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Matematisk Afdeling
Københavns Universitet
Gerd Grubb

2. Fourier expansions in higher dimensions

2.1 Multiple Fourier series.

The theory of Fourier expansions extends readily to higher dimensions. Here the complex formulation is advantageous, because it gives simpler formulas (allowing a better overview than when multiple products of cosines and sines occur everywhere).

Before presenting this, let us underline the fact that is put forward in [A04, Section 2.5], that any $2p$-periodic function f that is *square integrable* on the interval $[-p, p]$ can be expanded in a Fourier series, with coefficients determined by the Euler formulas on page 39. Moreover, the Bessel inequality and Parseval identity hold for f. Special cases are piecewise continuous functions, or just bounded (measurable) functions. It can be seen directly from the Euler formulas that the Fourier coefficients are bounded in n, but the Bessel inequality gives a still better information, namely that $a_n \to 0$ and $b_n \to 0$ for $n \to \infty$.

Now recall the complex formulation in one variable: It is based on the Euler identity, for $x \in \mathbb{R}$:

$$
\begin{aligned}
e^{ix} &= \cos x + i \sin x, \quad \text{hence} \\
\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), &\quad \sin x = \tfrac{1}{2i}(e^{ix} - e^{-ix}).
\end{aligned}
\tag{2.1}
$$

In the Fourier series of a $2p$-periodic function $f(x)$,

$$
f(x) = a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(\tfrac{n\pi}{p} x) + b_n \sin(\tfrac{n\pi}{p} x) \big),
\tag{2.2}
$$

we can insert the replacements

$$
\begin{aligned}
\cos(\tfrac{n\pi}{p} x) &= \tfrac{1}{2}(e^{i\frac{n\pi}{p}x} + e^{-i\frac{n\pi}{p}x}), \\
\sin(\tfrac{n\pi}{p} x) &= \tfrac{1}{2i}(e^{i\frac{n\pi}{p}x} - e^{-i\frac{n\pi}{p}x});
\end{aligned}
\tag{2.3}
$$

then

$$
s_N(x) = a_0 + \sum_{n=1}^{N} \big(a_n \cos(\tfrac{n\pi}{p} x) + b_n \sin(\tfrac{n\pi}{p} x) \big) = \sum_{m=-N}^{N} c_m e^{i\frac{m\pi}{p}x},
\tag{2.4}
$$

with

$$
c_0 = a_0, \quad c_n = \tfrac{1}{2}(a_n - i b_n), \quad c_{-n} = \tfrac{1}{2}(a_n + i b_n).
\tag{2.5}
$$

This justifies writing (2.2) as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{p}x}. \tag{2.6}$$

One has that

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-i\frac{n\pi}{p}x}\,dx \text{ for all } n \in \mathbb{Z}, \tag{2.7}$$

which holds also when complex-valued functions $f(x)$ are allowed (still with $x \in \mathbb{R}$). The Parseval identity is:

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2p}\int_{-p}^{p} |f(x)|^2\,dx. \tag{2.8}$$

Theorem 1.2 says in the complex formulation that when f is PC1C with period $2p$, then:

$$\begin{aligned} &\text{(i)} \quad c_n(f') = i\frac{n\pi}{p}c_n(f) \text{ for all } n \in \mathbb{Z},\\ &\text{(ii)} \quad \sum_{n=-\infty}^{\infty} |c_n| < \infty, \end{aligned} \tag{2.9}$$

and (iii) the Fourier series converges uniformly (and absolutely) to f.

It is not hard to extend the ideas to higher dimensions. For simplicity in the formulas we now let $p = \pi$ and leave to the reader to do the scaling when other lengths are needed.

On $\mathbb{R}^k$ with points denoted $x = (x_1, \dots, x_k)$ we consider functions $f(x)$ that have period 2π in each variable $x_1, \dots, x_k$. They are completely determined by their values on the cube $[-\pi, \pi]^k$. The elements of $\mathbb{Z}^k$ will be denoted $\mathbf{n} = (n_1, \dots, n_k)$, with length

$$\|\mathbf{n}\| = \sqrt{n_1^2 + \cdots + n_k^2}\,. \tag{2.10}$$

The functions

$$e^{i\mathbf{n}\cdot x} = e^{i(n_1x_1+\dots n_kx_k)}, \quad \mathbf{n} \in \mathbb{Z}^k, \tag{2.11}$$

are 2π-periodic in each variable x_j and satisfy

$$(e^{i\mathbf{n}\cdot x}, e^{i\mathbf{m}\cdot x}) = \begin{cases} 0 \text{ if } \mathbf{n} \neq \mathbf{m}, \\ (2\pi)^k \text{ if } \mathbf{n} = \mathbf{m}, \end{cases} \tag{2.12}$$

when we use the scalar product (inner product)

$$\begin{aligned} (f,g) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(x_1, x_2, \dots, x_k)\overline{g}(x_1, x_2, \dots, x_k)\,dx_1dx_2\dots dx_k \\ &= \int_{[-\pi,\pi]^k} f(x)\overline{g}(x)\,dx. \end{aligned}$$

For,

$$(e^{i\mathbf{n}\cdot x}, e^{i\mathbf{m}\cdot x}) = \int_{-\pi}^{\pi} e^{in_1x_1}e^{-im_1x_1}\,dx_1 \cdots \int_{-\pi}^{\pi} e^{in_kx_k}e^{-im_kx_k}\,dx_k;$$

here if $n_j \neq m_j$ for some j, the integral in x_j gives a factor 0; on the other hand if $n_j = m_j$ for all j, each integral over $[-\pi, \pi]$ contributes with a factor 2π.

It is shown on the basis of the one-dimensional result that a square integrable function $f(x)$ on $]-\pi, \pi]^k$, extended to be 2π-periodic in each variable $x_1, \dots, x_k$, can be expanded in a Fourier series

$$\begin{aligned} f(x) &\sim \sum_{\mathbf{n}\in\mathbb{Z}^k} c_{\mathbf{n}} e^{i\mathbf{n}\cdot x}, \text{ where} \\ c_{\mathbf{n}} &= \frac{1}{(2\pi)^k}\int_{[-\pi,\pi]^k} f(x)e^{-i\mathbf{n}\cdot x}\,dx, \end{aligned} \tag{2.13}$$

in such a way that the partial sum

$$s_N(x) = \sum_{\max\{|n_1|,\dots|n_k|\}\le N} c_{\mathbf{n}} e^{i\mathbf{n}\cdot x} \tag{2.14}$$

converges in the mean to $f(x)$, in the sense that

$$\int_{[-\pi,\pi]^k} |f(x) - s_N(x)|^2\,dx \to 0 \text{ for } N \to \infty. \tag{2.15}$$

Here the following Parseval identity holds:

$$\sum_{\mathbf{n}\in\mathbb{Z}^k} |c_{\mathbf{n}}|^2 = \frac{1}{(2\pi)^k}\int_{[-\pi,\pi]^k} |f(x)|^2\,dx. \tag{2.16}$$

As a corollary to the Parseval identity we see that $|c_{\mathbf{n}}| \to 0$ for $\|\mathbf{n}\| \to \infty$; this holds under the mere assumption that f is square integrable on $[-\pi, \pi]^k$. We give below some information on uniform convergence.

For $k = 2$, the formulation with cosine and sine is found from the above by noting that

$$e^{i(n_1x_1+n_2x_2)} = (\cos n_1x_1 + i\sin n_1x_1)(\cos n_2x_2 + i\sin n_2x_2).$$

For n_1 and $n_2 \in \mathbb{N}$ we can use this in the four terms

$$c_{(n_1,n_2)}e^{i(n_1x_1+n_2x_2)} + c_{(n_1,-n_2)}e^{i(n_1x_1-n_2x_2)} \\ + c_{(-n_1,n_2)}e^{i(-n_1x_1+n_2x_2)} + c_{(-n_1,-n_2)}e^{i(-n_1x_1-n_2x_2)}$$

and regroup them as a linear combination of $\cos n_1x_1 \cos n_2x_2$, $\cos n_1x_1 \sin n_2x_2$, $\sin n_1x_1 \cos n_2x_2$ and $\sin n_1x_1 \sin n_2x_2$. This is somewhat unmanageable, but it becomes more manageable when we restrict the attention to functions that are *odd* in x_1 as well as x_2; they only have sine terms

$$\begin{aligned} f(x) &\sim \sum_{n_1,n_2\in\mathbb{N}} b_{n_1,n_2} \sin n_1x_1 \sin n_2x_2, \text{ with} \\ b_{n_1,n_2} &= -c_{(n_1,n_2)} + c_{(n_1,-n_2)} + c_{(n_1,-n_2)} - c_{(-n_1,-n_2)}, \end{aligned} \tag{2.17}$$

since, in the calculation of (2.13), a cosine factor $\cos n_i x_i$ integrated together with f in the x_i-variable gives 0. Here

$$b_{n_1,n_2} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi f(x_1,x_2) \sin n_1 x_1 \sin n_2 x_2 \, dx_1 dx_2.$$

Note however that when one differentiates a sine series, cosine comes in again.

There is a general result:

Theorem 2.1.

1° *If $f(x)$ is 2π-periodic in each coordinate, and C^1, then for all $\mathbf{n} \in \mathbb{Z}^k$,*

$$c_{\mathbf{n}}\Big(\frac{\partial f}{\partial x_j}\Big) = i n_j c_{\mathbf{n}}(f), \quad j = 1, \dots, k, \tag{2.18}$$

and the Parseval identity for the derivatives implies

$$\sum_{\mathbf{n} \in \mathbb{Z}^k} \|\mathbf{n}\|^2 |c_{\mathbf{n}}(f)|^2 < \infty. \tag{2.19}$$

Moreover, if f is C^l for some $l \geq 1$, then

$$\sum_{\mathbf{n} \in \mathbb{Z}^k} \|\mathbf{n}\|^{2l} |c_{\mathbf{n}}(f)|^2 < \infty. \tag{2.20}$$

2° *For $k = 2$ or 3, if $f(x)$ is 2π-periodic in each coordinate and C^{l+2}, then*

$$\sum_{\mathbf{n} \in \mathbb{Z}^k} \|\mathbf{n}\|^{l} |c_{\mathbf{n}}(f)| < \infty. \tag{2.21}$$

The estimate (2.21) *implies that the Fourier series and it termwise differentiated series up to order l are uniformly convergent.*

Indications of proof. In 1°, the identity in (2.18) is shown by integration by parts (in the x_j-variable) in the formula for $c_{\mathbf{n}}(\frac{\partial f}{\partial x_j})$. Then the Parseval identity for $\frac{\partial f}{\partial x_j}$ implies the convergence of the series $\sum_{\mathbf{n}} |n_j|^2 |c_{\mathbf{n}}(f)|^2$. When we sum over j we find (2.19). When f is C^l, this can be applied for any succession of l partial derivatives, showing that

$$\sum_{\mathbf{n} \in \mathbb{Z}^k} |p(n_1, \dots, n_k)|^2 |c_{\mathbf{n}}(f)|^2 < \infty$$

for any polynomial p of degree l. Since $\|\mathbf{n}\|^{2l}$ is bounded by a sum of squares of such polynomials, the result (2.20) follows.

For 2°, note that it is here a question of series with $|c_{\mathbf{n}}|$ in the first power only. One can show that the series $\sum_{\mathbf{n} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}} \|\mathbf{n}\|^{-4}$ is convergent for $k = 2$ and 3 (this is related to the fact that $\int_{|x|\ge 1} |x|^{-4}\,dx < \infty$ in dimensions < 4; see also Exercise E2.3). Then we can do a trick as in the proof of Theorem 1.2:

$$\sum_{\|\mathbf{n}\|\le N} \|\mathbf{n}\|^l |c_{\mathbf{n}}| = \sum_{0<\|\mathbf{n}\|\le N} \|\mathbf{n}\|^{-2}\|\mathbf{n}\|^{l+2}|c_{\mathbf{n}}|$$
$$\le \tfrac12\Big(\sum_{0<\|\mathbf{n}\|\le N} \|\mathbf{n}\|^{-4} + \sum_{0<\|\mathbf{n}\|\le N} \|\mathbf{n}\|^{2(l+2)}|c_{\mathbf{n}}|^2\Big) \le C,$$

for all N, where we for the last series use that (2.20) holds with l replaced by $l+2$.

The estimate (2.21) implies uniform convergence of the termwise differentiated series up to order l, since $|e^{i\mathbf{n}\cdot x}| = 1$, and

$$\frac{\partial}{\partial x_j} e^{i\mathbf{n}\cdot x} = i n_j e^{i\mathbf{n}\cdot x}, \tag{2.22}$$

so (2.21) (times a constant) is a majorizing series for all those termwise derived series. □

Remark 2.2.

(a) There is a result along the lines of 2° also in higher dimensions. One can show that for general k, the series $\sum_{\mathbf{n}\in\mathbb{Z}^k\setminus\{\mathbf{0}\}} \|\mathbf{n}\|^{-2k'}$ is convergent when $2k' > k$. Then when the Fourier coefficients of f satisfy the estimate (2.20) with l replaced by $l+k'$, they will satisfy (2.21), by a version of the above trick.

This is actually an example of Sobolev's Theorem — prominent in the more advanced theory — that says that functions with finite Sobolev norm of order $l+k'$, some $k' > k/2$, are in C^l (here the squareroot of (2.20) plays the role of the l'th Sobolev norm).

(b) In part 1° of Theorem 2.1, the formulas (2.18)–(2.19) can be shown under slightly weaker assumptions. It suffices that f is continuous with square integrable first derivatives defined in some reasonable sense. For example, if $[-\pi,\pi]^k$ is divided into a finite number of polyedric subdomains, and the first derivatives of f are defined in each of these subdomains and extend to continuous functions on their closures, then (2.18) and (2.19) hold. We can call such derivatives *piecewise continuous* (although the notion could be defined also in more general situations). Similarly, if f is C^{l-1} and the l'th order derivatives are piecewise continuous, then (2.20) holds.

In Theorem 2.1 2°, it is then sufficient for (2.21) that f is C^{l+1} with piecewise continuous derivatives of order $l+2$.

2.2 The wave equation with initial data on a rectangle.

Using Theorem 2.1, we can justify the solution formula for the two-dimensional wave equation in [A04, Section 3.7] as follows (using the notation (x,y) for a point in $\mathbb{R}^2$):

Theorem 2.3. 1° *When $f(x,y)$ is C^2 and $g(x,y)$ is C^1 on $M = [0,a]\times[0,b]$, and f and g are zero on the boundary $\partial M = \{\,(x,y)\in M \mid x = 0$ or a, $y = 0$ or $b\,\}$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(|B_{mn}| + |B^*_{mn}|) < \infty. \tag{2.23}$$

Then the series in (4) converges uniformly on $M\times[0,\infty[$ to a continuous function $u(x,y,t)$ satisfying the boundary condition $u = 0$ for $(x,y)\in\partial M$ and the first initial condition $u = f$ for $t = 0$.

2° *When furthermore $f(x,y)$ is C^4 and $g(x,y)$ is C^3 on M, and*

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 g}{\partial x^2} \text{ and } \frac{\partial^2 g}{\partial y^2} \text{ are } 0 \text{ on } \partial M, \tag{2.24}$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2+n^2)(|B_{mn}|+|B^*_{mn}|) < \infty, \tag{2.25}$$

and $u(x,y,t)$ is C^2 on $M \times [0,\infty[$ and satisfies the wave equation and the initial- and boundary conditions.

Proof. When f and g satisfy the hypotheses in 1°, they extend to functions f^* resp. g^* on $\mathbb{R}^2$ that are odd in x with period $2a$, and odd in y with period $2b$, such that the extended functions are in C^1 on $\mathbb{R}^2$ and the second derivatives of f^* are piecewise continuous. The B_{mn} are the coefficients in the sine expansion of f in two variables as in (2.17), so by Theorem 2.1 2° with $l = 0$ and Remark 2.2(b), the series $\sum_{m,n} |B_{mn}|$ is convergent. The B^*_{mn} satisfy

$$B^*_{mn} = \frac{b^*_{mn}}{\lambda_{mn}}, \tag{2.26}$$

where the b^*_{mn} are the coefficients in the sine expansion of g in two variables. By 1° of Theorem 2.1,

$$\sum_{m,n\in\mathbb{N}} (m^2+n^2)|b^*_{mn}|^2 < \infty. \tag{2.27}$$

Since $\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$ clearly satisfies

$$c_1(m^2+n^2)^{\frac{1}{2}} \le \lambda_{mn} \le c_2(m^2+n^2)^{\frac{1}{2}} \tag{2.28}$$

with positive constants c_1 and c_2, we conclude from (2.26) and (2.27) that the series of $|B^*_{mn}|^2$ satisfies

$$\sum_{m,n\in\mathbb{N}} (m^2+n^2)^2|B^*_{mn}|^2 < \infty. \tag{2.29}$$

From this we deduce as in the proof of Theorem 2.1 2° that

$$\sum_{m,n\in\mathbb{N}} |B^*_{mn}| < \infty. \tag{2.30}$$

This completes the proof of (2.23), which implies uniform convergence as stated.

When furthermore the hypotheses of 2° are satisfied, the extensions f^* and g^* are C^3 on $\mathbb{R}^2$ and the fourth-order derivatives of f^* are piecewise continuous. It follows from Theorem 2.1 2° and Remark 2.2(b) that

$$\sum_{m,n}(m^2+n^2)|B_{mn}|<\infty, \quad \sum_{m,n}(m^2+n^2)^{\frac{1}{2}}|b^*_{mn}|<\infty. \tag{2.30}$$

Using again (2.26) and (2.28) we conclude that (2.25) holds. Then the termwise differentiated series up to order 2 converge uniformly on $M\times[0,\infty[$, since each differentiation in x or in y essentially gives a factor n or m, and each differentiation in t gives a factor λ_{mn}. Thus the differential equation and the remaining initial condition can be verified. □

The conditions on f and g in the theorem are *sufficient conditions* — one can weaken them a little and still get solvability — but at least they give some firm ground for the claim that the described procedure gives a solution of the problem posed. (For example, in Theorem 2.3 2°, one can allow the fourth-order derivatives of f and the third-order derivatives of g to be piecewise continuous on M.)

In Example 3.7.1, $h(x)=x(1-x)$ extends to an odd, 2-periodic function h^* whose first derivative is a continuous triangular function, and the second derivative is piecewise constant, having jumps at the period points $2n$, $n\in\mathbb{Z}$. Then the odd, 2-periodic extension of $f(x,y)$ is C^1 with piecewise continuous (in fact piecewise constant) second derivatives. Here Theorem 2.3 1° gives that the series for $u(x,y,t)$ converges uniformly (which is also clear from the formulas), but (2.24) is not satisfied, and the differential equation holds only in a generalized sense. (In a deeper analysis one can show that the solution is smooth in large areas, but that the irregularities in the initial value propagate along characteristic cones when when t increases.)

2.3 The heat equation with initial data on a rectangle.

The solution formulas for the heat problem [A04, page 161] can be checked in a similar way. Here we find, as for the one-dimensional heat equation, that the solution becomes C^∞ as soon as t becomes positive.

Theorem 2.4. 1° *When $f(x,y)$ is C^2 on $M=[0,a]\times[0,b]$, and is zero on the boundary ∂M, then*

$$\sum_{m,n\in\mathbb{N}}|A_{mn}|<\infty, \tag{2.31}$$

and the series in (13) *converges uniformly on $M\times[0,\infty[$ to a continuous function $u(x,y,t)$ satisfying the boundary condition $u=0$ for $(x,y)\in\partial M$ and the initial condition $u=f$ for $t=0$.*

2° *Assume merely that $f(x,y)$ is square integrable on M. Then the series in* (13) *and all the termwise differentiated series of arbitrarily high order converge uniformly on $M\times[\varepsilon,\infty[$, for any $\varepsilon>0$. In particular, the differential equation* (11) *and the boundary condition* (12) *are verified for $t>0$.*

Proof. Part 1° is shown in the same way as in Theorem 2.3; the expansion coefficients of f are now called A_{mn}, and the hypotheses assure that the odd periodic extension of f is C^1

with piecewise continuous second derivatives, so that (2.31) holds and defines a majorizing series for (13).

For part 2°, we observe that when $t \geq \varepsilon$, then

$$e^{-\lambda_{mn}^2 t} \leq e^{-\lambda_{mn}^2 \varepsilon} \leq e^{-c_1^2 \varepsilon (m^2+n^2)}, \tag{2.32}$$

cf. (2.28). For any $j, k, l \geq 0$, application of $\frac{\partial^j}{\partial x^j} \frac{\partial^k}{\partial y^k} \frac{\partial^l}{\partial t^l}$ termwise gives a series

$$\pm \sum_{m,n \in \mathbb{N}} A_{mn} \big(\frac{m\pi}{a}\big)^j \big(\frac{n\pi}{b}\big)^k \lambda_{mn}^{2l} \begin{matrix}\sin\\ \cos\end{matrix} \big(\frac{m\pi}{a} x\big) \begin{matrix}\sin\\ \cos\end{matrix} \big(\frac{n\pi}{b} y\big) e^{-\lambda_{mn}^2 t}. \tag{2.33}$$

Since the $|A_{mn}|$ are bounded by a constant (cf. (2.16)ff.), this is majorized by a convergent series

$$\sum_{m,n \in \mathbb{N}} c' (m^2+n^2)^{\frac{1}{2}j+\frac{1}{2}k+l} e^{-c_1^2 \varepsilon (m^2+n^2)} < \infty, \tag{2.34}$$

where the convergence follows e.g. since $(m^2+n^2)^{\frac{1}{2}j+\frac{1}{2}k+l} e^{-c_1^2 \varepsilon (m^2+n^2)} \leq c'' (m^2+n^2)^{-2}$. Thus u is C^∞ for $t > 0$. $\square$

When f is merely square integrable, one can say that the initial condition is verified in the sense that the series (13) for $t = 0$ converges in the mean to f on M.

2.4 The Poisson equation with zero boundary data.

For the solution of the Poisson equation on a rectangle put forward in [A04, Section 3.9], sufficient conditions for convergence can likewise be found from Theorem 2.1 ff. The discussion goes rather similarly to that in Theorems 2.3–4:

When the odd periodic extension f^* of f is C^1 with piecewise continuous second derivatives, the sinus coefficients A_{mn} of f satisfy (2.31). Then since $E_{mn} = -A_{mn}/\lambda_{mn}$, one has that

$$\sum_{m,n \in \mathbb{N}} (m^2+n^2) |E_{mn}| < \infty, \tag{2.35}$$

so the differential equation can be verified by termwise differentiation, and $u(x, y)$ is indeed a solution of the problem; it lies in $C^2(M)$.

The solution found in Example 3.9.1 is a generalized solution, and the calculations in Example 3.9.2 are quite formal (the series for u does not satisfy our criteria for termwise differentiation of order 2).

In more advanced treatments of the various differential equations, the theory of Sobolev spaces provides a more satisfactory framework than the spaces of C^l-functions.

2.5 Convergence analysis for the Laplace equation on a disk.

Consider the series solution established in [A04, Section 4.4] for the Laplace equation on a disk with radius a, with a prescribed boundary value f. We shall show that when f is merely square integrable, the series and all termwise differentiated series converge uniformly on the disks with the same center and radius $< a$. For simplicity in the formulation, let us take $a = 1$ and leave the scaling to the general case to the reader.

When $f(\theta)$ is square integrable on $[0, 2\pi]$, its Fourier coefficients are bounded:

$$|a_n| \le C,\ |b_n| \le C,\ \text{for all } n. \tag{2.36}$$

Let $\varepsilon \in]0,1[$. Then the series for u,

$$u(r,\theta) = a_0 + \sum_{n\in\mathbb{N}} r^n(a_n \cos n\theta + b_n \sin n\theta) \tag{2.37}$$

has the majorizing series, when $r \le 1-\varepsilon$:

$$\sum_{n\ge 0} C(1-\varepsilon)^n. \tag{2.38}$$

We can check termwise derivatives in r and θ, showing that each differentiation essentially gives a factor n, so that the series after k differentiations is majorized by a series

$$\sum_{n\ge 0} C'(1+n)^k(1-\varepsilon)^n. \tag{2.39}$$

Why is this convergent? Apply for example the quotient criterion, or note that $(1-\varepsilon)^n = e^{-sn}$, where $-s = \ln(1-\varepsilon) < 0$; here since the exponential function wins over any polynomial, $(1+n)^k e^{-sn} \le C''(1+n)^{-2}$ (as we have used before).

However, if we treat the (r,θ)-derivatives of u, we still have to worry about how the information carries over to the (x,y)-derivatives (in the original coordinates), especially how things fit together at $r=0$. But there is another point of view that gives the (x,y)-behavior directly:

When the Fourier series of f is written in the complex form

$$f(\theta) = \sum_{m=-\infty}^{\infty} c_m e^{im\theta}, \tag{2.40}$$

we get the series for u in the complex form, that we can reformulate further:

$$\begin{aligned} u(r,\theta) &= \sum_{m\in\mathbb{Z}} r^{|m|} c_m e^{im\theta} \\ &= \sum_{m\ge 0} c_m (re^{i\theta})^m + \sum_{m'>0} c_{-m'}(re^{-i\theta})^{m'} \\ &= \sum_{m\ge 0} c_m (x+iy)^m + \sum_{m'>0} c_{-m'}(x-iy)^{m'}, \end{aligned} \tag{2.41}$$

with bounded coefficients. Here, when (x,y) lies in the disk with radius $1-\varepsilon$, $x+iy$ and $x-iy$ both have absolute value $\le 1-\varepsilon$. Termwise differentiations in x and y give polynomials in m resp. m' as factors. Then we can again use series of the type in (2.39) as majorizing series, and find that all termwise derived series are uniformly convergent on the smaller disk. Thus u is C^∞ there and satisfies $\Delta u = 0$.

E2. Exercises

Exercise E2.1. Consider the cosine-sine formulation of the Fourier expansion of a function $f(x_1, x_2)$ that is 2π-periodic in each variable. You are asked to express the coefficients of the functions $\cos n_1 x_1 \cos n_2 x_2$ and $\cos n_1 x_1 \sin n_2 x_2$ in terms of the coefficients $c_{\mathbf{n}}$ in the series in (2.13). What is the constant term?

Exercise E2.2. Answer Exercise 3.8.12 in [A04], with the additional point:
(d) Show that the differential equation is verified for $z < c$, when $f(x, y)$ is square integrable.

Exercise E2.3. In the following, we identify $\mathbb{Z}^2$ with a subset of $\mathbb{R}^2$, namely with the points with integer coordinates (n_1, n_2).

(a) For each $l \in \mathbb{N}$, show that there are $8l$ of these integer points (n_1, n_2) on the boundary of the square $[-l, l] \times [-l, l]$, and that they satisfy

$$n_1^2 + n_2^2 \geq l^2.$$

(b) Show that there is a constant c such that

$$\sum_{|n_1| \leq l, |n_2| \leq l, (n_1, n_2) \neq (0,0)} \frac{1}{(n_1^2 + n_2^2)^2} \leq \sum_{1 \leq j \leq l} \frac{c}{l^3}$$

(c) Show that the series

$$\sum_{(n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(n_1^2 + n_2^2)^2}$$

is convergent.