§3. Boundary operators on Sobolev spaces

3.1 Density of smooth functions.

In the present chapter we investigate the Sobolev spaces $H^m(\Omega)$ and $H_0^m(\Omega)$ in detail for smooth sets Ω, showing in particular the statements on boundary values that were indicated in the preceding chapter. The basic step is the analysis of the case $\Omega = \mathbb{R}^n_+$ that we first present; the results will afterwards be carried over to more general sets by use of diffeomorphisms and truncation (reduction of the support by multiplication by smooth functions). (The presentation in the following has a certain overlap with [G03]; other presentations are given in Lions and Magenes [L-M68] and [A65].)

The first result we shall show is that smooth functions are dense in Sobolev spaces. Several techniques are involved here; we begin with *translation.*

For $j = 1, \dots, n$ and $h \in \mathbb{R}$ we define $\tau_{j,h}$ as the operator

$$\tau_{j,h} : f(x) \mapsto f(x - he_j) = f(x_1, \dots, x_{j-1}, x_j - h, x_{j+1}, \dots, x_n), \qquad (3.1)$$

here e_j is the j'th unit vector in $\mathbb{R}^n$. The operator maps a function f defined on a set $M \subset \mathbb{R}^n$ into a function $\tau_{j,h} f$ defined on the set

$$M + he_j = \{\, y = x + he_j \mid x \in M \,\}. \qquad (3.2)$$

Note that $\tau_{j,h}$ defines a *unitary operator* in $L_2(\mathbb{R}^n)$.

We shall also need the *restriction* operator r^+ from $\mathbb{R}^n$ to $\mathbb{R}^n_+$, and the restriction operator r^- from $\mathbb{R}^n$ to $\mathbb{R}^n_-$, where

$$\mathbb{R}^n_\pm = \{\, x \in \mathbb{R}^n \mid x_n \gtrless 0 \,\}.$$

Here we generally use the notation

$$r_\Omega u = u|_\Omega, \text{ and in particular } r^\pm = r_{\mathbb{R}^n_\pm}, \qquad (3.3)$$

defined for $u \in \mathcal{D}'(\mathbb{R}^n)$. Moreover we need the *extension by* 0 operators e_Ω, in particular $e^+ = e_{\mathbb{R}^n_+}$ and $e^- = e_{\mathbb{R}^n_-}$, defined for functions f on Ω by

$$e_\Omega f = \begin{cases} f & \text{on } \Omega, \\ 0 & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases} \qquad (3.4)$$

Lemma 3.1. 1° *When $f \in L_2(\mathbb{R}^n)$, then $\tau_{j,h} f \to f$ in $L_2(\mathbb{R}^n)$ for $h \to 0$. In particular, one has for any measurable set $M \subset \mathbb{R}^n$.*

$$\int_M |(\tau_{j,h} f)(x) - f(x)|^2 \, dx \to 0 \text{ for } h \to 0. \qquad (3.5)$$

2° *When f and $g \in L_2(\mathbb{R}^n)$, then*

$$(\tau_{j,-h}f, g)_{L_2(\mathbb{R}^n)} = (f, \tau_{j,h}g)_{L_2(\mathbb{R}^n)}. \tag{3.6}$$

In particular, if $h > 0$ and we set

$$\Omega_{-h} = \{\, x \in \mathbb{R}^n | x_n > -h \,\}, \tag{3.7}$$

then we have for $f \in L_2(\mathbb{R}^n_+)$ and $\varphi \in C_0^\infty(\Omega_{-h})$.

$$\int_{\Omega_{-h}} (\tau_{n,-h}f)(x)\varphi(x)\, dx = \int_{\Omega_{-h}} f(x)(\tau_{n,h}\varphi)(x)\, dx \tag{3.8}$$

(where extensions by 0 are understood).

Proof. For any $\varepsilon > 0$ there exists a function $g \in C_0^0(\mathbb{R}^n)$ (continuous with compact support) such that $\|f - g\|_{L_2(\mathbb{R}^n)} \le \varepsilon/3$. Then (with norms in $L_2(\mathbb{R}^n)$)

$$\begin{aligned} \|\tau_{j,h}f - f\| &\le \|\tau_{j,h}(f-g)\| + \|\tau_{j,h}g - g\| + \|g - f\| \\ &\le 2\varepsilon/3 + \|\tau_{j,h}g - g\|, \end{aligned}$$

where $\|\tau_{j,h}g - g\| \to 0$ for $h \to 0$ since g is uniformly continuous and the supports of g and the functions $\tau_{j,h}g$ ($h \in [-1,1]$) are contained in a fixed compact set. This shows 1°.

In 2°, (3.6) follows immediately from a change of variables in the integral. For (3.8) we observe that all the functions (understood as extended by 0 to $\mathbb{R}^n$) have their support in $\overline{\Omega}_{-h}$. $\square$

The spaces of smooth functions on $\overline{\mathbb{R}}^n_+$ that we shall use will be the following:

$$\begin{aligned} C^\infty_{(0)}(\overline{\mathbb{R}}^n_+) &= r^+ C_0^\infty(\mathbb{R}^n) \\ \mathcal{S}(\overline{\mathbb{R}}^n_+) &= r^+ \mathcal{S}(\mathbb{R}^n), \end{aligned} \tag{3.9}$$

where r^+ is defined in (3.3). $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ consists precisely of the functions on $\overline{\mathbb{R}}^n_+$ that are continuous and have continuous derivatives of all orders on $\overline{\mathbb{R}}^n_+$, and have compact support in $\overline{\mathbb{R}}^n_+$ (since every such function can be extended to a C^∞ function on $\mathbb{R}^n$; this is achieved by an old principle of Borel showing how to construct a C^∞ function with a given set of Taylor coefficients). We note that even though r^+ is the restriction to the open set $\mathbb{R}^n_+$ (which makes sense for arbitrary distributions in $\mathcal{D}'(\mathbb{R}^n)$), the functions in $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ and $\mathcal{S}(\overline{\mathbb{R}}^n_+)$ extend smoothly to the closed set $\overline{\mathbb{R}}^n_+$. When we consider (measurable) *functions*, it makes no difference whether we take the restriction to $\mathbb{R}^n_+$ or to $\overline{\mathbb{R}}^n_+$, but for *distributions* it would, since there exist nonzero distributions supported in the set $\{\, x \in \mathbb{R}^n \mid x_n = 0 \,\}$.

Theorem 3.2.
1° $C_0^\infty(\mathbb{R}^n)$ *is dense in* $H^m(\mathbb{R}^n)$ *for each integer* $m \geq 0$.
2° $C_{(0)}^\infty(\overline{\mathbb{R}}^n_+)$ *is dense in* $H^m(\mathbb{R}^n_+)$ *for each integer* $m \geq 0$.

Proof. Let us go directly to 2°, which has the most complicated proof. For $m = 0$ it is well-known that already $C_0^\infty(\mathbb{R}^n_+)$ is dense in $L_2(\mathbb{R}^n_+)$, so no more needs to be shown there. So let $m > 0$ and let $u \in H^m(\mathbb{R}^n_+)$. We construct an approximation by the following steps:

(i) *Translation.* The first trick is to push u a little bit *outside of* $\mathbb{R}^n_+$ using $\tau_{n,-h}$, to avoid having to deal with the boundary $\{ x_n = 0 \}$, where the derivatives of e^+u can give distributional terms. Let $h > 0$, and define Ω_{-h} by (3.7); for simplicity we write $\tau_{n,-h}$ as τ_{-h}. By Lemma 3.1 1°,

$$\|\tau_{-h} D^\alpha u - D^\alpha u\|_{L_2(\mathbb{R}^n_+)} \to 0 \text{ for } h \to 0, \text{ for all } |\alpha| \leq m. \tag{3.10}$$

Moreover we have by Lemma 3.1 2°, when $\varphi \in C_0^\infty(\Omega_{-h})$,

$$\begin{aligned}\langle \tau_{-h} D^\alpha u, \varphi \rangle_{\Omega_{-h}} &= \int_{\Omega_{-h}} (\tau_{-h} D^\alpha u)\varphi \, dx = \int_{\Omega_{-h}} (D^\alpha u)(\tau_h \varphi)\, dx \\ &= \int_{\Omega_{-h}} u(-D)^\alpha \tau_h \varphi \, dx = \int_{\Omega_{-h}} u\, \tau_h((-D)^\alpha \varphi)\, dx \\ &= \int_{\Omega_{-h}} (\tau_{-h} u)((-D)^\alpha \varphi)\, dx = \langle D^\alpha \tau_{-h} u, \varphi \rangle_{\Omega_{-h}},\end{aligned}$$

and hence $\tau_{-h} D^\alpha u = D^\alpha(\tau_{-h} u)$ as distributions on Ω_{-h}. It follows that $\tau_{-h} u \in H^m(\Omega_{-h})$ and that $\tau_{-h} u|_{\mathbb{R}^n_+}$ converges to u in $H^m(\mathbb{R}^n_+)$ for $h \to 0$.

(ii) *Truncation.* This is applied in order to get compact support in $\overline{\mathbb{R}}^n_+$. Let $\chi_N(x) = \chi(x/N)$, where χ is a cut-off function

$$\chi \in C_0^\infty(\mathbb{R}^n), 0 \leq \chi(x) \leq 1, \chi(x) = 1 \text{ for } |x| \leq 1, \chi(x) = 0 \text{ for } |x| \geq 2. \tag{3.11}$$

Consider $\chi_N \tau_{-h} u$. We have by the Leibniz formula, for any $|\alpha| \leq m$,

$$D^\alpha(\chi_N \tau_{-h} u) = \chi_N D^\alpha \tau_{-h} u + \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0}} \binom{\alpha}{\beta} D^\beta \chi_N D^{\alpha - \beta} \tau_{-h} u,$$

where the first term converges in $L_2(\Omega_{-h})$ to $D^\alpha \tau_{-h} u$ for $N \to \infty$ (by the Lebesgue theorem), and the other terms go to 0 in $L_2(\Omega_{-h})$, since $D^\beta \chi_N$ is $O(1/N)$ (or better), and $D^{\alpha-\beta}\tau_{-h} u$ is in $L_2(\Omega_{-h})$. It follows that

$$\|\chi_N \tau_{-h} u - \tau_{-h} u\|_{H^m(\Omega_{-h})} \to 0 \text{ for } N \to \infty; \tag{3.12}$$

then the convergence also holds in $H^m(\mathbb{R}^n_+)$ for the restrictions to $\mathbb{R}^n_+$.

(iii) *Mollification.* The final trick is to mollify the functions by convolution with h_j (letting $j \to \infty$), where

$$h_j(x) = j^n h(jx),$$
$$\text{with } h \in C_0^\infty(\mathbb{R}^n), h \geq 0, \int h(x)\,dx = 1 \text{ and } \operatorname{supp} h \subset \{\,|x| \leq 1\,\}, \tag{3.13}$$

so that h_j is supported in $\{\,|x| \leq 1/j\,\}$ and has $\int h_j(x)\,dx = 1$. Such convolutions are usually applied to distributions on $\mathbb{R}^n$, so we extend our functions by 0 to all of $\mathbb{R}^n$. This gives an undesired "jump" at the boundary of the original domain, but this does not matter, for we have assured by inserting τ_{-h} that for j large enough, the formulas will be good on $\mathbb{R}^n_+$. We consider

$$\begin{aligned} v_{j,N,h}(x) &= h_j * (e_{\Omega_{-h}} \chi_N \tau_{-h} u)(x) \\ &= \int_{|x-y|<1/j} h_j(x-y)\chi_N(y)(\tau_{-h}u)(y)\,dy, \end{aligned}$$

and note that for j large, say $j \geq 2/h$, the integral is defined in terms of the original functions when $x \in \mathbb{R}^n_+$, since $h_j(x-y)$ is then supported in Ω_{-h}. Then we get for all $\varphi \in C_0^\infty(\mathbb{R}^n_+)$ and all $|\alpha| \leq m$, by use of various changes in the order of integration,

$$\begin{aligned} \langle D^\alpha v_{j,N,h}, \varphi\rangle_{\mathbb{R}^n_+} &= \langle v_{j,N,h}, (-D)^\alpha \varphi\rangle_{\mathbb{R}^n_+} \\ &= \int_{x\in\mathbb{R}^n_+}\int_{y\in\Omega_{-h}} h_j(x-y)\chi_N(y)(\tau_{-h}u)(y)(-D_x)^\alpha\varphi(x)\,dydx \\ &= \int_{y\in\Omega_{-h}}\int_{x\in\mathbb{R}^n_+} [D_x^\alpha h_j(x-y)]\chi_N(y)(\tau_{-h}u)(y)\varphi(x)\,dxdy \\ &= \int_{y\in\Omega_{-h}}\int_{x\in\mathbb{R}^n_+} [(-D_y)^\alpha h_j(x-y)]\chi_N(y)(\tau_{-h}u)(y)\varphi(x)\,dxdy \\ &= \langle \chi_N(y)(\tau_{-h}u)(y), (-D_y)^\alpha \int_{x\in\mathbb{R}^n_+} h_j(x-y)\varphi(x)dx\rangle_{\Omega_{-h}} \\ &= \langle D_y^\alpha[\chi_N(y)(\tau_{-h}u)], \int_{x\in\mathbb{R}^n_+} h_j(x-y)\varphi(x)dx\rangle_{\Omega_{-h}} \\ &= \langle h_j * e_{\Omega_{-h}} D^\alpha(\chi_N\tau_{-h}u), \varphi\rangle_{\mathbb{R}^n_+}. \end{aligned}$$

This shows in particular that $r^+D^\alpha v_{j,N,h} \in L_2(\mathbb{R}^n_+)$; and since $h_j * g \to g$ in $L_2(\mathbb{R}^n)$ when $g \in L_2(\mathbb{R}_n)$ (cf. e.g. [G03]), we have moreover that

$$r^+D^\alpha v_{j,N,h} \to r^+D^\alpha(\chi_N\tau_{-h}u) \text{ in } L_2(\mathbb{R}^n_+) \text{ for } j \to \infty,$$

and hence

$$\|r^+ v_{j,N,h} - r^+ \chi_N \tau_{-h} u\|_{H^m(\mathbb{R}^n_+)} \to 0 \text{ for } j \to \infty. \tag{3.14}$$

Here $r^+ v_{j,N,h} \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, and we see that h, N and j can be chosen (in this order) so that $r^+ v_{j,N,h}$ is as close to u in $H^m(\mathbb{R}^n_+)$ as we want, cf. (3.10), (3.12), (3.14). This shows 2°.

The proof of 1° consists of the steps (ii) and (iii) alone. □

As an application of Theorem 3.2 we show:

Theorem 3.3. *There exists a continuous linear mapping $p_{(m)} : H^m(\mathbb{R}^n_+) \to H^m(\mathbb{R}^n)$ such that*

$$r^+ p_{(m)} u = u, \tag{3.15}$$

for $u \in H^m(\mathbb{R}^n_+)$.

Proof. For u in the dense subset $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ we define $p_{(m)} u$ as the function equal to u for $x_n \geq 0$ and equal to

$$(p_{(m)} u)(x', x_n) = \sum_{k=0}^{m-1} a_k u(x', -\lambda_k x_n) \text{ for } x_n < 0, \tag{3.16}$$

where $\lambda_0, \lambda_1, \dots, \lambda_{m-1}$ is a fixed set of (arbitrarily chosen) different positive numbers, and the a_k are determined such that they satisfy the equations

$$\sum_{k=0}^{m-1} \lambda_k^j a_k = (-1)^j \text{ for } j = 0, 1, \dots, m-1 \tag{3.17}$$

(the solution is unique since the determinant of the system of equations (3.17) is the Vandermonde determinant $\det(\lambda_k^j)_{j,k=0,\dots,m-1}$). The conditions (3.17) assure that $p_{(m)} u \in C^{m-1}(\mathbb{R}^n)$, the m'th derivatives being piecewise continuous (with a possible jump at $x_n = 0$). Moreover, $p_{(m)} u$ has compact support in $\mathbb{R}^n$. Then $p_{(m)} u \in H^m(\mathbb{R}^n)$, and its distribution derivatives up to order m coincide with the usual derivatives (defined on $\mathbb{R}^n$ for $|\alpha| \leq m-1$ and on $\mathbb{R}^n \setminus \{ x_n = 0 \}$ for $|\alpha| = m$), cf. e.g. [G03, Lemma 3.6]; and clearly also

$$\|p_{(m)} u\|_{H^m(\mathbb{R}^n)} \leq C \|u\|_{H^m(\mathbb{R}^n_+)} \tag{3.18}$$

with a suitable constant C depending on the choice of the λ_k. This defines $p_{(m)}$ in the desired way on $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$, and it extends to $H^m(\mathbb{R}^n_+)$ by closure. □

We note that it depends on the choice of the λ_k how $p_{(m)}$ treats supports of functions. For example if all λ_k are ≥ 1, $\operatorname{supp} u \subset \overline{\mathbb{R}}^n_+ \cap B(0,R)$ implies $\operatorname{supp} p_{(m)} u \subset B(0,R)$.

3.2 The trace operators on $\mathbb{R}^n_\pm$.

We shall now show how the first m boundary values are defined for the functions in $H^m(\mathbb{R}^n_\pm)$.

Let us define the partial Fourier transform (in the x' variable) for functions $u \in \mathcal{S}(\overline{\mathbb{R}}^n_+)$, by

$$\mathcal{F}_{x'\to\xi'}u = \acute{u}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} u(x', x_n)\, dx'. \tag{3.19}$$

For $u \in \mathcal{S}(\overline{\mathbb{R}}^n_+)$, the $H^m(\mathbb{R}^n_+)$ norm is equivalent with another expression defined as follows, with the notation $D' = (D_1, \dots, D_{n-1})$:

$$\begin{aligned}
\|u\|_m^2 &= \sum_{|\alpha|\le m} \|D^\alpha u\|_0^2 = \sum_{j=0}^m \sum_{|\beta|\le m-j} \|(D')^\beta D^j_{x_n} u\|_0^2 \\
&= (2\pi)^{-n+1} \int_{\mathbb{R}^{n-1}} \sum_{j=0}^m \sum_{|\beta|\le m-j} (\xi')^{2\beta} \int_0^\infty |D^j_{x_n} \acute{u}(\xi', x_n)|^2\, dx_n d\xi' \\
&\simeq (2\pi)^{-n+1} \int_{\mathbb{R}^{n-1}} \int_0^\infty \sum_{j=0}^m \langle\xi'\rangle^{2(m-j)} |D^j_{x_n} \acute{u}(\xi', x_n)|^2\, dx_n d\xi' \\
&\equiv \|u\|^2_{m,'} \Big[= \sum_{j=0}^m \sum_{|\beta|\le m-j} C_{m-j,\beta} \|(D')^\beta D^j_n u\|_0^2\Big],
\end{aligned} \tag{3.20}$$

since one has

$$\sum_{|\beta|\le k} (\xi')^{2\beta} \le \langle\xi'\rangle^{2k} = \sum_{|\beta|\le k} C_{k,\beta} (\xi')^{2\beta} \le C_k \sum_{|\beta|\le k} (\xi')^{2\beta}, \tag{3.21}$$

with positive integers $C_{k,\beta}$ and $C_k = \max_{|\beta|\le k} C_{k,\beta}$.

The partial Fourier transform (3.19) can also be given a sense for arbitrary $u \in H^m(\mathbb{R}^n_+)$, but we shall not do this in detail, since we primarily use the technique for smooth functions. The last lines in (3.20) can be used as a definition of the norm $\|u\|_{m,'}$ for arbitrary $u \in H^m(\mathbb{R}^n_+)$.

Theorem 3.4. *Let m be an integer > 0. For $0 \le j \le m-1$, the mapping*

$$\gamma_j \colon u(x', x_n) \mapsto D^j_{x_n} u(x', 0) \tag{3.22}$$

from $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ to $C^\infty_0(\mathbb{R}^{n-1})$ extends by continuity to a continuous linear mapping (also called γ_j) of $H^m(\mathbb{R}^n_+)$ into $H^{m-j-\frac12}(\mathbb{R}^{n-1})$.

Proof. For $v(t) \in C_0^\infty(\mathbb{R})$ one has the following fundamental inequality

$$\begin{aligned}|v(0)|^2 &= -\int_0^\infty \frac{d}{dt}[v(t)\bar{v}(t)]\,dt = -\int_0^\infty [v'(t)\bar{v}(t) + v(t)\overline{v'}(t)]\,dt \\ &\le 2\|v\|_{L_2(\mathbb{R}_+)} \|\frac{dv}{dt}\|_{L_2(\mathbb{R}_+)} \\ &\le a^2\|v\|^2_{L_2(\mathbb{R}_+)} + a^{-2}\|\frac{dv}{dt}\|^2_{L_2(\mathbb{R}_+)},\end{aligned} \tag{3.23}$$

valid for any $a > 0$. For general functions $u \in C_{(0)}^\infty(\overline{\mathbb{R}}^n_+)$ we apply a partial Fourier transform in x' (cf. (3.19)) and apply (3.23) with respect to x_n, setting $a = \langle\xi'\rangle^{\frac12}$ for each ξ' and using the norm in (1.9) for $H^{m-j-\frac12}(\mathbb{R}^{n-1})$:

$$\begin{aligned}\|\gamma_j u\|^2_{H^{m-j-\frac12}(\mathbb{R}^{n-1})} &= (2\pi)^{-n+1}\int_{\mathbb{R}^{n-1}} \langle\xi'\rangle^{2m-2j-1} |D^j_{x_n}\acute{u}(\xi',0)|^2 d\xi' \\ &\le (2\pi)^{-n+1}\int_{\mathbb{R}^{n-1}} \langle\xi'\rangle^{2m-2j-1} \int_0^\infty \big(a^2|D^j_{x_n}\acute{u}(\xi',x_n)|^2 \\ &\qquad + a^{-2}|D^{j+1}_{x_n}\acute{u}(\xi',x_n)|^2\big)\,dx_n d\xi' \\ &= (2\pi)^{-n+1}\int \big[\langle\xi'\rangle^{2(m-j)}|D^j_{x_n}\acute{u}(\xi',x_n)|^2 \\ &\qquad + \langle\xi'\rangle^{2(m-j-1)}|D^{j+1}_{x_n}\acute{u}(\xi',x_n)|^2\big]\,dx_n d\xi' \\ &\le \|u\|^2_{m,'} \le c\|u\|^2_m,\end{aligned} \tag{3.24}$$

cf. (3.20). This shows the continuity of the mapping, for the dense subset $C_{(0)}^\infty(\overline{\mathbb{R}}^n_+)$ of $H^m(\mathbb{R}^n_+)$, so γ_j can be extended to all of $H^m(\mathbb{R}^n_+)$ by closure. □

The mappings γ_j are called trace operators (or boundary operators).

The range space in Theorem 3.4 is optimal, for one can show that the mappings γ_j are surjective. In fact this holds for the whole *system* of trace operators γ_j with $j = 0, \dots, m-1$. Let us for each $m > 0$ define the *Cauchy trace operator* $\rho_{(m)}$ associated with the order m by

$$\rho_{(m)} = \begin{pmatrix}\gamma_0\\ \gamma_1\\ \vdots\\ \gamma_{m-1}\end{pmatrix} : H^m(\mathbb{R}^n_+) \to \prod_{j=0}^{m-1} H^{m-j-\frac12}(\mathbb{R}^{n-1}); \tag{3.25}$$

for $u \in H^m(\mathbb{R}^n_+)$ we call $\rho_{(m)}u$ the *Cauchy data* of u. (The space $\prod_{j=0}^{m-1} H^{m-j-\frac12}(\mathbb{R}^{n-1})$ is provided with the product norm, $\|\varphi\|_{\prod H^{m-j-\frac12}}$

$= \left(\|\varphi_0\|^2_{m-\frac12} + \cdots + \|\varphi_{m-1}\|^2_{\frac12}\right)^{\frac12}$.) By Theorem 3.4, the mapping $\rho_{(m)}$ in (3.25) is continuous, and we shall now show that it is surjective and has a continuous right inverse $\mathcal{K}_{(m)}$. Again we use that the density results from Section 3.1 allows us to define $\mathcal{K}_{(m)}$ in a convenient way on smooth functions first.

In preparation for a later development we shall define the inverse in such a way that the formulas cover both the trace operator from $\mathbb{R}^n_+$ to $\mathbb{R}^{n-1}$ ("from the right") and the trace operator from $\mathbb{R}^n_-$ to $\mathbb{R}^{n-1}$ ("from the left"), and in addition the "two-sided" operator from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$. Therefore we introduce the somewhat heavy notation:

$$\begin{aligned} &\widetilde{\gamma}_j\colon u(x',x_n) \mapsto D^j_{x_n} u(x',0) \text{ goes from } C_0^\infty(\mathbb{R}^n) \text{ to } C_0^\infty(\mathbb{R}^{n-1}),\\ &\gamma_j^\pm\colon u(x',x_n) \mapsto D^j_{x_n} u(x',0) \text{ go from } C_{(0)}^\infty(\overline{\mathbb{R}}^n_\pm) \text{ to } C_0^\infty(\mathbb{R}^{n-1}), \end{aligned} \tag{3.26}$$

(so here γ_j^+ is the mapping we called γ_j before). The proof of Theorem 3.4 extends immediately to show (by the appropriate modifications of (3.23)):

Theorem 3.5. *Let $m \in \mathbb{N}$. For $0 \le j < m$, the mappings* (3.26) *extend by continuity to continuous linear mappings from $H^m(\mathbb{R}^n)$ resp. $H^m(\mathbb{R}^n_\pm)$ to $H^{m-j-\frac12}(\mathbb{R}^{n-1})$.*

Note that γ_j^+ and γ_j^- are not completely analogous, for both are defined relative to $\partial/\partial x_n$ where the direction x_n points *into* $\mathbb{R}^n_+$ and *out of* $\mathbb{R}^n_-$. (This just shows up in certain coefficients in formulas where the operators are applied.)

We also define the Cauchy trace operators

$$\begin{aligned} \widetilde{\rho}_{(m)} &= \begin{pmatrix} \widetilde{\gamma}_0 \\ \vdots \\ \widetilde{\gamma}_{m-1} \end{pmatrix} : H^m(\mathbb{R}^n) \to \prod_{j=0}^{m-1} H^{m-j-\frac12}(\mathbb{R}^{n-1}),\\ \rho^\pm_{(m)} &= \begin{pmatrix} \gamma_0^\pm \\ \vdots \\ \gamma_{m-1}^\pm \end{pmatrix} : H^m(\mathbb{R}^n_\pm) \to \prod_{j=0}^{m-1} H^{m-j-\frac12}(\mathbb{R}^{n-1}), \end{aligned} \tag{3.27}$$

that are continuous linear operators. Note that since the continuity in Theorem 3.5 holds for any j and m with $m > j$, the above operators are likewise continuous:

$$\begin{aligned}\widetilde{\rho}_{(m)} &= \begin{pmatrix} \widetilde{\gamma}_0 \\ \vdots \\ \widetilde{\gamma}_{m-1} \end{pmatrix} : H^k(\mathbb{R}^n) \to \prod_{j=0}^{m-1} H^{k-j-\frac{1}{2}}(\mathbb{R}^{n-1}), \\ \rho^{\pm}_{(m)} &= \begin{pmatrix} \gamma_0^{\pm} \\ \vdots \\ \gamma_{m-1}^{\pm} \end{pmatrix} : H^k(\mathbb{R}^n_{\pm}) \to \prod_{j=0}^{m-1} H^{k-j-\frac{1}{2}}(\mathbb{R}^{n-1});\end{aligned} \tag{3.27$'$}$$

for any integer $k \geq m$.

Now the surjectivity will be shown.

Theorem 3.6. *Let $m \in \mathbb{N}$. Choose a function $\psi \in \mathcal{S}(\mathbb{R})$ with $\psi = 1$ on a neighborhood of 0. The operator*

$$\widetilde{\mathcal{K}}_{(m)} = \{ \widetilde{\mathcal{K}}_0, \dots, \widetilde{\mathcal{K}}_{m-1} \},$$

that sends $\varphi = \{ \varphi_0, \dots, \varphi_{m-1} \} \in \mathcal{S}(\mathbb{R}^{n-1})^m$ (column vector) over into

$$\begin{aligned}(\widetilde{\mathcal{K}}_{(m)}\varphi)(x) &= (2\pi)^{-n+1} \sum_{0 \leq j < m} \frac{(ix_n)^j}{j!} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \psi(\langle \xi' \rangle x_n) \hat{\varphi}_j(\xi')\, d\xi' \\ &= \sum_{j=0}^{m-1} \widetilde{\mathcal{K}}_j \varphi_j \in \mathcal{S}(\mathbb{R}^n),\end{aligned} \tag{3.28}$$

extends by continuity to a continuous linear operator (also called $\widetilde{\mathcal{K}}_{(m)}$) from $\prod_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\mathbb{R}^{n-1})$ to $H^m(\mathbb{R}^n)$, such that

$$\widetilde{\rho}_{(m)} \widetilde{\mathcal{K}}_{(m)} \varphi = \varphi \text{ for all } \varphi \in \prod_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\mathbb{R}^{n-1}). \tag{3.29}$$

It follows that $\widetilde{\rho}_{(m)} \colon H^m(\mathbb{R}^n) \to \prod_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\mathbb{R}^{n-1})$ is surjective.

For k integer $\geq m$, $\widetilde{\mathcal{K}}_{(m)}$ is also continuous:

$$\widetilde{\mathcal{K}}_{(m)} \colon \prod_{j=0}^{m-1} H^{k-j-\frac{1}{2}}(\mathbb{R}^{n-1}) \to H^k(\mathbb{R}^n), \tag{3.29$'$}$$

and the mapping in the first line of (3.27$'$) is surjective.

Proof. Since $\psi(0)=1$ and $\psi^{(k)}(0)=0$ for $k>0$, and $\widetilde{\gamma}_0 D^k_{x_n}(ix_n)^j/j! = \delta_{jk}$ (the Kronecker delta), one has for $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})^m$:

$$\begin{aligned}\widetilde{\gamma}_0\widetilde{\mathcal{K}}_{(m)}\varphi &= (2\pi)^{-n+1}\int e^{ix'\cdot\xi'}\hat{\varphi}_0(\xi')d\xi' = \varphi_0\\ \widetilde{\gamma}_1\widetilde{\mathcal{K}}_{(m)}\varphi &= \widetilde{\gamma}_0(2\pi)^{-n+1}\Big(\sum_{1\le j<m}\frac{(ix_n)^{j-1}}{(j-1)!}\int e^{ix'\cdot\xi'}\psi(\langle\xi'\rangle x_n)\hat{\varphi}_j(\xi')\,d\xi'\\ &\qquad+\sum_{0\le j<m}\frac{(ix_n)^j}{j!}\int e^{ix'\cdot\xi'}\psi'(\langle\xi'\rangle x_n)\langle\xi'\rangle\hat{\varphi}_j(\xi')\,d\xi'\Big)\\ &= \varphi_1,\end{aligned}$$

etc., which shows that (3.29) holds for $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})^m$. Moreover, we see from the fact that $\psi \in \mathcal{S}(\mathbb{R})$ and the $\varphi_j \in \mathcal{S}(\mathbb{R}^{n-1})$, that $x^\alpha D^\beta_x \sum_j \widetilde{\mathcal{K}}_j\varphi_j$ is bounded for all α and β, so $\sum_j \widetilde{\mathcal{K}}_j\varphi_j$ belongs to $\mathcal{S}(\mathbb{R}^n)$.

It remains to show that $\widetilde{\mathcal{K}}_j$ (applied to $\varphi_j \in \mathcal{S}(\mathbb{R}^{n-1})$) is continuous from $H^{m-j-\frac12}(\mathbb{R}^{n-1})$ to $H^m(\mathbb{R}^n)$. For then $\widetilde{\mathcal{K}}_{(m)}$ extends by continuity to a continuous mapping of $\prod_{j=0}^{m-1} H^{m-j-\frac12}(\mathbb{R}^{n-1})$ into $H^m(\mathbb{R}^n)$, and $\widetilde{\rho}_{(m)}\widetilde{\mathcal{K}}_{(m)}\varphi = \varphi$ remains valid there. For $k \ge m$, $\widetilde{\mathcal{K}}_{(m)}$ consists of the first m entries in $\widetilde{\mathcal{K}}_{(k)}$; so also the last statement in the theorem will follow.

Note first that the Fourier transform $\zeta_j(\xi_n,\sigma)$ of $(ix_n)^j\psi(\sigma x_n)$ satisfies, for $\sigma>0$,

$$\begin{aligned}\zeta_j(\xi_n,\sigma) &= \int_{\mathbb{R}} e^{-ix_n\xi_n}(ix_n)^j\psi(\sigma x_n)\,dx_n\\ &= \sigma^{-j-1}\int e^{-i(\sigma x_n)(\sigma^{-1}\xi_n)}(i\sigma x_n)^j\psi(\sigma x_n)\,d(\sigma x_n)\\ &= \sigma^{-j-1}\zeta_j(\frac{\xi_n}{\sigma},1).\end{aligned}\tag{3.30}$$

Then we find for all t, setting $\sigma = \langle\xi'\rangle$ and denoting $\xi_n/\langle\xi'\rangle = \eta_n$,

$$\begin{aligned}\int_{\mathbb{R}}\langle\xi\rangle^{2t}|\zeta_j(\xi_n,\langle\xi'\rangle)|^2d\xi_n &= \langle\xi'\rangle^{-2j-2}\int(1+|\xi'|^2+|\xi_n|^2)^t|\zeta_j(\eta_n,1)|^2d\xi_n\\ &= \langle\xi'\rangle^{-2j+2t-1}\int\langle\eta_n\rangle^{2t}|\zeta_j(\eta_n,1)|^2d\eta_n\\ &= c_{j,t}\langle\xi'\rangle^{-2j+2t-1},\end{aligned}$$

where $c_{j,t}$ is independent of ξ'. Since the Fourier transform of $\widetilde{\mathcal{K}}_j\varphi_j$ equals

$$\begin{aligned}\mathcal{F}_{x\to\xi}[\widetilde{\mathcal{K}}_j\varphi_j] &= \mathcal{F}_{x_n\to\xi_n}\mathcal{F}_{x'\to\xi'}[\widetilde{\mathcal{K}}_j\varphi_j] = \mathcal{F}_{x_n\to\xi_n}[\frac{(ix_n)^j}{j!}\psi(\langle\xi'\rangle x_n)\hat{\varphi}_j(\xi')]\\ &= \frac{1}{j!}\zeta_j(\xi_n,\langle\xi'\rangle)\hat{\varphi}_j(\xi'),\end{aligned}$$

we get for each j

$$
\begin{aligned}
\|\widetilde{\mathcal{K}}_j\varphi_j\|^2_{m,\wedge} &= (2\pi)^{-n}\int_{\mathbb{R}^n}\langle\xi\rangle^{2m}\big|\tfrac{1}{j!}\zeta_j(\xi_n,\langle\xi'\rangle)\hat{\varphi}_j(\xi')\big|^2 d\xi' d\xi_n\\
&= (2\pi)^{-n}c_{j,m}\left(\tfrac{1}{j!}\right)^2\int_{\mathbb{R}^{n-1}}\langle\xi'\rangle^{2m-2j-1}|\hat{\varphi}_j(\xi')|^2 d\xi' \qquad (3.31)\\
&= c'_{j,m}\|\varphi_j\|^2_{m-j-\frac12,\wedge},
\end{aligned}
$$

which shows the desired continuity. □

Remark 3.7. (3.31) shows in fact that $\widetilde{\mathcal{K}}_{(m)}$ by suitable normalizations can be defined as an isometry (but it is far from surjective)).

Let us now also define

$$
\mathcal{K}^{\pm}_{(m)} = r^{\pm}\widetilde{\mathcal{K}}_{(m)} = \{\,\mathcal{K}^{\pm}_j\,\}_{j=0}^{m-1} = \{\,r^{\pm}\widetilde{\mathcal{K}}_j\,\}_{j=0}^{m-1}, \tag{3.32}
$$

that are continuous

$$
\mathcal{K}^{\pm}_{(m)} = \prod_{j=0}^{m-1} H^{k-j-\frac12}(\mathbb{R}^{n-1}) \to H^k(\mathbb{R}^n_{\pm}), \quad \text{for } k \ge m. \tag{3.33}
$$

We obviously have for $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})^m$

$$
\rho^{\pm}_{(m)}\mathcal{K}^{\pm}_{(m)}\varphi = \varphi, \tag{3.34}
$$

and the validity of (3.34) extends to all $\varphi \in \prod_{j=0}^{m-1} H^{m-j-\frac12}(\mathbb{R}^{n-1})$. This shows:

Corollary 3.8. *For k integer $\ge m$, $\rho^{\pm}_{(m)}$ is surjective from $H^k(\mathbb{R}^n_{\pm})$ to $\prod_{j=0}^{m-1} H^{k-j-\frac12}(\mathbb{R}^{n-1})$ and has the continuous right inverse $\mathcal{K}^{\pm}_{(m)}$ defined by* (3.32). *$\mathcal{K}^{\pm}_{(m)}$ maps $\mathcal{S}(\mathbb{R}^{n-1})^m$ into $\mathcal{S}(\overline{\mathbb{R}}^n_{\pm})$.*

We mention without proof that one can extend the definition and continuity of $\widetilde{\mathcal{K}}_{(m)}$ and $\mathcal{K}^{\pm}_{(m)}$ in (3.29′) and (3.33) to all $k \in \mathbb{R}$. The definition and continuity of $\widetilde{\rho}_{(m)}$ and $\rho^{\pm}_{(m)}$ in (3.27′) can be extended to noninteger k, *but only for $k > m - \frac12$.*

Let us finally show that that the kernel of the mapping $\rho^{+}_{(m)} : H^m(\mathbb{R}^n_{+}) \to \prod_{j=0}^{m-1} H^{m-j-\frac12}(\mathbb{R}^{n-1})$ equals the space $H^m_0(\mathbb{R}^n_{+})$, cf. (1.31).

Theorem 3.9. *Let $m \in \mathbb{N}$. The following identity holds:*

$$H_0^m(\mathbb{R}_+^n) = \{ u \in H^m(\mathbb{R}_+^n) \mid \rho_{(m)}^+ u = 0 \}. \tag{3.35}$$

Proof. Since m is fixed, we can in the proof write ρ instead of $\rho_{(m)}^+$.

If $u \in C_0^\infty(\mathbb{R}_+^n)$, then of course $\rho u = 0$. Then since ρ is continuous with respect to the m-norm, and $H_0^m(\mathbb{R}_+^n)$ is the closure of $C_0^\infty(\mathbb{R}_+^n)$ in the m-norm, we see that $u \in H_0^m(\mathbb{R}_+^n)$ implies $\rho u = 0$.

The converse is harder to show. Let $u \in H^m(\mathbb{R}_+^n)$ be such that $\rho u = 0$, then we shall show that $u \in H_0^m(\mathbb{R}_+^n)$ by showing that for any $\varepsilon > 0$ there is a function $\varphi \in C_0^\infty(\mathbb{R}_+^n)$ with $\|u - \varphi\|_m \leq \varepsilon$.

First we use the density shown in Theorem 3.2. According to this, there is a sequence of functions $v_k \in C_{(0)}^\infty(\overline{\mathbb{R}}_+^n)$ such that $v_k \to u$ in $H^m(\mathbb{R}_+^n)$. For each k, $\rho v_k \in C_0^\infty(\mathbb{R}^{n-1})$, and since $\rho u = 0$, $\rho v_k \to 0$ in $\prod_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\mathbb{R}^{n-1})$. To this sequence we apply the right inverse $\mathcal{K}_{(m)}^+$ constructed above. Let $z_k = \mathcal{K}_{(m)}^+ \rho v_k$, then $\rho z_k = \rho v_k$, $z_k \in \mathcal{S}(\overline{\mathbb{R}}_+^n)$ and $z_k \to 0$ in $H^m(\mathbb{R}_+^n)$. Hence, if we set

$$w_k = v_k - z_k, \; k \in \mathbb{N},$$

we have that $w_k \in \mathcal{S}(\overline{\mathbb{R}}_+^n)$, $w_k \to u$ in $H^m(\mathbb{R}_+^n)$, and $\rho w_k = 0$ for all k. Take k so large that $\|w_k - u\|_m \leq \varepsilon/4$, and denote this w_k by w.

Now we shall approximate w by a function in $C_0^\infty(\mathbb{R}_+^n)$. As in Theorem 3.2 we shall use three tricks (but this time the translation pushes the support *into* $\mathbb{R}_+^n$!).

(i) *Truncation.* Consider $\chi_N w = \chi(x/N)w(x)$, where χ is as in (3.11). It is seen just as in the proof of Theorem 3.2 that $\chi_N w \to w$ in $H^m(\mathbb{R}_+^n)$. Choose N so large that $\|\chi_N w - w\| \leq \varepsilon/4$.

(ii) *Translation.* Consider $(\tau_{n,h}\chi_N w)(x) = \chi_N(x', x_n - h)w(x', x_n - h)$, briefly denoted $\tau_h \chi_N w$, for $h > 0$. Since $\rho w = 0$, $\tau_h \chi_N w$ (extended by 0 for $x_n < h$) is in $C^{m-1}(\mathbb{R}^n)$, and the m'th derivatives are continuous for $x_n > h$ and $x_n < h$, extendible to continuous functions for $x_n \geq h$ and for $x_n \leq h$, and the function has compact support. Then we have (as in Theorem 3.2) that $\tau_h \chi_N w \in H^m(\mathbb{R}^n)$, with distributional derivatives coinciding with the classical derivatives of order $\leq m$, defined except possibly at $x_n = h$. In view of Lemma 3.1 we can take h so small that

$$\|\chi_N w - \tau_h \chi_N w\|_{H^m(\mathbb{R}^n)} \leq \varepsilon/4.$$

(iii) *Mollification.* Consider $h_j * (\tau_h \chi_N w)$ (as usual, the extension by zero of $\tau_h \chi_N w$ for $x_n < h$ is understood). Then we have that $h_j * (\tau_h \chi_N w) \to \tau_h \chi_N w$ in $H^m(\mathbb{R}^n)$ for $j \to \infty$. For $j \geq 2/h$, the support of these C^∞ functions is contained in a compact subset of $\mathbb{R}^n_+$; so $r^+[h_j * (\tau_h \chi_N w)] \in C_0^\infty(\mathbb{R}^n_+)$. Then finally we take j so large (and greater than $2/h$) that $\|h_j * (\tau_h \chi_N w) - \tau_h \chi_N w\|_{H^m(\mathbb{R}^n)} \le \varepsilon/4$.

Altogether, we find (using also that $\|r^+ v\|_{H^m(\mathbb{R}^n_+)} \le \|v\|_{H^m(\mathbb{R}^n)}$ when $v \in H^m(\mathbb{R}^n)$) that

$$\|r^+[h_j * (\tau_h \chi_N w)] - u\|_{H^m(\mathbb{R}^n_+)} \le \varepsilon,$$

where $r^+[h_j * (\tau_h \chi_N w)] \in C_0^\infty(\mathbb{R}^n_+)$, which shows the theorem. $\square$

One finds in a completely analogous way that

$$H_0^m(\mathbb{R}^n_-) = \{\, u \in H^m(\mathbb{R}^n_-) \mid \gamma_0^- u = \dots = \gamma_{m-1}^- u = 0 \,\}.$$

Moreover we have

$$\begin{aligned} \{\, u \in H^m(\mathbb{R}^n) \mid \widetilde{\gamma}_0 u = \dots = \widetilde{\gamma}_{m-1} u = 0 \,\} & \\ = \text{ the closure of } C_0^\infty(\mathbb{R}^n_+ \cup \mathbb{R}^n_-) \text{ in } H^m(\mathbb{R}^n). & \quad (3.36) \end{aligned}$$

(The latter space could be called $H_0^m(\mathbb{R}^n_+ \cup \mathbb{R}^n_-)$, but that is not a standard notation.) In the proof of (3.36) one splits u into $r^+ u$ and $r^- u$, and one translates each of these pieces *into* $\mathbb{R}^n_+$ *resp.* $\mathbb{R}^n_-$.

3.3 Smooth domains with boundary.

We shall now extend the results to Sobolev spaces over domains Ω with a curved boundary. In the present notes we shall assume that the boundary is C^∞, but it is also possible to work with boundaries with a finite degree of smoothness (e.g. the book of Grisvard [Gri85] and its references deal with polygonal domains, and there is much recent work on domains with Lipschitz boundaries). For example, a box is a domain where $-\Delta$ can be treated by "separation of variables" methods, but the boundary is not very smooth because of the edges and corners.

We shall sometimes use the notation

$$-D = \overline{D} = +i\partial\,, \text{ then } (-D)^\alpha \text{ is briefly written } \overline{D}^\alpha. \quad (3.37)$$

A *diffeomorphism* κ (in our C^∞ setting) is a C^∞ mapping from an open set $U \subset \mathbb{R}^n$ to an open set $V \subset \mathbb{R}^n$.

$$\kappa\colon x \mapsto y = \kappa(x) = (\kappa_1(x), \dots, \kappa_n(x))\,,\ x \in U, \quad (3.38)$$

having a C^∞ inverse $\kappa^{-1} = \widetilde{\kappa}\colon y \mapsto x = (\widetilde{\kappa}_1(y), \dots, \widetilde{\kappa}_n(y))$. When $\kappa\colon U \to V$ is a C^∞ mapping, the inverse κ^{-1} exists as a C^∞ mapping in a neighborhood of a point $y = \kappa(x)$ if and only if the functional matrix (Jacobi matrix) $\kappa'(x)$ is regular; here

$$\kappa' = \begin{pmatrix} \frac{\partial \kappa_1}{\partial x_1} & \cdots & \frac{\partial \kappa_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \kappa_n}{\partial x_1} & \cdots & \frac{\partial \kappa_n}{\partial x_n} \end{pmatrix}, \quad \text{and we denote } J(x) = |\det \kappa'(x)|, \tag{3.39}$$

usually called the Jacobi determinant.

For the diffeomorphisms we consider here, it is convenient to assume that the derivatives of κ and $\widetilde{\kappa} = \kappa^{-1}$ *are bounded* on U resp. V. This is a natural assumption, satisfied in particular if κ is the identity ($\kappa(x) = x$). Generally, it can at least be obtained in a given situation by replacing U by a subset U' such that $\overline{U'}$ is compact in U (then $V' = \kappa(U')$ has $\overline{V'}$ compact in V).

A diffeomorphism $\kappa \colon U \to V$ defines a coordinate change for which one has the chain rule: When u is a differentiable function of $x \in U$, the composed function (called Tu)

$$(Tu)(y) = u(\widetilde{\kappa}(y)),\ y \in V, \tag{3.40}$$

is a differentiable function of $y \in V$, and

$$\frac{\partial (Tu)(y)}{\partial y_j} = \sum_{k=1}^{n} \frac{\partial u(x)}{\partial x_k} \frac{\partial \widetilde{\kappa}_k(y)}{\partial y_j}\Big|_{x=\widetilde{\kappa}(y)}. \tag{3.41}$$

The formula for coordinate changes extends to distributions (cf. e.g. [G03]) as follows: When $f \in L_{1,\,\mathrm{loc}}(U)$ and $\varphi \in C_0^\infty(U)$, then

$$\int_V (Tf)(y)(T\varphi)(y)dy = \int_U f(x)\varphi(x)J(x)dx, \tag{3.42}$$

and this is generalized to distributions $f \in \mathcal{D}'(U)$ by the formula (where $\psi = T\varphi$),

$$\langle Tf, \psi \rangle_V = \langle f, J \cdot T^{-1}\psi \rangle_U. \tag{3.43}$$

We shall also use the notation $\underline{u}(y)$ instead of $(Tu)(y)$.

Theorem 3.10. *Let m be an integer ≥ 0. When κ is a diffeomorphism of U onto V such that the derivatives of κ and $\widetilde{\kappa} = \kappa^{-1}$ are bounded, then T defined in* (3.40) *is a homeomorphism of $H^m(U)$ onto $H^m(V)$ (as well as from $C_0^\infty(U)$ to $C_0^\infty(V)$ and from $\mathcal{D}'(U)$ to $\mathcal{D}'(V)$).*

Proof. We first observe that T is a homeomorphism of $L_2(U)$ onto $L_2(V)$, since (3.42) holds for $f, \varphi \in L_2(U)$ with J and $1/J$ bounded. Let us also observe that T clearly maps $C_0^\infty(U)$ bijectively and continuously onto $C_0^\infty(V)$, with inverse T^{-1} defined in the analogous way from $\widetilde{\kappa}$. Then the definition (3.43) shows that the generalization maps $\mathcal{D}'(U)$ continuously onto $\mathcal{D}'(V)$ with inverse T^{-1} of the same type. We shall now show that T maps $H^m(U)$ continuously into $H^m(V)$.

Let $u \in H^m(U)$. Then we have in view of the various rules (cf. also (3.37)), for $|\alpha| \le m$ and $\psi \in C_0^\infty(V)$, $\varphi = T^{-1}\psi$,

$$\begin{aligned}\langle D_y^\alpha(Tu)(y), \psi(y)\rangle_V &= \langle Tu, \overline{D}_y^\alpha \psi\rangle_V \\ &= \langle u, J\cdot T^{-1}(\overline{D}_y^\alpha\psi)\rangle_U = \langle u, JT^{-1}(\overline{D}_y^\alpha T\varphi)\rangle_U \\ &= \langle u, \sum_{|\beta|\le|\alpha|} c_{\alpha\beta}(x)\overline{D}_x^\beta\varphi\rangle_U,\end{aligned}$$

with bounded C^∞ coefficients $c_{\alpha\beta}(x)$ that depend on κ (here (3.41) is used);

$$\begin{aligned}&= \langle \frac{1}{J}\sum_{|\beta|\le|\alpha|} D^\beta(c_{\alpha\beta}u), JT^{-1}\psi\rangle_U \\ &= \langle \sum_{|\beta|\le|\alpha|} c'_{\alpha\beta}D^\beta u, JT^{-1}\psi\rangle_U = \langle T(\sum_{|\beta|\le|\alpha|} c'_{\alpha\beta}D^\beta u), \psi\rangle_V,\end{aligned}$$

where the $c'_{\alpha\beta}$ are bounded C^∞ functions. Since $u \in H^m(U)$, the function $\sum_{|\beta|\le|\alpha|} c'_{\alpha\beta}D^\beta u$ is in $L_2(U)$, and T of this is in $L_2(V)$. It equals $D^\alpha Tu$, so we see that $D^\alpha Tu \in L_2(V)$ for $|\alpha| \le m$; thus T maps $H^m(U)$ into $H^m(V)$. The mapping is continuous, for one has for $|\alpha| \le m$,

$$\begin{aligned}\|D_y^\alpha(Tu)\|_{L_2(V)} &= \sup\{\, |\langle D_y^\alpha(Tu), \psi\rangle_V| \;\big|\; \psi \in C_0^\infty(V), \|\psi\| \le 1\} \\ &\le \sup\{\, |\langle \sum_{|\beta|\le|\alpha|} c'_{\alpha\beta}D^\beta u, JT^{-1}\psi\rangle_U \;\big|\; \|\psi\| \le 1\,\} \\ &\le C\|u\|_{H^m(U)},\end{aligned}$$

since $T^{-1}\colon L_2(V) \to L_2(U)$ is bounded, and the functions $J(x)$ and $c'_{\alpha\beta}(x)$ are bounded.

The analogous mapping associated with $\widetilde{\kappa}$ likewise sends $H^m(V)$ continuously into $H^m(U)$, and is precisely the inverse of T. $\square$

Remark 3.11. The theorem holds also for Sobolev spaces with noninteger or negative exponent. (Here $H^t(\Omega)$ is defined in general as the set of restrictions to Ω of distributions in $H^t(\mathbb{R}^n)$, with norm

$$\|u\|_{H^t(\Omega)} = \inf\{\, \|v\|_{H^t(\mathbb{R}^n)} \mid v \in H^t(\mathbb{R}^n),\ u = v|_\Omega\,\}; \tag{3.44}$$

one can show that this is consistent with (1.7) for t integer ≥ 0.) For noninteger exponents ≥ 0 one can either use a theorem on interpolation in Hilbert spaces, or a more direct calculation based on the fact that

$$\Big(\|u\|^2_{H^m(\mathbb{R}^n)} + \sum_{|\alpha|=m} \int_{\mathbb{R}^{2n}} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy\Big)^{\frac{1}{2}}$$

is an equivalent norm on $H^{m+s}(\mathbb{R}^n)$ for $s \in\,]0,1[\,$, m integer ≥ 0. For negative exponents, one can use the duality between $H^t(\mathbb{R}^n)$ and $H^{-t}(\mathbb{R}^n)$. Further details can be found e.g. in [L-M68] and in Hörmander [H63].

In order to define our smooth open sets Ω, we introduce the notation (for $a > 0$):

$$\begin{aligned} Q_a &= \{\, x \in \mathbb{R}^n \mid |x_j| < a \text{ for } j = 1, \dots, n \,\}, \\ Q'_a &= \{\, x' \in \mathbb{R}^{n-1} \mid |x_j| < a \text{ for } j = 1, \dots, n-1 \,\}, \\ Q^+_a &= \{\, x \in Q_a \mid x_n > 0 \,\}, \end{aligned} \tag{3.45}$$

describing cubes in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ with "radius" a and "center" 0. For arbitrary points $x_0 \in \mathbb{R}^n$ we define $Q_a(x_0) = \{\, x \in \mathbb{R}^n \mid x - x_0 \in Q_a \,\}$, etc.

Definition 3.12. *Let Ω be an open subset of $\mathbb{R}^n$. Ω is called* **smooth** *if for each boundary point $x_0 \in \partial\Omega$ there exists a neighborhood U of x_0, a number $a > 0$, and a diffeomorphism κ of U onto Q_a (cf. (3.38) ff.) such that*

$$\begin{aligned} \kappa(U \cap \Omega) &= \{\, y \in Q_a \mid y_n > 0 \,\}\ [= Q^+_a], \\ \kappa(U \cap \partial\Omega) &= \{\, y \in Q_a \mid y_n = 0 \,\}\ [= Q'_a], \\ \kappa(x_0) &= 0. \end{aligned} \tag{3.46}$$

The restriction of κ to the mapping from $U \cap \partial\Omega$ to Q'_a will be called λ, with inverse λ^{-1} called $\widetilde{\lambda}$.

The triple (κ, U, Q_a) is called a local coordinate system. Of course, Q_a can be replaced by other types of open sets (e.g. balls as in [G03]). In the following we often use the notation

$$\partial\Omega = \Gamma. \tag{3.47}$$

Observe that when $(\kappa_{(1)}, U_1, Q_a)$ and $(\kappa_{(2)}, U_2, Q_a)$ are given as in Definition 3.12, and $U_1 \cap U_2 \neq \emptyset$, then

$$\begin{gathered} \kappa_{(1)} \circ \widetilde{\kappa}_{(2)} \colon \kappa_{(2)}(U_1 \cap U_2) \to \kappa_{(1)}(U_1 \cap U_2) \text{ (subsets of } Q_a) \\ \text{is a diffeomorphism preserving the property } y_n = 0\ . \end{gathered} \tag{3.48}$$

In the following we assume for simplicity (unless other informations are given) that Ω is *bounded*. Then Γ can be covered by a *finite* family of open sets U as in Definition 3.12, so that a neighborhood of the boundary is described by a finite family of coordinate systems $(\kappa_{(j)}, U_j, Q_a)_{j=1}^N$. Now one can choose a system of functions $\{\,\psi_j\,\}_{j=0,\dots,N}$ such that, with $U_0 = \Omega$,

$$\psi_j \in C_0^\infty(U_j), \psi_j \geq 0\,, \text{ and } \sum_{j=0}^N \psi_j = 1 \text{ on a neighborhood of } \overline{\Omega}. \tag{3.49}$$

(As usual, we identify the functions in $C_0^\infty(U_j)$ with functions in $C_0^\infty(\mathbb{R}^n)$ having support in U_j.) This is a partition of unity for $\overline{\Omega}$, subordinate to the covering by the U_j, $j = 0, \dots, N$. Then for any functions u on Ω resp. v on Γ,

$$u = \sum_{j=0}^N \psi_j u, \quad v = \sum_{j=1}^N \psi_j v,$$

where the pieces $\psi_j u$ and $\psi_j v$ for $j \geq 1$ are carried over to functions on Q_a^+ resp. Q_a' by the diffeomorphisms $\kappa_{(j)}$.

It suffices for our general purposes to have one such family of local coordinate systems and partition of unity; all other choices must be compatible with it (by relations such as (3.48)).

Now it is not difficult to generalize Theorems 3.2 and 3.3 to Ω by the help of the local coordinates.

Theorem 3.13. *Let Ω be a smooth open bounded subset of $\mathbb{R}^n$, and let $m \in \mathbb{N}$. Then one has:*

1° *$C^\infty(\overline{\Omega})$ is dense in $H^m(\Omega)$.*

2° *There exists a continuous linear operator $p_{(m)} : H^m(\Omega) \to H^m(\mathbb{R}^n)$ such that (cf.* (3.3)*)*

$$r_\Omega p_{(m)} u = u \text{ for all } u \in H^m(\Omega). \tag{3.50}$$

Proof. 1°. Let $u \in H^m(\Omega)$, then with the above notation, $u = \sum_{j=0}^N u_j$ where $u_j = \psi_j u$. For $j = 0$ we can approximate u_0 in $H^m(\Omega)$ by a sequence of functions $\{\,\varphi_{0,k}\,\}_{k\in\mathbb{N}}$ in $C_0^\infty(\Omega)$, by taking e.g. $\varphi_{0,k} = h_k * u_0$ as in Theorem 3.2 (here u_0 should at first be considered as extended by 0 to $\mathbb{R}^n$, but $\varphi_{0,k} \in C_0^\infty(\Omega)$ when $k \geq 2/\delta$, where δ is the distance from $\operatorname{supp} u_0$ to $\partial\Omega$). For $j \geq 1$, let $v_j = u_j \circ \widetilde{\kappa}_{(j)}$, then $v_j \in H^m(Q_a^+)$ (by Theorem 3.10) and its support is a closed subset of $\{\, y \in Q_a \mid y_n \geq 0\,\}$, so that it can be identified with its extension by 0 to all of $\mathbb{R}^n_+$, which belongs to $H^m(\mathbb{R}^n_+)$. By Theorem 3.2, there is a sequence $\{\,\varphi_{j,k}\,\}_{k\in\mathbb{N}}$ in $C_{(0)}^\infty(\overline{\mathbb{R}}^n_+)$ converging to v_j in $H^m(\mathbb{R}^n_+)$.

Let $\eta_j \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\eta_j \subset Q_a$ and $\eta_j = 1$ on $\operatorname{supp} v_j$, then since $\eta_j v_j = v_j$,

$$\|\eta_j\varphi_{j,k} - v_j\|^2_{H^m(\mathbb{R}^n_+)} = \|\eta_j\varphi_{j,k} - \eta_j v_j\|^2_{H^m(Q_a^+)}$$
$$= \sum_{|\alpha|\le m} \|D^\alpha(\eta_j(\varphi_{j,k} - v_j))\|^2_{L_2(Q_a^+)} \to 0$$

for $k \to \infty$. Now let

$$u_k = \varphi_{0,k} + \sum_{j=1}^N (\eta_j\varphi_{j,k}) \circ \kappa_{(j)},$$

then $u_k \in C^\infty(\overline{\Omega})$ and $u_k \to u$ in $H^m(\Omega)$.

The proof of 2° goes in a similar way. Here we apply Theorem 3.3 to each of the functions v_j with $j > 0$ (choosing the extension operator $p_{(m),j}$ so that $p_{(m),j}v_j$ is supported in Q_a); then the sum of the extensions, carried back by use of the $\kappa_{(j)}$, is an extension of the original function u; this defines a linear, continuous extension operator $p_{(m)}$. □

A generalization of the trace theorems requires a little more explanation; in particular we have to define Sobolev spaces over Γ. In the following, we refer to a fixed choice of local coordinates $(\kappa_{(j)}, U_j, Q_a)_{j=1}^N$, and leave it as an exercise for the reader to verify that the qualitative definitions, we give, do not depend on the particular choice.

Each of the diffeomorphisms $\kappa_{(j)}: U_j \to Q_a$ induces a mapping $\lambda_{(j)}: U_j \cap \Gamma \to Q'_a$ with inverse $\widetilde{\lambda}_{(j)}: Q'_a \to U_j \cap \Gamma$ (consisting of n C^∞ maps sending $y' \in Q'_a$ to $x = (\widetilde{\lambda}_{(j)1}(y'), \dots, \widetilde{\lambda}_{(j)n}(y'))$. $\lambda_{(j)}$ and $\widetilde{\lambda}_{(j)}$ do not satisfy our definition of diffeomorphisms, since $U_j \cap \Gamma$ is generally not an open subset of $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$, but the composition

$$\lambda_{(1)} \circ \widetilde{\lambda}_{(2)} : \lambda_{(2)}(U_1 \cap U_2 \cap \Gamma) \to \lambda_{(1)}(U_1 \cap U_2 \cap \Gamma), \tag{3.51}$$

that goes from one subset of Q'_a to another, is a diffeomorphism (in $n-1$ variables). We shall view them as diffeomorphisms in a more general sense. (This kind of observation is the starting point in a more general description of C^∞ manifolds, that we however refrain from taking up systematically.)

The Lebesgue measure on $\mathbb{R}^n$ induces a surface measure $d\sigma$ on Γ that is used in integrations over Γ. Moreover, there is defined a normal vector at each $x \in \Gamma$; here we always take the *unit normal vector* $\vec{n}(x)$ *directed towards the interior of* Ω,

$$\vec{n}(x) = (n_1(x), \dots, n_n(x)), \tag{3.52}$$

$\vec{n}(x)$ can be defined from the local coordinate systems as the vector orthogonal to the tangent vectors $\partial\widetilde{\lambda}_{(j)}(y')/\partial y_k, k = 1, \dots, n-1$. In its dependence on x it is a C^∞ vector field (each $n_k(x)$ depends C^∞ on $y' \in Q'_a$ in each local coordinate system). The surface measure $d\sigma$ is defined precisely such that one has the Gauss formula, for $k = 1, \dots, n$,

$$\int_{x\in\Omega} \partial_{x_k} u(x)\,dx = -\int_{x\in\Gamma} n_k(x)u(x)\,d\sigma. \tag{3.53}$$

Example 3.15. For concreteness, consider the case where Ω is represented as the set

$$\Omega = \{\,(x', x_n) \mid x_n > f(x')\,\},$$

where $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a C^∞ function. (We here let x' run in $\mathbb{R}^{n-1}$ for simplicity, leaving local formulations to the reader.) Here the mapping

$$\kappa : (x', x_n) \mapsto (x', x_n - f(x')),$$

is a diffeomorphism mapping Ω onto $\mathbb{R}^n_+$ and mapping $\Gamma = \partial\Omega$ onto $\mathbb{R}^{n-1}$ (with $\kappa^{-1}\colon (x', x_n) \mapsto (x', x_n + f(x'))$). The interior unit normal at the point $(x', f(x'))$ is

$$\vec{n}(x) = \frac{(-\partial_{x_1} f(x'), \dots, -\partial_{x_{n-1}} f(x'), 1)}{\sqrt{(\partial_{x_1} f)^2 + \cdots + (\partial_{x_{n-1}} f)^2 + 1}}.$$

The surface measure on Γ is defined such that one has

$$\int_{(x',f(x'))\in\Gamma} u(x', f(x'))\,d\sigma = \int_{x'\in\mathbb{R}^{n-1}} u(x', f(x'))J_\sigma(x')\,dx',$$
$$\text{where } J_\sigma(x') = \sqrt{(\partial_{x_1} f(x'))^2 + \cdots + (\partial_{x_{n-1}} f(x'))^2 + 1},$$

for this is consistent with the Gauss formula (3.53) for ∂_{x_n} (which, by the coordinate change κ, is reduced to a simple integration of a derivative).

The example can be used for general Ω in the neighborhood of each boundary point, by a suitable placement of the x_n-axis.

Now let us describe some function spaces (and distribution spaces) over Γ. We identify $L_{p,\,\mathrm{loc}}(\Gamma)$ (for $p \geq 1$) with the space of functions u on Γ such that $u(\widetilde{\lambda}_{(j)}(y')) \in L_{p,\,\mathrm{loc}}(Q'_a)$ in each local coordinate system. (Then the functions are only determined "almost everywhere" with respect to the measure $d\sigma$.) This has a sense also if Γ is unbounded. When Γ is bounded, $L_{p,\,\mathrm{loc}}(\Gamma) = L_p(\Gamma)$, with the norm

$$\|u\|_{L_p(\Gamma)} = \left(\int_\Gamma |u(x)|^p d\sigma\right)^{1/p}, \tag{3.54}$$

that can also be written in terms of the local coordinates, by use of a partition of unity. The spaces $L_p(\Gamma)$ are Banach spaces; in particular, $L_2(\Gamma)$ is a Hilbert space with respect to the scalar product

$$(f,g)_\Gamma = \int_\Gamma f(x)\overline{g}(x)d\sigma. \tag{3.55}$$

The spaces $C^k(\Gamma)$ $(0 \le k \le \infty)$ are defined as the spaces of functions on Γ that are C^k functions in the local coordinate systems. This has a meaning also if Γ is unbounded; when Γ is bounded, the spaces $C^k(\Gamma)$ for $k < \infty$ are Banach spaces, whose norm can be taken as

$$\begin{aligned}&\|u\|_{C^k(\Gamma)}\\ &\quad = \sup\{\, |D^\alpha_{y'}(\psi_j u)(\widetilde{\lambda}_{(j)}(y'))| \;\big|\; y' \in Q'_a, |\alpha| \le k, j = 1,\dots,N\,\}\end{aligned} \tag{3.56}$$

(another system of local coordinates and partition of unity will give a norm equivalent with this). The whole set of these norms, for $k = 1, 2, \dots$, describes the Fréchet topology on $C^\infty(\Gamma)$.

Differential operators on $C^\infty(\Gamma)$ are defined by the help of the local coordinate systems:

A differential operator P of order r is a linear operator $P\colon C^\infty(\Gamma) \to C^\infty(\Gamma)$ such that for each $j = 1, \dots, N$,

$$(Pu)(\widetilde{\lambda}_{(j)}(y')) = \sum_{|\alpha| \le r} a_{j,\alpha}(y') D^\alpha_{y'}(u(\widetilde{\lambda}_{(j)}(y'))), \tag{3.57}$$

with coefficients $a_{j,\alpha} \in C^\infty(Q'_a)$. — Sometimes one needs to *construct* a differential operator on Γ from differential operators given in Q'_a for each j. Here, if P_j is given as a differential operator of order r in Q'_a for each $j = 1, \dots, N$, then P, defined for $u \in C^\infty(\Gamma)$ by the formula

$$Pu = \sum_{j=1}^N P_j\big((\psi_j u) \circ \widetilde{\lambda}_{(j)}\big) \circ \lambda_{(j)} \tag{3.58}$$

is a differential operator of order r on Γ.

The space $\mathcal{D}'(\Gamma)$ of distributions on Γ is defined as usual as the space of continuous linear functionals u on $C^\infty(\Gamma)$, with the notation $\langle u, \varphi\rangle_\Gamma$ for $u \in \mathcal{D}'(\Gamma)$, $\varphi \in C^\infty(\Gamma)$. Here we identify $L_1(\Gamma)$ with a subspace of $\mathcal{D}'(\Gamma)$ by the formula

$$\langle u, \varphi\rangle_\Gamma = \int_\Gamma u\varphi\, d\sigma \text{ when } u \in L_1(\Gamma),\ \varphi \in C^\infty(\Gamma). \tag{3.59}$$

(We note that since Γ is bounded, $L_p(\Gamma) \subset L_1(\Gamma)$ for all $p \geq 1$.) The distributions on Γ can be carried over to distributions on Q'_a, for $j = 1, \dots, N$, by rules for coordinate changes imitating those that hold for functions in $L_1(\Gamma)$. Since Γ is bounded and is described by a finite set of local coordinate systems one finds that the distributions in $\mathcal{D}'(\Gamma)$ have *finite order* (i.e., for each $u \in \mathcal{D}'(\Gamma)$ there is an N so that u in the local coordinates can be written as a sum of derivatives up to order N, of continuous functions).

With these preparatory remarks, we now define $H^s(\Gamma)$ for each $s \in \mathbb{R}$ by use of the local coordinate systems and a partition of unity as follows:

$$\begin{aligned} H^s(\Gamma) = \{\, u \in \mathcal{D}'(\Gamma) \mid (\psi_j u)(\widetilde{\lambda}_{(j)}(y')) \in H^s(\mathbb{R}^{n-1}) \,\}, \\ \text{with norm} \\ \|u\|_s = \Big(\sum_{j=1}^{N} \|(\psi_j u)(\widetilde{\lambda}_{(j)}(y'))\|^2_{H^s(\mathbb{R}^{n-1})} \Big)^{\frac{1}{2}}. \end{aligned} \tag{3.60}$$

The value of the norm clearly depends on the choice of local coordinates and partition of unity, but one can show using Remark 3.11 that the space $H^s(\Gamma)$ is independent of this choice. (One regards $H^s(\Gamma)$ as a "Hilbertable" space rather than as a Hilbert space.) Actually, one often uses the norm definition in (3.60) for $s \geq 0$ only and sets

$$\|u\|_{-s} = \sup\{ \frac{|\langle u, \bar{\varphi} \rangle|}{\|\varphi\|_s} \mid \varphi \in C^\infty(\Gamma)\,,\ \varphi \neq 0 \,\}; \tag{3.61}$$

this gives a norm on $H^{-s}(\Gamma)$, since $C^\infty(\Gamma)$ is dense in all $H^t(\Gamma)$ spaces and there is an identification

$$H^{-s}(\Gamma) = (H^s(\Gamma))^*. \tag{3.62}$$

For $H^0(\Gamma)$ it is preferable to use the $L_2(\Gamma)$ norm (3.54).

Since the distributions in $\mathcal{D}'(\Gamma)$ are of finite order, and $C^0(\Gamma) \subset L_2(\Gamma)$, one has that

$$\mathcal{D}'(\Gamma) = \bigcup_{t \in \mathbb{R}} H^t(\Gamma). \tag{3.63}$$

3.4 The trace operators on smooth domains.

The first trace operator γ_0 is of course defined for $u \in C^0(\overline{\Omega})$ simply by

$$\gamma_0 u = u|_\Gamma. \tag{3.64}$$

For the definition of γ_0 on Sobolev spaces, and for the definition of normal derivatives, we need a more careful description of Ω near Γ. For this purpose we introduce some particularly suitable local coordinate systems (by a standard construction from differential geometry, included here for completeness).

Let $\vec{n}(x)$ be the unit normal vector field defined for $x \in \Gamma$, oriented towards Ω. For $z \in \Gamma$ and $\delta > 0$, let $L_{z,\delta}$ be the line interval

$$L_{z,\delta} = \{\, x = z + t\vec{n}(z) \mid t \in]-\delta, \delta[\,\}, \tag{3.65}$$

and let $\Sigma_\delta = \bigcup_{z\in\Gamma} L_{z,\delta}$. For a local coordinate system $(\kappa_{(j)}, U_j, Q_a)$, consider the mapping

$$\underset{\in Q'_a\times\mathbb{R}}{(y',t)} \mapsto (z,t) = \underset{\in(\Gamma\cap U_j)\times\mathbb{R}}{(\widetilde{\lambda}_{(j)}(y'),t)} \mapsto x = \underset{\in\mathbb{R}^n}{z + t\vec{n}(z)}, \tag{3.66}$$

where $z = \widetilde{\lambda}_{(j)}(y') = (z_1(y'), \dots, z_n(y'))$; it is a C^∞ map of $Q'_a \times \mathbb{R}$ into $\mathbb{R}^n$, temporarily denoted $\widetilde{\mu}$. Observe that at a fixed $y'_0 \in Q'_a$,

$$\begin{aligned} \frac{\partial}{\partial y_l}\big(z_k(y') + tn_k(z(y'))\big)|_{y'=y'_0,t=0} &= \frac{\partial z_k}{\partial y_l}(y'_0), \\ \frac{\partial}{\partial t}\big(z_k(y') + tn_k(z(y'))\big)|_{y'=y'_0,t=0} &= n_k(z(y'_0)), \end{aligned} \tag{3.67}$$

for $l = 1, \dots, n-1$ and $k = 1, \dots, n$, where $\partial z/\partial y_l = (\partial z_1/\partial y_l, \dots, \partial z_n/\partial y_l)$ for each l is a vector in the tangent plane to Γ at the point $z(y'_0)$. The vectors $\partial z/\partial y_1, \dots, \partial z/\partial y_{n-1}$ are linearly independent since they are the first $n-1$ columns in the functional matrix of $\widetilde{\kappa}_{(j)}$ at $(y'_0, 0)$; they span the tangent plane. Then also the n vectors $\partial z/\partial y_1, \dots, \partial z/\partial y_{n-1}$ and $\vec{n}(z)$ (considered at $z(y'_0)$) are linearly independent. In view of (3.67) they equal the columns in the functional matrix of the mapping $\widetilde{\mu}\colon (y',t) \mapsto x$ (cf. (3.66)) evaluated at $(y'_0, 0)$. Thus this functional matrix is invertible at $(y'_0, 0)$. The invertibility extends to a neighborhood, so we can take $r > 0$ and $\delta > 0$ so small that the mapping $\widetilde{\mu}$ is a diffeomorphism from $Q'_r(y'_0)\times]-\delta, \delta[$ to $U'_j = \{\, z(y') + t\vec{n}(z(y')) | y' \in Q'_r(y'_0), |t| < \delta \,\}$, with $U'_j \subset U_j$; in particular, U'_j is an open neighborhood of $z(y'_0)$. This can be carried out for every $y'_0 \in Q'_a$ (in fact every smaller closed cube $\overline{Q'_{a'}}$ with $a' \in]0, a[$ is covered by a finite family of such open cubes $Q'_r(y'_0)$). With a similar procedure for the other coordinate systems we obtain altogether a new family of local coordinate systems

$$\begin{aligned} Q'_{r_l}(y_l)\times]-\delta_l, \delta_l[&\overset{\text{(i)}}{\to} V'_l \times]-\delta_l, \delta_l[\\ &\overset{\text{(ii)}}{\to} V_l = \{\, z + t\vec{n}(z) \mid z \in V'_l, |t| < \delta_l \,\}, \end{aligned} \tag{3.68}$$

where the mapping (i) is of the form $\widetilde{\lambda}_{(j)} \times \mathrm{id}\colon (y', t) \mapsto (z, t)$ for some j, and (ii) maps (z, t) into $z + t\vec{n}(z)$; the full mapping will be called $\widetilde{\mu}_{(l)}$. It suffices to let l run in a finite set $l = 1, \dots, l_0$ in order to cover Γ by such sets V_l', in view of the compactness of Γ.

These coordinate systems are convenient because the t-coordinate represents the normal coordinate, for each l. They can be made still more convenient in the following way: We claim that if we replace all the δ_l's by a sufficiently small positive δ, then the *globally defined mapping*

$$\begin{aligned} \sigma\colon \Gamma \times \,]-\delta, \delta[\, \to \Sigma_\delta = \{\, z + t\vec{n}(z) \mid z \in \Gamma, |t| < \delta \,\}, \\ \text{where } \sigma(z, t) = z + t\vec{n}(z), \end{aligned} \tag{3.69}$$

is a diffeomorphism (in a general sense). Indeed, consider numbers $\delta > 0$ smaller than all the δ_l. Clearly, σ in (3.69) is defined in such a way that it is, for δ small enough, a diffeomorphism in the neighborhood of each point, where it can be described by a composition of $\lambda_{(j)} \times \mathrm{id}$ (recall that $\lambda_{(j)}\colon U_j \cap \Gamma \to Q_a'$) and $\widetilde{\mu}_{(l)}$ (for suitable j and l). Now if (3.69) were not injective for any $\delta > 0$, there would for each $n \in \mathbb{N}$ exist two different points (z_n, t_n) and $(w_n, s_n) \in \Gamma \times \,]-1/n, 1/n[$ such that

$$z_n + t_n\vec{n}(z_n) - w_n - s_n\vec{n}(w_n) = 0. \tag{3.70}$$

Clearly $t_n \to 0$ and $s_n \to 0$ for $n \to \infty$, and by the compactness of Γ we can (by taking subsequences) obtain that $z_n \to z_0$, $w_n \to w_0$ for $n \to \infty$. In the limit we get $z_0 = w_0$ from (3.70). But then, since σ is a bijection in the neighborhood of $(z_0, 0)$, we conclude that $z_n = w_n$ and $t_n = s_n$ for all sufficiently large n, which shows the contradiction.

Let us sum up what we have shown (in a slightly modified notation):

Theorem 3.16. *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$. There exists a family of local coordinate systems $\mu_{(j)}\colon U_j \to Q_a' \times \,]-\delta, \delta[\, = \mu_{(j)}(U_j)$, $j = 1, \dots, N$, $\bigcup_{j=1}^{N} U_j \supset \Gamma$, with the properties:*

(i) *$\mu_{(j)}$ maps $U_j \cap \Gamma$ onto $Q_a' \times \{\,0\,\}$ (we denote this restricted mapping by $\lambda_{(j)}$ and its inverse by $\widetilde{\lambda}_{(j)}$);*

(ii) *the inverse of $\mu_{(j)}$, denoted $\widetilde{\mu}_{(j)}$, is of the form*

$$\widetilde{\mu}_{(j)}\colon (y', t) \mapsto z + t\vec{n}(z), \; z = \widetilde{\lambda}_{(j)}(y'), \tag{3.71}$$

where $\vec{n}(z)$ is the interior unit normal to Γ at z;

(iii) *the mapping* (3.69) *can be expressed locally by* $\sigma = \widetilde{\mu}_{(j)} \circ (\lambda_{(j)} \times \mathrm{id})$*; it is a diffeomorphism of* $\Gamma \times]-\delta, \delta[$ *onto* $\Sigma_\delta = \bigcup_{j=1}^{N} U_j$.

We call such families of local coordinate systems *special local coordinates.* Σ_δ is often called *a tubular neighborhood* of Γ.

Consider the last coordinate in the mapping $\mu_{(j)}$, denoted

$$\rho_{(j)} \colon U_j \to]-\delta, \delta[\,. \tag{3.72}$$

When $(\mu_{(k)}, U_k, Q'_a \times]-\delta, \delta[\,)$ is another coordinate system in the above family, $\rho_{(j)}(x)$ must equal $\rho_{(k)}(x)$ for $x \in U_j \cap U_k$, since σ is bijective. Then in fact the $\rho_{(j)}$, taken together, define globally a mapping

$$\rho \colon \Sigma_\delta = \textstyle\bigcup_{j=1}^{N} U_j \to]-\delta, \delta[\,; \tag{3.73}$$

it is C^∞ and has the property that Γ is precisely the set

$$\Gamma = \{\, x \in \Sigma_\delta \mid \rho(x) = 0 \,\}. \tag{3.74}$$

We note moreover that the gradient of ρ equals $\vec{n}(x)$ at $x \in \Gamma$, for it is normal to Γ, since

$$0 = \frac{\partial}{\partial y_l} t = \frac{\partial}{\partial y_l}(\rho \circ \widetilde{\mu}_{(j)}(y', t)) = \sum_{k=1}^{n} \frac{\partial \rho}{\partial x_k} \frac{\partial}{\partial y_l} \widetilde{\mu}_{(j),k}(y', t),$$

where the vectors $\partial \widetilde{\mu}_{(j)} / \partial y_l$ span the tangent space, and it has length 1 since

$$1 = \frac{\partial}{\partial t} t = \frac{\partial}{\partial t}(\rho \circ \widetilde{\mu}_{(j)}(y', t)) = \sum_{k=1}^{n} \frac{\partial \rho}{\partial x_k} \frac{\partial}{\partial t} \widetilde{\mu}_{(j),k}(y', t),$$

where $\frac{\partial}{\partial t} \widetilde{\mu}_{(j)}(y', t) = \frac{\partial}{\partial t}(z + t\vec{n}(z)) = \vec{n}(z)$ with length 1. The sets $\Gamma_s = \{\, x \in \Sigma_\delta \mid \rho(x) = s \,\}$ are smooth manifolds for $|s| < \delta$, usually called the *parallel surfaces* to Γ. The mappings $\mu_{(j)}$ define local coordinates on the Γ_s by restriction to $t = s$.

When there is given a family of special local coordinates as in Theorem 3.16, we usually choose an associated partition of unity to be on the following convenient form: When $x = \widetilde{\mu}_{(j)}(y', t)$, we let

$$\psi_j(x) = \varphi_j(\widetilde{\lambda}_{(j)}(y'))\zeta(t), \tag{3.75}$$

where $\zeta \in C_0^\infty(\,]-\delta, \delta[\,)$, $\zeta(t) = 1$ for $|t| < \delta/2$, say, and the functions $\varphi_j \circ \widetilde{\lambda}_{(j)} \in C_0^\infty(Q'_a)$ are chosen such that $\sum_{j=1}^{N} \varphi_j(x) = 1$ for $x \in \Gamma$; then also $\sum_{j=1}^{N}\psi_j(x)=1$ for $x\in\Sigma_{\delta/2}$. The whole system can be supplied with a function $\psi_0\in C_0^\infty(\Omega)$, such that $\sum_{j=0}^{N}\psi_j=1$ on $\Omega\cup\Sigma_{\delta/2}$.

We now *extend the normal vector* field $\vec{n}$, given on Γ, *to* Σ_δ *by the definition*

$$\vec{n}(x)=\vec{n}(z)\ \text{ when } x=z+t\vec{n}(z),\ z\in\Gamma \text{ and } |t|<\delta, \tag{3.76}$$

this is clearly a C^∞ vector field. Here for $t=s$, $\vec{n}(x)$ is the normal to Γ_s at x and equals the gradient of $\rho(x)$.

Example 3.17. Of course one could extend the normal vector field to a neighborhood of Γ in other smooth ways, e.g. in Example 3.15 one could simply take $\vec{n}(x',f(x')+s)=\vec{n}(x',f(x'))$ for $s\neq 0$.

The reader is encouraged to make a calculation of what the special local coordinates look like, when Γ is given as in Example 3.15, for various choices of f.

The definition (3.76) is convenient, because *differentiation along* $\vec{n}(x)$ takes a particularly simple form. Let $D_{\vec{n}}$ be the first order differential operator defined on Σ_δ by

$$(D_{\vec{n}}u)(x)=\sum_{k=1}^{n}n_k(x)D_{x_k}u(x) \tag{3.77}$$

(the differentiation along the vector field $\vec{n}$, multiplied by $-i$). The special coordinates in Theorem 3.16 are such that

$$D_{\vec{n}}u(x)\ \text{goes over to}\ D_t\underline{u}(y',t), \tag{3.78}$$

when

$$\underline{u}(y',t)=u(\widetilde{\mu}_{(j)}(y',t)).$$

For, by the chain rule,

$$\partial_t\underline{u}(y',t)=\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}(x)\frac{\partial}{\partial t}\widetilde{\mu}_{(j),k}(y',t),$$

where $\partial_t\widetilde{\mu}_{(j)}(y',t)=\partial_t(z+t\vec{n}(z))=\vec{n}(z)=\vec{n}(x)$, cf. (3.76). Then we can simply read $D_{\vec{n}}$ as the derivative D_t in the representation of Σ_δ as $\Gamma\times\,]-\delta,\delta[$ in (3.69), without even having to refer to the local coordinates. $D_{\vec{n}}u$ is called *the normal derivative* of u.

Note that if u is constant along the lines $L_{z,\delta}$ (cf. (3.65)), then $D_{\vec{n}}u$ is 0, since $\underline{u}$ is independent of t then. This holds in particular for the coordinates $n_k(x)$ of $\vec{n}$, since $\vec{n}$ is chosen to be constant along the lines $L_{z,\delta}$:

$$D_{\vec{n}}n_k(x)=0 \text{ for } k=1,\dots,n. \tag{3.79}$$

We can also define higher normal derivatives $(D_{\vec{n}})^k u$, and note that in view of (3.79),

$$\begin{aligned}(D_{\vec{n}})^k u &= (\sum_{j=1}^n n_j D_j)\cdots(\sum_{l=1}^n n_l D_l)(\sum_{m=1}^n n_m D_m)u \\ &= (\sum_{j=1}^n n_j D_j)\cdots\sum_{l=1}^n\sum_{m=1}^n n_l n_m D_l D_m u \\ &= \sum_{j=1}^n\cdots\sum_{l=1}^n\sum_{m=1}^n (n_j\cdots n_l n_m)(D_j\cdots D_l D_m)u \\ &= \sum_{|\alpha|=k}\frac{k\,!}{\alpha\,!}n^\alpha D^\alpha u\end{aligned} \tag{3.80}$$

(by the "multi-nomial" formula

$$(\sum_{j=1}^n a_j)^k = \sum_{|\alpha|=k}\frac{k\,!}{\alpha\,!}a^\alpha, \tag{3.81}$$

used in a formal way). $(D_{\vec{n}})^k$ is called *the normal derivative of order* k, and we have again:

$$(D_{\vec{n}})^k u(x) \text{ corresponds to } D_t^k \underline{u}(y',t) \text{ in special coordinates.} \tag{3.82}$$

An arbitrary differential operator (1.13) can be decomposed in the form

$$A = \sum_{k=0}^r S_k D_{\vec{n}}^k \text{ or } \sum_{k=0}^r D_{\vec{n}}^k S_k', \tag{3.83}$$

where the S_k and S_k' are differential operators of order $\le r-k$ that are *tangential*, in the sense that they in the special local coordinates only contain derivatives with respect to y' (but have coefficients depending on $y=(y',t)$),

$$S_k \longleftrightarrow \sum_{|\beta|\le r-k} b_\beta(y',t)D_{y'}^\beta \text{ in local coordinates.}$$

The representations in (3.83) are obtained straightforwardly from the form of A in the special local coordinates.

The coordinate t will usually be called y_n from now on.

When S is a tangential operator, represented in special local coordinates by

$$S \longleftrightarrow \sum_{|\beta|\le m} s_\beta(y', y_n) D_{y'}^\beta \text{ for each } j,$$

then we denote by S_Γ the operator obtained by restriction to $y_n = 0$, i.e.

$$S_\Gamma \longleftrightarrow \sum_{|\beta|\le m} s_\beta(y', 0) D_{y'}^\beta \text{ for each } j; \tag{3.84}$$

the indexation by Γ is sometimes also used to denote the (localized) operator on the right hand side.

We now define the higher order trace operators by

$$\gamma_j u = \gamma_0 (D_{\vec{n}})^j u\,, \text{ for } u \in C^\infty(\overline{\Omega}), \tag{3.85}$$

(where only the values of u on Σ_δ play a role); and we note that in the special local coordinates, γ_j carries over to precisely the operator that was called γ_j in Section 3.1, see (3.22).

Theorem 3.18. *Let Ω be an open, smooth bounded subset of $\mathbb{R}^n$ with boundary $\partial\Omega = \Gamma$ and define the normal vector field $\vec{n}$ as above. Let m be a positive integer.*

1° For each integer $j \in [0, m-1]$, the j'th normal derivative $\gamma_j \colon C^\infty(\overline{\Omega}) \to C^\infty(\Gamma)$ extends to a continuous linear mapping $\gamma_j \colon H^m(\Omega) \to H^{m-j-\frac12}(\Gamma)$.

2° The Cauchy operator defined in this way,

$$\rho_{(m)} = \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{m-1} \end{pmatrix} : H^k(\Omega) \to \prod_{j=0}^{m-1} H^{k-j-\frac12}(\Gamma), \quad k \ge m \tag{3.86}$$

(k integer), has a right inverse $\mathcal{K}_{(m)}$ that is continuous:

$$\mathcal{K}_{(m)} \colon \prod_{j=0}^{m-1} H^{k-j-\frac12}(\Gamma) \to H^k(\Omega), \quad k \ge m; \tag{3.86$'$}$$

so $\rho_{(m)}$ in (3.86) is surjective. Moreover, $\mathcal{K}_{(m)}$ maps $C^\infty(\Gamma)^m$ into $C^\infty(\overline{\Omega})$.

3° The kernel of $\rho_{(m)}$ in (3.86) for $k = m$ is precisely $H_0^m(\Omega)$ (defined by (1.31)).

Proof. We use the special local coordinates from Theorem 3.16 and an associated partition of unity as in (3.75) ff. Since $\rho_{(m)}$ and $\mathcal{K}_{(m)}$ consist of the first m entries in $\rho_{(k)}$ resp. $\mathcal{K}_{(k)}$, the results for general k follow fom the case $k=m$.

For $u \in C^\infty(\overline{\Omega})$ we have according to (3.60)

$$\|\gamma_j u\|_{m-j-\frac12} = \Big(\sum_{k=1}^N \|(\psi_k \gamma_j u)(\widetilde{\lambda}_{(k)}(y'))\|^2_{m-j-\frac12} \Big)^{\frac12} .$$

Now let $\psi_k' \in C_0^\infty(U_k)$ be such that it is 1 on a neighborhood of $\operatorname{supp}\psi_k$, and is constant in the normal direction near Γ (in the local coordinates (y', y_n) it should be constant in y_n for small y_n), then we can write

$$\psi_k \gamma_j u = \psi_k \gamma_j \psi_k' u,$$

and use the estimate shown in Theorem 3.5 (cf. also Theorem 3.10),

$$\|(\psi_k \gamma_j \psi_k' u)(\widetilde{\lambda}_{(k)}(y'))\|_{m-j-\frac12} \le C\|(\psi_k' u)(\widetilde{\mu}_{(k)}(y))\|_m \le C'\|\psi_k' u\|_m$$

which gives 1° by summation over k.

The right inverse $\mathcal{K}_{(m)}$ is constructed by use of Corollary 3.8. For a given $v \in \prod_{j=0}^{m-1} H^{m-j-\frac12}(\Gamma)$ we set

$$\mathcal{K}_{(m)} v = \sum_{k=1}^N \psi_k' \mathcal{K}_{(m),k} \psi_k v \tag{3.87}$$

(with ψ_k' and ψ_k as above); here $\psi_k' \mathcal{K}_{(m),k} \psi_k v$ is defined by applying $\mathcal{K}^+_{(m)}$ from Corollary 3.8 to $\psi_k v$ (carried over to local coordinates), multiplying by ψ_k' (in the local coordinate system) and afterwards carrying back to U_k. (Of course this could be explained by a formula, as in the following. A difficulty is that $\mathcal{K}_{(m)}$ tends to spread out the support, but this is handled by the cut-off functions.) In view of the mentioned properties of ψ_k', we now have, in the local coordinates,

$$(\rho_{(m)} \psi_k' \mathcal{K}_{(m),k} \psi_k v) \circ \widetilde{\mu}_{(k)} = \underline{\psi}_k' \rho^+_{(m)} \mathcal{K}^+_{(m)} \underline{\psi}_k \underline{v} = \underline{\psi}_k' \underline{\psi}_k \underline{v} = \underline{\psi_k v}$$

(where $\underline{\psi}_k'$ is evaluated for $y_n = 0$). Carrying this over to U_k and summing over k, we obtain 2°.

Also 3° is shown by use of the special local coordinates and a partition of unity. Here we observe that when $\gamma_0 u = \dots = \gamma_{m-1} u = 0$, then also $\gamma_0 D_x^\alpha u = 0$ for all $|\alpha| \le m-1$, which is seen immediately from the representation in local coordinates. We moreover find that $\rho_{(m)} \psi_k u = 0$, by use of the

Leibniz formula. Now we get an approximating $C_0^\infty(\Omega)$ function by adding the approximating $C_0^\infty(\Omega \cap U_k)$ functions we can construct for each U_k by use of Theorem 3.9 in local coordinates for $k = 1, \dots, N$ (and by use of step (iii) in the proof of Theorem 3.2 for the piece $\psi_0 u$ supported in a compact subset of Ω). $\square$

With reference to the above analysis we can finally give the following more concrete interpretation of the variational construction in Theorem 2.7 for a strongly elliptic operator A. Here we denote the first m traces in relation to the order $2m$ by γ,

$$\rho_{(m)} u = \gamma u. \tag{3.88}$$

Theorem 3.19. *Let Ω be an open, smooth bounded subset of $\mathbb{R}^n$, and let A and $s(u, v)$ be given as in Theorem 2.7, with coefficients in $C^\infty(\overline{\Omega})$. The operator A_γ that acts like A and has the domain*

$$\begin{aligned} D(A_\gamma) &= D(A_{\max}) \cap H_0^m(\Omega) \\ &= \{\, u \in D(A_{\max}) \cap H^m(\Omega) \mid \gamma_0 u = \dots = \gamma_{m-1} u = 0 \,\} \end{aligned} \tag{3.89}$$

has its spectrum and numerical range in a sector W (as in (1.42)). In particular if $-\mu \in \mathbb{C} \setminus W$, the Dirichlet problem

$$\begin{aligned} Au + \mu u &= f \text{ in } \Omega, \\ \gamma u &= 0 \text{ on } \Gamma, \end{aligned} \tag{3.90}$$

has for every $f \in L_2(\Omega)$ one and only one solution in $H^m(\Omega) \cap D(A_{\max})$.

The statement on the Laplace operator in Theorem 1.6 can also be made more concrete in this way.

Moreover, the operator $\mathcal{K}_{(m)}$ allows us to treat also the following nonhomogeneous Dirichlet problem

$$\begin{aligned} Au + \mu u &= f \text{ in } \Omega, \\ \gamma u &= \varphi \text{ on } \Gamma, \end{aligned} \tag{3.91}$$

where f is given in $L_2(\Omega)$, φ is given in $\prod_{j=0}^{m-1} H^{2m-j-\frac{1}{2}}(\Gamma)$, and u is sought in $D(A_{\max}) \cap H^m(\Omega)$. For, let u be such a solution, and set

$$v = u - \mathcal{K}_{(m)} \varphi. \tag{3.92}$$

Since $\mathcal{K}_{(m)} \varphi \in H^{2m}(\Omega)$ by (3.29′), we see that $v \in D(A_{\max}) \cap H^m(\Omega)$; and since $\gamma v = \gamma u - \varphi = 0$, v in fact belongs to $D(A_{\max}) \cap H_0^m(\Omega)$. Then v solves

$$\begin{aligned} Av + \mu v &= f - (A + \mu) \mathcal{K}_{(m)} \varphi \text{ in } \Omega, \\ \gamma v &= 0 \text{ on } \Gamma. \end{aligned} \tag{3.93}$$

Conversely, if v is a solution in $D(A_{\max}) \cap H_0^m(\Omega)$ of (3.93), then $u = v + \mathcal{K}_{(m)}\varphi$ satisfies (3.91) and belongs to $D(A_{\max}) \cap H^m(\Omega)$. When $-\mu \in \mathbb{C} \setminus W$ as in Theorem 3.19, there is existence and uniqueness of the solution of (3.93), hence also of (3.91). Here u can be described explicitly as

$$u = (A_\gamma + \mu)^{-1}(f - (A + \mu)\mathcal{K}_{(m)}\varphi) + \mathcal{K}_{(m)}\varphi. \tag{3.94}$$

We have shown:

Corollary 3.20. *Assumptions as in Theorem* 3.19. *The problem* (3.91)), where f is given in $L_2(\Omega)$, φ is given in $\prod_{j=0}^{m-1} H^{2m-j-\frac{1}{2}}(\Gamma)$, and u is sought in $D(A_{\max}) \cap H^m(\Omega)$, has the unique solution (3.94), when $-\mu \in \mathbb{C} \setminus W$.

The assumption that φ should lie in $\prod_{j=0}^{m-1} H^{2m-j-\frac{1}{2}}(\Gamma)$ (and not just in $\prod_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\Gamma)$) was made in order to assure that $\mathcal{K}_{(m)}\varphi \in D(A_{\max})$. It may seem restrictive. But in fact we shall show later that $D(A_\gamma)$ is contained in $H^{2m}(\Omega)$ and not just in $H^m(\Omega)$. Then it is natural to take the boundary values in the range space of γ on $H^{2m}(\Omega)$.