by use of the Fourier transform $\mathcal{F}$:

$$\mathcal{F}u = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx;\ \text{ then}$$
$$(-\Delta)^a u = \operatorname{Op}(|\xi|^{2a})u = \mathcal{F}^{-1}(|\xi|^{2a}\hat{u}(\xi)).$$

This shows that it is a pseudodifferential operator (ps.d.o.) of order $2a$.

One of the difficulties with the operator is that it is *nonlocal*, in contrast to differential operators. This is problematic when one wants to study it on a bounded set; what is really meant? Here comes one of the interpretations:

For a bounded open set $\Omega \subset \mathbb{R}^n$ with some smoothness, we consider the problem

$$(-\Delta)^a u = f \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega};$$

this is called the homogeneous Dirichlet problem.

It is known that there is a solvability for $f \in L_2(\Omega)$, by a variational construction; here $u \in \dot{H}^a(\overline{\Omega})$ (the functions in $H^a(\mathbb{R}^n)$ with support in $\overline{\Omega}$). What more can be said about the regularity of u?

Through the times, the results on the regularity of the solutions were somewhat sparse.

• Vishik, Eskin, Shamir 1960's. E.g., $u \in \dot{H}^{2a}(\overline{\Omega})$ when $a < \frac{1}{2}$.
• When f and Ω are C^∞, there is some analysis of the behavior of solutions at $\partial\Omega$ by Eskin '81, Hörmander '85, Bennish '93, Chkadua and Duduchava '01.

Recent activity:

• Ros-Oton and Serra (JMPA '14) showed by potential theoretic and integral operator methods, when Ω is $C^{1,1}$, that for small $\alpha > 0$,

$$f \in L_\infty(\Omega) \implies u \in d^a C^\alpha(\overline{\Omega}), \qquad d(x) = \operatorname{dist}(x, \partial\Omega).$$

Moreover, $u \in C^a(\overline{\Omega})$. Lifted to $\alpha = a - \varepsilon$ later. They gave further results recently in low-regularity situations.

• G (Adv.Math. '15) presented a new systematic theory of ps.d.o. boundary problems covering $(-\Delta)^a$, using unpublished ideas of Hörmander. Further developed in Anal.PDE '14. When Ω is C^∞,

$$f \in L_\infty(\Omega) \implies u \in d^a C^a(\overline{\Omega}), \quad a \neq \tfrac{1}{2},$$

$$f \in C^t(\overline{\Omega}) \implies u \in d^a C^{a+t}(\overline{\Omega}), \text{ for } t > 0,\ a+t, 2a+t \notin \mathbb{N},$$

$$f \in C^\infty(\overline{\Omega}) \iff u \in d^a C^\infty(\overline{\Omega}). \qquad \text{(Hörmander book'85)}$$

In the excepted cases there is a correction $-\varepsilon$ in the Hölder exponent.

Optimal in the case of smooth Ω. The factor d^a is necessarily there.

The regularity study was a prerequisite for the proof of an integration by parts formula by Ros-Oton and Serra ARMA'14:

$$2\int_\Omega (x \cdot \nabla u)\,(-\Delta)^a u\,dx = (2a-n)\int_\Omega u(-\Delta)^a u\,dx \\ + \Gamma(1+a)^2 \int_{\partial\Omega} x \cdot \nu\,\gamma_0(d^{-a}u)^2\,d\sigma;$$

here $\nu(x)$ is the interior normal to $\partial\Omega$ at $x \in \partial\Omega$, and γ_0 denotes taking the boundary value from inside Ω.

This leads to a Pohozaev formula for solutions of the Dirichlet problem with nonlinear right-hand side $f(u)$, useful in uniqueness questions.

The formula has been generalized to a larger class of positive translation-invariant integral operators with homogeneous convolution kernels, by R.-O. and S. jointly with Valdinoci (arXiv Feb.'15).

We have very recently (arXiv Nov.'15) found out how to extend the formula to a corresponding class of pseudodifferential operators that are moreover allowed to be x-dependent (not translation invariant); here we assume smoothness of Ω and of the x-dependence.

In the lectures I will try to give you the background and mechanisms for these results.

Plan:

1. The pseudodifferential calculus on $\mathbb{R}^n$.
2. The model Dirichlet problem on $\mathbb{R}^n_+$.
3. General Dirichlet problems on sets Ω.
4. Nonhomogeneous boundary conditions.
5. Integration by parts and a Pohozaev formula.

1. The pseudodifferential calculus on $\mathbb{R}^n$

Pseudodifferential operators were introduced in the 1960's as a generalization of singular integral operators (Calderon, Zygmund, Seeley, Giraud, Mikhlin . . . , and particularly Kohn, Nirenberg, Hörmander.)

It generalizes the use of the Fourier transformation to x-dependent operators. Recall that the Fourier transform

$$\mathcal{F}u = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx;$$

is bijective on $\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty \mid |x^\alpha D^\beta u| \le C, \text{ all } \alpha, \beta\}$ (the Schwartz space), and extends to a bijection on $L_2(\mathbb{R}^n)$, with a similar inverse.

We use multi-index notation, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $D^\alpha = (-i)^{|\alpha|} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

A very important property is that $\mathcal{F}$ sends the *differential operator* D^α over into the *multiplication operator* ξ^α. Imitating this idea, when we have a function $p(\xi)$, the **pseudodifferential operator** P with *symbol* p is the operator

$$Pu = \mathcal{F}^{-1}\big(p(\xi)\hat{u}(\xi)\big) = \mathrm{Op}(p)u.$$

For example,

$$\Delta = \mathrm{Op}(-|\xi|^2), \quad (-\Delta)^a = \mathrm{Op}(|\xi|^{2a}).$$

This is simple and easy when the symbols only depend on ξ, for example,

$$\mathrm{Op}(p(\xi))\,\mathrm{Op}(q(\xi)) = \mathrm{Op}(p(\xi)q(\xi)). \qquad (*)$$

But now we extend the definition to x-dependent symbols $p(x,\xi)$,

$$(Pu)(x) = \mathcal{F}^{-1}\big(p(x,\xi)\hat{u}(\xi)\big) = \mathrm{Op}(p(x,\xi))u,$$

and this is more delicate. We no longer have $(*)$, but, with a good choice of the symbol classes (given below),

$$\mathrm{Op}(p(x,\xi))\,\mathrm{Op}(q(x,\xi)) - \mathrm{Op}(p(x,\xi)q(x,\xi)) \text{ is of } \textit{lower order.} \qquad (**)$$

Definition. S^m is the space of *symbols* $p(x,\xi)$ *of order* m, satisfying

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C\langle\xi\rangle^{m-|\alpha|}, \text{ for } x,\xi \in \mathbb{R}^n,$$

all α, β. p is called *classical*, when there is a sequence of symbols $p_j(x,\xi)$, homogeneous of degree $m-j$ in ξ for $|\xi| \ge 1$, such that

$$|D_x^\beta D_\xi^\alpha (p - \sum\nolimits_{j<M} p_j)| \le C\langle\xi\rangle^{m-|a|-M},$$

for all α, β, M. Here $\langle\xi\rangle = (|\xi|^2+1)^{\frac{1}{2}}$.

When p is of order m, $P = \mathrm{Op}(p(x,\xi))$ is continuous:

$$P\colon H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n), \text{ all } s \in \mathbb{R};$$

recall that $H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle\xi\rangle^s \hat{u} \in L_2(\mathbb{R}^n)\}$.

So P has better continuity properties, the lower m is. In particular, when $p \in S^{-\infty} = \bigcap_m S^m$, P maps any H^s into C^∞; is a "smoothing operator".

For compositions, one has that $PQ = R + \mathcal{S}$, where $\mathcal{S}$ is smoothing, and the symbol of R is the *Leibniz product*:

$$p(x,\xi)\#q(x,\xi) \sim \sum\nolimits_{\alpha\in\mathbb{N}_0^n} \partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi)/\alpha!,$$

modulo symbols in $S^{-\infty}$.

In particular, when p and q are of orders m_1, m_2,

$$p\#q = p\cdot q + s_1, \text{ where } s_1 \sim \sum\nolimits_{|\alpha|\geq 1} \partial_\xi^\alpha p D_x^\alpha q/\alpha!;$$

here s_1 is of order $m_1 + m_2 - 1$. This shows $(**)$.

Technical difficulties in the theory of ps.d.o.s:

1) Series expansions are usually not convergent, but hold in an *asymptotic sense*. (Like Taylor expansions of non-analytic functions.)

2) Some formulas just hold *modulo smoothing operators*.

3) Integrals need interpretation as *oscillatory integrals*.

Somewhat sophisticated, but can be made useful with some care. Introduction e.g. in G'09, Springer GTM book.

When p is classical with expansion $p \sim \sum_{j \in \mathbb{N}_0} p_j$, the first term p_0 (homogeneous in ξ of degree m) is called the *principal symbol*. P and p are said to be *elliptic*, when $p_0(x,\xi) \neq 0$ for $|\xi| \geq 1$; *strongly elliptic* when $\operatorname{Re} p_0(x,\xi) > 0$ for $|\xi| \geq 1$. Many properties are governed by the principal symbol, and a study of the x-independent operator $\operatorname{Op}(p_0(x_0,\xi))$ for fixed x_0 is usually a pilot project for the study of the full operator.

The symbol will be said to be **even**, when

$$\partial_x^\beta \partial_\xi^\alpha p_j(x,-\xi) = (-1)^{-j-|\alpha|} \partial_x^\beta \partial_\xi^\alpha p_j(x,\xi) \text{ for } |\xi| \geq 1, \text{ all } j, \alpha, \beta.$$

We shall consider operators satisfying: *P is a classical strongly elliptic ps.d.o. of order $2a$ with even symbol, $0 < a < 1$.*

This includes $(-\Delta)^a$ (there is a trick to handle that the symbol $|\xi|^{2a}$ is not quite smooth at 0), and $A(x,D)^a$, where $A(x,D)$ is a second-order strongly elliptic differential operator with smooth coefficients. In particular, $(-\Delta + m^2)^a$.

2. The model Dirichlet problem on $\mathbb{R}^n_+$

Now consider a subset Ω of $\mathbb{R}^n$, either a bounded open C^∞-subset, or $\mathbb{R}^n_+ = \{x = (x', x_n) \mid x' \in \mathbb{R}^{n-1}, x_n > 0\}$. In either case we denote by r^+ the restriction operator from $\mathbb{R}^n$ to Ω, and by e^+ the extension-by-zero operator

$$e^+ f(x) = \begin{cases} f(x), \text{ when } x \in \Omega, \\ 0 \text{ when } x \in \complement\Omega (= \mathbb{R}^n \setminus \Omega). \end{cases}$$

The analogous operators for $\complement\overline{\Omega}$ are called r^- and e^-.

The properties of classical ps.d.o.s are preserved under C^∞-coordinate changes, and there is an invariant definition of the principal symbol (as a section of the cotangent bundle). Therefore situations with arbitrary smooth subsets Ω can usually be reduced to situations with $\mathbb{R}^n_+$, by use of local coordinates and partitions of unity.

We shall now discuss the Dirichlet problem, aiming to see why the factor $d(x)^a$, $d(x) = \text{dist}(x, \partial\Omega)$, enters in the description of solutions.

A model case.

Consider $P = (-\Delta + 1)^a$ on $\mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$. Here $d(x) = x_n$.

The symbol of $(-\Delta+1)^a$ equals $(|\xi|^2+1)^a = \langle\xi\rangle^{2a}$. It has a factorization

$$\langle\xi\rangle^{2a} = (\langle\xi'\rangle^2+\xi_n^2)^a = (\langle\xi'\rangle - i\xi_n)^a(\langle\xi'\rangle + i\xi_n)^a.$$

Set $\chi^t_\pm(\xi',\xi_n) = (\langle\xi'\rangle \pm i\xi_n)^t$, and define $\Xi^t_\pm = \mathrm{Op}(\chi^t_\pm)$. Then

$$(-\Delta+1)^a = \Xi^a_- \, \Xi^a_+.$$

The powers z^t, $z\in\mathbb{C}$, are defined to be real for $z>0$. The operators $\Xi^t_\pm$ are a kind of generalized ps.d.o.s; their symbols satisfy

$$|\partial^{\alpha'}_{\xi'}\partial^{a_n}_{\xi_n}\chi^t_\pm(\xi',\xi_n)| \le C(\langle\xi'\rangle^{t-|\alpha'|} + \langle\xi\rangle^{t-|\alpha'|})\langle\xi\rangle^{-\alpha_n},$$

but not the full set of ps.d.o. estimates in ξ.
They are invertible, with $(\Xi^t_\pm)^{-1} = \Xi^{-t}_\pm$. Moreover, $(\Xi^t_+)^* = \Xi^t_-$.
With $\mathbb{C}_\pm = \{z\in\mathbb{C} \mid \operatorname{Im} z \gtrless 0\}$, we observe that $\chi^t_+(\xi',\xi_n)$ *extends holomorphically into* $\mathbb{C}_-$ *as a function of* ξ_n (and $\chi^t_-(\xi',\xi_n)$ extends holomorphically into $\mathbb{C}_+$ as a function of ξ_n).
Hence, by the Paley-Wiener theorem, $\tilde\chi^t_+(\xi',x_n) = \mathcal{F}^{-1}_{\xi_n\to x_n}\chi^t_+(\xi',\xi_n)$ is supported for $x_n\ge 0$, and therefore the operator Ξ^t_+ (which in the x_n-direction is a convolution with $\tilde\chi^t_+(\xi',x_n)$) *preserves support in* $\overline{\mathbb{R}}^n_+$.

Introduce the two families of Sobolev spaces

$$\overline{H}^s(\Omega) = r^+ H^s(\mathbb{R}^n),$$
$$\dot{H}^s(\overline{\Omega}) = \{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\},$$

(used presently with $\Omega = \mathbb{R}^n_+$). Here $\overline{H}^s(\Omega)$ and $\dot{H}^{-s}(\overline{\Omega})$ are dual spaces of one another. (The notation with dots and lines stems from works of Hörmander.)

Because of the support preserving property of Ξ^t_+,

$$\Xi^t_+ : \dot{H}^s(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} \dot{H}^{s-t}(\overline{\mathbb{R}}^n_+), \text{ all } s,$$

with inverse Ξ^{-t}_+.

This mapping Ξ^t_+ has the adjoint $r^+\Xi^t_- e^+$, mapping in the dual scale of spaces:

$$r^+\Xi^t_- e^+ : \overline{H}^{t-s}(\mathbb{R}^n_+) \xrightarrow{\sim} \overline{H}^{-s}(\mathbb{R}^n_+),$$

with inverse $r^+\Xi^{-t}_- e^+$.

Now we show how to solve the model Dirichlet problem

$$r^+(-\Delta+1)^a u = f \text{ on } \mathbb{R}^n_+, \quad \operatorname{supp} u \subset \overline{\mathbb{R}}^n_+. \qquad (1)$$

Say, f is given in $\overline{H}^t(\mathbb{R}^n_+)$ for some $t \geq 0$, and u is sought in $\dot{H}^a(\overline{\mathbb{R}}^n_+)$. By the factorization,

$$r^+(-\Delta+1)^a u = r^+\Xi^a_-\Xi^a_+ u = r^+\Xi^a_-(e^+r^+ + e^-r^-)\Xi^a_+ u = r^+\Xi^a_- e^+r^+\Xi^a_+ u,$$

since $r^-\Xi^a_+ u = 0$. Then since $(r^+\Xi^a_- e^+)^{-1} = r^+\Xi^{-a}_- e^+$ on the $\overline{H}^s$-scales, (1) can be reduced to

$$r^+\Xi^a_+ u = r^+\Xi^{-a}_- e^+ f, \quad \operatorname{supp} u \subset \overline{\mathbb{R}}^n_+, \qquad (2)$$

where $r^+\Xi^{-a}_- e^+ f \in \overline{H}^{t+a}(\mathbb{R}^n_+)$. By a moment's thought, this has the unique solution

$$u = \Xi^{-a}_+ e^+(r^+\Xi^{-a}_- e^+ f).$$

Thus (1) has the unique solution u, and it lies in

$$\Xi^{-a}_+(e^+\overline{H}^{t+a}(\mathbb{R}^n_+)) \equiv H^{a(t+2a)}(\overline{\mathbb{R}}^n_+), \textit{ Hörmander's space.}$$

What is this space? If $t + a < \frac{1}{2}$, it is simply $\dot{H}^{t+2a}(\overline{\mathbb{R}}^n_+)$. But when $t + a > \frac{1}{2}$, $e^+\overline{H}^{t+a}(\mathbb{R}^n_+)$ has a *jump* at $x_n = 0$; this gives rise to a singularity at $x_n = 0$ when Ξ^{-a}_+ is applied. We can calculate:

Special Fourier transformation formula: For $s > -1$, $\sigma > 0$ (e.g. $\langle\xi'\rangle$),

$$\mathcal{F}^{-1}_{\xi_n\to x_n}(\sigma + i\xi_n)^{-s-1} = \tfrac{1}{\Gamma(s+1)} e^+ r^+ x_n^s e^{-\sigma x_n}. \qquad (\star)$$

(Note that $e^+ r^+$ corresponds to multiplying with the Heaviside function.) To study the example $\Xi_+^{-a}(e^+\overline{H}^1(\mathbb{R}^n_+)) = H^{a(1+a)}(\overline{\mathbb{R}}^n_+)$, let $v \in \overline{H}^1(\mathbb{R}^n_+)$. Let $\varphi = \gamma_0 v$ (the boundary value), it is in $H^{\frac12}(\mathbb{R}^{n-1})$. Let

$$v_0 = \mathcal{F}^{-1}_{\xi\to x}(\hat\varphi(\xi')(\langle\xi'\rangle + i\xi_n)^{-1}) = \mathcal{F}^{-1}_{\xi'\to x'}(\hat\varphi(\xi')e^+ r^+ e^{-\langle\xi'\rangle x_n}) \in e^+\overline{H}^1(\mathbb{R}^n_+).$$

Then v and v_0 have the same boundary value φ (from $\mathbb{R}^n_+$), and the rest $v' = e^+ v - v_0$ belongs to $\dot H^1(\overline{\mathbb{R}}^n_+)$.

When we apply Ξ_+^{-a}, v' is mapped into $\dot H^{1+a}(\overline{\mathbb{R}}^n_+)$. For $v_0 \in e^+\overline{H}^1(\mathbb{R}^n_+)$ we find, using $(\star)$:

$$\Xi_+^{-a} v_0 = \mathcal{F}^{-1}(\hat\varphi(\xi')(\langle\xi'\rangle + i\xi_n)^{-1-a}) = \tfrac{1}{\Gamma(a+1)} x_n^a v_0.$$

Hence

$$H^{a(1+a)}(\overline{\mathbb{R}}^n_+) \subset e^+ x_n^a \overline{H}^1(\mathbb{R}^n_+) + \dot H^{1+a}(\overline{\mathbb{R}}^n_+).$$

A similar analysis gives more generally, when $t + a > \frac12$,

$$H^{a(t+2a)}(\overline{\mathbb{R}}^n_+) \subset e^+ x_n^a \overline{H}^{t+a}(\mathbb{R}^n_+) + \dot H^{t+2a\,(-\varepsilon)}(\overline{\mathbb{R}}^n_+),$$

$(-\varepsilon)$ active if $t + a - \frac12 \in \mathbb{N}$. Moreover, $H^{a(t+2a)}(\overline{\mathbb{R}}^n_+) \subset H^{t+2a}_{\mathrm{loc}}(\mathbb{R}^n_+)$.

To sum up, we have found:

Theorem 1. *The Dirichlet problem for* $(-\Delta+1)^a$ *on* $\mathbb{R}^n_+$ *is uniquely solvable; here when* $t \geq 0$,

$$f \in \overline{H}^t(\mathbb{R}^n_+) \implies u \in H^{a(t+2a)}(\overline{\mathbb{R}}^n_+),$$

$$\textit{with } H^{a(t+2a)}(\overline{\mathbb{R}}^n_+) \begin{cases} = \dot{H}^{t+2a}(\overline{\mathbb{R}}^n_+) \textit{ if } -\frac{1}{2} < t+a < \frac{1}{2}, \\ \subset e^+ x_n^a \overline{H}^{t+a}(\mathbb{R}^n_+) + \dot{H}^{t+2a\,(-\varepsilon)}(\overline{\mathbb{R}}^n_+) \textit{ if } t+a > \frac{1}{2}. \end{cases}$$

The case $-\frac{1}{2} < t+a < \frac{1}{2}$ is covered in Eskin's book '81.

As a corollary for $t \to \infty$, we see that the solutions satisfy:

$f \in C^\infty(\overline{\mathbb{R}}^n_+)$ with bounded support $\implies u \in e^+ x_n^a C^\infty(\overline{\mathbb{R}}^n_+)$.

This was just a model case, and there remains to make the ideas work for general operators P and general domains Ω.

Remark. For $P = (-\Delta)^a$ we are relying on the complex factorization $|\xi|^{2a} = (|\xi'| - i\xi_n)^a (|\xi'| + i\xi_n)^a$ in the model case of operators defined in terms of ξ_n. In contrast, the basic argument of Ros-Oton and Serra relies on the factorization $|\xi|^{2a} = |\xi|^a |\xi|^a$, as real symbols, with a different complicated analysis of the operators $\mathrm{Op}(|\xi|^a)$ in terms of ξ_n.

Now we shall treat a general ps.d.o. P satisfying our assumptions, and a general $\Omega \subset \mathbb{R}^n_+$. First there is an auxiliary theorem (G CPDE'90):

Theorem 2. 1° *For any $t \in \mathbb{R}$ there exist pseudodifferential operators $\Lambda^t_\pm$ of order t with symbols $\lambda^t_\pm(\xi', \xi_n)$, homogeneous of degree t for $|\xi| \geq 1$ and invertible, with $\overline{\lambda^t_+} = \lambda^t_-$, such that λ^t_+ extends holomorphically into $\mathbb{C}_-$ as a function of ξ_n, and λ^t_- extends holomorphically into $\mathbb{C}_+$ as a function of ξ_n .*
2° *Moreover, when Ω is smooth bounded $\subset \mathbb{R}^n$, there exist ps.d.o. families $\Lambda^{(t)}_\pm$ for $t \in \mathbb{R}$, elliptic of order t and invertible with inverse $\Lambda^{(-t)}_\pm$, such that the symbols in local coordinates at $\partial\Omega$ are like those of $\Lambda^t_\pm$.*

The first family of operators $\Lambda^t_\pm$ then have all the nice mapping properties relative to $\mathbb{R}^n_+ \subset \mathbb{R}^n$ that the $\Xi^t_\pm$ had, with the advantage of being true ps.d.o.s so that the general calculus applies to them.
The second family of operators $\Lambda^{(t)}_\pm$ have the mapping properties relative to the embedding $\Omega \subset \mathbb{R}^n$, for all s:

$$\Lambda^{(t)}_+ : \dot{H}^s(\overline{\Omega}) \xrightarrow{\sim} \dot{H}^{s-t}(\overline{\Omega}), \quad r^+\Lambda^{(t)}_- e^+ : \overline{H}^s(\Omega) \xrightarrow{\sim} \overline{H}^{s-t}(\Omega),$$

with inverses $\Lambda^{(-t)}_+$ resp. $r^+\Lambda^{(-t)}_- e^+$.

Consider the Dirichlet problem on Ω for a given P,

$$r^+ Pu = f \text{ on } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega}. \qquad (3)$$

Define

$$Q = \Lambda_-^{(-a)} P \Lambda_+^{(-a)}.$$

Q is of order 0, and thanks to the evenness of P it satisfies the so-called *0-transmission property:*

$$\partial_x^\beta \partial_\xi^\alpha q_j(x, -\nu(x)) = (-1)^{-j-|\alpha|} \partial_x^\beta \partial_\xi^\alpha q_j(x, \nu(x))$$

at the boundary points; $x \in \partial\Omega$ and $\nu(x)$ is the interior normal at x.

For operators like Q, there has existed a boundary value calculus for many years, the *Boutet de Monvel calculus*, initiated in BdM Acta'71 and further developed in e.g. G Duke'84, CPDE'90, book'96.

In the present case we can moreover show:

Theorem 3. *Under our hypotheses on P, the principal symbol q_0 of Q has a factorization, in local coordinates at $\partial\Omega$ (with normal direction ξ_n):*

$$q_0(x, \xi', \xi_n) = q_0^-(x, \xi', \xi_n) q_0^+(x, \xi', \xi_n),$$

where $q_0^\pm$ are homogeneous of degree 0, and q_0^+ extends holomorphically into $\mathbb{C}_-$ as a function of ξ_n, q_0^- extends holomorphically into $\mathbb{C}_+$ as a function of ξ_n.

Using this, we get from the Boutet de Monvel calculus for Q:

Proposition 4. *The operator r^+Qe^+ maps*

$$r^+Qe^+ \colon \overline{H}^s(\Omega) \to \overline{H}^s(\Omega), \text{ for all } s > -\tfrac{1}{2},$$

as a Fredholm operator with smooth kernel and cokernel.
It has the regularity property: When $v \in \overline{H}^\sigma(\Omega)$ with $\sigma > -\frac{1}{2}$, $r^+Qe^+v \in \overline{H}^s(\Omega) \implies v \in \overline{H}^s(\Omega)$.

This leads to solvability of the Dirichlet problem for P, by use of the support-preservation properties of the families $\Lambda_\pm^{(t)}$, as follows.
The question is: For given $f \in \overline{H}^t(\Omega)$, find $u \in \dot{H}^a(\overline{\Omega})$ such that (3) holds. To find this, insert $P = \Lambda_-^{(a)} Q \Lambda_+^{(a)}$ in (3), then

$$r^+P = r^+\Lambda_-^{(a)} Q \Lambda_+^{(a)} = r^+\Lambda_-^{(a)}(e^+r^+ + e^-r^-)Q\Lambda_+^{(a)} = (r^+\Lambda_-^{(a)}e^+)r^+Q\Lambda_+^{(a)},$$

since $r^+\Lambda_-^{(a)}e^- = 0$, and (3) can be written as

$$(r^+\Lambda_-^{(a)}e^+)r^+Q\Lambda_+^{(a)}u = f. \qquad (4)$$

Define

$$g = (r^+\Lambda_-^{(-a)}e^+)f \in \overline{H}^{t+a}(\Omega), \quad \text{then } f = (r^+\Lambda_-^{(a)}e^+)g;$$
$$v = r^+\Lambda_+^{(a)}u \in L_2(\Omega), \quad \text{then } u = \Lambda_+^{(-a)}e^+v.$$

Hereby the problem reduces to: Find $v \in L_2(\Omega)$ such that

$$(r^+ Q e^+) v = g, \text{ given in } \overline{H}^{t+a}(\Omega). \qquad (5)$$

Call $(r^+ Q e^+) = Q_+$ for short. By Proposition 4, Q_+ has a parametrix (almost-inverse) $\widetilde{Q_+}$, such that

$$Q_+ \widetilde{Q_+} = I + \mathcal{S}_1, \quad \widetilde{Q_+} Q_+ = I + \mathcal{S}_2,$$

with smoothing operators $\mathcal{S}_1$ and $\mathcal{S}_2$ of finite rank.

Hereby $v = \widetilde{Q_+} g$ solves (5) in a Fredholm sense. Then, by insertion,

$$u = \Lambda_+^{(-a)} e^+ v = \Lambda_+^{(-a)} e^+ \widetilde{Q_+} (r^+ \Lambda_-^{(-a)} e^+) f$$

solves the original problem (3) in a Fredholm sense.

Introduce the Hörmander spaces:

$$H^{a(s)}(\overline{\Omega}) = \Lambda_+^{(-a)} e^+ \overline{H}^{s-a}(\Omega).$$

We can show using the model case discussed earlier, that they satisfy:

$$H^{a(s)}(\overline{\Omega}) \begin{cases} = \dot{H}^s(\overline{\Omega}) \text{ if } -\frac{1}{2} < s - a < \frac{1}{2}, \\ \subset e^+ d^a \overline{H}^{s-a}(\Omega) + \dot{H}^{s\,(-\varepsilon)}(\overline{\Omega}) \text{ if } s - a > \frac{1}{2}, \end{cases}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$, $(-\varepsilon)$ is active if $s - a - \frac{1}{2} \in \mathbb{N}$.

Then we finally conclude:

Theorem 5. *r^+P is Fredholm*

$$r^+P\colon H^{a(t+2a)}(\overline{\Omega}) \to \overline{H}^t(\Omega), \text{ for all } t > -\tfrac{1}{2}.$$

There is the regularity property: When $u \in \dot{H}^\sigma(\overline{\Omega})$ with $\sigma > -\frac{1}{2}$, then $r^+Pu \in \overline{H}^t(\Omega) \implies u \in H^{a(t+2a)}(\overline{\Omega})$.

The theorem can be extended to other scales of function spaces, e.g.:

• The Sobolev scale H^s_p, $1 < p < \infty$, and the Besov and Triebel-Lizorkin scales B^s_{pq}, F^s_{pq}, for large sets of indices.

• The Hölder-Zygmund scale $C^s_* = B^s_{\infty,\infty}$ (here C^s_* equals the Hölder space C^s when $s \in \mathbb{R}_+ \setminus \mathbb{N}$).

Thus for example:

$$f \in C^t(\overline{\Omega}) \implies u \in e^+d^aC^{a+t}(\overline{\Omega}), \text{ for } t \geq 0,\ a+t, 2a+t \notin \mathbb{N},$$
$$f \in C^\infty(\overline{\Omega}) \implies u \in e^+d^aC^\infty(\overline{\Omega});$$

with ε subtracted from the Hölder exponent in the excepted cases.

4. Nonhomogeneous boundary conditions

Up to now we have studied the socalled homogeneous Dirichlet problem.

Question: Is there a nontrivial "Dirichlet boundary value" on $\partial\Omega$, such that the problem represents the case where that value is zero?

To give a simple explanation, consider the C^∞-situation. Define

$$\mathcal{E}_a(\overline{\Omega}) = e^+(d^a C^\infty(\overline{\Omega})).$$

One can show that $\mathcal{E}_a(\overline{\Omega})$ is dense in $H^{a(s)}(\overline{\Omega})$ for all $s > a - \frac{1}{2}$, and that

$$H^{a(s)}(\overline{\Omega}) \text{ converges to } \mathcal{E}_a(\overline{\Omega}) \text{ for } s \to \infty.$$

Take $\Omega = \mathbb{R}^n_+$. When $u \in \mathcal{E}_a(\overline{\mathbb{R}}^n_+)$ then $u = x_n^a v$ with $v \in e^+ C^\infty(\overline{\mathbb{R}}^n_+)$, and

$$u(x) = x_n^a v(x',0) + x_n^{a+1}\partial_n v(x',0) + \tfrac{1}{2} x_n^{a+2}\partial_n^2 v(x',0) + \dots \text{ for } x_n > 0,$$

by Taylor expansion of v.
When $u \in \mathcal{E}_{a-1}(\overline{\mathbb{R}}^n_+)$ then $u = x_n^{a-1} w$ with $w \in e^+ C^\infty(\overline{\mathbb{R}}^n_+)$, and

$$u(x) = x_n^{a-1} w(x',0) + x_n^a \partial_n w(x',0) + \tfrac{1}{2} x_n^{a+1}\partial_n^2 w(x',0) + \dots .$$

We see that $\mathcal{E}_{a-1}(\overline{\mathbb{R}}^n_+) \supset \mathcal{E}_a(\overline{\mathbb{R}}^n_+)$, differing just by the term $x_n^{a-1} w(x',0)$. Hereby

$$u \in \mathcal{E}_{a-1}(\overline{\mathbb{R}}^n_+) \text{ is in } \mathcal{E}_a(\overline{\mathbb{R}}^n_+) \iff \big(w(x',0) =\big)\, \gamma_0(x_n^{1-a} u) = 0.$$

This also holds for the H-scales: $H^{(a-1)(s)}(\overline{\mathbb{R}}^n_+) \supset H^{a(s)}(\overline{\mathbb{R}}^n_+)$, and

$$u \in H^{(a-1)(s)}(\overline{\mathbb{R}}^n_+) \text{ is in } H^{a(s)}(\overline{\mathbb{R}}^n_+) \iff \gamma_0(x_n^{1-a} u) = 0.$$

For general Ω replace x_n by $d(x)$; then we likewise have $H^{(a-1)(s)}(\overline{\Omega}) \supset H^{a(s)}(\overline{\Omega})$, and

$$u \in H^{(a-1)(s)}(\overline{\Omega}) \text{ is in } H^{a(s)}(\overline{\Omega}) \iff \gamma_0(d^{1-a} u) = 0.$$

Definition. *For $u \in H^{(a-1)(s)}(\overline{\Omega})$, the Dirichlet resp. Neumann boundary values are defined as:*

$$\gamma_0(d^{1-a}u) \in H^{s-a+\frac12}(\partial\Omega), \ \textit{when } s > a - \tfrac12 \ \textit{resp.}$$
$$\gamma_1(d^{1-a}u) = \gamma_0(\partial_\nu(d^{1-a}u)) \in H^{s-a-\frac12}(\partial\Omega), \ \textit{when } s > a + \tfrac12.$$

One can check that when u is in the smaller space $H^{a(s)}(\overline{\Omega})$ (i.e., when $\gamma_0(d^{1-a}u)$ vanishes), then $\gamma_1(d^{1-a}u) = \gamma_0(d^{-a}u)$.
All this allows to set up nonhomogeneous Dirichlet resp. Neumann problems, where u is sought in $H^{(a-1)(s)}(\overline{\Omega})$ (for suitable s):

$$\text{Dirichlet problem}: \begin{cases} r^+Pu = f \text{ on } \Omega, \\ \operatorname{supp} u \subset \overline{\Omega}, \\ \gamma_0(d^{1-a}u) = \varphi \text{ on } \partial\Omega. \end{cases}$$

$$\text{Neumann problem}: \begin{cases} r^+Pu = f \text{ on } \Omega, \\ \operatorname{supp} u \subset \overline{\Omega}, \\ \gamma_1(d^{1-a}u) = \psi \text{ on } \partial\Omega. \end{cases}$$

The data are given with $f \in \overline{H}^{s-2a}(\Omega)$, $\varphi \in H^{s-a+\frac{1}{2}}(\partial\Omega)$, $\psi \in H^{s-a-\frac{1}{2}}(\partial\Omega)$.
One finds that the nonhomogeneous Dirichlet problem is Fredholm solvable in these spaces.
The Neumann problem is Fredholm solvable at least when P has principal symbol $\sim |\xi|^{2a}$. (G in A&PDE '14)

Note here that when $u \in \mathcal{E}_{a-1}(\overline{\Omega})$, say, is such that $\gamma_0(d^{1-a}u) = \varphi \neq 0$ at $x_0 \in \partial\Omega$, then $u(x)$ blows up like d^{a-1} when $x \to x_0$. The solutions with nonzero Dirichlet data are "large" in this sense (also observed by Abatangelo, arXiv '13).

These Dirichlet and Neumann problems have *local* boundary conditions.

There exist other boundary value problems with good solvability. For example a certain "nonlocal Neumann problem" for $(-\Delta)^a$ (Dipierro, Ros-Oton and Valdinoci, arXiv'14), where a homogeneous condition linking the value on Ω with the value on $\complement\Omega$ is imposed.

There also exist some quite different operators associated with $(-\Delta)^a$, namely the "spectral fractional Laplacians". They are the a'th powers (defined by spectral theory) of the standard realizations $-\Delta_{\mathrm{Dir}}$ and $-\Delta_{\mathrm{Neu}}$ on Ω. Here the acting operator is different from $r^+(-\Delta)^a$ for $0 < a < 1$, and the domains differ in general.
For example, the domain of the spectral fractional Dirichlet operator differs from that of our Dirichlet operator (for $f \in L_2(\Omega)$) when $a \geq \frac{1}{2}$.
Regularity studies in Caffarelli-Stinga AnnIHP'16 and G MathNach'16.

5. Integration by parts and a Pohozaev formula

Let us first focus on the reduction to the boundary of the expression

$$I \equiv \int_\Omega Pu\,\partial_j \bar u'\,dx + \int_\Omega \partial_j u\,\overline{P^* u'}\,dx.$$

Ros-Oton and Serra showed when $P = (-\Delta)^a$ (ARMA'14), and jointly with Valdinoci for more general x-independent selfadjoint positive homogeneous P (arXiv'15), that

$$I = \Gamma(a+1)^2 \int_{\partial\Omega} s_0(x)\nu_j(x)\gamma_0\big(\tfrac{u}{d^a}\big)\gamma_0\big(\tfrac{\bar u'}{d^a}\big)\,d\sigma,$$

when u, u' solve the Dirichlet problem (3) with $r^+Pu, r^+Pu' \in C^{0,1}(\overline{\Omega})$. Here $\nu_j(x)$ is the j'th component of the interior normal ν at $x \in \partial\Omega$, and $s_0(x) = p_0(x, \nu(x))$, where p_0 is the (principal) symbol of P. Ω is $C^{1,1}$. It is a striking formula, since it is exact and turns I into a *local* expression, and it is remarkable by pointing to the role of the boundary value $\gamma_0(\frac{u}{d^a})$; the Neumann value when the Dirichlet value is 0. It generalizes the easy formula for $a = 1$:

$$\int_\Omega [(-\Delta u)\,\partial_j \bar u' + \partial_j u(-\Delta \bar u')]\,dx = \int_{\partial\Omega} \nu_j\,\gamma_1 u\,\gamma_1 \bar u'\,d\sigma.$$

We set out to understand their formula in the context of ps.d.o.s, and to generalize it to cases where P is x-dependent and not necessarily selfadjoint, positive, homogeneous. We take Ω smooth. The result is:

Theorem 6. *Let P be a classical ps.d.o. of order $2a$ ($0<a<1$) with even symbol, elliptic avoiding a ray. Then for solutions u, u' of the Dirichlet problem* (3),

$$I = \Gamma(a+1)^2 \int_{\partial\Omega} s_0 \nu_j \gamma_0\big(\tfrac{u}{d^a}\big)\gamma_0\big(\tfrac{\bar u'}{d^a}\big)\, d\sigma + \int_\Omega [P,\partial_j] u\, \bar u'\, dx,$$

where $[P,\partial_j]$ is the commutator $P\partial_j - \partial_j P$.

It suffices that r^+Pu, r^+Pu' are in $C^{1-a+\varepsilon}(\overline\Omega)$ or $\overline{H}^{\frac12 - a+\varepsilon}(\Omega)$.

For u, u' of low regularity, the integrals in I are understood as Sobolev space dualities. "Elliptic avoiding a ray" means that $p_0(x,\xi) \in \mathbb{C}\setminus\{z = re^{i\theta} \mid r \ge 0\}$ for some θ. It holds e.g. when $\operatorname{Re} p_0 > 0$ for $\xi \ne 0$, strong ellipticity, which is usually assumed when you study heat equations $\partial_t u + Pu = 0$.

The **model case** is $(-\Delta+1)^a$ on $\mathbb{R}^n_+$, $j=n$. To fix the ideas, take $u, u' \in H^{a(1+a)}(\overline{\mathbb{R}}^n_+)$ (then $r^+Pu, r^+Pu' \in \overline{H}^{1-a}(\mathbb{R}^n_+)$). There is an auxiliary result:

Proposition 7. *Let $u, u' \in H^{a(a+1)}(\overline{\mathbb{R}}^n_+)$. Let $w = r^+\Xi^a_+ u$, $w' = r^+\Xi^a_+ u'$; they are in $\overline{H}^1(\mathbb{R}^n_+)$. Then*

$$\int_{\mathbb{R}^n_+} \Xi^a_- e^+ w\, \partial_n \bar{u}'\, dx = \int_{\mathbb{R}^{n-1}} \gamma_0 w\, \gamma_0 \bar{w}'\, dx' + \int_{\mathbb{R}^n_+} w\, \partial_n \bar{w}'\, dx.$$

In this nontrivial calculation it enters that $r^+\partial_n u' \in r^+\partial_n x_n^a \overline{H}^1(\mathbb{R}^n_+)$ $+\dot{H}^a(\overline{\mathbb{R}}^n_+)$, having an x_n^{a-1}-singularity at the boundary.
Since $r^+(-\Delta+1)^a u = r^+\Xi^a_- e^+ r^+\Xi^a_+ u = r^+\Xi^a_- e^+ w$, we find

$$\begin{aligned} I &\equiv \int_{\mathbb{R}^n_+} (-\Delta+1)^a u\, \partial_n \bar{u}'\, dx + \int_{\mathbb{R}^n_+} \partial_n u\, (-\Delta+1)^a \bar{u}'\, dx \\ &= \int_{\mathbb{R}^n_+} \Xi^a_- e^+ w\, \partial_n \bar{u}'\, dx + \int_{\mathbb{R}^n_+} \partial_n u\, \overline{\Xi^a_- e^+ w'}\, dx \\ &= 2\int_{\mathbb{R}^{n-1}} \gamma_0 w\, \gamma_0 \bar{w}'\, dx' + \int_{\mathbb{R}^n_+} (w\, \partial_n \bar{w}' + \partial_n w\, \bar{w}')\, dx \\ &= \int_{\mathbb{R}^{n-1}} \gamma_0 w\, \gamma_0 \bar{w}'\, dx'. \end{aligned}$$

One has moreover that $w = r^+\Xi^a_+ u \implies \gamma_0 w = \Gamma(a+1)\gamma_0(\frac{u}{x_n^a})$. This implies Theorem 6 in the model case.

For general P, still with $\Omega = \mathbb{R}^n_+$, a factorization is used:

$$P = \Lambda^a_- Q \Lambda^a_+ = \Lambda^a_- Q_0^- Q_0^+ \Lambda^a_+ + \mathcal{S},$$

where $Q_0^\pm$ are defined from the factors $q_0^\pm$ in the principal symbol of Q (Theorem 3). Here $Q_0^\pm$ are generalized ps.d.o.s of order 0, and $\mathcal{S}$ is a generalized ps.d.o. of order $2a - 1$.
For bounded smooth sets Ω one combines the result for $\mathbb{R}^n_+$ with calculations in local coordinates. Extensive details in arXiv:1511.03901.

Corollary 8. *For functions $u, u' \in H^{a(s)}(\overline{\Omega})$ ($s > a + \frac{1}{2}$) there holds*

$$\int_\Omega (Pu\,(x \cdot \nabla \bar{u}') + (x \cdot \nabla u)\,\overline{P^* u'})\,dx =$$

$$\Gamma(a+1)^2 \int_{\partial\Omega} (x \cdot \nu) s_0 \gamma_0(\tfrac{u}{d^a})\,\gamma_0(\tfrac{\bar{u}'}{d^a})\,d\sigma - n \int_\Omega Pu\,\bar{u}'\,dx + \int_\Omega [P, x \cdot \nabla] u\,\bar{u}'\,dx.$$

Here

$$[P, x \cdot \nabla] = P_1 - P_2, \quad P_1 = \mathrm{Op}(\xi \cdot \nabla_\xi p(x,\xi)), \quad P_2 = \mathrm{Op}(x \cdot \nabla_x p(x,\xi)).$$

P_2 is new, P_1 is new when p is not homogeneous in ξ.

This has implications for the *nonlinear Dirichlet problem*

$$r^+Pu = f(u), \quad \operatorname{supp} u \subset \overline{\Omega}, \qquad (6)$$

with $f(s) \in C^{0,1}(\mathbb{R})$. Let $F(t) = \int_0^t f(s)\,ds$.

Corollary 9. *Let P be selfadjoint. For the bounded real solutions of* (6) *there holds the Pohozaev formula:*

$$-2n\int_\Omega F(u)\,dx + n\int_\Omega f(u)\,u\,dx$$
$$= \Gamma(1+a)^2\int_{\partial\Omega}(x\cdot\nu)\,s_0\gamma_0(\tfrac{u}{d^a})^2\,d\sigma + \int_\Omega [P, x\cdot\nabla]u\,u\,dx.$$

In particular, when P is x-independent,

$$-2n\int_\Omega F(u)\,dx + n\int_\Omega f(u)\,u\,dx$$
$$= \Gamma(1+a)^2\int_{\partial\Omega}(x\cdot\nu)\,s_0\gamma_0(\tfrac{u}{d^a})^2\,d\sigma + \int_\Omega P_1 u\,u\,dx.$$

Let us end with two small applications.

Example 1. Let $P = (-\Delta + m^2)^a$. For the eigenvalue problem

$$r^+ Pu = \lambda u, \quad \operatorname{supp} u \subset \overline{\Omega},$$

any λ, one can derive that $\gamma_0(\frac{u}{d^a}) = 0$ implies $u \equiv 0$. Hence:
There are no eigenfunctions with vanishing of both Dirichlet and Neumann data.

Example 2. For $P = (-\Delta + m^2)^a$, consider the problem

$$r^+ Pu = \operatorname{sign} u|u|^r, \quad \operatorname{supp} u \subset \overline{\Omega},$$

on a star-shaped domain Ω.
When $r \geq \frac{n+2a}{n-2a}$, the *critical* and *supercritical* cases, one can derive that:
There are no nontrivial solutions.

The results in both examples follow from an analysis of the signs of terms in the symbols of P and P_1.

Applications of the formula in x-dependent cases will also need sign analyses of the various terms derived from the symbol of P. As far as I know, this is an open question.