

Analysis of invariants associated with spectral boundary problems for elliptic operators

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Introduction

Boundary conditions defined by pseudodifferential projections — also called spectral boundary conditions —

$$(1) \quad \Pi u|_{\partial X} = 0$$

for Dirac operators D were first used by Atiyah, Patodi and Singer in their important paper [APS1], which also introduced the eta invariant. Such problems have since then been studied in numerous other works (of which a section is found in the references to this survey). The operator $(D_\Pi)^* D_\Pi$ is a realization of the Laplace operator $D^* D$ with complementing projection boundary conditions

$$(2) \quad \Pi u|_{\partial X} = 0, \quad \Pi^\perp D u|_{\partial X} = 0.$$

More generally, one can consider a second order operator P of Laplace-type together with a boundary condition similar to (2); such problems have been studied by the author in [G6] (for the motivation in physics, see the introduction there and Vassiliev [V1, V2]).

In this survey paper we shall give an account of recent results concerning some of the basic geometric invariants associated with such operators. They are defined in analysis as coefficients in trace expansions for associated heat operators or resolvents, expanded in powers and logarithms of the time variable t or spectral variable λ ; they can also be defined from the pole structure of meromorphic extensions of associated zeta and eta functions.

We shall consider two basic questions: 1) What happens to the coefficients (or specific ones of them) when the boundary projection is changed? 2) What happens to the coefficients when the interior operator is changed?

In both cases, we focus particularly on the coefficients of logarithmic terms and the global coefficients “behind” them.

REMARK. The use of the terminology “spectral boundary condition”, earlier applied to boundary conditions defined by the positive eigenprojection of a first-order elliptic differential operator, is motivated here by the fact that *any* orthogonal pseudodifferential projection is the positive eigenprojection for a first-order elliptic pseudodifferential operator (see Theorem 1.2 below).

1. Pseudodifferential operators on closed manifolds

1.1 Trace expansions.

To prepare for the case of manifolds with boundary, we first recall some results for closed manifolds.

Consider an n -dimensional compact C^∞ manifold X without boundary. Let A be a classical (also called one-step polyhomogeneous) pseudodifferential operator (ψ do) of order $\nu \in \mathbb{R}$, acting on the sections of a C^∞ vector bundle E over X of dimension n_1 . Let P be a classical elliptic ψ do of positive integer order m , likewise acting in E and such that the principal symbol has no eigenvalues on \mathbb{R}_- . As shown by Grubb and Seeley in [GS1, Th. 2.7] for (1.1), with the transition to the essentially equivalent statements (1.2) and (1.3) accounted for e.g. in [GS2], a calculation in local coordinates gives trace expansions (we denote $\{0, 1, 2, \dots\} = \mathbb{N}$):

$$(1.1) \quad \mathrm{Tr}(A(P - \lambda)^{-N}) \sim \sum_{j \in \mathbb{N}} \tilde{c}_j (-\lambda)^{\frac{\nu+n-j}{m}-N} + \sum_{k \in \mathbb{N}} (\tilde{c}'_k \log(-\lambda) + \tilde{c}''_k) (-\lambda)^{-k-N},$$

$$(1.2) \quad \Gamma(s) \mathrm{Tr}(AP^{-s}) \sim \sum_{j \in \mathbb{N}} \frac{c_j}{s + \frac{j-\nu-n}{m}} - \frac{\mathrm{Tr}(A\Pi_0(P))}{s} + \sum_{k \in \mathbb{N}} \left(\frac{c'_k}{(s+k)^2} + \frac{c''_k}{s+k} \right),$$

$$(1.3) \quad \mathrm{Tr}(Ae^{-tP}) \sim \sum_{j \in \mathbb{N}} c_j t^{\frac{j-\nu-n}{m}} + \sum_{k \in \mathbb{N}} (-c'_k \log t + c''_k) t^k.$$

In (1.1), $N > \frac{\nu+n}{m}$ and $\lambda \rightarrow \infty$ on rays in an open subsector of \mathbb{C} containing \mathbb{R}_- . (1.2) means that $\Gamma(s) \mathrm{Tr}(AP^{-s})$, defined in a standard way for $\mathrm{Re} s > \frac{\nu+n}{m}$, extends meromorphically to \mathbb{C} with the pole structure indicated in the right hand side. Here $\Pi_0(P)$ is the orthogonal projection onto the nullspace $V_0(P)$ of P (on which P^{-s} is taken to be zero). (1.3) holds when the resolvent for large λ is defined for $|\arg \lambda - \pi| \leq \frac{\pi}{2} + \varepsilon$, some $\varepsilon > 0$; here $t \rightarrow 0+$ and the coefficients are *the same* as those in (1.2). There are universal nonzero proportionality factors linking the coefficients \tilde{c}_j and c_j , \tilde{c}'_k and c'_k , resp. \tilde{c}''_k and c''_k .

A remarkable feature is the presence of the series over k that appears when A and P are not differential operators. Each of the coefficients \tilde{c}_j and \tilde{c}'_k comes from a specific homogeneous term in the symbol of $A(P - \lambda)^{-N}$, whereas the coefficients \tilde{c}''_k depend on the full symbol. Thus the coefficients \tilde{c}_j and \tilde{c}'_k depend each on a finite set of homogeneous terms in the symbols of A and P ; we call such coefficients ‘locally determined’ (or ‘local’), while the \tilde{c}''_k are called ‘global’. When $\nu \notin \mathbb{Z}$, the \tilde{c}'_k vanish. When $\nu \in \mathbb{Z}$ and $(j - n - \nu)/m$ is an integer $k \geq 0$, both \tilde{c}_j and \tilde{c}'_k contribute to the power $(-\lambda)^{-k-N}$; their sum is independent of the choice of local coordinates, whereas the splitting in \tilde{c}_j and \tilde{c}'_k depends in a well-defined way on the symbol structure in local coordinates (see [GS1, Th. 2.1] or [G7, Th. 1.3]).

Particular attention has been paid to the first two coefficients \tilde{c}'_0 and \tilde{c}''_0 in the series over k . Here \tilde{c}'_0 is proportional to the *noncommutative residue* of A , as introduced by Wodzicki [W] and Guillemin [Gu]:

$$(1.4) \quad \text{res } A = m \cdot \text{Res}_{s=0} \text{Tr}(AP^{-s}) = m \cdot c'_0;$$

it is an integral of the $(-n)$ th order symbol of A . As for \tilde{c}''_0 , it is only the full coefficient of $(-\lambda)^{-N}$, called $C_0(A, P)$,

$$(1.5) \quad C_0(A, P) = \tilde{c}''_0 + \tilde{c}_{\nu+n},$$

that is determined from the operators A and P when ν is integer $\geq -n$; the partition in two terms depends on the choice of local coordinates. For convenience, we define $\tilde{c}_{\nu+n} = 0$ when $\nu \notin \mathbb{Z}$ or $\nu < -n$. Then we have in local coordinates:

$$(1.6) \quad \tilde{c}'_0 = c'_0, \quad \tilde{c}''_0 = c''_0, \quad \tilde{c}_{\nu+n} = c_{\nu+n}.$$

The coefficient $C_0(A, P)$ equals the canonical trace $\text{TR } A$ introduced by Kontsevich and Vishik [KV] when $\nu < -n$ (then it is simply $\text{Tr } A$), when $\nu \notin \mathbb{Z}$ (cf. also Lesch [L], [G7]), and when A and P have symbols of even-even parity (see below) and the dimension of the manifold is odd. The cases of [KV] are supplied by the case of even-odd symbols on manifolds of even dimension in [G7] in this volume, where we derive the results in all cases by resolvent expansion techniques.

We say (as in [G5], [G7]) that a classical ψ do Q of order $r \in \mathbb{Z}$ with symbol $q \sim \sum_{l \in \mathbb{N}} q_{r-l}(x, \xi)$ (q_{r-l} homogeneous of degree $r-l$ in ξ for $|\xi| \geq 1$) in a local coordinate system has **even-even** alternating parity (in short: is even-even), when the symbol terms with even (resp. odd) degree are even (resp. odd) in ξ :

$$(1.7) \quad \partial_x^\beta \partial_\xi^\alpha q_{r-l}(x, -\xi) = (-1)^{r-l-|\alpha|} \partial_x^\beta \partial_\xi^\alpha q_{r-l}(x, \xi) \text{ for } |\xi| \geq 1, \text{ all } \alpha, \beta.$$

The operator (or symbol) is said to have **even-odd** alternating parity in the reversed situation where the symbol terms with even (resp. odd) degree are odd (resp. even) in ξ :

$$(1.8) \quad \partial_x^\beta \partial_\xi^\alpha q_{r-l}(x, -\xi) = (-1)^{r-l-1-|\alpha|} \partial_x^\beta \partial_\xi^\alpha q_{r-l}(x, \xi) \text{ for } |\xi| \geq 1.$$

[KV] calls the even-even symbols “odd-class”, studying them on odd-dimensional manifolds.

More generally, it can be shown that

$$(1.9) \quad C_0(A, P) - C_0(A, P') \text{ and } C_0([A, A'], P) \text{ are locally determined}$$

(we call $C_0(A, P)$ a quasi-trace then). In [G7] this is deduced from resolvent expansions. Using more functional calculus, involving properties of complex powers and logarithms of P , one can show residue formulas for these expressions; this was done for the first expression in Okikiolu [O] and [KV], and for both expressions in Melrose and Nistor [MN], with extensions to the b -calculus. In the latter work and in sequels to it, $C_0(A, P)$ is called a regularized trace of A and denoted e.g. $\widehat{\text{Tr}} A$, $\text{Tr}_P(A)$. It is studied from a physics point of view in Cardona, Ducourtioux, Magnot, Paycha [CDMP], [CDP], where it is called a weighted trace.

A generalization to boundary value problems (in the Boutet de Monvel calculus) has been worked out in a joint work with Schrohe [GSc], on the basis of resolvent considerations (the complex powers are not inside the calculus).

By division by $\Gamma(s)$ in (1.2), one finds the pole structure of the generalized zeta function $\zeta(A, P, s) = \text{Tr}(AP^{-s})$ (note that the double poles become simple). In particular, it has the Laurent expansion at zero, when $\Pi_0(P) = 0$:

$$(1.10) \quad \zeta(A, P, s) \sim \frac{1}{s} C_{-1}(A, P) + C_0(A, P) + \sum_{l \geq 1} C_l(A, P) s^l,$$

where $C_0(A, P)$ is defined above and $C_{-1}(A, P) = c'_0$. When $\Pi_0(P) \neq 0$, (1.10) holds with $C_0(A, P)$ replaced by $C_0(A, P) - \text{Tr}(A\Pi_0(P))$.

1.2 Special cases.

When $A = I$, we get the ordinary zeta function for P , $\zeta(P, s) = \zeta(I, P, s)$. Here $c'_0 = 0$ and c''_0 is locally determined; this also holds for $\zeta(A, P, s)$ when A is a differential operator, cf. [GS1, Th. 2.7]. So $C_0(A, P)$ is locally determined when A is a differential operator. Note that when $\Pi_0(P) = 0$, $\zeta(P, 0)$ is local because it equals $C_0(I, P)$. In the case $A = I$, the term $C_1(I, P)$ in (1.10) equals minus the “zeta-determinant” $\log \det P$. The higher Laurent coefficients $C_j(A, P)$ are described in [G7].

Another interesting special case is where $A = C$ and $P = |C|$ for a selfadjoint elliptic first-order ψ do C . Let

$$(1.11) \quad C' = C + \Pi_0(C).$$

Here we can define the eta function for C ,

$$(1.12) \quad \eta(C, s) = \text{Tr}(C|C|^{-s-1}) = \zeta(A_C, |C|, s), \text{ with } A_C = C|C'|^{-1}.$$

By (1.2), it has the pole structure (note that C is zero on the nullspace of $|C|$):

$$(1.13) \quad \eta(C, s) \sim \frac{1}{\Gamma(s)} \left[\sum_{j \in \mathbb{N}} \frac{d_j}{s + j - n} + \sum_{k \in \mathbb{N}} \left(\frac{d''_k}{(s + k)^2} + \frac{d''_k}{s + k} \right) \right],$$

where, as usual, the division by the gamma factor makes the double poles simple. In particular, the behavior at $s = 0$ is:

$$(1.14) \quad \eta(C, s) \sim \frac{1}{s} C_{-1}(A_C, |C|) + C_0(A_C, |C|) + O(s), \text{ with} \\ C_{-1}(A_C, |C|) = d'_0, \quad C_0(A_C, |C|) = d_n + d''_0.$$

We recall from [APS2] and Gilkey [Gi] the nontrivial result that the eta residue always vanishes:

THEOREM 1.1. *For any first-order selfadjoint elliptic ψ do C , $d'_0 = 0$ in (1.13)–(1.14).*

Thus $\eta(C, s)$ has a finite value $d_n + d''_0$ at 0. In [APS1], the *eta invariant* was defined as this value plus the kernel dimension of C ;

$$(1.15) \quad \eta_C = \eta(C, 0) + \dim V_0(C) = d_n + d''_0 + \dim V_0(C);$$

addition of other integers than $\dim V_0(C)$ can be relevant, see (2.33)–(2.34) below.

A third special case is where A is a pseudodifferential projection, i.e., a classical ψ do Π on X of order 0 satisfying $\Pi^2 = \Pi$. Here we have:

THEOREM 1.2.

(i) For any ψ do projection Π , $\text{res } \Pi = 0$; in other words, $C_{-1}(\Pi, P) = 0$ for arbitrary P .

(ii) If Π is the positive eigenprojection for a selfadjoint first-order elliptic ψ do C , then in the expansions (1.2), (1.10)ff. for $A = \Pi$, $P = |C|$,

$$(1.16) \quad C_0(\Pi, |C|) = \frac{1}{2}\eta(C, 0) + \frac{1}{2}\zeta(|C|, 0),$$

where $\zeta(|C|, 0) = C_0(I, |C|) - \dim V_0(C)$; here $C_0(I, |C|)$ is locally determined.

(iii) For any orthogonal ψ do projection Π , there exists a selfadjoint first-order elliptic invertible ψ do C such that Π is its positive eigenprojection.

PROOF. (i) was shown in [**W**, **Cor. 7.12**]. The statement there pertains to orthogonal projections, but for any general ψ do projection Π there is an orthogonal projection Π_{ort} with the same range and with $\Pi = R^{-1}\Pi_{\text{ort}}R$ for a suitable ψ do R (see e.g. [**G6**, **Prop. 4.8**] for details and references), so $\text{res } \Pi = \text{res } \Pi_{\text{ort}} = 0$.

In (ii), the positive eigenprojection means the orthogonal projection onto the closure of the span of eigensections with positive eigenvalues. Here $\Pi = \frac{1}{2}(C + |C|)|C'|^{-1}$ (with notation (1.11)), so in view of (1.14),

$$(1.17) \quad \begin{aligned} \zeta(\Pi, |C|, s) &= \frac{1}{2}\zeta(C|C'|^{-1}, |C|, s) + \frac{1}{2}\zeta(I - \Pi_0(C), |C|, s) \\ &= \frac{1}{2}\eta(C, s) + \frac{1}{2}\zeta(|C|, s), \end{aligned}$$

since $|C|^{-s}$ is taken to be 0 on $V_0(C)$. The residue at 0 vanishes in view of Theorem 1.1 and the information on zeta functions recalled in the beginning of this section. The value of the left-hand side at zero is $\zeta(\Pi, |C|, 0) = C_0(\Pi, |C|)$, since $\Pi \Pi_0(|C|) = 0$; and the value of the right-hand side is $\frac{1}{2}\eta(C, 0) + \frac{1}{2}\zeta(|C|, 0)$.

For (iii), choose an auxiliary first-order elliptic selfadjoint ψ do C_1 and set $C''' = \Pi C_1 \Pi - \Pi^\perp C_1 \Pi^\perp$; it can be modified on its nullspace to play the role of C (more details e.g. in [**G6**, **Prop. 4.8**]).

With (iii), we see that (i) is a rather direct consequence of Theorem 1.1 and the calculations for (ii); the argument of [**W**] for (i) is closely related to this. \square

If in Theorem 1.2 (ii), C is a differential operator, then $C|C'|^{-1}$ is even-odd of order 0 and C^2 is even-even, so if n is even, the canonical trace of $C|C'|^{-1}$ is defined according to [**G7**], and

$$(1.18) \quad \eta(C, 0) = \zeta(C|C'|^{-1}, |C|, 0) = \zeta(C|C'|^{-1}, C^2, 0) = \text{TR}(C|C'|^{-1}).$$

Then (1.16) takes the form

$$(1.19) \quad C_0(\Pi, |C|) = \frac{1}{2} \text{TR}(C|C'|^{-1}) + \frac{1}{2}\zeta(|C|, 0).$$

2. Spectral boundary problems for first- and second-order operators

2.1 First-order operators.

We now turn to boundary value problems for elliptic differential operators, with ψ do projections in the boundary condition. The new results in this and the following chapter are published in [G6].

For first-order elliptic operators in vector bundles, it is not always possible to get a solvable problem modulo finite dimensional spaces (a Fredholm realization) by imposing a local boundary condition. In their study of Dirac operators, Atiyah, Patodi and Singer [APS1] therefore considered a boundary condition with a ψ do projection defined from the tangential part of the interior operator. The general choices of projections leading to Fredholm realizations were described by Seeley in [S]; he called them *well-posed* boundary conditions (also for cases of higher order operators). The well-posed problems associated with first-order operators generalizing Dirac operators are described in detail in [G2].

Let us recall the set-up: We consider a first-order differential operator D from $C^\infty(X, E)$ to $C^\infty(X, E_1)$, where E and E_1 are Hermitian n_1 -dimensional vector bundles over a compact n -dimensional C^∞ manifold X with boundary $\partial X = X'$. E and E_1 have Hermitian metrics, and X has a smooth volume element, defining Hilbert space structures on the sections, $L_2(E)$, $L_2(E_1)$. The restrictions of E and E_1 to the boundary X' are denoted E' resp. E'_1 . A neighborhood of X' in X has the form $X_c = X' \times [0, c[$, and there E, E_1 are isomorphic to the pull-backs of E' resp. E'_1 . We let x_n denote the coordinate in $[0, c[$. $L_2(E')$ and $L_2(E'_1)$ are defined with respect to the volume element $v(x', 0)dx'$ on X' induced by the element $v(x', x_n)dx'dx_n$ on X .

D may always be written in the following form over X_c :

$$(2.1) \quad D = \sigma(\partial_{x_n} + A_1),$$

where σ is an isomorphism from $E|_{X_c}$ to $E_1|_{X_c}$, and A_1 for each x_n is an elliptic operator in the x' -variable. We shall study operators that resemble Dirac operators in their structure. D will be said to be of *product type*, when σ is independent of x_n and is unitary from E' to E'_1 , and $A_1 = A$ independent of x_n and formally selfadjoint; here the product measure $dx'dx_n$ is used on X_c . We say that D is of *non-product type* when σ is still independent of x_n and unitary, but the condition on A_1 is relaxed to:

$$(2.2) \quad A_1 = A + x_n A_{11} + A_{10},$$

where A is as above and the A_{1j} are smooth x_n -dependent tangential differential operators in x' of order $\leq j$. In [GS1], [G2], these operators of product type and of non-product type were said to be “of Dirac-type”. In some other works, that notation is reserved for operators that moreover satisfy

$$(2.3) \quad E = E_1, \quad \sigma^2 = -I, \quad \sigma A = -A\sigma,$$

D is formally selfadjoint on X and D^2 is principally scalar; such assumptions will here only be made in special cases.

Integration by parts shows that the formal adjoint D^* equals

$$(2.4) \quad D^* = (-\partial_{x_n} + A'_1)\sigma^*, \quad A'_1 = A + x_n A_{11}^* + A'_{10}, \quad \text{on } X_c,$$

for some morphism (zero-order operator) A'_{10} .

We shall also use the notation

$$(2.5) \quad D^0 = \sigma(\partial_{x_n} + A), \quad D^{0'} = (-\partial_{x_n} + A)\sigma^*;$$

these operators have a meaning on $X^0 = X' \times \overline{\mathbb{R}}_+$; and $D^{0'}$ is the formal adjoint of the operator D^0 going from $L_2(E^0)$ to $L_2(E_1^0)$, where E^0 and E_1^0 are the liftings of E' resp. E'_1 to X^0 , and the product measure is used.

Sometimes we consider, along with D satisfying (2.1)–(2.2), a product type operator D_0 on X , which is like D^0 on X_c and is extended to an elliptic operator on X (e.g. by patching it together with D).

By $V_{>}$, V_{\geq} , $V_{<}$ or V_{\leq} we denote the subspaces of $L^2(E')$ spanned by the eigenvectors of A corresponding to eigenvalues which are > 0 , ≥ 0 , < 0 , or ≤ 0 . (For precision one can write $V_{>}(A)$, etc.) V_0 is the nullspace of A . The corresponding projections are denoted $\Pi_{>}$, Π_{\geq} , etc. (note that $\Pi_{\geq} = \Pi_{>} + \Pi_0$ and $\Pi_{<} = I - \Pi_{\geq}$). They are pseudodifferential operators (ψ do's) of order 0; Π_0 has finite rank and is a ψ do of order $-\infty$. We set

$$(2.7) \quad |A| = (A^2)^{\frac{1}{2}}, \quad A' = A + \Pi_0, \quad \text{so that } |A'| = |A| + \Pi_0 \text{ and} \\ \Pi_{>} = \frac{1}{2} \frac{|A|+A}{|A'|} = \frac{1}{2} \frac{A}{|A|} + \frac{1}{2} - \frac{1}{2}\Pi_0.$$

Together with the equation $Du = f$, we consider a boundary condition

$$(2.8) \quad \Pi\gamma_0 u = 0,$$

where $\gamma_0 u = u|_{X'}$, defining the realizations D_{Π} and D_{Π}^0 , acting like D resp. D^0 and with domain

$$(2.9) \quad D(D_{\Pi}) \text{ resp. } D(D_{\Pi}^0) = \{ u \in H^1(E) \text{ resp. } H^1(E^0) \mid \Pi\gamma_0 u = 0 \};$$

we denote by $H^s(E)$ the Sobolev space of order s . [APS1] considered the case where $\Pi = \Pi_{\geq}(A)$, but increasingly general projections have been studied through the years. The most general case is where Π is a pseudodifferential projection that is *well-posed* with respect to D (cf. Seeley [S] or [G2]). This means that when we at each (x', ξ') in the cotangent sphere bundle of X' denote by $N^+(x', \xi') \subset \mathbb{C}^{n_1}$ the space of boundary values of null-solutions of the model operator (defined from the principal symbol d^0 of D),

$$(2.10) \quad N^+(x', \xi') = \{ z(0) \in \mathbb{C}^{n_1} \mid d^0(x', 0, \xi', D_{x_n})z(x_n) = 0, z(x_n) \in L_2(\mathbb{R}_+)^{n_1} \},$$

then the principal symbol $\pi^0(x', \xi')$ of Π maps $N^+(x', \xi')$ bijectively onto the range of $\pi^0(x', \xi')$ in \mathbb{C}^{n_1} . Equivalently, the model problem with homogeneous boundary condition is uniquely solvable in $L_2(\mathbb{R}_+)^{n_1}$.

EXAMPLE 2.1. For $|\xi'| \geq 1$, the space $N^+(x', \xi')$ equals the positive eigenspace for $d^0(x', \xi')$, i.e., the range of the principal symbol $\pi_{>}^0(x', \xi')$ of $\Pi_{>}(A)$, so $\Pi_{>}(A)$ is well-posed for D ; this is the case considered in [APS1]. Various finite rank perturbations of $\Pi_{>}(A)$ were considered in Douglas-Wojciechowsky [DW], Müller [M], Dai and Freed [DF], Grubb and Seeley [GS1, GS2]. Booss-Bavnbek and Wojciechowsky, cf. e.g. [BW], pointed to the interest of studying the exact Calderón projector which differs from $\Pi_{>}(A)$ by an operator of order $-\infty$ in the product

case; [Woj] treated quite general perturbations of order $-\infty$. Brüning and Lesch [BL] studied a principally different family of pseudodifferential projections that we shall here denote $\Pi(\theta)$, and finally [G2, G4] included all well-posed projections in the study.

2.2 Second-order operators.

As noted in [S], [G2] (and in the proof of Theorem 1.2 (i) above), it is no restriction to assume that Π is an orthogonal projection. In view of Green's formula

$$(2.11) \quad (Du, v)_X - (u, D^*v)_X = -(\sigma\gamma_0u, \gamma_0v)_{X'},$$

and elliptic regularity, the adjoint $(D_\Pi)^*$ is the realization of D^* defined by the boundary condition $\Pi^\perp\sigma^*\gamma_0v = 0$ (associated with the well-posed projection $\Pi' = \sigma\Pi^\perp\sigma^*$ for D^*). It follows that D^*D is of the form (on X_c):

$$(2.12) \quad P = -\partial_{x_n}^2 + P' + x_nP_2 + P_1,$$

with $P' = A^2$, the P_j being x_n -dependent differential operators of order j in $E|_{X_c}$, and that $D_\Pi^*D_\Pi$ is the realization of D^*D defined by the boundary condition

$$(2.13) \quad \Pi\gamma_0u = 0, \quad \Pi^\perp(\gamma_1u + A_1(0)\gamma_0u) = 0.$$

The study of spectral invariants of D_Π can to a large extent be based on the study of the second-order realization $D_\Pi^*D_\Pi$.

Recently, there has been an interest in studying similar second-order problems for their own sake, with a view to applications in brane theory (see Vassiliev [V1], [V2] and the introduction in [G6]), so let us look at a slightly more general situation:

P is an elliptic second-order partial differential operator in E , of the form (2.11) on X_c , with P' being an elliptic selfadjoint nonnegative second-order differential operator in E' (independent of x_n), and the P_j as described above. It is considered together with the boundary condition

$$(2.14) \quad Tu = 0, \quad \text{where } Tu = \{\Pi_1\gamma_0u, \Pi_2(\gamma_1u + B\gamma_0u)\},$$

where Π_1 is a ψ do projection operator and B is a first-order ψ do, both acting in E' , and $\Pi_2 = I - \Pi_1$. We denote by P_T the realization of P defined by this boundary condition; it acts like P and has the domain

$$(2.15) \quad D(P_T) = \{u \in H^2(E) \mid Tu = 0\}.$$

Here $D_\Pi^*D_\Pi = P_T$ in the special case where $P = D^*D$, $\Pi_1 = \Pi$ and is orthogonal, and $B = A_1(0)$.

We denote $\mathfrak{A} = (P' - \lambda)^{\frac{1}{2}} = (P' + \mu^2)^{\frac{1}{2}}$; here λ runs in $\mathbb{C} \setminus \overline{\mathbb{R}}_+$, and $\mu = (-\lambda)^{\frac{1}{2}}$ runs in $\{\mu \mid \operatorname{Re} \mu > 0\}$. The principal symbol is $\mathfrak{a}^0(x', \xi', \mu) = (p'^0(x', \xi') + \mu^2)^{\frac{1}{2}}$. We can assume that X is smoothly imbedded in a closed n -dimensional manifold \tilde{X} , provided with a vector bundle \tilde{E} such that $E = \tilde{E}|_X$, and such that P is defined in \tilde{E} with similar properties. Let us denote

$$(2.16) \quad \Gamma_\theta = \{\mu \in \mathbb{C} \setminus \{0\} \mid |\arg \mu| < \theta\}.$$

The following result was proved in [G6, Sect. 2]:

THEOREM 2.2. Assume (H1) and (H2):

(H1) The principal symbols of Π_1 and P' commute.

(H2) There is a $\theta \in]0, \frac{\pi}{2}]$ such that, with $b^h(x', \xi')$ and $\pi_i^h(x', \xi')$ denoting the strictly homogeneous principal symbols of B and the Π_i ,

$$(2.17) \quad \mathbf{a}^0 - \pi_2^h b^h \pi_2^h \text{ is invertible for } \xi' \in \mathbb{R}^n, \mu \in \Gamma_\theta \cup \{0\} \text{ with } (\xi', \mu) \neq (0, 0).$$

Then for each $\theta' \in]0, \theta[$ there is an $r = r(\theta') \geq 0$ such that when $|\arg \mu| \leq \theta'$ and $|\mu| \geq r$, $P_T + \mu^2 = P_T - \lambda$ is a bijection from $D(P_T)$ to $L_2(E)$ with inverse $(P_T - \lambda)^{-1} = R_T(\lambda)$;

$$(2.18) \quad R_T(\lambda) = Q(\lambda)_+ + G(\lambda),$$

where $Q(\lambda) = (P - \lambda)^{-1}$ on \tilde{X} and $G(\lambda)$ is a singular Green operator belonging to the parameter-dependent calculus of **[G3]**, with symbol in $\mathcal{S}^{0,0,-3}(\Gamma_\theta, \mathcal{S}_{++})$.

The original result also gave a more precise formula for $G(\lambda)$ in terms of the given operators; we shall consider some consequences of this later.

Using the general machinery of **[G3]** (or more specific calculations as in **[GS1]**, **[G2]**), one deduces the existence of the following general trace expansions:

THEOREM 2.3. Assumptions as in Theorem 2.2. Let F be a differential operator in E of order m and let $N > \frac{n+m}{2}$. Then $FR_T^N(\lambda)$ is trace-class and the trace has an expansion for $|\lambda| \rightarrow \infty$ with $\arg \lambda \in]\pi - 2\theta, \pi + 2\theta[$ (uniformly in closed subsectors):

$$(2.19) \quad \begin{aligned} & \text{Tr}(FR_T^N(\lambda)) \\ & \sim \sum_{-n \leq k < 0} \tilde{a}_k(F)(-\lambda)^{\frac{m-k}{2}-N} + \sum_{k \geq 0} (\tilde{a}'_k(F) \log(-\lambda) + \tilde{a}''_k(F))(-\lambda)^{\frac{m-k}{2}-N}, \end{aligned}$$

with locally determined coefficients \tilde{a}_k and \tilde{a}'_k . If m is odd, $\tilde{a}_{-n} = 0$.

Here, if F is tangential (differentiates only with respect to x') on X_c , the log-coefficients \tilde{a}'_k with $0 \leq k < m$ vanish, and the \tilde{a}''_k with $0 \leq k < m$ are locally determined.

In these formulas, the notation differs slightly from that of (1.1): we have collected the coefficients of each power of $-\lambda$ in one term, and adapted the indexation of all coefficients to the way k enters in the powers.

The expansion can be translated (as in **[G6]**) to a statement on the meromorphic extension of $\text{Tr}(FP_T^{-s})$; let us write the result in the cases where $F = \varphi$, a morphism (or “smearing function”), or $F = D_1$, a first-order differential operator:

$$(2.20) \quad \begin{aligned} & \Gamma(s) \text{Tr}(\varphi P_T^{-s}) \\ & \sim \sum_{-n \leq k < 0} \frac{a_k(\varphi)}{s + \frac{k}{2}} - \frac{\text{Tr}(\varphi \Pi_0(P_T))}{s} + \sum_{k=0}^{\infty} \left(\frac{a'_k(\varphi)}{(s + \frac{k}{2})^2} + \frac{a''_k(\varphi)}{s + \frac{k}{2}} \right); \\ & \Gamma(s) \text{Tr}(D_1 P_T^{-s}) \\ & \sim \sum_{-n < k < 0} \frac{a_k(D_1)}{s + \frac{k-1}{2}} - \frac{\text{Tr}(D_1 \Pi_0(P_T))}{s} + \sum_{k=0}^{\infty} \left(\frac{a'_k(D_1)}{(s + \frac{k-1}{2})^2} + \frac{a''_k(D_1)}{s + \frac{k-1}{2}} \right); \end{aligned}$$

the last expansion can also be written in the more customary form (with $s = \frac{s'+1}{2}$):

$$(2.21) \quad \text{Tr}(D_1 P_T^{-\frac{s'+1}{2}}) \\ \sim \frac{1}{\Gamma(\frac{s'+1}{2})} \left[\sum_{-n < k < 0} \frac{2a_k(D_1)}{s' + k} - \frac{2 \text{Tr}(D_1 \Pi_0(P_T))}{s' + 1} + \sum_{k=0}^{\infty} \left(\frac{4a'_k(D_1)}{(s' + k)^2} + \frac{2a''_k(D_1)}{s' + k} \right) \right].$$

(The formula (1.13) was simpler, since $|C'|^{-1}$ could be taken into the operator in front.) The coefficients are related to those in the resolvent expansions by universal nonzero proportionality factors; in particular,

$$(2.22) \quad \tilde{a}'_0(F) = a'_0(F); \quad \tilde{a}''_0(F) = a''_0(F).$$

If $\theta > \frac{\pi}{4}$ in (H2) so that the “heat operator” e^{-tP_T} exists, there is a trace expansion of $F e^{-tP_T}$ in the spirit of (1.3).

In (2.20), $\text{Tr}(\varphi P_T^{-s})$ is a generalized zeta function, also denoted $\zeta(\varphi, P_T, s)$, and $\text{Tr}(D_1 P_T^{-\frac{s+1}{2}})$ in (2.21) is somewhat like an eta function.

2.3 Consequences for first-order operators.

It is accounted for in [G6] how $P_T = D_{\Pi}^* D_{\Pi}$ enters as a special case in the above theorems, when the principal symbols of A^2 and Π commute (this holds in particular when A^2 is principally scalar, i.e., the principal symbol is scalar). Here the well-posedness of Π is in a certain sense equivalent with (H2). So, our basic assumptions on D and Π are as follows:

HYPOTHESIS (H3). *D is as described in (2.1)ff., of product type or non-product type. Π is an orthogonal ψ do projection in $L_2(E')$ that is well-posed for D , and the principal symbols of Π and A^2 commute.*

Then $\zeta(\varphi, P_T, s)$ equals $\zeta(\varphi, D_{\Pi}^* D_{\Pi}, s)$, also denoted $\zeta(D_{\Pi}^* D_{\Pi}, s)$ if $\varphi = I$. We can let $D_1 = \psi D$ for some morphism ψ from E_1 to E , defining

$$(2.23) \quad \text{Tr}(D_1 P_T^{-\frac{s+1}{2}}) = \text{Tr}(\psi D (D_{\Pi}^* D_{\Pi})^{-\frac{s+1}{2}}) = \eta(\psi, D_{\Pi}, s),$$

an eta function of D_{Π} . The existence of the above expansions for $\text{Tr}(\varphi (D_{\Pi}^* D_{\Pi} - \lambda)^{-N})$, $\text{Tr}(\psi D (D_{\Pi}^* D_{\Pi} - \lambda)^{-N})$, $\zeta(\varphi, D_{\Pi}^* D_{\Pi}, s)$ and $\eta(\psi, D_{\Pi}, s)$ is known from [G2] for general Π (with special choices of Π treated in earlier works); we repeat them here for clarity:

$$\begin{aligned} & \text{Tr}(\varphi (D_{\Pi}^* D_{\Pi} - \lambda)^{-N}) \\ & \sim \sum_{-n \leq k < 0} \tilde{a}_k(\varphi) (-\lambda)^{-\frac{k}{2} - N} + \sum_{k \geq 0} (\tilde{a}'_k(\varphi) \log(-\lambda) + \tilde{a}''_k(\varphi)) (-\lambda)^{-\frac{k}{2} - N}, \\ & \text{Tr}(D_1 (D_{\Pi}^* D_{\Pi} - \lambda)^{-N}) \\ & \sim \sum_{-n < k < 0} \tilde{a}_k(D_1) (-\lambda)^{\frac{1-k}{2} - N} + \sum_{k \geq 0} (\tilde{a}'_k(D_1) \log(-\lambda) + \tilde{a}''_k(D_1)) (-\lambda)^{\frac{1-k}{2} - N}, \end{aligned}$$

$$\begin{aligned}
& \Gamma(s) \operatorname{Tr}(\varphi(D_{\Pi}^* D_{\Pi})^{-s}) \\
& \sim \sum_{-n \leq k < 0} \frac{a_k(\varphi)}{s + \frac{k}{2}} - \frac{\operatorname{Tr}(\varphi \Pi_0(D_{\Pi}))}{s} + \sum_{k=0}^{\infty} \left(\frac{a'_k(\varphi)}{(s + \frac{k}{2})^2} + \frac{a''_k(\varphi)}{s + \frac{k}{2}} \right), \\
(2.24) \quad & \operatorname{Tr}(D_1(D_{\Pi}^* D_{\Pi})^{-\frac{s+1}{2}}) \\
& \sim \frac{1}{\Gamma(\frac{s+1}{2})} \left[\sum_{-n < k < 0} \frac{2a_k(D_1)}{s+k} + \sum_{k=0}^{\infty} \left(\frac{4a'_k(D_1)}{(s+k)^2} + \frac{2a''_k(D_1)}{s+k} \right) \right];
\end{aligned}$$

where we used that $D_1 = \psi D$ vanishes on $V_0(D_{\Pi})$.

The coefficient analysis for P_T described below will allow some new conclusions on these special cases also.

2.4 Analysis of the zero'th coefficients.

The formulas (2.20) and (2.21) show in particular how the zeta function and eta-like function behave near $s = 0$:

$$\begin{aligned}
(2.25) \quad & \operatorname{Tr}(\varphi P_T^{-s}) = a'_0(\varphi)s^{-1} + (a''_0(\varphi) - \operatorname{Tr}(\varphi \Pi_0(P_T)))s^0 + O(s), \\
& \operatorname{Tr}(D_1 P_T^{-\frac{s+1}{2}}) = \pi^{-\frac{1}{2}} 2a'_0(D_1)s^{-2} + \pi^{-\frac{1}{2}} 4a''_0(D_1)s^{-1} + O(1),
\end{aligned}$$

for $s \rightarrow 0$. An important question in this context is what we can say about the value, or the vanishing, of the coefficients in the Laurent expansions (2.25). It is here that we use the more precise description of the singular Green part $G(\lambda)$ of $R_T(\lambda)$ mentioned after the theorem.

We recall from the general theory of pseudodifferential boundary operators (ψ -do's) that when a singular Green operator $G = \operatorname{OPG}(g(x', \xi, \eta_n, \mu))$ is trace-class on \mathbb{R}_+^n , its trace equals the \mathbb{R}^{n-1} -trace of the ψ do on \mathbb{R}^{n-1} called the *normal trace* of G , $\operatorname{tr}_n G$; it is the operator with symbol

$$(\operatorname{tr}_n g)(x', \xi', \mu) = \int g(x', \xi', \xi_n, \xi_n, \mu) d\xi_n.$$

Then the trace expansion of G is obtained by applying the rules for the boundaryless manifold \mathbb{R}^{n-1} to $\operatorname{tr}_n G$. We also observe that

$$(2.26) \quad R_T^N = (Q^N)_+ + G^{(N)} = \frac{1}{(N-1)!} \partial_{\lambda}^{N-1} R_T = \frac{1}{(N-1)!} \partial_{\lambda}^{N-1} Q_+ + \frac{1}{(N-1)!} \partial_{\lambda}^{N-1} G,$$

where $G^{(N)} = \frac{1}{(N-1)!} \partial_{\lambda}^{N-1} G$ is a singular Green operator of class 0 with symbol in $\mathcal{S}^{0,0,-2N-1}(\Gamma_{\theta}, \mathcal{S}_{++})$.

The ψ do $\varphi Q^N(\lambda)_+$ has a trace expansion without logarithmic or nonlocal terms:

$$(2.27) \quad \operatorname{Tr} \varphi Q^N(\lambda)_+ \sim \sum_{k \geq -n, k+n \text{ even}} c_k(\varphi) (-\lambda)^{-\frac{k}{2}-N}.$$

The crucial information on $G(\lambda)$ we shall use is shown in [G6, Sect. 4]:

THEOREM 2.4. *Let φ be a morphism in E , independent of x_n on X_c (its restriction to X' likewise denoted φ). For $G^{(N)}(\lambda)$ from (2.26), cut down to X_c , we have that*

$$(2.28) \quad \operatorname{tr}_n \varphi G^{(N)}(\lambda) = \frac{1}{2} \varphi \Pi_2 (P' - \lambda)^{-N} + S_1(\lambda) + S_2(\lambda),$$

where S_1 and S_2 have trace expansions of the form

(2.29)

$$\begin{aligned} \mathrm{Tr}_{X'} S_1(\lambda) &\sim \sum_{k \geq 1-n} s_{1,k}(-\lambda)^{-\frac{k}{2}-N}, \\ \mathrm{Tr}_{X'} S_2(\lambda) &\sim \sum_{1-n \leq k \leq 0} s_{2,k}(-\lambda)^{-\frac{k}{2}-N} + \sum_{k \geq 1} (s'_{2,k} \log(-\lambda) + s''_{2,k})(-\lambda)^{-\frac{k}{2}-N}; \end{aligned}$$

here the $s_{i,k}$ and $s'_{i,k}$ are locally determined.

On interior coordinate patches, the trace of $G^{(N)}(\lambda)$ is $O(\lambda^{-M})$, for any M . Thus the only contributions to $a'_0(\varphi)$ and the only nonlocal contributions to $a''_0(\varphi)$ in (2.20), (2.25) come from the first term in the right-hand side of (2.28)! And this is a function whose expansion we know very well from the case of closed manifolds. In fact, $\frac{1}{2}\varphi\Pi_2(P' - \lambda)^{-N}$ has an expansion:

$$(2.30) \quad \begin{aligned} &\mathrm{Tr}\left(\frac{1}{2}\varphi\Pi_2(P' - \lambda)^{-N}\right) \\ &\sim \sum_{1-n \leq k < 0} \tilde{c}_k(-\lambda)^{\frac{k}{2}-N} + \sum_{k \geq 0, k \text{ even}} (\tilde{c}'_k \log(-\lambda) + \tilde{c}''_k)(-\lambda)^{-\frac{k}{2}-N}, \end{aligned}$$

by (1.1) (with $A = \frac{1}{2}\varphi\Pi_2$ of order 0, $\dim X' = n - 1$, and a regrouping and change in the indexation as indicated after (2.19)). Here, since $\mathrm{res} \varphi = 0$,

$$(2.31) \quad \tilde{c}'_0 = \frac{1}{2} \mathrm{res}\left(\frac{1}{2}\varphi\Pi_2\right) = -\frac{1}{4} \mathrm{res}(\varphi\Pi_1).$$

We conclude immediately, in view of Theorem 1.2 (i):

THEOREM 2.5. *Assumptions of Theorem 2.3. One has in general that $a'_0(I) = 0$, and $a'_0(\varphi) = -\frac{1}{4} \mathrm{res}(\varphi\Pi_1)$, in (2.25).*

In particular, $\zeta(D_\Pi^ D_\Pi, s)$ is regular at $s = 0$ for all choices of Π .*

Note that B does not enter in the value. Also, the result for $\zeta(D_\Pi^* D_\Pi, s)$ shows that the regularity at $s = 0$ is preserved under perturbations of Π by operators of order ≤ -1 , since such perturbations preserve well-posedness. (This was known earlier for perturbations of order $\leq -n$, [G2].)

REMARK 2.6. When φ is nontrivial, there is another sufficient condition for the vanishing of $a'_0(\varphi)$ (apart from the possibility that $\varphi\Pi_1$ could be a projection): When Π_1 is the positive eigenprojection for a selfadjoint differential operator C and $n - 1$ is even, then

$$\mathrm{res}(\varphi\Pi_1) = \mathrm{res}\left(\frac{1}{2}\varphi(I + C|C'|^{-1})\right) = \frac{1}{2} \mathrm{res}(\varphi C|C'|^{-1})$$

vanishes since $\varphi C|C'|^{-1}$ has even-odd parity, cf. (1.8). Also perturbations of Π_1 of order $\leq -n$ are allowed, since they do not interfere with the residue.

REMARK 2.7. Concerning \tilde{c}''_0 , we observe: When Π_1 is the positive eigenprojection for a first-order selfadjoint invertible elliptic ψ do C , then in (2.30) with $\varphi = I$,

$$(2.32) \quad \begin{aligned} \tilde{c}''_0 &= C_0\left(\frac{1}{2}\Pi_2, P'\right) = -\frac{1}{2}C_0(\Pi_1, P') + \text{local terms} \\ &= -\frac{1}{2}C_0(\Pi_1, |C|) + \text{local terms} = -\frac{1}{4}\eta(C, 0) + \text{local terms}, \end{aligned}$$

by (1.9), (1.16) and the fact that $\zeta(|C|, 0) = C_0(I, |C|)$ is local. The case where C has a nontrivial nullspace $V_0(C)$ is analyzed in [G6, Th. 4.9, Cor. 5.4–5.5]. Here it is found e.g. that if $V_0(C) = V'_0 \oplus V''_0$ (orthogonal decomposition) and $\Pi_1 = \Pi_{>}(C) + \Pi_{V'_0}$ (the latter denoting the orthogonal projection onto V'_0), then

$$(2.33) \quad \begin{aligned} \zeta(P_T, 0) &= -\frac{1}{4}\eta_{C, V'_0} - \dim V_0(P_T) + \text{local contributions}, \\ \text{with } \eta_{C, V'_0} &= \eta(C, 0) + \dim V'_0 - \dim V''_0. \end{aligned}$$

Moreover, one has for D_Π with such Π , that

$$(2.34) \quad \text{index } D_\Pi = -\frac{1}{2}\eta_{C, V'_0} + \text{local contributions},$$

which allows the remarkable observation that the “non-locality” depends only on the projection, not on the interior operator.

Similarly to Theorem 2.4, one has for $D_1G(\lambda)$:

THEOREM 2.8. *Let D_1 be a first-order differential operator on X , of the form $D_1 = \psi(\partial_{x_n} + B_1)$ on X_c , where B_1 is tangential and ψ is a morphism in E , independent of x_n . Then for $G^{(N)}(\lambda)$, cut down to X_c , we have that*

$$(2.35) \quad \text{tr}_n(D_1G^{(N)}(\lambda)) = -\frac{1}{2}\psi\Pi_2\frac{1}{(N-1)!}\partial_\lambda^{N-1}(P' - \lambda)^{-\frac{1}{2}} + \tilde{S}_1(\lambda) + \tilde{S}_2(\lambda),$$

where \tilde{S}_1 and \tilde{S}_2 have trace expansions of the form:

$$(2.36) \quad \begin{aligned} \text{Tr}_{X'} \tilde{S}_1(\lambda) &\sim \sum_{k \geq 1-n} \tilde{s}_{1,k}(-\lambda)^{\frac{1-k}{2}-N}, \\ \text{Tr}_{X'} \tilde{S}_2(\lambda) &\sim \sum_{1-n \leq k \leq 0} \tilde{s}_{2,k}(-\lambda)^{\frac{1-k}{2}-N} + \sum_{k \geq 1} (\tilde{s}'_{2,k} \log(-\lambda) + \tilde{s}''_{2,k})(-\lambda)^{\frac{1-k}{2}-N}; \end{aligned}$$

here the $\tilde{s}_{i,k}$ and $\tilde{s}'_{i,k}$ are locally determined.

An application of the calculus of [GS1] gives that

$$(2.37) \quad \begin{aligned} \text{Tr}_{X'}(-\frac{1}{2}\psi\Pi_2\frac{1}{(N-1)!}\partial_\lambda^{N-1}(P' - \lambda)^{-\frac{1}{2}}) \\ \sim \sum_{1-n \leq k < 0} \tilde{d}_k(\psi)(-\lambda)^{\frac{1-k}{2}-N} + \sum_{k \geq 0} (\tilde{d}'_k \log(-\lambda) + \tilde{d}''_k)(-\lambda)^{\frac{1-k}{2}-N}, \end{aligned}$$

where an analysis as in [G5, pf. of Th. 5.2] shows that

$$(2.38) \quad \tilde{d}'_0 = -\alpha \text{res}(\psi\Pi_2) = \alpha \text{res}(\psi\Pi_1),$$

with a universal nonzero factor α . This is the only contribution to $a'_0(D_1)$, so we conclude:

THEOREM 2.9. *Assumptions as in Theorem 2.7. In (2.25),*

$$(2.39) \quad a'_0(D_1) = \alpha \text{res}(\psi\Pi_1),$$

with a universal nonzero factor α . Here $a'_0(D_1)$ vanishes if $\psi\Pi_1$ is a projection.

In particular, $\eta(\psi, D_\Pi, s)$ has a simple pole at 0 if $\psi\sigma\Pi$ is a projection, e.g. if $\psi = \sigma^*$.

As in Remark 2.6, another sufficient condition for the vanishing of $a'_0(D_1)$ is that Π_1 is the positive eigenprojection of a differential operator and n is odd. Also perturbations of Π_1 by operators of order $\leq -n$ are allowed here.

3. Results under further symmetry conditions, perturbations of the boundary projection

3.1 Results for zeta functions.

In the following, we take Π_1 equal to an orthogonal pseudodifferential projection Π (so that $\Pi_2 = \Pi^\perp$). We consider the case where there exists a unitary morphism σ in E such that

$$(3.1) \quad \sigma^2 = -I, \quad \sigma P' = P' \sigma, \quad \Pi^\perp = -\sigma \Pi \sigma.$$

THEOREM 3.1. *Let (3.1) hold. Then*

$$(3.2) \quad \begin{aligned} \operatorname{Tr}_{X'}(\tfrac{1}{2}\Pi^\perp \frac{\partial_\lambda^{m-1}}{(m-1)!}(P' - \lambda)^{-1}) &= \tfrac{1}{4} \operatorname{Tr}_{X'}(\frac{\partial_\lambda^{m-1}}{(m-1)!}(P' - \lambda)^{-1}), \\ -\operatorname{Tr}_{X'}(\tfrac{1}{2}\sigma\Pi^\perp \frac{\partial_\lambda^{m-1}}{(m-1)!}(P' - \lambda)^{-\frac{1}{2}}) &= -\tfrac{1}{4} \operatorname{Tr}_{X'}(\sigma \frac{\partial_\lambda^{m-1}}{(m-1)!}(P' - \lambda)^{-\frac{1}{2}}). \end{aligned}$$

Let $\Pi_1 = \Pi$, $\Pi_2 = \Pi^\perp$. Then in (2.30) with $\varphi = I$, and in (2.37) with $\psi = \sigma$, all log-terms vanish and all the remaining coefficients are locally determined. In particular, \tilde{c}''_0 is locally determined (from the symbol of P'), \tilde{d}''_0 vanishes, and \tilde{d}''_0 is locally determined (from the symbol of P' and σ).

It follows that in (2.20), (2.25) with $\varphi = I$ and $D_1 = \sigma(\partial_{x_n} + B_1)$,

$$(3.3) \quad \begin{aligned} \tilde{a}''_0(D_1) &= a''_0(D_1) = 0, \\ \tilde{a}''_0(I), a''_0(I), \tilde{a}''_0(D_1) \text{ and } a''_0(D_1) &\text{ are locally determined.} \end{aligned}$$

In the proof, the identities in (3.2) are obtained by linearity and cyclic permutation in the trace formulas. Now since the operators $(P' - \lambda)^{-a} = (P' + \mu^2)^{-a}$, $a \in \mathbb{N}$ or $\mathbb{N} + \frac{1}{2}$, are strongly polyhomogeneous in (ξ', μ) , the traces have expansions without logs and with only local coefficients, by [GS1].

More precisely, $a''_0(I)$ in this case depends on the symbol of P , on σ , and on the first n strictly homogeneous terms in the symbols of Π and B ; and $a''_0(D_1)$ depends on the mentioned symbols together with that of B_1 .

We shall pursue this result for the traces arising from D_Π in cases with self-adjointness properties. Here we are interested in truly selfadjoint product cases as well as in nonproduct cases where D is principally selfadjoint at X' . Assume that $E = E_1$. Along with D we consider a product type operator D_0 , defined as after (2.5).

In addition to the requirements that σ be unitary and A be selfadjoint, we now assume (2.3), which means that D_0 is formally selfadjoint on X_c when this is provided with the “product” volume element $v(x', 0)dx'dx_n$. (If D_0 is selfadjoint on X , we call this a *selfadjoint product case*.)

When Π is an orthogonal projection in $L_2(E')$, it is well-posed for D if and only if it is so for D_0 . For D_0 in selfadjoint product cases, some choices of Π will lead to selfadjoint realizations $D_{0,\Pi}$, namely (in view of (2.11)) those for which

$$(3.4) \quad \Pi = -\sigma\Pi^\perp\sigma.$$

The properties (2.3) and (3.4) imply (3.1) with $P' = A^2$, so we can apply Theorem 3.1 to $D_\Pi^*D_\Pi$ (and $D_{0,\Pi}^2$).

As pointed out in the appendix A.1 of Douglas and Wojciechowski [DW], it follows from Ch. 17 (by Palais and Seeley) of the Palais seminar [P] that when (2.3) holds and n is odd, there exists a subspace L of $V_0(A)$ such that $\sigma L \perp L$ and $V_0(A) = L \oplus \sigma L$. Müller showed in [M] (cf. (1.6)ff. and Prop. 4.26 there) that such L can be found in any dimension. Denoting the orthogonal projection onto L by Π_L , we have that

$$(3.5) \quad \Pi_+ = \Pi_{>}(A) + \Pi_L$$

satisfies (3.4). The projections $\Pi(\theta)$ introduced by Brüning and Lesch [BL] likewise satisfy (3.4). These projections commute with A , so Hypothesis (H3) is satisfied.

Theorem 3.1 implies immediately:

COROLLARY 3.2. *When D and Π satisfy (H3) and in addition (2.3) and (3.4), then in (2.24) with $\varphi = I$,*

$$(3.6) \quad \tilde{a}_0''(I) (= a_0''(I)) \text{ is locally determined.}$$

This has an interesting consequence for perturbations of Π :

THEOREM 3.3. *In addition to the hypotheses of Corollary 3.2, assume that*

$$(3.7) \quad \Pi = \bar{\Pi} + \mathcal{S},$$

where $\bar{\Pi}$ is a fixed well-posed projection satisfying (3.4) and \mathcal{S} is of order $\leq -n$. ($\bar{\Pi}$ can in particular be taken as Π_+ in (3.5) or $\Pi(\theta)$ from [BL].)

Then the $\tilde{a}_0''(I)$ -terms (and $a_0''(I)$ -terms) in (2.24) for $D_\Pi^*D_\Pi$ and $D_{\bar{\Pi}}^*D_{\bar{\Pi}}$ are the same,

$$(3.8) \quad \tilde{a}_0''(I)(D_\Pi^*D_\Pi) = \tilde{a}_0''(I)(D_{\bar{\Pi}}^*D_{\bar{\Pi}}).$$

It follows that

$$(3.9) \quad \zeta(D_\Pi^*D_\Pi, 0) + \dim V_0(D_\Pi) = \zeta(D_{\bar{\Pi}}^*D_{\bar{\Pi}}, 0) + \dim V_0(D_{\bar{\Pi}});$$

in particular

$$(3.10) \quad \zeta(D_\Pi^*D_\Pi, 0) = \zeta(D_{\bar{\Pi}}^*D_{\bar{\Pi}}, 0) \pmod{\mathbb{Z}}.$$

The argument in the proof is that since these constants $\tilde{a}_0''(I)$ are locally determined, they depend, besides on D , only on the first n homogeneous terms in the symbols of the projections, and these are the same for Π and $\bar{\Pi}$.

The result of the theorem was shown in [Woj] for the case where $D = D_0$ in a selfadjoint product case, $\Pi = \Pi_+$ and \mathcal{S} is of order $-\infty$, assuming that D_{0,Π_+} is invertible. The hypothesis on invertibility was removed by Lee in the appendix of

[PW]; he shows moreover that $\zeta(D_{0,\Pi_+}^2, 0) + \dim V_0(D_{0,\Pi_+}) = 0$, so we conclude that

$$(3.11) \quad \zeta(D_{0,\Pi}^2, 0) + \dim V_0(D_{0,\Pi}) = 0, \text{ when } \Pi = \Pi_+ + \mathcal{S}.$$

3.2 Results for eta functions.

There are also such perturbation results for the eta function $\eta(D_\Pi, s)$, the meromorphic extension of $\text{Tr}(D(D_\Pi^* D_\Pi)^{-\frac{s+1}{2}})$, when (2.3) and (3.4) hold:

COROLLARY 3.4. *Assumptions of Corollary 3.2. In (2.24) with $D_1 = D$, one has that $\tilde{a}'_0(D) = a'_0(D) = 0$, and*

$$(3.12) \quad \tilde{a}''_0(D) (= a''_0(D)) \text{ is locally determined.}$$

In other words, the double pole of $\eta(D_\Pi, s)$ at 0 vanishes and the residue at 0 is locally determined.

We underline that the hypotheses, besides (2.3), (3.4), only contain requirements on principal symbols (namely the well-posedness of Π for D and the commutativity of the principal symbols of Π and A^2). So the result implies in particular that *the vanishing of the double pole of the eta function is invariant under perturbations of Π of order -1 (respecting (3.4))*. Earlier results have dealt with perturbations of Π_+ of order $-\infty$ [Woj], or perturbations of general Π of order $-n$ [G4].

Now consider the simple pole of $\eta(D_\Pi, s)$ at 0. Here we can generalize the result of Wojciechowski [Woj] on the regularity of the eta function after a perturbation of order $-\infty$, to perturbations of order $-n$ of general $\bar{\Pi}$:

THEOREM 3.5. *Assumptions of Theorem 3.3.*

In (2.24) with $D_1 = D$, the $\tilde{a}''_0(D)$ -terms (and $a''_0(D)$ -terms) for $D_{\bar{\Pi}}^ D_{\bar{\Pi}}$ and $D_\Pi^* D_\Pi$ are the same:*

$$(3.14) \quad \tilde{a}''_0(D)(D_{\bar{\Pi}}^* D_{\bar{\Pi}}) = \tilde{a}''_0(D)(D_\Pi^* D_\Pi);$$

in other words, $\text{Res}_{s=0} \eta(D_\Pi, s) = \text{Res}_{s=0} \eta(D_{\bar{\Pi}}, s)$.

In particular, if $\tilde{a}''_0(D)(D_{\bar{\Pi}}^ D_{\bar{\Pi}}) = 0$ (this holds for Π_+ and for certain $\Pi(\theta)$ if D equals D_0 in a selfadjoint product case), then $\tilde{a}''_0(D)(D_\Pi^* D_\Pi) = 0$, i.e., the eta function $\eta(D_\Pi, s)$ is regular at 0.*

The argument is again that the local determinedness implies that symbol changes below the first n terms in the projection do not enter in the constants.

The eta regularity for the case $\bar{\Pi} = \Pi_+$, D equal to D_0 and selfadjoint on X with product volume element on X_c , was shown in [DW91] under the assumptions n odd and D compatible; this was extended to general n and not necessarily compatible D in Müller [M]. It was shown for certain $\Pi(\theta)$ in [BL, Th. 3.12].

The result on the regularity of the eta function at $s = 0$ for $(-n)$ -order perturbations of the product case with $\bar{\Pi} = \Pi_+$ has been obtained independently by Lei [Le] at the same time as our result, by another analysis based on heat operator formulas.

We refer to [G6] for further discussions of \tilde{a}''_0 . There are some general results in [G4] on the behavior of the other coefficients under perturbations of Π .

4. Perturbation of the interior operator

4.1 General perturbation results.

In this chapter, we discuss the behavior of all the logarithmic and nonlocal coefficients in (2.24) (not just the leading ones) under perturbations of D . In particular, it is interesting to compare with the special situation of a product-type operator with Π equal to $\Pi_{>}(A)$ plus a projection in the nullspace of A . In that case, when n is even, there are no logarithmic terms at integer powers except possibly for $k = 0$ if $\varphi \neq I$ (so they occur only at half-integer powers $(-\lambda)^{-k-\frac{1}{2}}$); when n is odd, there are no logarithmic terms at all. (This is known from [GS2].) The results we now present are proved in [G5].

We have two kinds of results. One kind is a general statement in the non-product case, that when $D_1 - D_2$ vanishes to a certain order at ∂X , then the log-coefficients a'_k up to a certain index are preserved, and the a''_k -coefficients appearing together with them are perturbed only by local terms. The arguments can be used also when comparing the trace expansions for a general D with those for an associated D_0 of product type, under suitable hypotheses on the volume form.

The other kind of result is concerned with perturbations of the product-case by tangential operators commuting with A . Here it is found that in odd dimensions, there is still a vanishing of all the log-coefficients; on the other hand nontrivial log-coefficients can be expected at both integer and half-integer powers when n is even.

We fix the boundary projection; it can be a general well-posed projection in the first kind of result, and in the second kind it is taken equal to $\Pi_{\geq}(A)$. (Its perturbations follow the rules from Chapters 2 and 3, and from [G4].)

Consider first the general non-product case. When D is given in the form (2.1) with (2.2) on X_c , we can write, in the notation of [G5],

$$(4.1) \quad A_1 = A + x_n P_1 + P_0,$$

where P_1 is first-order tangential and P_0 is of order 0 and *constant in x_n* (since $A_{10} = A_{10}|_{x_n=0} + x_n A'_{10}$ where the last term may be absorbed in $x_n A_{11}$). The formal adjoint is

$$(4.2) \quad D^* = (-\partial_{x_n} + A'_1)\sigma^*, \quad A'_1 = A + x_n P_1^* + P_0', \text{ on } X_c,$$

where $P_0' = P_0^* - v^{-1}\partial_{x_n}v$; here $v(x)$ is the function entering in the volume form $v(x)dx$. Conditions on the volume form are needed when D is compared with D_0 (coupled with the volume form $v(x', 0)dx'dx_n$) in Theorem 4.5 below.

Let D_1 and D_2 be two first-order elliptic operators on X of non-product type (as in (2.1) with (4.1)), with the same σ and provided with the same well-posed boundary condition $\Pi\gamma_0 u = 0$. Let $D_{1,\Pi}$ and $D_{2,\Pi}$ be the realizations defined by the boundary condition $\Pi\gamma_0 u = 0$, and let $\Delta_i = D_i^* D_i$, $\Delta_{i,B} = D_{i,\Pi}^* D_{i,\Pi}$. Let $Q_{i,\lambda}$ be parametrices of the Δ_i on a neighbouring manifold \tilde{X} , and denote the resolvents of the $\Delta_{i,B}$ by $R_{i,\lambda} = (\Delta_{i,B} - \lambda)^{-1}$.

We have the following general perturbation result:

THEOREM 4.1. *Let l be the largest nonnegative integer such that*

$$(4.3) \quad D_1 - D_2 = x_n^l \bar{P}_l \text{ on } X_c,$$

for some tangential x_n -dependent first-order differential operator \overline{P}_l . Let F be a differential operator in E of order m' and let $N > \frac{n+m'}{2}$. Consider

$$(4.4) \quad F(R_{2,\lambda}^N - R_{1,\lambda}^N) = (F(Q_{2,\lambda}^N - Q_{1,\lambda}^N))_+ + F\overline{G}_\lambda^{(N)}.$$

The ψ do part has an asymptotic trace expansion

$$(4.5) \quad \text{Tr}[(F(Q_{2,\lambda}^N - Q_{1,\lambda}^N))_+] \sim \sum_{-n \leq k < \infty} \tilde{p}_k(-\lambda)^{\frac{m'-k}{2}-N},$$

where $\tilde{p}_k = 0$ for $k - m' + n$ odd.

The s.g.o. part has an asymptotic trace expansion

$$(4.6) \quad \text{Tr}[F\overline{G}_\lambda^{(N)}] \sim \sum_{-n+1+l \leq k < k_0} \tilde{g}_k(-\lambda)^{\frac{m'-k}{2}-N} + \sum_{k \geq k_0} (\tilde{g}'_k \log(-\lambda) + \tilde{g}''_k)(-\lambda)^{\frac{m'-k}{2}-N},$$

where

$$(4.7) \quad k_0 = l + 1 \text{ when } F \text{ is general, } k_0 = m' + l + 1 \text{ when } F \text{ is tangential on } X_c.$$

It follows that

$$(4.8) \quad \text{Tr}[F(R_{2,\lambda}^N - R_{1,\lambda}^N)] \sim \sum_{-n \leq k < k_0} \tilde{c}_k(-\lambda)^{\frac{m'-k}{2}-N} + \sum_{k \geq k_0} (\tilde{c}'_k \log(-\lambda) + \tilde{c}''_k)(-\lambda)^{\frac{m'-k}{2}-N},$$

with k_0 as above. For $k \leq l - n$, the \tilde{c}_k vanish when $k - m' + n$ is odd.

The coefficients \tilde{c}_k and \tilde{c}'_k are locally determined.

The results carry over to similar results for the heat operators and power operators associated with the $\Delta_{i,B}$. Alternatively, we can formulate the results as follows:

COROLLARY 4.2. *Hypotheses and definitions as in Theorem 4.1. For the trace expansions*

$$(4.9) \quad \begin{aligned} \text{Tr}(FR_{1,\lambda}^N) &\sim \sum_{-n \leq k < 0} a_k(-\lambda)^{\frac{m'-k}{2}-N} + \sum_{k \geq 0} (\tilde{a}'_k \log(-\lambda) + \tilde{a}''_k)(-\lambda)^{\frac{m'-k}{2}-N}, \\ \text{Tr}(Fe^{-t\Delta_{1,B}}) &\sim \sum_{-n \leq k < 0} a_k t^{\frac{k-m'}{2}} + \sum_{k \geq 0} (-a'_k \log t + a''_k) t^{\frac{k-m'}{2}}, \\ \text{Tr}(F\Delta_{1,B}^{-s}) &\sim \sum_{-n \leq k < 0} \frac{a_k}{s + \frac{k-m'}{2}} - \frac{\text{Tr}(F\Pi_0(D_{1,\Pi}))}{s} \\ &\quad + \sum_{k \geq 0} \left(\frac{a'_k}{(s + \frac{k-m'}{2})^2} + \frac{a''_k}{s + \frac{k-m'}{2}} \right) \end{aligned}$$

(the summation limit 0 replaced by m' if F is tangential), the replacement of D_1 by D_2 leaves the coefficients \tilde{a}'_k and a'_k invariant for $k < k_0$. The other coefficients

with $k < k_0$ are modified only by local terms; those with $k \leq l - n$ and $k - m' + n$ odd are invariant.

There are similar results for expansions associated with $D_{i,\Pi}R_{i,\lambda}$, $D_{i,\Pi}e^{-t\Delta_{i,B}}$ and $D_{i,\Pi}\Delta_{i,B}^{-s}$ (here the index Π on the factor in front can be omitted since the operators map into the domain):

THEOREM 4.3. *Hypotheses and definitions as in Theorem 4.1. Let ψ be a morphism from E_1 to E , and let $N > (n + m' + 1)/2$. For the trace expansions*

$$(4.10) \quad \begin{aligned} \mathrm{Tr}(F\psi D_1 R_{1,\lambda}^N) &\sim \sum_{-n \leq k < 0} \tilde{b}_k(-\lambda)^{\frac{m'+1-k}{2}-N} \\ &\quad + \sum_{k \geq 0} (\tilde{b}'_k \log(-\lambda) + \tilde{b}''_k)(-\lambda)^{\frac{m'+1-k}{2}-N}, \\ \mathrm{Tr}(F\psi D_1 e^{-t\Delta_{1,B}}) &\sim \sum_{-n \leq k < 0} b_k t^{\frac{k-m'-1}{2}} + \sum_{k \geq 0} (-b'_k \log t + b''_k) t^{\frac{k-m'-1}{2}}, \\ \mathrm{Tr}(F\psi D_1 \Delta_{1,B}^{-s}) &\sim \sum_{-n \leq k < 0} \frac{b_k}{s + \frac{k-m'-1}{2}} \\ &\quad + \sum_{k \geq 0} \left(\frac{b'_k}{(s + \frac{k-m'-1}{2})^2} + \frac{b''_k}{s + \frac{k-m'-1}{2}} \right) \end{aligned}$$

(the summation limit 0 replaced by m' if F is tangential), the replacement of D_1 by D_2 leaves the coefficients \tilde{b}'_k and b'_k invariant for $k < k_0$. The other coefficients with $k < k_0$ are modified only by local terms; those with $k \leq l - n$ and $k - m' + n$ even are invariant.

Because of the factor D_i in front, this is not a special case of Theorem 4.1. Let us also mention, for the case $F = I$, the more customary formulation of the third expansion, as in the last line of (2.24):

COROLLARY 4.4. *Hypotheses and definitions as in Theorem 4.1. Let ψ be a morphism from E_1 to E , and let $k_0 = l + 1$. In the eta function expansion*

$$(4.11) \quad \begin{aligned} \eta(\psi, D_{1,\Pi}, s) &= \mathrm{Tr}(\psi D_1 (D_{1,\Pi}^* D_{1,\Pi})^{-\frac{s+1}{2}}) \\ &\sim \frac{1}{\Gamma(\frac{s+1}{2})} \left[\sum_{-n < k < 0} \frac{2b_k}{s+k} + \sum_{k=0}^{\infty} \left(\frac{4b'_k}{(s+k)^2} + \frac{2b''_k}{s+k} \right) \right], \end{aligned}$$

a replacement of D_1 by D_2 leaves the coefficients \tilde{b}'_k and b'_k invariant for $k < k_0$. The other coefficients with $k < k_0$ are modified only by local terms; those with $k \leq l - n$ and $k + n$ even are invariant.

The proofs are given in [G5]; here we incorporate $D_{i,\Pi}$ and $D_{i,\Pi}^*$ in larger skew-selfadjoint matrices

$$(4.12) \quad \mathcal{D}_{i,B} = \begin{pmatrix} 0 & -D_{i,\Pi}^* \\ D_{i,\Pi} & 0 \end{pmatrix},$$

(as in [GS1]) and study the difference of their resolvents, using the calculus of [G3] to handle the resulting singular Green operator term and to find its expansion properties.

4.2 Comparison with the product case.

We can also compare the expansions for a given D of non-product type (2.1), (4.1), with the expansions for an operator D_0 of product type having the form D^0 (2.5) on X_c . Here the volume form $v(x) dx$ for D is replaced by the volume form $v(x', 0) dx$ for D_0 on X_c , so the preceding results cannot immediately be applied. However, if for some $l \geq 1$,

$$(4.13) \quad P_0 = 0, \quad x_n P_1 = x_n^l P_l, \quad \partial_{x_n}^j v(x', 0) = 0 \text{ for } 1 \leq j \leq l,$$

then D^* can be written in the form

$$(4.14) \quad D^* = (-\partial_{x_n} + A + x_n^l P_l') \sigma^* \text{ on } X_c;$$

here P_l and P_l' are first-order tangential differential operators. Then the method of proof of the preceding results extends to show:

THEOREM 4.5. *Consider (4.9) with F equal to a morphism φ , and (4.10) with $F = I$ (so $m' = 0$).*

1° (The case $l = 1$.) *Assume that $P_0 = 0$ and $\partial_{x_n} v(x', 0) = 0$. Then the coefficients a'_0, a'_1 (and $\tilde{a}'_0, \tilde{a}'_1$) in (4.9) are the same for the expansions defined for D_Π and for $D_{0,\Pi}$. The coefficients a''_0, a''_1 (and $\tilde{a}''_0, \tilde{a}''_1$) differ in the two cases only by local terms.*

Moreover, in (4.10), the coefficients b'_0, b'_1 (and $\tilde{b}'_0, \tilde{b}'_1$) are the same for D_Π and for $D_{0,\Pi}$. The coefficients b''_0, b''_1 (and $\tilde{b}''_0, \tilde{b}''_1$) differ in the two cases only by local terms.

2° (The general case $l \geq 1$.) *Assume that (4.13) holds. Then in (4.9), (4.10), the coefficients a'_k and b'_k for $0 \leq k \leq l$ (as well as \tilde{a}'_k and \tilde{b}'_k for $0 \leq k \leq l$) are preserved when D_Π is replaced by $D_{0,\Pi}$. The nonlocal coefficients behind them, $a''_k, b''_k, \tilde{a}''_k, \tilde{b}''_k$ with $0 \leq k \leq l$ are only locally perturbed.*

We also have the result that when $D - D_0 = x_n P_1 + P_0$ on X_c , the zero-order operator P_0 not necessarily being 0, then a'_0 is the same for D_Π and $D_{0,\Pi}$, and a''_0 differs only by local terms. This was known from [G1, GS1] in cases where Π equals $\Pi_{\geq}(A)$ or certain finite rank perturbations of it. However, a'_1 will in general depend on P_0 , as demonstrated in [G5, Rem. 3.10].

4.3 Perturbation of the product case by commuting operators.

The study of perturbations of the product case that commute with A is somewhat different; here one can use functional calculus for the operators near the boundary, expressing them as functions of A (continuing the line of [GS2]). The traces we study are reduced to traces of pseudodifferential operators on the boundary, built up of A and its eigenprojections. Parity considerations play a great role, because of the fact that A and its integer powers have even-even parity, whereas $|A|$ has even-odd parity, cf. (1.7)–(1.8). By working out detailed formulas for the resolvent and its iterates we were able to show in [G5]:

THEOREM 4.6. Assume that D is a perturbation of D^0 as in (4.1) on X_c such that the zero-order x_n -independent operator (morphism) P_0 commutes with A , and in the Taylor expansions on X_c ,

$$(4.15) \quad x_n P_1(x_n) = \sum_{1 \leq k \leq K} x_n^k P_{1k} + x_n^{K+1} P'_{K+1}(x_n) \text{ for any } K,$$

the tangential x_n -independent first-order differential operators P_{1k} commute with A . The product measure is used on X_c , and D is provided with the boundary condition

$$(4.16) \quad \Pi_{\geq}(A)\gamma_0 u = 0.$$

Let F be a differential operator in E of order m' , and let $N > \frac{n+m'}{2}$.

If n is odd, the resolvent and heat operator, resp. gamma times zeta function, associated with Δ_B have trace expansions without logarithms, resp. meromorphic extensions without double poles:

$$(4.17) \quad \begin{aligned} \text{Tr}(F(\Delta_B - \lambda)^{-N}) &\sim \sum_{-n \leq k < \infty} \tilde{a}_k(-\lambda)^{\frac{m'-k}{2}-N}, \\ \text{Tr}(F e^{-t\Delta_B}) &\sim \sum_{-n \leq k < \infty} a_k t^{\frac{k-m'}{2}}, \\ \Gamma(s)\zeta(F, \Delta_B, s) &\equiv \Gamma(s) \text{Tr}(F\Delta_B^{-s}) \sim \sum_{-n \leq k < \infty} \frac{a_k}{s + \frac{k-m'}{2}} - \frac{\text{Tr}(F\Pi_0(\Delta_B))}{s}, \end{aligned}$$

where the coefficients are locally determined for $-n \leq k < 0$ (for $-n \leq k < m'$ if F is tangential). Here \tilde{a}_{-n} and a_{-n} vanish if m' is odd.

There are similar results for $\text{Tr}(F\psi D(\Delta_B - \lambda)^{-N})$ and its associated heat trace and power trace (an eta-function), where ψ is a morphism from E_1 to E .

In the case of a manifold of even dimension n , we first show that for an x_n -independent and tangential differential operator F taken together with a product case operator D_0 , logarithmic terms can appear at most at the power $(-\lambda)^{-N}$ and the half-powers $(-\lambda)^{-N-k-\frac{1}{2}}$, $k \in \mathbb{N}$; this is for the expansion of $\text{Tr}(F(\Delta_B - \lambda)^{-N})$, and there are corresponding statements for the other trace expansions.

But now, when D_0 is replaced by a perturbation D as in Theorem 4.6 (or when F is x_n -dependent or non-tangential), logs can appear at both integer and half-integer powers in general.

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