Boundary Conditions

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We have already heard several talks on Lars Hörmander's mathematical contributions, so I need not try to repeat the great lines.

Instead I will focus on a small corner of special interest for me, namely works related to boundary value problems.
The examples will demonstrate how much I have learned from him.

Together with this, I shall recall some interactions between Lund and Copenhagen, and the people involved.

On boundary conditions there are some useful early observations:

• The presentation of L_2-Sobolev spaces (with anisotropy) over $\overline{\mathbb{R}}^n_+ = \{x \mid x_n \geq 0\}$ in the book [H63]: (here $\langle\xi\rangle = (1+|\xi|^2)^{\frac12}$, $\xi' = (\xi_1 \dots, \xi_{n-1})$)

$$\text{When } u \in H_{(s,t)}(\mathbb{R}^n) \iff \langle\xi\rangle^s\langle\xi'\rangle^t\hat{u}(\xi) \in L_2(\mathbb{R}^n),$$

$H_{(s,t)}(\overline{\mathbb{R}}^n_+)$ consists of distributions *restricted* from $H_{(s,t)}(\mathbb{R}^n)$,

$\overset{\circ}{H}_{(s,t)}(\overline{\mathbb{R}}^n_+)$ consists of $H_{(s,t)}$-distributions *supported* in $\overline{\mathbb{R}}^n_+$.

One with (s,t) is dual to the other with $(-s,-t)$. In the books [H83-85], $H_{(s,t)}(\overline{\mathbb{R}}^n_+)$ was relabelled $\overline{H}_{(s,t)}(\mathbb{R}^n_+)$. For $t=0$, replace $H_{(s,0)}$ by $H_{(s)}$. In particular, $\overset{\circ}{H}_{(\frac12)}(\overline{\mathbb{R}}^n_+)$ is the natural (e.g. interpolating) space between $\overset{\circ}{H}_{(1)}(\overline{\mathbb{R}}^n_+)$ and $\overset{\circ}{H}_{(0)}(\overline{\mathbb{R}}^n_+)$, in contrast to Lions and Magenes's definition (book '68) of $H_0^{\frac12}(\mathbb{R}^n_+)$ by closure of $C_0^\infty(\mathbb{R}^n_+)$ in $H^{\frac12}(\mathbb{R}^n_+)$. They denoted $\overset{\circ}{H}_{(\frac12)}(\overline{\mathbb{R}}^n_+)$ by $H_{00}^{\frac12}(\mathbb{R}^n_+)$, and have some remarks on the monstrous (teratological) interpolation results one gets with $H_0^{\frac12}(\mathbb{R}^n_+)$.

• In [H63], "partial hypoellipticity at a boundary" shows by use of $H_{(s,t)}$-spaces that for solutions of

$$Au = f \text{ in } \mathbb{R}^n_+, \quad A = D^m_{x_n} + \sum_{\alpha_n < m} a_\alpha D^\alpha,$$

the regularity of normal derivatives of u at $\{x_n = 0\}$ can be derived from the regularity of f and the tangential derivatives of u.

• In Annals '66 there is a construction of the Calderón projector (proposed by Calderón in '63), applied to non-elliptic boundary conditions where it motivates hypoelliptic ψdo's.
Good details, but harder to read than Seeley's more geometric presentation in Am.J.Math. '66.
The case of Douglis-Nirenberg elliptic systems (mixed-order systems, e.g. the Stokes operator) was "left to the reader" and I took it up, complicated because of the compatibility conditions needed for the Cauchy data (G, JFA '77).

• The AMS Proc. on Singular Integrals paper (printed '67) introducing $S^m_{\varrho,\delta}$-symbols was interesting already by the definition of $S^m_{1,0}$-symbols. For me, a precise lemma on singular sequences, constructed to determine norms modulo compact perturbations, was useful since I could generalize it to determine the essential spectrum of boundary value problems for Douglis-Nirenberg elliptic systems, such as the Stokes system (joint work with Geymonat '73).

• There is a seminal paper in Arch.Rat.Mech. '76 on the boundary problem of Geodesy, using Nash-Moser estimates to treat the inverse boundary value problem of detecting the surface of the earth from measurements of the gravity potential w and the gravity vector grad w (the Molodenskiĭ problem).
A Danish colleague C.C. Tcherning, professor in Geodesy, told me later how the literature on the Molodenskiĭ problem was divided into "before Hörmander" and "after Hörmander".
Additional work on the nonlinear problem by Nash-Moser techniques still takes place in combination with numerical analysis tools: E. Stephan, H. Gimperlein, preprint 2013.

We organised in Denmark a "hypoelliptic study-group", from 1970 on, with participants from Copenhagen, DTU and Århus Universities, studying Lars' and other people's papers.
The culmination was when Lars came to a workshop in Århus May 1972. Besides the Danish participants, J. and A. Unterberger, C. Zuily and M. Derridj took part in the meeting.

Also in 1972, I was invited to be the Faculty opponent to Johannes Sjöstrand's thesis, which took place May 1972.

Anders Melin had a lektor position in Copenhagen in 1974 and part of 1975.

Karl Gustav Andersson had a lektor position in Copenhagen in the academic year 1978-79.

Interactions in the eighties

The year that Johannes Sjöstrand became professor in Lund, 1985, we started the Danish-Swedish Analysis Seminar. Organized by Lars Hörmander, Anders Melin and me (and Johannes until he moved to France). Some speakers:

Oct. 25, 1985 in Copenhagen: E. Balslev, J. Sjöstrand, E. Skibsted, A. Melin.

Dec. 6, 1985 in Copenhagen: H. Izosaki, H. Schlichtkrull, A. Grigis, G. Grubb.

Feb. 14, 1986 in Copenhagen: A. Jensen, G. Métivier, M. Beals, N. Dencker.

Mar. 21, 1986 in Lund: M. Ikawa, S. Alinhac, H. Eliasson, G. Lebeau.

May 7, 1986 in Copenhagen: R. Melrose, L. Hörmander, B. Helffer, T. Branson.

Sep. 19, 1986 in Lund: J. Nourrigat, D. Robert, M. Murata, J. Brüning.

Nov. 21, 1986 in Copenhagen: E.T. Poulsen, B. Thaller, B. Durhuus, L.-E. Lundberg.

Jan. 23, 1987 in Lund: Y. Meyer, C. Zuily, J.-M. Bismut, B. Branner.

Mar. 6, 1987 in Copenhagen: V. Enss, H. Kitada, B. Dahlberg, N.J. Kokholm.

May 8, 1987 in Lund: J.-M. Bony, I.M. Sigal, E. Trubowitz.

May 9, 1987 in Lund: R.B. Melrose, J.-C. Guillot, A. Laptev.

May 26, 1988 in Copenhagen: R. Beals, J.J. Kohn, S. Baouendi, A. Jensen.

May 11, 1989 in Lund: A. Unterberger, R. Temam, P. Perry, P. Sjögren.

May 12, 1989 in Lund: H. Brezis, W. Müller, B. Ørsted, C. Sogge.

The abstracts were collected in two small reports in the Copenhagen University Report Series from '87 and '89.

For a number of years in the 80's and 90's, Lars, Anders and I were members of the committee organizing the yearly French meeting on Equations Dérivées Partielles, which in those years usually took place in Saint-Jean-de-Monts.

The "hypoelliptic study group" was followed by a study group around '81 and '82 in Copenhagen where we read the manuscripts of Lars' forthcoming books, in order to make comments and look for inaccuracies. The latter were rare, but there were many suggestions for elaborations and explanations.

In particular, Niels Jørgen Kokholm made a substantial effort with the books, and he went to study with Lars in '82-'83. Here he worked out a microlocal analysis of deficiency indices of PDE with polynomial coefficients.
Later he wrote a thesis in Copenhagen '88, on a problem from chemistry (Thor A. Bak), and worked with me on L_p ψdbo estimates.

Peter Gilkey had a longer stay in Lund giving lectures (late 80's?). On the question of heat trace asymptotics for Dirac operators with ψdo boundary conditions, Lars referred him to me, since I had (finite) ψdbo heat trace expansions in my book from '86.
The index term, not covered by the book, was only known in product cases (Atiyah, Patodi and Singer '75), but I found a way to couple my analysis with theirs to reach it in the nonproduct case (CPDE '92).
Later I had joint works with Seeley on full expansions with log-terms, Inv. '95 and JGA '96.

Lars and I did a small joint work, on the transmission property, G-H Math.Scand. 1990:

When X is an n-dimensional C^∞-manifold and Y is an open subset with smooth boundary, a ψdo P on X of order m is said to have the *transmission property* with respect to Y, when

$$P_Y = r_Y P e_Y$$

maps $C^\infty(\overline{Y})$ into $C^\infty(\overline{Y})$. Here e_Y extends by zero in $X \setminus Y$, and r_Y restricts to Y.

Consider $X = \mathbb{R}^n$, $Y = \mathbb{R}^n_+$, and denote $\partial_\xi^\alpha \partial_x^\beta p = p^{(\alpha)}_{(\beta)}$. It was known from Boutet de Monvel '71 that the property for classical integer-order operators P is characterized by a parity in ξ_n: *For all* l, α, β,

$$(p_{m-l})^{(\alpha)}_{(\beta)}(x', 0, \xi', -\xi_n) = (-1)^{m-l-|\alpha|}(p_{m-l})^{(\alpha)}_{(\beta)}(x', 0, \xi', \xi_n). \quad (1)$$

For $p \in S^m_{1,0}$, $m \in \mathbb{Z}$, a sufficient condition for the transmission property was given by Boutet de Monvel, generalizing (1) as a statement on the asymptotic behavior for $\xi_n \to \pm\infty$.

The new thing in '90 was that we treated $p \in S^m_{\varrho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ with $0 \le \delta < \varrho \le 1$, allowing arbitrary $m \in \mathbb{R}$. Here we showed a necessary and sufficient condition for the transmission property wrt $\mathbb{R}^n_+$, namely:

For all α, β,

$$a_{\alpha,\beta}(x', z_n) = \mathcal{F}^{-1}_{\xi_n \to z_n} p^{(\alpha)}_{(\beta)}(x', 0, 0, \xi_n) \quad (2)$$

extends to a function in $C^\infty(\overline{\mathbb{R}}^n_+)$.

Moreover, when the condition holds uniformly in x', $P_{\mathbb{R}^n_+}$ is continuous

$$\begin{aligned} &P_{\mathbb{R}^n_+} : \overline{H}_{(s)}(\mathbb{R}^n_+) \to \overline{H}_{(s-m-c_\varrho)}(\mathbb{R}^n_+) \text{ for } s > -\tfrac{1}{2}, \text{ where} \\ &\quad c_\varrho = 0 \text{ if } \varrho = 1, \text{ or } |s| < \tfrac{1}{2}, \\ &\quad c_\varrho = (1-\varrho)(s - \tfrac{1}{2}) \text{ if } \varrho < 1, s > \tfrac{1}{2}, \\ &\quad c_\varrho > 0 \text{ if } \varrho < 1, s = \tfrac{1}{2}. \end{aligned}$$

This property was derived from an analysis of Poisson operators $Kv = r_{\mathbb{R}^n_+} P(v(x') \otimes \delta_{x_n})$. It may be of interest for studies of ψdo's with nonsmooth x-dependence. Here there is a technique of M. Taylor of replacing a Hölder-class symbol p of type $S^m_{1,0}$ by a sum $p = p^\# + p^\flat$, where $p^\#$ is smooth in $S^m_{1,\delta}$ for some $\delta > 0$, and $p^\flat$ is of lower order.

Most ψdo's do *not* have the transmission property, but some can be classified as giving specific singularities at a boundary.
In a lecture note from Princeton '65 based on works of Vishik and Eskin, Lars analysed classical symbols satisfying a "modified transmission condition" of type μ (where Boutet de Monvel's case is $\mu = 0$):

$$(p_{m-l})^{(\alpha)}_{(\beta)}(x', 0, \xi', -\xi_n) = e^{\pi i(m-l-2\mu-|\alpha|)}(p_{m-l})^{(\alpha)}_{(\beta)}(x', 0, \xi', \xi_n), \quad (2)$$

This is satisfied by $(-\Delta)^a$ with $\mu = a$, $m = 2a$.
The notes show the Fredholm property for Dirichlet realizations, mapping

$$P\colon H_{\mu(s)} \to \overline{H}_{(s-m)}(\Omega), \text{ all } s > \mu - \tfrac{1}{2}; \text{ where} \qquad (3)$$

$$u \in H_{\mu(s)}(\mathbb{R}^n_+) \iff r^+ \operatorname{Op}((\langle\xi'\rangle + i\xi_n)^\mu)u \in \overline{H}_{s-\mu}(\mathbb{R}^n_+) \text{ if } u \in \overset{\circ}{H}_{(\mu-\frac{1}{2})}(\overline{\mathbb{R}}^n_+).$$

Such spaces have an x_n^μ-singularity at the boundary: $\bigcap_s H_{\mu(s)}(\Omega)$ $= x_n^\mu C^\infty(\overline{\Omega})$. For $0 < \mu < 1$, $s > n/2$, $H_{\mu(s)} \subset x_n^\mu C^\tau$ for a small τ (follows from a potential-theoretic study by Ros-Oton and Serra 2012).
And $H_{\mu(\mu)} = \overset{\circ}{H}_{(\mu)}$; here $\overset{\circ}{H}_{(\frac{1}{2})} = \{u \in \overline{H}_{(\frac{1}{2})}; x_n^{-\frac{1}{2}} u \in L^2\}$.
Interesting that (3) was shown for variable-coeff. ψdo's. For P of type a, elliptic of order $2a$, the Dirichlet domain is *the same* as that of $(-\Delta)^a$.
A hint of the theory is given in [H83-85] around Theorem 18.2.18.

Interactions in the nineties

The festive meetings in 1995. Each meeting lasted three days, the first one in Copenhagen with a reception in my home, and the two other days in Lund with a dinner at Grand Hotel there.

March 17, 1995 in Copenhagen: L. Nirenberg, H. Brezis, S. Alinhac, J. Råde.

March 18, 1995 in Lund: R.B. Melrose, B. Helffer, D. Robert, J.J. Duistermaat, G. Grubb.

March 19, 1995 in Lund: J.-M. Bony, J.-Y. Chemin, G. Lebeau, G. Métivier.

May 19, 1995 in Copenhagen: S. Agmon, J. Sjöstrand, E.B. Davies, G. Vodev, A. Melin.

May 20, 1995 in Lund: F. Treves, J.-M. Trépreau, L. Boutet de Monvel, V. Guillemin, E. Skibsted, J.P. Solovej.

March 21, 1995 in Lund: J. Brüning, M. Beals, C. Sogge, P. Gérard.

A proceedings volume in a Birkhäuser series was printed after this occasion, where Lars himself proofread and straightened up all the manuscripts in the appropriate $\TeX$-style (assisted by Anders Melin).

The Danish-Swedish seminar has been continued through the years, somewhat irregularly, and now takes place under the name of Øresund Seminar in Analysis.

Since this talk is given under the heading *Pseudodifferential Operators*, let me end by telling about my own and some colleagues' continued efforts within this field.

My book from '86, 2. ed. '96, treated ψdo boundary problems with a *parameter*, with applications e.g. to parabolic problems. One outcome was solvability results for the Navier-Stokes problem in anisotropic L_p function spaces, in joint works with Solonnikov 1987-95.

Helmut Abels extended the theory to ψdbo's with *nonsmooth x-dependence* 2005, with applications to Navier-Stokes problems with a free boundary (necessarily nonsmooth).

For the ψdo theory on open sets, methods for C^τ-smoothness in x ($\tau > 0$) had been developed by Kumano-go and Nagase '78, J. Marschall '87 and M. Taylor '91, and others. One needs $\tau > 0$ to have the effect of a *principal symbol* of the given order plus a remainder of lower order.

Abels and I jointly with I. Wood applied the nonsmooth ψdbo-theory to construct resolvents of boundary value problems for 2' order strongly elliptic operators on bounded domains $\Omega \subset \mathbb{R}^n$ (preprint 2010). Here we found that it suffices for the domain to be $B^{3/2}_{p,2}$, $p > 2(n-1)$ (includes all $C^{3/2+\varepsilon}$-domains).

One of our results is a *Krein resolvent formula* within the calculus:

$$A_{\nu,C}^{-1} - A_\gamma^{-1} = K_\gamma L^{-1}(K'_\gamma)^*.$$

- A_γ is the Dirichlet realization,
- $A_{\nu,C}$ is the realization of a Robin boundary condition $\nu u = C\gamma u$,
- K_γ is the Poisson operator solving the Dirichlet problem,
- $L = C - P_{\gamma,\nu}$, where $P_{\gamma,\nu}$ is the Dirichlet-to-Neumann operator νK_γ, a ψdo on $\partial\Omega$. All with C^τ-smoothness for a $\tau > 0$.

I have recently proved *spectral asymptotics* formulas for such operators, and for nonsmooth *singular Green operators* G in general (to appear in CPDE):

$$s_j(G) \sim c(g^0) j^{-(n-1)/t}, \text{ for } j \to \infty,$$

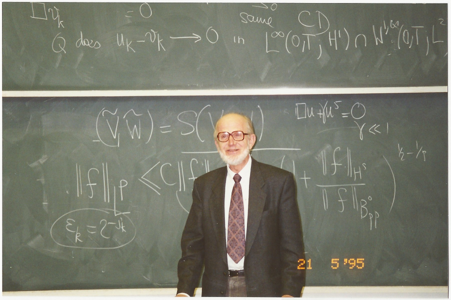
when G is of order $-t < 0$ and class 0, with principal symbol g^0. Here $s_j(G) = \mu_j(G^*G)^{\frac{1}{2}}$, j-th eigenvalue.

Krein formulas have been shown by Gesztesy and Mitrea 2011 for the Laplacian on quasi-convex Lipschitz domains (also containing all $C^{3/2+\varepsilon}$-domains), but without spectral results.

This is just one direction where the tools and techniques of pseudodifferential operators have developed. The other speakers here have told you of other very exciting directions.

Let us keep this legacy of Lars' work alive and useful also in the future!

To end the talk, here are two snapshots from the celebrations in 1995.



(A. Melin, S. Agmon, Lars Hörmander, F. Treves, K.G. Andersson, J. Boman)